Physics 129: Particle Physics
Lecture 12: Towards a Relativistic Theory

Oct 6, 2020

- Suggested Reading:
  - Thomson 3.1-3.3, 4.1-4.3
  - Griffiths 6.1-6.3,7.1-7.2
  - Robert Littlejohn's notes for Physics 221:
    - Klein-Gordon Eq:
      http://bohr.physics.berkeley.edu/classes/221/notes/kleing.pdf
    - Dirac Eq:
      http://bohr.physics.berkeley.edu/classes/221/notes/dirac.pdf

Plan for this week

- Until now: Calculations based on non-relativisitic quantum mechanics
- This week, Introduce building blocks for fully relativistic treatment
- Separate calculations into kinematic term (density of states and conservation of energy-momentum) and dynamics term (H_{int})
- Kinematics:
  - Wave functions with appropriate relativistic normalization
  - Lorentz invariant expression for Density of States
  - Formulae written in covariant form so that Lorentz invariance is manifest
- Dynamics:
  - Replace Schrodinger Equation with equation that puts space and time on equal footing
    - Klein-Gordon equation for spinless particles
    - Dirac equation for massive particles with spin
  - Re-visit perturbation theory using Lorentz invariant notation
    - Feynman diagrams
- Next week: Apply these new tools to Quantum Electrodynamics

Reminder of non-relativistic result: Fermi's Golden Rule for Decays

- Decay rate (in natural units)

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_f)$$

- where:
  - Γ_{fi} is transition rate (number of transitions per unit time) from $|i\rangle$ to $|f\rangle$
  - $T_{fi} = \langle\psi_f\big| V |\psi_i\rangle$ is called the matrix element
  - $\rho(E_f)$ is density of states (aka "phase space factor")
- To make this Lorentz Invariant, we'll need to make small redefinitions of each piece of the expression

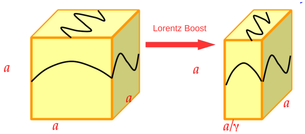
Imposing Lorentz Invariance: Wave Function Normalization

- Non-relativistic normalization:

$$\int \psi^* \psi dV = 1$$

  So, wf includes normalization $1/\sqrt{V}$

- This normalization not Lorentz invariant



$$dV \propto \gamma$$

- To make normalization invariant need normalization proportional to $\sqrt{\gamma}$
- Since $\gamma = E/m$, normalize wf to E particles per unit volume
- Standard normalization:

$$\int \psi^* \psi dV = 2E$$

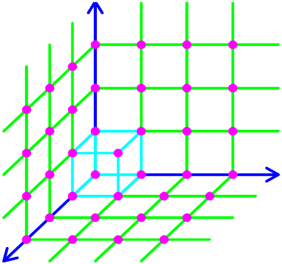
  Reason for the factor of 2 will become clear later

- Define newly normalized wf

$$\psi' = \sqrt{2E}\psi$$

Reminder: Density of States in NR QM

- In NR QM, density of states obtained by putting wf in box



$$p_x = \frac{2\pi n_x}{L} \quad p_y = \frac{2\pi n_y}{L} \quad p_z = \frac{2\pi n_z}{L}$$

- Normalizing to one particle per $[n_x.n_y, n_z]$ state:

$$\begin{aligned} dN_i &= \frac{total\ phase\ space}{elemental\ volume} \\ &= \frac{\left(\frac{1}{V}\int dp_x dp_y dp_z\right)}{\left(\frac{(2\pi)^3}{V}\right)} \\ &= \frac{1}{(2\pi)^3}\int d^3p \end{aligned}$$

- For n particles:

$$dN_n = \frac{1}{(2\pi)^{3n}}\int \prod_{i=1}^{n-1} d^3p$$

- Density of states:

$$\rho(E_f) \quad = \quad \frac{dN_n}{dE_f} = \frac{1}{(2\pi)^{3n}}\frac{d}{dE}\int \prod_{i=1}^{n-1} d^3p$$

Imposing Lorentz Invariance: Density of States

- From previous page

$$\rho(E_f) \quad = \quad \frac{dN_n}{dE_f} = \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i-1}^{n-1} d^3 p_i$$

