TLS - The Physics of a Two-Level-System: Introduction to Quantum Information Processing using the IBM Quantum Experience

Physics 111B: Advanced Experimentation Laboratory

University of California, Berkeley

Contents

1 Quantum Information Processing with the IBM Quantum Experience Description

- Pre-requisites: Physics 137A

- Programming experience in Phython is highly recommended

Not that it is relevant, but: There is absolutely NO eating or drinking in the 111-Lab anywhere, except in rooms 282 & 286 LeConte on the bench with the BLUE stripe around it. Thank You, the Staff.

This lab will be graded 30% on theory, 50% on technique, and 20% on analysis. For more information, see the **Advanced Lab Syllabus**.

Comments: E-mail **Winthrop Williams**

2 Before the 1st Day of Lab

Complete the TLS Pre Lab found in the Signature Sheet for this lab. Print the signature sheet, discuss the experiment and pre-lab questions and answers with any faculty member or GSI, and receive their signature. In the course of the lab there will be examination points where you must STOP and get a GSI or professor to verify your understanding and/or verify proper experimental setup. You cannot skip these checkpoints, and must receive signatures demonstrating that you've consulted the staff. Some experiments may have mid lab questions that must be completed by specific days of the experiment. The completed Signature Sheet MUST be submitted as the first page of your lab report. Quick links to the checkpoint questions are found here:

1. Take a look at these references: [1].
2. Go to the website of the IBM quantum experience (`https://quantum-computing.ibm.com/`). Familiarize yourself with the environment and get an account.
3. Install phython3 including qiskit, numpy and matplotlib. In addition, install jupyter notebook on your computer. You will need this software to access and program the IBM quantum experience. If you have trouble with installing or accessing any of these tools, please contact the teaching staff.
4. Last day of the experiment please fill out the TLS **Experiment Evaluation**

You should keep a laboratory notebook. The notebook should contain a detailed record of everything that was done and how/why it was done, as well as all of the data and analysis, also with plenty of how/why entries. This will aid you when you write your report.

3 Objectives

- Learn "everything" about the dynamics of a quantum mechanical two-level system
- Learn about the basics of quantum information processing
- Learn about how to control quantum computers
- Learn about noise in quantum mechanics and in particular in quantum computers

4 Introduction

Quantum computers promise to harness entanglement and the superposition principle of quantum mechanics to accomplish computational tasks difficult to perform on classical computers. The basic idea is to extend the notion of bits to quantum bits allowing for superpositions of 0's and 1's. Thus, the memory (or even the program) in a quantum computer can be in multiple states "at the same time". Loosely speaking the

superposition principle allows quantum computers to explore many realizations of a computation simultaneously and thus can solve certain problems faster than their classical counterparts. Consider, for instance, a function $f(x) \in \{1, 0\}$ returning a single bit depending on an input $x \in \{1, 0\}$. There exist only four different functions f_i which can do this: $f_0(x) = 0$ ($f(x)$ is always 0), $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = \text{NOT}x$. We set now the goal to decide whether the function $f_i(x)$ is constant or not. In a classical setting, we need to call the unknown function $f_i(x)$ at least twice to find out whether the function is constant or not. However, if we use a quantum computer, we can call the function with the argument in a superposition $x = (|0\rangle + |1\rangle)/\sqrt{2}$ and—simplistically speaking—evaluate whether the two values are the same by interfering the results. This is the essence of the Deutsch-Josza algorithm [2], one of the first quantum algorithms discovered. This improvement by a factor of two may not sound like much, but the advantages become much evident when the Deutsch-Josza algorithm is generalized to more bits. Other notable quantum algorithms are the Bernstein-Vazirani algorithm, Shor's factoring algorithm for integers promising an exponential speed-up as compared to all known factoring algorithms and Grover's search algorithm of unstructured data bases yielding a quadratic speedup with the number of elements in the data base. In recent years, researchers have also found a number of other applications such as solving for the ground state of the molecules or condensed matter problems by emulating the corresponding Hamiltonians.

With these applications in mind, researchers have been trying to establish technologies to implement quantum computers. For this, we must direct our attention first on how the quantum information is encoded, or in other words, what is the physical representation of the quantum version of a bit, i.e. a qubit?

