PORTFOLIO

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(3.2) 11

Theorem 1. *For any integer a, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$*

Let a be an integer, and $a^2 - 1$ be even. So, there exists an integer i such that:

$$a^2 - 1 = 2i.$$

To help us in our proof, we introduce and prove Lemma 2 and Lemma 3.

Lemma 2. *if $a^2 - 1$ is an even integer, then a^2 is an odd integer.*

Proof. let $a^2 - 1$ be an even integer, so there exists an integer i such that:

$$a^2 - 1 = 2i.$$

Then, we add 1 the equation above getting:

$$a^2 = 2i + 1.$$

So, a^2 is an odd integer, because i is an integer. □

Lemma 3. *if a^2 is an odd integer, then a is an odd integer.*

Proof. We will show the contrapositive. If a is an even integer, then a^2 is an even integer. Let a be an even integer. So there exists an integer m such that:

$$a = 2m$$

. Then we square the equation above and factor 2 to find:

$$\begin{aligned} a^2 &= 4m^2 \\ &= 2(2m^2). \end{aligned}$$

Since m is an integer and integers are closed under multiplication, a^2 is an even integer. So, we have shown that if a^2 is an odd integer then a is an odd integer. □

Now we will continue with the proof of Theorem 1.

Proof. So, we have shown that a is an odd integer. So, there exists an integer j such that $a = 2j + 1$. We continue by substituting $2j + 1$ for a in $a^2 - 1$.

So,

$$\begin{aligned} a^2 - 1 &= (2j + 1)^2 - 1 \\ &= 4j^2 + 4j \\ &= 4(j^2 + j). \end{aligned}$$

By the closure properties of integers, $j^2 + j$ is an integer, so 4 divides $a^2 - 1$.

□

(3.4) 12b

Theorem 4. *Let x and y be real numbers, then:*

$$|x - y| \geq |x| - |y|$$

.

Proof. Let x and y be real numbers. We begin by substituting 0 in the form of $(-y) + y$ to get:

$$|x| - |y| = |x - y + y| - |y|.$$

Then, we add $|y|$ to the equation above:

$$|x| = |x - y + y|.$$

Then, we use the triangle inequality found in Theorem 3.25 on page 137 from our textbook.

Theorem 5 (The Triangle Inequality)**.** *Let a and b be real numbers, then*

$$|a + b| \leq |a| + |b|.$$

We apply the triangle inequality to $|(x - y) + y|$ by substituting $(x - y)$ for a and y for b to get:

$$|(x - y) + y| \leq |x - y| + |y|.$$

Then we substitute x for $(x - y) + y$ to get:

$$|x| \leq |x - y| + |y|.$$

Finally, we subtract $|y|$ to find:

$$|x| - |y| \leq |x - y|.$$

□

(4.1) 8b

Theorem 6. *For each natural number n, 6 divides $(n^3 - n)$.*

To help use in our proof, we introduce and prove Lemma 7.

Lemma 7. *For each natural number n, 6 divides $3n^2 + 3n$.*

Proof. We will use mathematical induction. To show the base case, let $n = 1$. So:

$$3n^2 + 3n = 6$$

The integer 1 exists so that $6 = (1)(6)$. so, the base case holds.

We continue by showing the inductive step. We want to show that if 6 divides $3k^2 + 3k$, then 6 divides $3(k+1)^2 + 3(k+1)$.

Let 6 divide $3k^2 + 3k$, so there exists an integer z such that:

$$3k^2 + 3k = 6z.$$

We want to show that there exists an integer a such that

$$3(k+1)^2 + 3(k+1) = 6a.$$

We begin by simplifying the left side of the equation above:

$$\begin{aligned} 3(k+1)^2 + 3(k+1) &= 3(k^2 + 2k + 1) + 3k + 3 \\ &= 3(k^2 + k) + 3k + 3 + 3k + 3 \\ &= 3(k^2 + k) + 6k + 6. \end{aligned}$$

Since we assume that 6 divides $3(k^2 + k)$, we can substitute $6z$ so that:

$$3(k+1)^2 + 3(k+1) = 6z + 6k + 6.$$

Then, we factor out 6 so:

$$3(k+1)^2 + 3(k+1) = 6(z + k + 1).$$

By the closure properties of integers, $(z + k + 1)$ is an integer, so 6 divides $3(k+1)^2 + 3(k+1)$.

