In order to prove that Fuzzy SAT (FSAT) is NP-Complete, we must show that

i.) FSAT $\in$ NP

AND

ii.) FSAT $\in$ NP-Hard

We will prove (i) directly and fairly trivially, before proving (ii) by reducing a known NP-Complete problem (SAT) to FSAT. Specifically, we will show that:

$$SAT \leq_p FSAT$$

Further, we will also provide a generalized polynomial-time reduction which will allow us to reduce SAT to a general instance of FSAT whereby at most h of the n variables may be assigned "?" (as opposed to ½).

Lemma 1.1: FSAT $\in$ NP

Proof:

We know that FSAT $\in$ NP iff $\exists$ an efficient certifier, we'll call it B, which can verify a proposed satisfying assignment in polynomial time. One such B could simply iterate over each clause, ensure that each clause evaluates to true or fuzzy, and return false if any don't. B would then return true iff $|F| \leq \frac{n}{2}$ where $F = \{x \in X : x = ?\}$.

B's complexity is linear in number an length of the clauses ∎

Lemma 1.2: SAT $\leq_p$ FSAT

Proof:

We will now provide a polynomial-time Karp reduction of SAT to FSAT.

SAT Input		FSAT Input
$C = C_1 \wedge C_2 \wedge \ldots \wedge C_k$	$\leq_p$	$C' = C_1 \wedge C_2 \wedge \ldots \wedge C_k \wedge C_{k+1} \wedge C_{k+2}$
$X = \{x_1, \ldots, x_n\}$		$X' = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}\}$

The reduction's steps will be as follows:

1.) add n vars to X (append to preserve order of first n)

2.) add 2 clauses to C

  i.) $C_{k+1} = x_{n+1} \vee x_{n+2} \vee \ldots \vee x_{2n}$

  ii.) $C_{k+2} = \overline{x}_{n+1} \vee \overline{x}_{n+2} \vee \ldots \vee \overline{x}_{2n}$

3.) answer "yes" to SAT, given C, X, iff FSAT, given C', X', answers "yes". Else, answer "no".

This reduction works because adding C_{k+1} and its negation, C_{k+2}, will force $\{x_{n+1}, \ldots, x_{2n}\}$ to be assigned to "?", eating up all of the allowed fuzzy assignments. The problem thus reduces to SAT as $\{x_1, \ldots, x_n\}$ may only take on boolean $b \in \{0,1\}$.

Observation 1.1: FSAT is restricted to $\frac{n}{2}$ fuzzy assignments, and we doubled the size of X, and hence n, to reduce SAT to it.

Corollary 1.1: We can reduce to the generalized FSAT outlined in the preamble, with $\frac{n}{h}$ possible "?" assignments in X, by slightly modifying our reduction. Where we double $|X|$ when $h=2$, we can just increase $|X|$ by a factor of h in the general FSAT, and letting C_{k+1} be disjunction of all added $x \in X$, and $C_{k+2} = C_{k+1}$

Thm 1.1: FSAT $\in$ NP-Complete

Proof: By Lemma's 1.1 and 1.2, $\{FSAT \in NP \wedge SAT \leq_p FSAT\}$.

By Thm. 8.16 from Kleinberg & Tardos, SAT $\in$ NP-Complete.

By Thm. 8.14 from K&T if SAT $\in$ NP-Complete and FSAT $\in$ NP, then $\{SAT \leq_p FSAT\} \rightarrow \{FSAT \in NP\text{-Complete}\}$.

All requirements of 8.14 are true, therefore FSAT $\in$ NP-Complete ∎