A qubit is a two-state system such as a spin-1/2 particle. When measured we can find the system either in the spin-up or in the spin-down state, or using quantum information language either in the $|0\rangle$ or $|1\rangle$ state with the wavefunction being represented as $|\Psi_{\text{qubit}}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. From this general definition it is clear that a qubits can take a variety of physical forms, for instance, we can encode them in the following physical objects:

- spin of a single electron confined, for example, in a quantum dot, on film of superfluid Helium, or held with by other means.
- the polarization degree-of-freedom of photons.
- two electronic states of an ion, often confined with electrical fields or in the lattice of a solid.
- two states of a neutral atom held with optical tweezers.
- direction of a current in superconducting circuits.
- finite nuclear spin in a molecule or a solid.

If you want to read more about these "quantum" technologies and get an overview you may want to consult [3].

Now that we have the qubits, we need to think about how to execute computations. Thinking about how classical computers work, we come up with the following tasks: (1) Initialize the qubits, (2) perform operations on the register, i.e. execute the algorithm, and (3) read out the result. The challenge will be to execute arbitrary operations while keeping the quantum information, i.e. the superpositions, intact. The fundamental problem is that we need to control the qubits, that is we need to interact with them, but at the same time we are not (and nobody else is) allowed to learn anything about their state before read-out (Q: why is nobody allowed to learn the state of the quantum register?). More formally, we require that decoherence due to noise is sufficiently small such that after the time it takes for the algorithm to run, the information stored in the qubits has not decayed.

Goal of this laboratory will be to famaliarize yourself with the dynamics of qubits, i.e. two-level systems, and to learn how to characterize their fundamental properties, such as their energy splitting, decoherence rates, as well as how to control them, i.e. to explore how to control a real qubit.

Good references to check out are:

- Norman Ramsey's Nobel lecture on Ramsey spectroscopy https://www.nobelprize.org/uploads/2018/06/ramsey-lecture.pdf

- Townsend, A Modern Approach to Quantum Mechanics, Chapter 4.3 and 4.4.
- Gross, Marx, Deppe: slides about controlling two-level-systems. http://www.wmi.badw.de/teaching/Lecturenotes/AS/AS2013_Chapter10_2_Slides.pdf
- Daniel A Steck, University of Oregon, starting at page 151: a pretty complete and rigorous treatment https://atomoptics-nas.uoregon.edu/~dsteck/teaching/quantum-optics/quantum-optics-notes.pdf

5 Background

5.1 Qubits and entanglement

As discussed above the most general state of a qubit is

$$|\Psi_{\text{qubit}}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle)\,. \tag{1}$$

The most general transformation (=computation) of this state into other (valid) quantum states are unitary transformations, i.e. transformations which leave orthogonal states orthogonal as well as do not change the length of the state vector. These transformations $\hat{U}$ belong to the special unitary group SU(2). As physicists, we often think of these transformations "generated" by some Hamiltonian $\hat{H}$, i.e. we have $\hat{U} = e^{-i\hat{H}t_0/\hbar}$ with the dynamics governed by the time evolution operator $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$.

For single qubits, the situation is relatively straightforward (as long as there is no decoherence), but things become much more interesting if multiple qubits are involved. The central concept here is that qubits can be entangled, i.e. the total quantum state can not be written as a tensor product of the single qubit states. Mathematically, a state of two quantum systems 1 ond 2 is called entangled if we can not write the total state $|\Psi\rangle_{\text{tot}}$ as $|\Psi\rangle_1 \bigotimes |\Psi\rangle_2$. Otherwise the state is separable, i.e. we have $|\Psi\rangle_{\text{tot}} = |\Psi\rangle_1 \bigotimes$ $ket\Psi_2$. (Q: is the state $(|00\rangle + |01\rangle)/\sqrt{2}$ entangled? How about $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$, how about $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$?)

In order to get a glimpse of why quantum information can be so powerful, let us consider the number of basis states of the Hilbert space associated with n qubits. Above you have seen that for two qubits, we have already four basis states. If we add a third qubit, it adds one more possibility for each state of the two qubit system, that is we double the number of basis states. Thus, the total number of basis states for an n qubit system is 2^n. This explains why it can be so hard to calculate even a moderately complex quantum dynamics of 50 two-level systems and why even small scale quantum computers of a hundred qubits could be quite valuable research tools.