So we have shown by mathematical induction, that for every natural number n, 6 divides $3n^3 + 3n$. □

Next, we will continue with the proof of Theorem 6.

Proof. We will use mathematical induction to prove Theorem 6. To show the base case, let $n = 1$. Then:

$$\begin{aligned}(n^3 - n) &= (1 - 1)\\ &= 0.\end{aligned}$$

The integer 0 exists so that $(6)(0) = 0$. Therefore, 6 divides 6 and the base case holds.

We continue by showing the inductive step: if 6 divides $k^3 - k$, then 6 divides $(k+1)^3 - (k+1)$. Let 6 divide $k^3 - k$, so there exists an integer i such that:

$$6i = k^3 - k.$$

We want to show that there exists an integer z such that:

$$6z = (k+1)^3 - k - 1.$$

We begin by expanding and simplifying the right side of the equation above:

$$\begin{aligned}(k+1)^3 - k - 1 &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= k^3 - k + 3k^2 + 3k.\end{aligned}$$

By our inductive hypothesis, we know that 6 divides $k^3 - k$ so we know that the integer q exists so that:

$$k^3 - k = 6q.$$

We continue by substituting $6q$ for $k^3 - k$ so:

$$(k+1)^3 - k - 1 = 6q + 3k^2 + 3k.$$

Here, we use lemma 2 to say that there exists an integer p such that:

$$6p = 3k^2 + 3k.$$

Then, we substitute $6p$ for $3k^2 + 3k$ to get:

$$\begin{aligned}(k+1)^3 - k - 1 &= 6q + 6p\\ &= 6(q + p).\end{aligned}$$

By the closure properties of integers, $(q + p)$ is an integer, so 6 divides $(k+1)^3 - k - 1$.

So we have shown by mathematical induction, that for every natural number n, 6 divides $(n^3 - n)$.

□

(4.2) 6

Let $y = ln(x)$. Then, $\frac{dy}{dx} = \frac{1}{x}$, $\frac{d^2y}{dx^2} = \frac{-1}{x^2}$, $\frac{d^3y}{dx^3} = \frac{1}{x^3}$, and $\frac{d^4y}{dx^4} = \frac{-1}{x^4}$.

Based on these examples we make the following conjecture:

Conjecture 8. *The nth derivative of $y = lnx$ follows the formula:*

$$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Proof. We will use mathematical induction.

First, we will show the base case. Let $n = 1$, so our proposed formula produces:

$$\begin{aligned} \frac{(-1)^0(0)!}{x^1} &= \frac{(1)(1)}{x} \\ &= \frac{1}{x}. \end{aligned}$$

Which equals the first derivative of $y = lnx$, so the base case holds.

Next, we will show the inductive step. We want to show that if $\frac{d^ky}{dx^k} = \frac{(-1)^{k-1}(k-1)!}{x^k}$, then $\frac{d^{k+1}y}{dx^{k+1}} = \frac{(-1)^{k+1-1}(k+1-1)!}{x^{k+1}}$.

Let $\frac{d^ky}{dx^k} = \frac{(-1)^{k-1}(k-1)!}{x^k}$. We are showing that $\frac{d^{k+1}y}{dx^{k+1}} = \frac{(-1)^k(k)!}{x^{k+1}}$.

Next, we will rewrite the hypothesis:

$$\frac{(-1)^{k-1}(k-1)!}{x^k} = (-1)^{k-1}(k-1)!x^{-k}$$

.

Then, we differentiate $\frac{d^ky}{dx^k}$ to find that:

$$\frac{d^{k+1}y}{dx^{k+1}} = -k(-1)^{k-1}(k-1)!x^{-k-1}.$$

Next, we simplify:

$$\begin{aligned} -k(-1)^{k-1}(k-1)!x^{-k-1} &= (-1)(-1)^{k-1}(k)(k-1)!x^{(-1)(k+1)} \\ &= \frac{(-1)^kk!}{x^{k+1}}. \end{aligned}$$

So, we have shown that the nth derivative of $y = lnx$ is $\frac{(-1)^{n-1}(n-1)!}{x^n}$. □

(5.3) 10

Let A and B be subsets of some universal set U.

