"a **proposition** is a statement"
Conjuction (∧): And, but, plus, as well, too : TFFF

Disjunction (∨): or : TTTF

Conditional implication (→): if, in the case that, given that, provided that, so long as, when, where, whenever, should : TFTT

"if": "We will contact you if there is an issue with the payment" is I→P

"only if": "We will contact you only if the case that there is an issue with the payment" is C→I

Biconditional Implication (↔): if and only if : TFFT

"Sufficient": "Matching fingerprints are a sufficient condition to establish the presence of the suspect in the room." is F→R

"Necessary": "Being plugged in is a necessary condition for the device to be on." O→P

"Necessary and Sufficient": ↔

A formula of propositional logic is a **contradiction** if and onlyif no truth assignment satisfies it.
A formula of propositional logic is **satisfiable** if and only ifthere is at least one truth assignment that satisfies it.
A formula of propositional logic is a **tautology** if and only if it is satisfied by every truth assignment. To prove that a formula is a tautology using tables, you must give every row of the table.
A formula of propositional logic that has at least one satisfying assignment and at least one non-satisfying assignment is called a **contingency**. To prove that a formula is a contingency, you must give two truth assignments: one satisfying, and one not satisfying. To prove that a formula is not a contingency using tables, you must give an entire table, showing either that every assignment satisfies the formula or no assignment does.

A set of formulas is **consistent** if and only if there exists at least one truth assignment that satisfies all of the formulas in the set. An argument is **valid** if and only if every truth assignment that satisfies all the premises also satisfies the conclusion.

Commutative Laws	Association Laws	DeMorgan's Laws	Distributive Laws	Idempotence Laws	Absorption
p∧q ≡ q∧p (∧-Commutative)	p∧(q∧r) ≡ (p∧q)∧r	¬(p∧q) ≡ ¬p∨¬q	p∧(q∨r) ≡ (p∧q)∨(p∧r)	p∧p ≡ p	p∧(q∨p) ≡ p
p∨q ≡ q∨p (∨-Commutative)	p∨(q∨r) ≡ (p∨q)∨r	¬(p∨q) ≡ ¬p∧¬q	p∨(q∧r) ≡ (p∨q)∧(p∨r)	p∨p ≡ p	

Material Implication	Bi-Implication	Exclusive Disjunction
p→q ≡ ¬p∨q	p↔q ≡ (p∧q)∨(¬p∧¬q)	p⊕q ≡ (p∧¬q)∨(¬p∧q)

<u>Setlist Notation</u>
A = {0,1,2,3}

<u>Set builder notation</u>
S = {n^2 | n is an integer ∧ n ≥ 20 ∧ n ≤ 500}

<u>Special Sets of Numbers</u>
<u>Natural Numbers</u>: ℕ = {0,1,2,3,...}
<u>Integers</u>: ℤ = {...,-2,-1,0,1,2,...}
<u>Rational Numbers</u>: ℚ = {$\frac{p}{q}$ | p ∈ ℤ ∧ q ∈ ℤ ∧ q ≠ 0}
= any # that can be written as a ratio/fraction/quotiant of integers

The universal set (or the universe) is the set of all the things we currently care about. We often use a cursive capital 𝕌 for the universe.

Equivalence Laws for FOL
¬∀xp(x) ≡ ∃x¬p(x) (Universal negation – DeMorgan)
¬∃p(x) ≡ ∀x¬p(x) (Existential negation – DeMorgan)

<u>Quantifier movement</u>
∀y(p(x)→r(x, y)) ≡ p(x)→∀yr(x, y)
∃y(p(x)→r(x, y)) ≡ p(x)→∃yr(x, y)
∀y(p(x)∧r(x, y)) ≡ p(x)∧∀yr(x, y)
∃y(p(x)∧r(x, y)) ≡ p(x)∧∃yr(x, y)