5.2 The Bloch sphere

In this lab, however, we first focus on the fundamental concept, so lets try to get some intuition about the dynamics of a single qubit. Usually (challenge: can you come up with a scenario where this is not the case?), the overall phase γ in Eq. 1 can not be observed. Therefore, we use the two angles, θ and ϕ, to parametrize the quantum state of a qubit. This allows us to visualize the quantum state as a point on a sphere of unit radius and we will identify the state in Eq. 1 with the unit vector

$$\vec{R} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}\,. \tag{2}$$

The main reason why the Bloch sphere is so important is that it allows us to intuitively grasp nearly all dynamics of a qubit. Further, we can think of the Bloch vector as the direction the spin of a spin-1/2 system

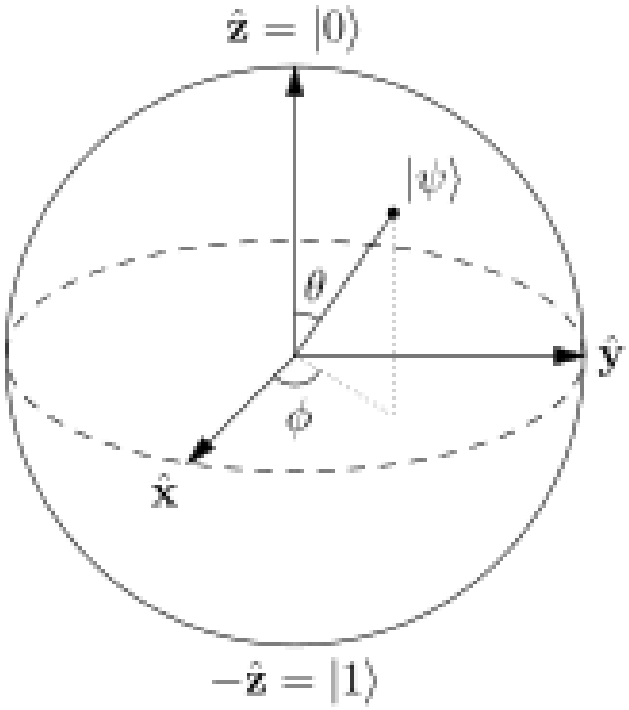

Figure 1: The quantum state of a qubit can be visualized as a point on the Bloch Sphere. Image unaltered taken from https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg, Smite-Meister / CC BY-SA (https://creativecommons.org/licenses/by-sa/3.0).

such as an electron, i.e. we can think of it representing a concrete physical quantity allowing us to make connection to classical physics, i.e. that of a magnetic moment in a magnetic field.

We now can get an idea from the Bloch sphere which operations on a qubit are allowed. In particular, all transformations taking a valid qubit state to another should be allowed, i.e. we can rotate the qubit around any axis. How could we implement such operations physically? The Bloch picture provides a useful hint. Identifying the Bloch vector with the direction of a magnetic moment, we are led to the idea that we can apply a magnetic field along the chosen axis for a given amount of time corresponding to the angle we would like to rotate the Bloch vector. Consider for instance, an electron with its spin initialized at the northpole of the Bloch sphere, i.e. in the $|0\rangle \equiv |\uparrow\rangle$ state, which we would like to rotate around the y-axis. A magnetic field along the y lets the electron spin precess around the y-axis, first transforming the $|0\rangle$ state to $(|0\rangle + |1\rangle)/\sqrt{2}$ and then to the $|1\rangle$ state. If we would like to rotate the spin by 2π, we will have to choose the strength of the magnetic field for a given time t_0 such that $\omega_{\text{Larmor}} t_0 = 2\pi$, where we have introduce the Larmor precession frequency $\omega_{\text{Larmor}} = \mu B$.

To complete this analogy we have to also understand what happens on the Bloch sphere when we measure the spin along the z-axis, or along the x-axis, or along any other axis for that matter. The answer is that the expectation of the spin along a particular axis is a projection onto this axis divided by 2 (try to verify this from Eqs. 1 and 2).