Theorem 9. *$A - B$ and $A \cap B$ are disjoint sets.*

Proof. We will show that $(A - B) \cap (A \cap B) = \emptyset$

By Theorem 5.20 from page 248 in our book, $A - B = A \cap B^c$. Let x be an arbitrary element of $A - B$, so $x \in A$ and $x \in B^c$.

By the definition of relative complements, if $x \in B^c$, then $x \notin B$. So, $x \in A$ and $x \notin B$.

By the definition of intersections, $x \notin (A \cap B)$. Therefore, $(A - B) \cap (A \cap B) = \emptyset$ and $A - B$ and $A \cap B$ are disjoint sets. □

Theorem 10. $A = (A - B) \cup (A \cap B)$

Proof. Again, we will use Theorem 5.20 from our book to say:

$$(A - B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B).$$

Then, we use the distributive property:

$$(A \cap B^c) \cup (A \cap B) = A \cap (B^c \cup B).$$

Notice that in parenthesis, we have $(B^c \cup B)$. For any $x \in U$, either $x \in B$ or $x \notin B$. So, $x \in B$ or $x \in B^c$. Therefore, every element of U is in $(B^c \cup B)$. U is the universal set, so there are no elements outside of U, so:

$$(B^c \cup B) = U.$$

Now, we have:

$$\begin{aligned}(A - B) \cup (A \cap B) &= A \cap (B^c \cup B).\\ &= A \cap U\end{aligned}$$

It is a property of the universal set that for any set $X \subseteq U$, $X \cap U = X$. We apply this property to find that:

$$(A - B) \cup (A \cap B) = A.$$

□

(5.4) 7

Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$.

Then, $A \times B \neq B \times A$.

Explanation: We can compute $A \times B$ and $B \times A$ to show that they are unequal: $A \times B = \{(1, 2)\}$, while $B \times A = \{(2, 1)\}$.

Two ordered pairs are equal if and only if the first entries equal and the second entries equal. $1 \neq 2$ and $2 \neq 1$, so $(1, 2) \neq (2, 1)$. Therefore, $A \times B \neq B \times A$.

Using the same sets A, B, and C:

$(A \times B) \times C \neq A \times (B \times C)$.

Explanation: Similarly, we can compute the cross products to show they are unequal: $A \times B = \{(1, 2)\}$, $(A \times B) \times C = \{((1, 2), 3)\}$, while $B \times C = \{(2, 3)\}$ and $A \times (B \times C) = \{(1, (2, 3))\}$.

So, the ordered pair (1,2) is an element of $(A \times B) \times C$, but $(1, 2)$ is not an element of $A \times (B \times C)$, so $(A \times B) \times C \neq A \times (B \times C)$.

(6.3) 8a

Let f be a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, where:

$$f(m,n) = 2m + n.$$

Theorem 11. *f is sujective and not injective.*

Proof. To show that f is not injective we will present a counterexample where $(m_1, n_1) = (1, 2)$ and $(m_2, n_2) = (2, 0)$.

$$\begin{aligned} f(m_1, n_1) &= f(1, 2) \\ &= (2)(1) + (2)(1) \\ &= 4. \end{aligned}$$

$$\begin{aligned} f(m_2, n_2) &= f(2, 0) \\ &= (2)(2) + (0)(1) \\ &= 4. \end{aligned}$$

So, $(m_1, n_1) \neq (m_2, n_2)$ but $f(m_1, n_1) = f(m_2, n_2)$. Therefore, f is not injective.

Next, we will show that f is surjective. Let $z \in \mathbb{Z}$. Then, $(0, z)$ exists in $\mathbb{Z} \times \mathbb{Z}$ so that $f(0, z) = z$. So, f is surjective. $\square$

(6.5) 6c

Let f be a function from A to B and g be a function from B to A. Let I_A and I_B denote the identity functions on A and B respectively.

Theorem 12. *If $g \circ f = I_A$ and $f \circ g = I_B$, then f and g are bijections and $g = f^{-1}$.*

Proof. Let $x \in A$ so, there exists $y \in B$ such that $f(x) = y$. We assume that $g \circ f = I_A$. So:

$$\begin{aligned} g \circ f &= I_A \\ &= g(f(x)) \\ &= g(y) \\ &= x. \end{aligned}$$

So, g is surjective.

