Homework 13

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Question 1

A) $2 + 4 + 6 + 8 = 20$
B) $1 + 3 + 8 + 17 = 29$
C) $1/1 + 1/2 + 1/3 = 11/6$
D) $5 * 4 * 3 * 2 * 1 = 120$
E) $(5 * 4 * 3 * 2 * 1)/(4 * 3 * 2 * 1) = 120/24$
F) $100!/99! = 100$
G) $100!/98! = 9900$

Question 2

A) *Proof* (induction on n)
(Base step, n = 7)
$3^7 = 2187 < 5040 = 7!$

(induction step)
assume that $3^k < k!$ for any natural number $k \geq 7$
(Goal: $3^{k+1} < (k+1)!$)
$3^{k+1} = 3^k * 3$
$< k! * 3$
Because $k > 6$, we know that $k + 1 > 3$
$k! * 3 < k! * (k + 1)$
$= (k + 1)!$
Therefore, $3^{k+1} < (k + 1)!$ $\square$

B) *Proof* (induction on n)
(Base step, n = 2)
$3^2 = 9 > 4 = 2^2$

(induction step)
assume that $3^k > k^2$ for any $k \geq 2$
(Goal: $3^{k+1} > (k + 1)^2$)
$3^{k+1} = 3^k * 3^1$
$= 3^k * 3$

$> k^2 * 3$
$= k^2 + k^2 + k^2$
Since $k \geq 2$, $k^2 \geq 4$
$k^2 + k^2 + k^2 > (k+1) * (k+1) = (k+1)^2$
Therefore, $3^{k+1} > (k+1)^2$ $\square$

C) *Proof* (induction on n)
(Base step, n = 1)

$$\sum_{i=1}^{n} 2i = 2 * 1 = 2 = 1(1+1) = n(n+1)$$

(induction step)
assume $\sum_{i=1}^{k} 2i = k(k+1)$ for some natural number k ≥ 1

$$\sum_{i=1}^{k+1} 2i = (\sum_{i=1}^{k} 2i) + 2(k+1) = k(k+1) + 2(k+1)$$

$k(k+1) + 2(k+1) = k^2 + k + 2k + 2 = k^2 + 3k + 2 = (k+1)(k+2) = (k+1)((k+1)+1)\square$

D) *Proof* (induction on n)
(Base step, n = 1)

$$\sum_{i=1}^{n} 2^{i-1} = 2^{1-1} = 2^0 = 1 = 2^1 - 1 = 2^n - 1$$

(induction step)
$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ for some natural number k ≥ 1

$$\sum_{i=1}^{k+1} 2^{i-1} = (\sum_{i=1}^{k} 2^{i-1}) + 2^{(k+1)-1} = 2^k - 1 + 2^{(k+1)-1}$$

$2^k - 1 + 2^{(k+1)-1} = 2^k - 1 + 2^k = 2^{k+1} - 1$ $\square$

E) *Proof* (induction on n)
(Base step, n = 1)

$$\sum_{i=0}^{n} i! * i = 0! * 0 + 1! * 1 = 1 = (1+1)! - 1 = (n+1)! - 1$$

(induction step)
$\sum_{i=1}^{k} i! * i = (k+1)! - 1$ for some natural number k

$$\sum_{i=0}^{k+1} i! * i = (\sum_{i=0}^{k}) + (k+1)! * (k+1) = (k+1)! - 1 + (k+1)! * (k+1)$$

$(k+1)! - 1 + (k+1)! * (k+1) = (k+1)!(k+1)!(k+1) - 1 = 2(k+1)!(k+1) - 1 = (2k+2)!(k+1) - 1$
$= (2k^2 + 4k + 2)! - 1 = (k+2)! - 1\Box$

F) *Proof* (induction on n)
(Base step, n = 1)
$n^2 - 3n = 1^2 - 3(1) = 1 - 3 = -2 and -2 is even because -2 = 2(-1)$
(induction step)
Assume that $k^2 - 3k$ is even for some natural number k. This means that $k^2 - 3k = 2j$ for some integer j.
$(k+1)^2 - 3(k+1) = (k+1)(k+1) - 3(k+1)$
$= (k+1)^2 - 3k - 3$
$= k^2 + 2k + 1 - 3k - 3$
$= k^2 - 3k + 2k + 1 - 3$
$= 2j + 2k - 2$
$= 2(j + k - 1)$
Since j + k - 1 is an integer, this means that $(k+1)^2 - 3(k+1)$ is even. $\Box$