The Bloch picture, however, is not perfect. You may have noticed that the angle θ in Eq. 1 is divided by 2. For instance, somewhat confounding fact is that the Bloch vector of orthogonal states of a wavefunction are not orthogonal. Maybe more interestingly, a rotation of $|0\rangle$ around the y-axis by $\theta = 2\pi$ will yield $-|0\rangle$, but in the Bloch picture the state returns to its original state $|0\rangle$. A question you may wonder now is whether this has any (observable) consequences, after all only the modulus of the wavefunction amplitudes is relevant for the measurement result. If you wonder about this you may want to have look at the end of Ref. [4].

5.3 Josephson junction based qubits

In this laboratory we will use the IBM quantum experience, a quantum computer which uses qubits formed from superconducting circuits and Josephson junctions. Here we will not be able to describe the inner

workings of these qubits (a nice rounded reference talking about a lot of aspects seems to be the arXiv article by Mahdi Naghiloo https://arxiv.org/pdf/1904.09291.pdf), but rather make a few conceptional remarks. The general idea is to design a circuit with as little dissipation as possible to maintain quantum coherence, i.e. we want the system dynamics to be as unitary as possible. Consider for instance a resonator made from a capacitance C in parallel with an inductance L. If we cool this system to sufficiently low temperatures, the quantized nature of this harmonic oscillator will become important. In quantum computers using superconducting qubits, the resonance frequency of the resonator will be of order $\omega = 2\pi \times 5$ GHz. Thus, if we cool the system well below a temperature such that $k_B T \ll \hbar\omega$ (k_B is the Boltzmann constant), the system will eventually relax to the ground state of the resonator. Thus we have initialized the system already in a well-defined (=pure) quantum state, i.e. the $|0\rangle$-state.