Next, let $b \in B$, so exists $a \in A$ such that: $g(b) = a$. We assume that $f \circ g = I_B$, so:

$$\begin{aligned} f \circ g &= I_B \\ &= f(g(b)) \\ &= f(a) \\ &= b. \end{aligned}$$

So, f is surjective.

Next, let $l, m \in A$ and $l = m$. We have shown that g is surjective, so there exist $p, q \in B$ such that $g(p) = l$ and $g(q) = m$. Since $f \circ g = I_B$:

$$\begin{aligned} f(g(p)) &= f(l) \\ &= p. \end{aligned}$$

Similarly:

$$\begin{aligned} f(g(q)) &= f(m) \\ &= q. \end{aligned}$$

Since $l = m$, $f(l) = f(m)$. So, $p = q$. Therefore, g is injective.

Next, let $p, q \in B$ and $p = q$. We have shown that f is surjective, so there exist $l, m \in A$ such that $f(l) = p$ and $f(m) = q$.
Since $g \circ f = I_A$,

$$\begin{aligned} g(f(l)) &= g(p) \\ &= l. \end{aligned}$$

Similarly:

$$\begin{aligned} g(f(m)) &= g(q) \\ &= m. \end{aligned}$$

Since $p = q$, $g(p) = g(q)$. So, $l = m$. Therefore, f is injective.

Finally, we will show that $g = f^{-1}$. We assume that $f \circ g = I_B$, so:

$$f(g(x)) = x.$$

We have shown that f ans g and bijections, so:

$$\begin{aligned} f^{-1}(f(g(x))) &= f^{-1}(x) \\ &= g(x). \end{aligned}$$

So, we have shown that $f^{-1}(x) = g(x)$.

$\square$

(7.2) 11

Let U be a finite, nonempty set. Let $P(U)$ be the power set of U. Let A and B be elements of $P(U)$. Let $\sim$ be a relation on $P(U)$, where $A \sim B$ if and only if $A \cap B = \emptyset$.

Theorem 13. *The relation $\sim$ is symmetric, intransitive, and irreflexive.*

Proof. The relation $\sim$ is symmetric because intersections are associative.
So, $A \cap B = \emptyset$ implies that $B \cap A = \emptyset$. Therefore, $A \sim B$ implies $B \sim A$ and $\sim$ is symmetric.

The relation $\sim$ is intransitive. We will show a counterexample. Let $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{2, 5\}$. So, $A \cap B = \emptyset$, so $A \sim B$. Similarly, $B \cap C = \emptyset$, so $B \sim C$. But, $A \cap C = \{2\}$, so A does not relate to C. Therefore, $\sim$ is intransitive.

Next, $\sim$ is irreflexive, because for any set A, $A \cap A = A$. So, whenever A is nonempty, A will not be related to itself by $\sim$.

$\square$

(9.1) 5c

Let A and B be sets.

Theorem 14. *If $A \cap B$ is a infinite set, then A is an infinite set.*

Proof. Let $A \cap B$ be an infinite set and suppose that A is a finite set.

First, we will show that $A \cap B$ is a subset of A. Let $x \in A \cap B$, then x is an element of A and B. So, every element of $A \cap B$ is an element of A. Therfore, $A \cap B$ is a subset of A.

Every subset of a finite subset is finite, so if A is finite then $A \cap B$ is finite. But, we assumed that $A \cap B$ is infinite. Therefore, we have found a contradiction proving that if $A \cap B$ is an infinite set, then A is an infinite set. $\square$

Research Proposal

Our proposed project is to analyze the sets of all possible domains and codomains of the rules of functions as a relation. Let A and B be sets and f be a function from A to B. Let A_i and B_i be arbitrary elements of the power sets of A and B respectively. Let R be a relation where $A_i \sim B_i$ if and only if the rule from f forms a function from A_i to B_i.

Background

In MATH 3000 Transition to Higher Math, I learned that functions are mappings from a domain to a codomain where every element of the domain is associated with one element of the codomain.