<u>Quantifier Independence</u>
∀x∀yp(x, y) ≡ ∀y∀xp(x, y)
∃x∃yp(x, y) ≡ ∃y∃xp(x, y)

<u>Distribution</u>
∀x(p(x)∧q(x)) ≡ ∀xp(x)∧∀xq(x)
∃x(p(x)∨q(x)) ≡ ∃xp(x)∨∃xq(x)

<u>Null Quantification</u>
∀xp(y) ≡ p(y) where x is not free in p(y)
∃xp(y) ≡ p(y) where x is not free in p(y)

Universal Proofs (for sets)
<u>Claim</u>: For all sets A, B, and C, A∩B⊆B∪C
Proof: Choose sets A, B, and C.
— Choose x ∈ A∩B
— So x∈A and x∈B
— Since x∈B, we know x∈B∪C
Therefore A∩B⊆B∪C□

<u>Claim</u>: For all sets A, B, and C, if A⊆B then C\B⊆C\A
Proof: Choose sets A, B, and C, and assume A⊆B
— Choose x∈C\B
— So x∈C and x∉B
—— Suppose towards a contradiction taht x∈A
—— Then we can apply A⊆B to show that x∈B
— But this contradicts our earlier deduction that x∉B, so x must not be a member of A
— This, together with x∈C allows us to conclude that x∈C\A
Therefore C\B⊆C\A □

<u>Claim</u>: For all sets A, B, and C, if A⊆C, then A∪B⊆C∪B
Proof: Choose sets A, B, and C, and assume A⊆C
— Choose x ∈ A∪B
— So x∈A or x∈B
—— **Case 1:** Suppose x∈A
—— In this case, we can apply A⊆C to get x∈C
—— And weaken to get x∈C∪B
—— **Case 2:** Suppose x∈B
—— From this x∈C∪B
— In either case, we've proven x∈C∪B
Therefore A∪B⊆C∪B

Existential Claims
Def: x is **odd** if there exists an integer n that $x = 2n + 1$
Def: x is **divisible by** y if there exists an integer n that $x = n * y$
Def: x is **rational** if there exists an integer p and q that $x = p/q$ and q≠0

Claim: 10 times any interger is even
Proof: Choose an integer n. — let m = 5n. since 5 and n are integers so is m — because 10n = 2*5n = 2m we know 10n is even — therefore 10 times any integer is even□

Reflexive Proof
Claim: E is reflexive (E = {(n, m) | n+m is even}
Proof: Choose an integer n.
$n + n = 2n$
Since n is an integer, this means n + n is even, hence E(n, n)□

Symmetric Proof
Claim: E is symmetric (E = {(n, m) | n = m is even} Proof: Choose intefgers n and m and assume E(n, m)
So n - m is even — so there exists an integer k such that n - m = 2k —m - n = -(n - m) = -2k = 2 * (-k) — Since k is an int so is -k — therefore m - n is even so E(m, n)

Transitive Proof
Claim: E is transitive (E = {(n, m) | n - m is even}
Proof: Choose integers x, y, and z and assume E(x, y) and E(y, z)
So l - m and m - n are both even
Thus there exist integers j and k such that l - m = 2j and m - n = 2k
(l-m)+(m - n) = 2j + 2k
l - n = 2(j+k)
Since j and k are integers, j + k is too
Therefore l - n is even, hence E(l, n)□

Anti-symmetric Proof
Claim: P is antisymmetric where p = s, t s is a prefix of t
Proof: Choose strings s and t and assume P(s, t) and P(t, s) — s is prefix of t and t is prefix of s — exists strings u and v that t = s + u and s = t + v — thus t = (t + v) + u — v and u must be empty otherewise length longer than length of t — so t = s + u = s □

Example 5.26. Prove that for every $n \geq 4$, $2^n > 3n$.