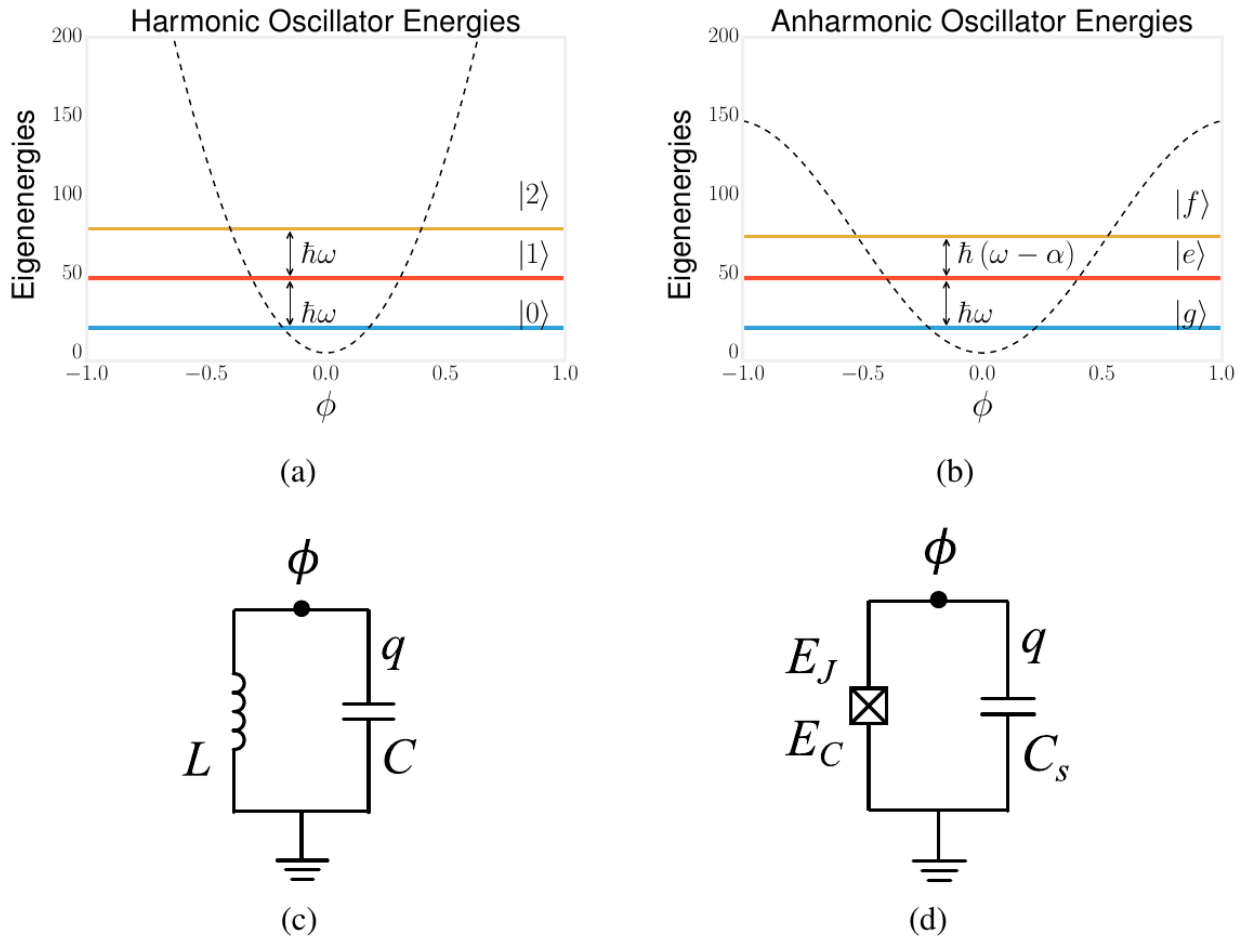

Figure 2: Figure reproduced from Nicholas Materise arXiv:1708.07000 [quant-ph], https://arxiv.org/abs/1708.07000. Comparison of the quantum harmonic oscillator with anharmonic oscillator. (a) and (b) give the eigenenergies of the two oscillators, where the horizontal lines are the eigenenergies and the dashed lines represent notional potentials. (c) and (d) are the corresponding circuit models for an LC circuit and a transmon qubit. Φ and q are the canonical variables flux and charge, respectively. They can also be thought of the current and voltage in the circuit. In a) L is the inductance which is replaced in b) by the non-linear inductance of a Josephson Junction with Josephson energy E_J and charging energy E_C. One way to think of the inductance of a Josephson junction is inductance means that the system tries to resist a change of the current. In this case it means that changing the current will require that the phase of the wavefunction changes (see Eq. 3 and thus inducing a voltage across the Josephson junction.

We have now one problem left on our way to construct a quantum bit. This problem becomes apparent when we consider how to manipulate this quantum system to place it in the $|1\rangle$ state. For this we must give the system some energy. The canonical method is to apply an oscillating current/voltage in resonance with the LC oscillator, i.e. of frequency ω. Thus, the resonator can absorb quanta of energy $\hbar\omega$, i.e. be placed in some superposition of the form $\alpha|0\rangle + \beta|1\rangle$. However, the radiation is also resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition and so forth (see Fig. 2. Thus, we will continue to excite the state $|1\rangle$ to the $|2\rangle$ state and so forth. Thus, we will create larger and larger excitations, populating more and more states. However, a qubit is composed of only two basis states. How do we escape from this dilemma?

The root of our problem is the harmonicity of the resonator, i.e. if the microwave field would become non-resonant once the resonator is excited the problem will go away. Motivated by this we introduce a non-linear element. This non-linear element in superconducting qubits is a Josephson junction, i.e. the current depends in a non-linear fashion on the applied voltage. In particular, the Josephson relation for the current I through the junction is

$$I = I_c sin\varphi\,. \tag{3}$$

Here, I_c is the critical current and φ is the phase difference between the wavefunctions of the electrons in the two leads of the Josephson junction. The time evolution of the phase difference will be governed by the energy difference of the electrons between the two leads, i.e. it will evolve according to $\phi(t) = \Delta E/\hbar t = 2eUt$ where e is the electric charge and U is the voltage between the two leads. As a consequence the current will rapidly oscillate if we apply a small voltage between the leads. This nonlinearity leads to a non-uniform splitting between the various energy eigenstates of the oscillator (see Fig. 2). Thus, the transition frequency between the $|1\rangle$ and $|2\rangle$ states may be signinficantly lower than that between the $|0\rangle$ and $|1\rangle$. In other words, the drive initially resonant with the $|0\rangle \leftrightarrow |1\rangle$ transition will become non-resonant if the (nonlinear) resonator is excited from the $|0\rangle$ to the $|1\rangle$ state and excitation to higher lying states is strongly suppressed. As a consequence we have restricted all dynamics to $\{|0\rangle, |1\rangle\}$ manifold, i.e. a two-level system. Next we will discuss the dynamics of such a two-level system.