Functions are composed of a domain, a codomain, and a rule. Rules can form functions with many domains and codomains. For example, the identity rule can form a function from integers to integers, real number to real numbers, and integers to real numbers.

In MATH 3000, I also learned that relations can be functions, and can be injective, surjective, reflexive, symmetric, and transitive. When does the relation R fulfill these qualities of relations? I also learned about countable and uncountable sets. When are the domain and range of the relation R countable and uncountable?

Proposed Project

In our project, we want to provide conjectures and proofs on the following questions about the relation R:

Question 1: When is R reflexive, transitive, and symmetric?

Question 2: When is R injective and surjective?

Question 3: When is R a function?

Question 4: When is the domain of R finite, countably infinite, and uncountably infinite?

Question 5: When is the codomain of R finite, countably infinite, and uncountably infinite?

We hope to prove conjectures about our questions by only making assumptions about the injectivity and surjectivity of f and the cardinality of the sets A and B.

To help us generate proper conjectures we will answer our questions about specific functions when A and B are finite, countably infinite, and uncountably infinite, and when f is bijective, injective but not surjective, surjective but not injective, and neither injective nor surjective.

For example, we might start with the identity function on the set of natural numbers to help us generate a conjecture about the case where f is bijective from two countably infinite sets.

Similarly, we can answer these questions about some constant function on the set of real numbers to help us generate a conjecture about the case where f is neither injective nor surjective and the domain and codomain are uncountably infinite.

We anticipate that we will try to prove our conjectures directly and by contradiction.

Article Review

In 1950, Kennith Arrow published a paper titled *A Difficulty in the Concept of Social Welfare* which contains the proof of a theorem now known as "Arrow's Impossibility Theorem." In basic terms, Arrow's theorem says that there is no way to aggregate the preferences of individuals into a social preference that satisfies some basic notions of fairness and reasonableness. We will prepare for the formal presentation of Arrow's Impossibility Theorem and its proof in a simple case by introducing some definitions and notation.

We will start by explaining how the preferences of individuals and societies can be described, and explaining the axioms of preferences.

Let n be the natural number equal to the number of participating stake holders in a society. Let i be an arbitrary element of the natural numbers between 1 and n. Let S denote the set of states of the world that a society needs to choose between.

We will use the relation R_i to symbolize the weak preferences of agent i. For any x, y in S, xR_iy if and only if agent i weakly prefers x over y.

Similarly, we will use the relation R to symbolize the weak preferences of the society. For any x, y in S, xRy if and only if the society weakly prefers x over y.

The following axioms apply to the social preference relation R and the individual preference relations R_i.

Axiom 1: R is transitive. So, for any x, y, z in S if xRy and yRz, then xRz
Axiom 2: R is complete. So, for any x, y in S either xRy or yRx.

Next, we will describe strict preference and indifference in terms of R and R_i.

Definition 15. For the strict preference relation P, xPy is the negation of yRx.

The strict preference xPy is read, "x is strictly preferred over y." Since preference relations are transitive and complete, we can describe preferences as an ordering of S where the ordering R can be described as an ordered n-tuple. This will help us to simplify notation when we describe agents' strict preferences on multiple pairs from S.

Definition 16. For the indifference relation I, xIy means xRy and yRx.

The indifference xIy is read, "x is indifferent to y."

Definition 17 (Social Welfare Function)**.** A *social welfare function* is a function from the set of all sets of possible (complete and transitive) individual orderings to the set of all possible (complete and transitive) social orderings. So, a social welfare function maps each possible set of individual orderings, $\{R_1, ..., R_n\}$, to a single social ordering, R.

Social welfare functions are central to any area of life that demands collective decision making. One example of a social welfare function is a voting mechanism. There are many ways to take the preferences of political participants and make decisions about leadership and policy. Some examples include: majority, plurality, pairwise run offs, dictatorship, and an electoral college.

Axiom 3: Let x, y be in S and xR_iy, for every i, then if each R_i changes in a way that ranks x at least as high, for each i, then the social ordering of x and y is the same, xRy.

On Axiom 3, Arrow says:

> Since we are trying to describe social "welfare" and not some sort of "illfare," we must assume that the social welfare function is such that the social ordering responds positively to alterations in individual values or at least not negatively (Arrow 336).

