Equivalence Laws for FOL

$\neg\forall x p(x) \equiv \exists x \neg p(x)$ (Universal negation – DeMorgan)
$\neg\exists p(x) \equiv \forall x \neg p(x)$ (Existential negation – DeMorgan)

<u>Quantifier movement</u>
$\forall y(p(x)\rightarrow r(x, y)) \equiv p(x)\rightarrow\forall y r(x, y)$
$\exists y(p(x)\rightarrow r(x, y)) \equiv p(x)\rightarrow\exists y r(x, y)$
$\forall y(p(x)\wedge r(x, y)) \equiv p(x)\wedge\forall y r(x, y)$
$\exists y(p(x)\wedge r(x, y)) \equiv p(x)\wedge\exists y r(x, y)$

<u>Quantifier Independence</u>
$\forall x\forall y p(x, y) \equiv \forall y\forall x p(x, y)$
$\exists x\exists y p(x, y) \equiv \exists y\exists x p(x, y)$

<u>Distribution</u>
$\forall x(p(x)\wedge q(x)) \equiv \forall x p(x)\wedge\forall x q(x)$
$\exists x(p(x)\vee q(x)) \equiv \exists x p(x)\vee\exists x q(x)$

<u>Null Quantification</u>
$\forall x p(y) \equiv p(y)$ where x is not free in p(y)
$\exists x p(y) \equiv p(y)$ where x is not free in p(y)

Universal Proofs (for sets)

<u>Claim</u>: For all sets A, B, and C, $A\cap B\subseteq B\cup C$
Proof: Choose sets A, B, and C.
— Choose $x \in A\cap B$
— So $x\in A$ and $x\in B$
— Since $x\in B$, we know $x\in B\cup C$
Therefore $A\cap B\subseteq B\cup C\square$

<u>Claim</u>: For all sets A, B, and C, if $A\subseteq B$ then $C\setminus B\subseteq C\setminus A$
Proof: Choose sets A, B, and C, and assume $A\subseteq B$
— Choose $x\in C\setminus B$
— So $x\in C$ and $x\notin B$
—— Suppose towards a contradiction taht $x\in A$
—— Then we can apply $A\subseteq B$ to show that $x\in B$
— But this contradicts our earlier deduction that $x\notin B$, so x must not be a member of A
— This, together with $x\in C$ allows us to conclude that $x\in C\setminus A$
Therefore $C\setminus B\subseteq C\setminus A$ $\square$

<u>Claim</u>: For all sets A, B, and C, if $A\subseteq C$, then $A\cup B\subseteq C\cup B$
Proof: Choose sets A, B, and C, and assume $A\subseteq C$
— Choose $x \in A\cup B$
— So $x\in A$ or $x\in B$
—— **Case 1:** Suppose $x\in A$

—— In this case, we can apply A⊆C to get x∈C
—— And weaken to get x∈C∪B
—— **Case 2:** Suppose x∈B
—— From this x∈C∪B
— In either case, we've proven x∈C∪B
Therefore A∪B⊆C∪B

Existential Claims
Def: x is **odd** if there exists an integer n that $x = 2n + 1$
Def: x is **divisible by** y if there exists an integer n that $x = n * y$
Def: x is **rational** if there exists an integer p and q that $x = p/q$ and q≠0

Claim: 10 times any interger is even
Proof: Choose an integer n. — let m = 5n. since 5 and n are integers so is m — because 10n = 2*5n = 2m we know 10n is even — therefore 10 times any integer is even□

Reflexive Proof
Claim: E is reflexive (E = {(n, m) | n+m is even}
Proof: Choose an integer n.
$n + n = 2n$
Since n is an integer, this means n + n is even, hence E(n, n)□

Symmetric Proof
Claim: E is symmetric (E = {(n, m) | n = m is even} Proof: Choose intefgers n and m and assume E(n, m)
So n - m is even — so there exists an integer k such that n - m = 2k —m - n = -(n - m) = -2k = 2 * (-k) — Since k is an int so is -k — therefore m - n is even so E(m, n)

Transitive Proof
Claim: E is transitive (E = {(n, m) | n - m is even}
Proof: Choose integers x, y, and z and assume E(x, y) and E(y, z)
So l - m and m - n are both even
Thus there exist integers j and k such that l - m = 2j and m - n = 2k
(l-m)+(m - n) = 2j + 2k
l - n = 2(j+k)
Since j and k are integers, j + k is too
Therefore l - n is even, hence E(l, n)□

Anti-symmetric Proof
Claim: P is antisymmetric where p = s, t s is a prefix of t
Proof: Choose strings s and t and assume P(s, t) and P(t, s) — s is prefix of t and t is prefix of s — exists strings u and v that t = s + u and s = t + v — thus t = (t + v) + u — v and u must be empty otherewise length longer than length of t — so t = s + u = s□