5.4 Control of two-level systems using microwave fields

In this section, we will give an overview of how to solve for the dynamics of a single qubit (the full derivation can be found in Appendix A). We will focus on a two-level system, such as an atom or a Josephson junction qubit irradiated by linearly polarized light. The derivation uses a couple of useful tricks connecting back to the Bloch picture and the behavior of a spin-1/2 system with a magnetic moment in a magnetic field.

Before we do this, it is useful to think a bit more about the dynamics of a classical magnetic moment: consider a classical magnetic moment pointing along a strong and constant magnetic field. How can we flip the magnetic moment by 180 degrees using only a weak magnetic field (this will correspond to the operation $|0\rangle \leftrightarrow |1\rangle$, i.e. a `not`-operation)?

The probably, easiest way to figure this is out is to think about a magnetic moment aligned with the strong field B_z in the z direction and to consider an additional weak field perpendicular to the z direction. What happens if that weak field tips the magnetic moment? How does the weak field need to evolve in time in order to continue tipping the magnetic moment still precessing around the strong field? A useful way to visualize what is needed is to put yourself in a frame rotating at the Larmor precession frequency $\omega_L = \frac{e}{2m}B_z$ of the magnetic moment given by the strong magnetic field. In this frame the magnetic moment is fixed and the strong magnetic field does not cause any precession. Thus it is as if $B_z = 0$. In other words, we have transformed its dynamics away and we can think that all the dynamics is happening due to the weak field perpendicular to the z-axis. Now what we need to flip the spin is a magnetic field pointing into a fixed direction letting the magnetic moment precess around this axis. Upon transforming back into the laboratory frame we see that the what we need is a weak magnetic field rotating at the precession frequency due to the strong field in the z-axis. This is circularly polarized radiation with resonant frequency ω_L.

To treat the quantum mechanical situation, we will start by writing down the Hamilton operator for our two-level system. It is convenient to start in the energy eigenbasis of the unperturbed qubit. That is if the qubit is placed in these states, it stays there and acquires only a phase with respect to the other qubit state.

We will call these states $|0\rangle$ and $|1\rangle$ (or spin up/down). Thus, the energy of the system will be given by

$$\hat{H}_{\text{qubit}} = \frac{\hbar\omega}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) = \frac{\hbar\omega}{2}\hat{\sigma}_z \,, \tag{4}$$

where we have introduced the Pauli operator σ_z along the z-axis and $\hbar\omega$ is the energy difference between both states. We note that according to the Schrödinger equation, $i\hbar\delta_t |\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$, we have the time evolution $|0\rangle \to e^{-i\omega t/2}|0\rangle$ and $|1\rangle \to e^{i\omega t/2}|1\rangle$, i.e. the energy difference between the two states is indeed $\hbar\omega$. In order to drive transitions between the two energy eigenstates, we consider now an electromagnetic radiation field at frequency $\omega_{\text{rad}} \approx \omega$ coupling those two states. If we assume that the qubit has a magnetic moment μ, we are lead to the idea that the coupling is mediated by a magnetic field pointing in the $x-y$ plane. Based on the discussion above on the classical intuition, we also surmise that the magnetic field needs to oscillate, i.e. we need radiation resonant with the energy difference. Thus, we postulate that we need a Hamiltonian of the form $\hat{H}_{\text{rad}} = -\hat{\mu}_x B_x \cos(\omega_{\text{rad}} t + \varphi)$, i.e. we need to send in photons providing with energy $E = \hbar\omega_{\text{rad}}$ matching the energy difference of the qubit $\hbar\omega$. For an abstract qubit, we may write this Hamiltonian as $\hat{H} = \hbar\Omega\cos(\omega_{\text{rad}} t + \varphi)(|0\rangle\langle 1| + |1\rangle\langle 0|)$ with $\Omega = \hat{\mu}_x B_x/\hbar$ (note that $\hat{\mu}_x$ is proportional to the dipole operator $|0\rangle\langle 1| + |1\rangle\langle 0|$). Intuitively, the $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$ terms introduce coupling between $|0\rangle$ and $|1\rangle$ with strength $\hbar\Omega$ where Ω will turn out to be the so-called Rabi frequency.