Next, we will define some potential characteristics of social welfare functions.

Definition 18 (Imposition)**.** A social welfare function is *imposed* if and only if for some x, y in S, xRy regardless of the set of individual orderings

A social welfare function is imposed if and only if it is not surjective. A non-surjective social welfare function will be imposed, because there will be some R in the set of all possible (complete and transitive) social orderings of S that is never mapped to. So, for some x, y in S, xRy regardless of the individual orderings. Conversely, if a social welfare function is imposed, then there is some possible social ordering, R, that is never mapped to. Therefore, the social welfare function is not onto the set of a possible orderings of S.

Although we will continue to use the term "imposed", it should be helpful to keep this logical equivalence in mind when we prove Theorem 22.

Definition 19 (Dictatorship)**.** A social welfare function is *dictatorial* if and only if there exists an agent i where for every possible pair of states, x and y, xP_iy implies xPy regardless of the preferences of all other individuals.

A social welfare function is dictatorial if and only if it is the projection of some R_i in the set of all possible individual orderings. Again, we will continue to use the term "dictatorial," but this fact should be helpful when we prove Theorem 22.

Definition 20 (Independence of Irrelevant Alternatives)**.** Let R_1, R_2 and R'_1, R'_2 be two sets of individual preferences on S. Then, a social welfare function is *independent of irrelevant alternatives* if for all individuals i and for all states x and y, xR_iy if and only if xR'_iy then the social choice made is the same whether the individual orderings are R_1, R_2, or R'_1, R'_2.

If a social welfare function is dependent on irrelevant alternatives, then the social evaluation of some x compared to some y will depend on the individual evaluations of some z.

Now, we have the background to formally present Arrow's Impossibility Theorem and to prove a simpler case of it.

Theorem 21 (Arrow's Impossibility Theorem)**.** *If there are at least three alternatives, there does not exist a social welfare function that can be all of the following:*
i. Non-Imposing
ii. Non-Dictatorial
iii. Independent of Irrelevant Alternatives

We will show Arrow's Impossibility Theorem holds in the case where there are 3 distinct options and 2 participants.

Theorem 22. *If there are* 3 *mutually exclusive alternatives and 2 participants, then there does not exist a social welfare function that can be all of the following:*
i. Non-Imposing
ii. Non-Dictatorial
iii. Independent of Irrelevant Alternatives

Proof. Let $S = \{x, y, z\}$ and x', y', z' denote distinct arbitrary elements of S. Let agent 1 and agent 2 be the only participating stakeholders in the society.

We will show that if there are three distinct alternatives and two participants, then no social welfare function can be non-imposing, non-dictatorial, and independent of irrelevant alternatives, by contradiction.

By axiom 1 (transitivity) and axiom 2 (completeness), agent 1 and agent 2 have transitive preferences on S. Let f denote a social welfare function that is non-imposing, non-dictatorial, and independent of irrelevant alternatives.

To help us in our proof, we introduce Lemma 23, Lemma 24, and Lemma 25.

Lemma 23. *if $x'P_1y'$ and $x'P_2y'$ then $x'Py'$*

Proof. By assumption, f is non-imposed, so there exist R_1 and R_2 such that for some x', y', $x'Py'$

Let R_i' be a variation of R_i where the only difference is that x' is made to be (if necessary) the most preferred element of S for agent i. So, $x'P_1'y'$ and $x'P_2'y'$.

By axiom 3, the society should still prefer x' over y', so $x'P'y'$.

Since f is assumed to be independent of irrelevant alternatives, how x' and y' relate on P only depends on how x' and y' relate for each agent.

So, regardless of how z' relates to x' and y', the society will still prefer x' over y'. Therefore, $x'Py'$.

∎

Lemma 24. *For any x', y' in S, if $x'P_1y'$, $y'P_2x'$, and $x'Py'$, then whenever $x'P_1y'$, $x'Py'$*

Proof. Let R_1 be an ordering where $x'P_1y'$. Let R_2 be any ordering. Let R_1' be the same as R_1, and R_2' be derived from R_2 by making x' the lowest rated element of S, while maintaining the relative positions of the other elements of S. Finally, let $x'P'y'$.