- As with wf, this is not Lorentz invarient

$$\begin{aligned} p_x' &= px \\ p_y' &= py \\ p_z' &= \gamma\left(p_z - \beta E\right) \\ E &= \gamma\left(p_z - \beta E\right) \end{aligned}$$

- You will prove on Homework 7 that

$$\frac{dp_z}{E} \quad = \quad \frac{dp_z'}{dE'}$$

- Therefore, we can obtain a Lorentz invariant density of states (aka Lorentz invariant phase space or LIPS):

$$\rho(E) = \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{E_i}$$

Imposing Lorentz Invariance: Energy and Momentum Conservation

- From previous page, LIPS:

$$\rho(E) = \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{E_i}$$

- Product is over $n-1$ particles, since energy-momentum conservation constrains particle n
- Add δ-fn's and take product to n.
- The δ-fn's are:

$$\text{energy conservation}: \qquad \int dE_n \delta(E_{init} - \sum_i^n E_i) = 1$$

$$\text{momentum conservation}: \qquad \int d^3\vec{p}_n \delta(\vec{p}_{init} - \sum_i \vec{p}_i) = 1$$

- The density of states becomes:

$$\rho(E) \quad = \quad \frac{1}{(2\pi)^{3n}} \int \frac{d^3 p_i}{E_i} \delta(\vec{p}_{init} - \sum_i \vec{p}_i) \delta(E_{init} - \sum_i^n E_i)$$

Putting it all together: Fermi's Golden Rule Revisited

- Start with NRQM FGR:

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_f)$$

- Redefine matrix element to factor out energies from initial and final state:

$$\left|M_{if}\right|^2 = \left|T_{if}\right|^2 \ (2E_{init}) \left(\prod_{i=1}^{n} 2E_i\right)^2$$

- Golden rule becomes

$$\Gamma_{fi} = (2\pi)^4 \frac{1}{2E_{init}} \int \left|M_{if}\right|^2 \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 \, (2E)} \delta\left(\vec{p}_{init} - \sum_i \vec{p}_i\right) \delta\left(E_{init} - \sum_i^n E_i\right)$$

- Note:
  - $\left|M_{if}\right|^2$ is calculated using Lorentz invariant wf. It is Lorentz invariant
  - $d^3p/(2\pi)^3E$ is LIPS for each final state particle
  - Energy-momentum conservation in the δ-fns
  - Γ_{fi} is inversely proportional to energy of initial state. This is what we expect in case of decaying particle whose decay rate is slowed by time dilation

We'll look at invariant scattering cross sections next class lecture

Replacing the Schrodinger eq with Lorentz invariant dynamics

- Although it is a bit ugly, we have everything we need to caluclate LIPS
- How about the matrix element?
- This requires us to know $\langle \psi_f | V | \psi_i \rangle$
- But solutions for ψ can only be found for relativistic case if we replace Schrodinger eq with Lorentz invariant equation of motion
- Start with relativistic expression

$$E^2 = p^2 + m^2$$

  - Second order in space and time: Klein-Gordon Eq
  - Works fine for spinless particles
- But when we try to introduce massive particles with spin, it doesn't work
  - When we Lorentz boost $\vec{\mu} \cdot \vec{p}$ can change sign
  - Can't write Lorentz invariant cross sections
  - Need to move to an equation where spin is intrinsically part of the equations of motion
  - Can be done with diff eq first order in space and time
  - This is the Dirac eq

The Klein-Gordon Equation (QED without spin)

- Relativistic energy-momentum conservation

$$E^2 = p^2 + m^2$$

  becomes in operator form for a wave function ϕ

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

- Multiply by ϕ^* on the right and then subtract complex conjugate equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left[i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)\right] + \nabla \cdot \left[-i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)\right] \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot j \end{aligned}$$

  The continuity equation (for probability)

- Using free particle solutions $\phi = N e^{ip\cdot x - iEt}$ (N is the normalization) identify

$$\begin{aligned} \rho &\equiv 2E|N|^2 \\ \vec{j} &\equiv 2p|N|^2 \\ j^\mu &\equiv (\rho, \vec{j}) \end{aligned}$$

Klein-Gordon equation: Negative Energy Solutions

- Since $E = \pm\sqrt{p^2 + m^2}$, negative energy solutions exist
  - $\rho \equiv 2E|N|^2 \rightarrow$ negative probability density ρ
- Resolve this problem by multiplying by charge q
- Now j^μ is a four-vector current
  - The negative energy states are redefined as states of opposite electric charge

Introduction of relativity requires introduction of anti-particle states!

