wdc22

a.) We will prove that RSP $\in$ NP-Complete. to do this, we must show:

AND i.) RSP $\in$ NP
ii.) RSP $\in$ NP-Hard

We will first prove (i) directly, then prove (ii) by reducing the Vertex Cover (VC) proplem to RSP in polynomial time using a cook reduction.

Lemma 4.1: RSP $\in$ NP

Proof: whether an outputted $S \subseteq W$, $|S| \leq k$ for a given $G=(V,E)$ and $W \subseteq V$ satisfies RSP can be checked in polynomial time by taking the set difference $V - \{S \cup B\}$ where $B = \{v \in V : (u,v) \in E \wedge u \in S\}$. If that difference is empty, S is valid, else not. This is a polynomial-time operation.

Lemma 4.2: $VC \leq_p RSP$

Proof: We will now give a cook reduction that requires at most $\binom{n}{k}$ calls to RSP "black box" where $n = |V|$, k from input.

VC Input		RSP Input
$G=(V,E)$	$\leq_p$	$G'=(V',E')$
k = max cover size		$W \subseteq V$, $|W| = k$

by steps on following page.

4 (a)

wdc22

def VC-RSP Reduction

```
Init: V' := V, E' := E, G' = G

for each distinct W ⊆ V of size k:
    let F = {(u,v) ∈ E: u ∉ W ∧ v ∉ W}
    for each (u,v) ∈ F
        let t be a node
        V' = V' ∪ {t}
        E' = E' ∪ {(u,t), (t,v)}
    end for
    answer "yes" iff RSP(G' = (V', E'), W, k) answers "yes."
    else, reset V' = V, E' = E and continue
end for
answer "no", as RSP was never satisfied.
end def
```

Claim 4.1: VC-RSP Reduction is $O(n^k)$ polynomial complexity.

Proof: There are $\binom{n}{k}$ possible k-sized subsets W of V. This is because we have $n = |V|$ vertices, and $k = |W|$ ways to select W. This is polynomial in n because k is a constant, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \leq \frac{n^k}{k!} \leq n^k, \text{ therefore } O(n^k)$$

because for each $\binom{n}{k}$ possible W, we only perform polynomial operations to construct E', V' and then call RSP. So, $O(n^k)$ calls to RSP, and $O(n^k)$ complexity. ∎

Claim 4.2: VC-RSP makses RSP answer "yes" iff VC is "yes"

Proof: By adding interstitial nodes for edges without an endpoint in W, we ensure that RSP will only be able to reach all nodes in E', and hence find a vertex cover, if that W is a vertex cover ■

Proof Lemma 4.2: Our reduction is polynomial by Claim 4.1 and correct by Claim 4.2, therefore we have given a polynomial cook reduction of VC to RSP ■

Thm 4.1: RSP ∈ NP-Complete

Proof: by lemmas 4.1, 4.2, we have satisfied (i) and (ii) as defined in the preamble, therefore we have shown RSP to be NP-Complete ■

b.) We will now show that Sensor Placement (SP) is NP-Complete. In order to do this, we must show:

i.) $SP \in NP$
ii.) $SP \in NP$-Hard

We will first prove (i) directly, before proving (ii) by reducing Restricted Sensor Placement (RSP), which we have shown in (4a) to be NP-Complete, to SP via a polynomial-time reduction.

Lemma 4.3: $SP \in NP$

Proof: we will show this by describing an efficient (polynomial-complexity) certifier, which we shall call B, for SP. Given a potentially satisfying placement $S \subseteq V$, B will iterate over S and return "yes" iff there are no nodes x s.t. $(u,x) \in E$ $u,x \in V$, $\{u \notin S \wedge x \notin S\}$. If such an x exists, B will return "no" ∎

Lemma 4.4: $RSP \leq_p SP$

Proof: We will now provide a polynomial-time Karp reduction from RSP to SP:

```
def RSP-SP Reduction
  let F={(u,v)∈E: u∈W ∨ v∈W}
  if RSP answers "yes" given G'=(V,F) and k as inputs
      answer "yes"
  else
      answer "no"
end def
```

RSP Inputs		SP Inputs
$G=(V,E)$	$\leq_p$	$G'=(V,F)$
$W \subseteq V$, k		k

4 (b)

Clearly, this reduction is polynomial, as it requires only scanning over E, and adding $e \in E$ to F which satisfy F's construction. This operation is $O(m)$ linear in $m = |E|$.

Now we will prove its correctness, i.e. that $SP(G' = (V,F), k)$ answers "yes" iff $RSP(G = (V,E), W, k)$ is "yes". By constructing F to contain only edges incident to members of W, we ensure that only vertices which were in W can be considered for sensing vertices outside of W in the general SP. So, if $\exists$ a valid sensor placement of at most size k in $G' = (V,F)$, SP will find it and return "yes."

Conversely, if no valid sensor placement exists within W, F will be constructed such that there will be no $e \in F$ incident to both (what used to be) W and the arbitrary sensor which is "out of reach" of (what used to be) W. This will cause SP to return "no", as it won't be able to find a k-sized subset of V which satisfies SP's constraints ■

Thm 4.2: $SP \in$ NP-Complete

Proof: By lemma's 4.3 and 4.4, we see that $SP \in$ NP-Complete and $RSP \leq_p SP$ (which implies $SP \in$ NP-Hard, as we showed $RSP \in$ NP-Complete in (4a)). We have now satisfied (i) and (ii) in the preamble, thus SP is NP-Complete ■