So the only difference between R_1, R_2, and R_1', R_2' is that x' is rated higher by R_2 than R_2'. By axiom 3, that implies that $x'Py'$.

□

Lemma 25. *If $x'P_1y'$ and $y'P_2x'$,then $x'Iy'$*

Proof. We will show Lemma 25 by contradiction. Suppose that for some R_1 and R_2 and for some x', y', $x'P_1y'$, $y'P_2x'$, but not $x'Iy'$. So, $x'Py'$, or $y'Px'$.

We will assume that $x'Py'$ and find a contradiction. We wont show the case where $y'Px'$, because it follows the same line of reasoning (x' and y' are arbitrary).

Suppose that $x'Py'$, $x'P_1y'$, and $y'P_2x'$. By assumption, f is independent of irrelevant alternatives, so if $x'P_1y'$ and $y'P_2x'$, then $x'Py'$ (there is no way to change the relative rating of z' to change the rating of x' compared to y').

Result 1: If $x'P_1y'$ and $y'P_2x'$, then $x'Py'$.

We will show that Result 1 leads to a contradiction.

Suppose agent 1 has the ordering (x', y', z'), and agent 2 has the ordering (y, z, x). So by assumption, $y'P_2x'$. By result 1, $x'Py'$. Both agents prefer y' over z', so by Lemma 23, $y'Pz'$. P is transitive, so if $x'Py'$, and $y'Pz'$, then $x'Pz'$.

Result 2: If $x'P_1z'$ and $z'P_2x'$, then $x'Pz'$.

Suppose that agent 1 has the ordering (y, x, z), and agent 2 has the ordering (z, y, x). By Lemma 23, $x'Pz'$. By Result 2, $x'Pz'$. By Lemma 23, $y'Px'$. Therefore, by axiom 1 (transitivity), yPz.

Result 3: If $y'P_1z'$ and $z'P_2y'$, then $y'Pz'$.

Suppose that agent 1 has the ordering (y, z, x), and agent 2 has the ordering z, x, y. By Lemma 23, $z'Px'$. By Result 3, $y'Pz'$. Therefore, by axiom 1 (transitivity), $y'Px'$.

Result 4: If $y'P_1x'$ and $x'P_2y'$, then $y'Px'$.

Suppose that agent 1 has the ordering (z, y, x) and agent 2 has the ordering (x, z, y). By Lemma 23, $z'Py'$. By Result 4, $y'Px'$. Therefore, by axiom 1 (transitivity), $z'Px'$.

Result 5: If $z'P_1x'$, $x'P_2z'$, then $z'Px'$.

Suppose that agent 1 has the ordering (z, x, y) and agent 2 has the ordering (x, y, z). By Result 5, $z'Px'$. By Lemma 23, $x'Py'$. So, by axiom 1(transitivity), $z'Py'$.

Result 6: If $z'P_1y'$ and $y'P_2z'$, then $z'Py'$.

By Result 1 and Lemma 24, if $x'P_1y'$, then $x'Py'$. So, agent 1 is a dictator, but we have assumed that the social welfare function is non-dictatorial. Therefore, we have found a contradiction proving that if $x'P_1y'$ and $y'P_2x'$, then $x'Iy'$. $\square$

Now we can prove Theorem 22. Suppose that agent 1 has the ordering (x, y, z) and agent 2 has the ordering (z, x, y). By Lemma 23, $x'Py'$. By assumption, $z'P_2y'$ and $y'P_1z$. By Lemma 25, $y'Iz'$. So, $x'Pz'$. But, $x'P_1z'$ and $z'P_2x'$ which, by Lemma 25, implies $x'Iz'$.

We have found a contradiction, because z' cannot be both preferred over x' and indifferent to x'. Therefore, we have shown that there cannot exist a social welfare function that is non-dictatorial, non-imposed, and independent of irrelevant alternatives (given axioms 1,2, and 3). $\square$

References

[1] Kennith J Arrow *A Difficulty in the Concept of Social Welfare* August 1950: *The Journal of Political Economy* Vol. 58, No. 4. pp.328-346 https://web.archive.org/web/20110720090207/http://gatton.uky.edu/Faculty/hoytw/751/articles/arrow.pdf