Moving to Covariant Notation

- From previous page, Klein-Gordeon eq:

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

- Moving to the covariant notation we discussed in lecture 2:

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = 0$$

where

$$\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

- We'll use this compact notation for most calculations, but will begin discussion of Dirac eq with explict space and time derivatives since that is more familar to most of you
- But will quickly switch to covariant notation because equations become unreadable otherwise

Dirac Equation (I)

- Want an equation that is linear in both space and time

$$\hat{E}\psi = \left(\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}} + \beta m\right)\psi$$

- Can be written in terms of energy and momentum operators

$$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi$$

- If solutions represent relativistic particles, they must also satisfy the relativistic eq: $E^2 = p^2 + m^2$. "Square" the equation to give

$$\frac{\partial^2}{\partial t^2}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi$$

- Work out the expression you get multiplying this through, or look in Thomson (pg 83)
- To satisfy $E^2 - p^2 = m^2$, α and β must satisfy certain conditions

  (see next page)

Dirac Equation (II): Conditions on α and β

- α and β must satisfy:

$$\begin{aligned} \alpha_x^2 = \alpha_y^2 = \alpha_z^2 &= I \\ \alpha_j\beta + \beta\alpha_j &= 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j &= 0 \quad (j \neq k) \end{aligned}$$

- This cannot be satisfied if α_i and β are numbers
  - They must be matrices with trace 0 and eigenvalues ± 1
  - They must also be Hermitian: $\alpha^\dagger = \alpha$, $\beta^\dagger = \beta$
- Lowest dimension where this can be achieved is a 4×4 matrix
- This 4×4 matrix equation is the Dirac Equation

  (See next page)

Dirac Equation: (III)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where

$$\begin{aligned}
\gamma_\mu &= (\beta, \beta\vec{\alpha}) \\
\beta &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\vec{\alpha} &= \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}
\end{aligned}$$

- 4-component spinors (<u>not</u> a 4-vector!)
- Four free particle solutions
  - Two particle states (helicity)
  - Reinterpret negative energy states as two antiparticle states
- $j^\mu = -e\overline{\psi}\gamma^\mu\psi$: conserved charge and current

Properties of γ Matices

- We have already seen:

$$\begin{aligned}\alpha_x^2 = \alpha_y^2 = \alpha_z^2 &= I \\ \alpha_j\beta + \beta\alpha_j &= 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j &= 0 \quad (j \neq k)\end{aligned}$$

- This translates to:

$$\begin{aligned}\left(\gamma^0\right)^2 = \beta^2 &= 1 \\ \left(\gamma^i\right)^2 = -\alpha_i\beta\beta\alpha_i &= -\alpha_i^2 = -1\end{aligned}$$

- Full set of relations are:

$$\begin{aligned}\left(\gamma^0\right)^2 &= 1 \\ \left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 &= -1 \\ \gamma^0\gamma^i + \gamma^i\gamma^0 &= 0 \\ \gamma^i\gamma^j + \gamma^j\gamma^i &= 0 \quad (i \neq j)\end{aligned}$$

- Or more compactly: $\{\gamma^\mu\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

Pauli-Dirac Representation

- γ-matrices defined:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Can write four-vector current

$$\begin{aligned} j^\mu &= \left(\rho, \vec{j}\right) = \psi^\dagger \gamma^0 \gamma^\mu \psi \\ &= \overline{\psi} \gamma^\mu \psi \end{aligned}$$

  where $\overline{\psi} = \psi^\dagger \gamma^0$
- Continuity Equation: $\partial_\mu j^\mu = 0$

Solutions to the Dirac Equation

- Normalized solutions to the Dirac Eq

$$\psi = u(E,\vec{p}) = e^{+i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu - m)\, u = 0$$

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- Antiparticle solutions:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu + m)\, v = 0$$

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$