Thus, the total Hamiltonian of the system becomes:

$$\hat{H} = \hat{H}_{\text{qubit}} + \hat{H}_{\text{rad}} = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\Omega\cos(\omega_{\text{rad}} t + \varphi)\hat{\sigma}_x \,. \tag{5}$$

Mathematically, the corresponding Schrödinger equation can be solved exact for the slightly different case of circular polarization, i.e. $\hat{H}_{\text{rad}} = \hbar\frac{\Omega}{2}\left(\cos(\omega_{\text{rad}} t + \varphi)\hat{\sigma}_x + \sin(\omega_{\text{rad}} t + \varphi)\hat{\sigma}_y\right)$, or by making the approximations $\{\Omega, |\omega_{\text{rad}} - \omega|\} \ll \omega$. The solutions can be obtained either by solving the differential equation or by using a basis transformation. In Appendix A, we choose the latter as it is much more insightful and the developed tricks and intuition carry over to more complicated situations whereas using differential equations will get rapidly unwieldy. The main trick used in Appendix A is to transform into a frame in which the interaction term $\hat{H}_{\text{rad}}$, and consequently, the total Hamiltonian becomes time-independent. This will then allow for straightforward integration of the Schrödinger equation, i.e. using that $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$.

This whole procedure carried out in Appendix A can be understood with the Bloch sphere analogy as is discussed in the introduction of this section. Guided by this classical intuition, we transform into a frame rotating with $\omega_{\text{rad}} = \omega = \mu B_z/\hbar$ where we are assuming to be exactly on resonance for the moment. Thus, the rotation of the frame matches exactly the free precession frequency of the spin and we expect that any qubit state will have no time evolution as shown mathematically in Appendix A. It is as if there is no magnetic background field anymore, i.e. in the rotating frame $B_{z,\text{rot}} = 0$. However, if we switch on a circularly polarized resonant field

$$\vec{B}_{\text{rad}}(t) = B_1\left(\cos(\omega_{\text{rad}} t + \varphi)\vec{e}_x + \sin(\omega_{\text{rad}} t + \varphi)\vec{e}_y\right) \,, \tag{6}$$

it rotates also with the same frequency as the qubit, i.e. in the rotating frame its orientation is constant (for linear polarized radiation we need to discard the so-called counter rotating term, see Appendix A). In the rotating frame, the spin will now start precessing around the axis defined by the coupling field and its phase φ. The situation is only slightly more complicated if $\omega \neq \omega_{\text{rad}}$, i.e. the electromagnetic field detuned by $\Delta = \omega_{\text{rad}} - \omega$ from resonance. In this, case the bare qubit has in the frame rotating with the radiation still some time evolution with frequency Δ, i.e. $B_{z,\text{rot}} = \hbar\Delta/\mu$ but the excitation field is constant. As a consequence, the spin will precess around the axis given by $(B_1, 0, B_{z,\text{rot}})$, i.e. $(\Omega, 0, \Delta)$, with effective Rabi frequency $\sqrt{\Omega^2 + \Delta^2}$ where we have assumed the phase $\varphi = 0$ (see Appendix A).

How would we change the rotation axis in the rotating frame? For this, we note that we assumed that the initial axis of the radiation field coincides with the x-axis, i.e. the sin term in in Eq. 6 vanishes at $t = 0$. Going back to Eq. 6, we see that all we have to do is to shift the phase φ of the drive thus providing a different axis for the system to rotate around. This will be key to later understand Ramsey measurements and the various single-qubit operations.