Proof. (induction on n)
(Base step, $n = 4$):
$2^4 = 16 > 6 = 3 \cdot 4$

(Induction step):
Assume that $2^k > 3k$ for some $k \geq 4$.
(Goal: $2^{k+1} > 3(k+1)$)

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2^1 \\ &= 2^k \cdot 2 \\ &> 3k \cdot 2 && \text{(by the induction hypothesis)} \\ &= 3k + 3k \end{aligned}$$

Since $k \geq 4$, we know that $3k \geq 12$, and more importantly, $3k > 3$. so:

$$3k + 3k > 3k + 3 = 3(k+1)$$

Therefore $2^{k+1} > 3(k+1)$. □

(Induction Step):
Assume that for some $k > 3$, $2^k < k!$.
(I will show: $2^{k+1} < (k+1)!$.)

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &< k! \cdot 2 && \text{(by IH)} \end{aligned}$$

Because $k > 3$, we know that $k + 1 > 2$.
($k + 1 > 4$ is also true, but we this is all we need.)

$$\begin{aligned} k! \cdot 2 &< k! \cdot (k+1) && \text{(because } k+1 > 2) \\ &= (k+1)! \end{aligned}$$

Therefore $2^{k+1} < (k+1)!$. □

Example 5.35. Prove that for any $n > 0$, $\sum_{i=1} (2i - 1) = n^2$

Proof. (Induct on n)
(Base step $n = 1$):
$\sum_{i=1}^{1} (2i - 1) = 2 \cdot 1 - 1 = 1 = 1^2$

(Induction step):
Assume $\sum_{i=1}^{k} (2i - 1) = k^2$ for some $k > 0$.
$\sum_{i=1}^{k+1} (2i - 1) = \left(\sum_{i=1}^{k} (2i - 1) \right) + 2(k+1) - 1 = k^2 + 2(k+1) - 1$ (by the induction hypothesis)
$k^2 + 2(k+1) - 1 = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$ □

Example 5.28. Prove that for any $n > 0$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. (induction on n)
(base step, $n = 1$):
$1 = \frac{2}{2} = \frac{1(1+1)}{2}$

(induction step):
Suppose that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ for some $k > 0$.
(will prove: $1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+1+1)}{2}$)

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{(by IH)} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} && \text{(factoring out } (k+1)) \\ &== \frac{(k+1)(k+1+1)}{2} \end{aligned}$$

Therefore $1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+1+1)}{2}$. □

A = {(x, y) | $3x = 2y + 1$ and $g(x) = 7x - 2$
Proof.
First Requirement:
Choose $x \in \mathbb{R}$.
Let $y = \frac{3x-3}{2}$.
Since 2, 3, and x are real numbers, so is y.

$$\begin{aligned} 2y + 1 &= 2 \cdot \left(\frac{3x - 3}{2} \right) + 1 \\ &= (3x - 3) + 1 \\ &= 3x - 2 \end{aligned}$$

This means that $A(x, y)$.
Second Requirement:
Choose real numbers x, y_1, and y_2 and assume that $A(x, y_1)$ and $A(x, y_2)$.
So we know $3x - 2 = 2y_1 + 1$ and $3x - 2 = 2y_2 + 1$.

$$\begin{aligned} 2y_1 + 1 &= 2y_2 + 1 \\ 2y_1 &= 2y_2 \\ y_1 &= y_2 \end{aligned}$$

□

Claim. g is one-to-one.

Proof.
Choose $x_1, x_2 \in \mathbb{R}$ and assume $g(x_1) = g(x_2)$.
So $7x_1 - 2 = 7x_2 - 2$.
$7x_1 = 7x_2$
$x_1 = x_2$ □

Claim. g is onto.

Proof.
Choose $y \in \mathbb{R}$.
Let $x = \frac{y+2}{7}$.
Since y, 2, and 7 are all real numbers, so is x.

$$\begin{aligned} x &= \frac{y+2}{7} \\ 7x &= y + 2 \\ 7x - 2 &= y \end{aligned}$$

Hence $g(x) = y$. □