We will now summarize the key results from Appendix A formally. First, we assume that the qubit is initialized in the $|0\rangle$ state. When driven on resonance, i.e. $\omega = \omega_{\text{rad}}$, the qubit will oscillate between the $|0\rangle$ and $|1\rangle$ state, that is the probability $P_{|1\rangle} = |\langle \Psi(t)_{\text{qubit}}|1\rangle|^2$ to find the qubit in $|1\rangle$ will evolve as

$$P_{|1\rangle} = \sin \Omega t \,. \tag{7}$$

In the literature this behavior is also known as Rabi flopping with Rabi frequency Ω. If we introduce a detuning $\Delta = \omega - \omega_{\text{rad}}$, we find that the population exchange is not complete anymore, but that the Rabi flopping speeds up

$$P_{|1\rangle} = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin\left(\sqrt{\Omega^2 + \Delta^2} t\right) \,. \tag{8}$$

For quantum computing, we are also interested in how the operations act on a quantum state $\alpha|0\rangle + \beta|1\rangle$, Thus, we either need to write out the corresponding unitary transformations from the solutions in Appendix A or just inspect how it acts on the $|1\rangle$, too. Doing the former, introducing the rotation angle $\theta = -\Omega t$, we find[1]:

$$R_x(\theta) = e^{i\frac{\theta}{2}\sigma_x} = \cos\tfrac{\theta}{2} I + i \sin\tfrac{\theta}{2}\sigma_x = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{9}$$

$$R_y(\theta) = e^{i\frac{\theta}{2}\sigma_y} = \cos\tfrac{\theta}{2} I + i \sin\tfrac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{10}$$

$$R_z(\theta) = e^{i\frac{\theta}{2}\sigma_z} = \cos\tfrac{\theta}{2} I + i \sin\tfrac{\theta}{2}\sigma_z = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{11}$$

where the Pauli operators/matricies σ_i are defined as follows:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

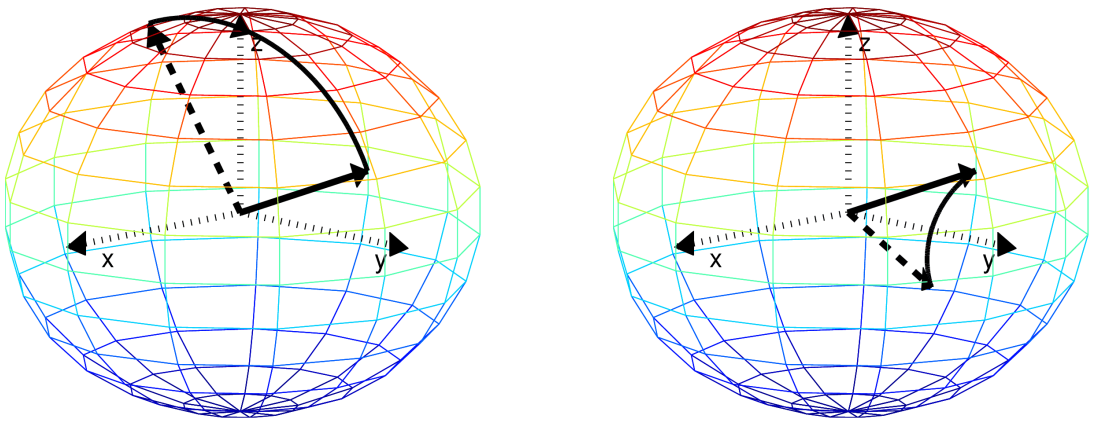

Figure 3: Rotation around the X and the Y axis visualized on the Bloch-sphere.

There is actually a more general and compact way by writing this as:

$$R(\theta, \hat{n}) = \exp\left(\frac{i}{\hbar}\vec{\sigma}\cdot\vec{n}\,\theta\right) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) + in_z \sin\left(\frac{\theta}{2}\right) & (in_x + n_y)\sin\left(\frac{\theta}{2}\right) \\ (in_x - n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{13}$$

, where $\vec{n}$ is a unit vector describing the (arbitrary) rotation axis and $\vec{\sigma}\cdot\vec{n} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$. Note that we used a different convention than what is typically used in physics, i.e. the solution to the Schrödinger

[1]Note that we introduced here a minus sign for the angle θ to obtain time evolution operators consistent with quantum information conventions where the time evolution operator is $U = e^{iHt}$ instead of e^{-iHt}.

equation would usually be $\exp\left(-i\hat{H}t\right)$, so it may be more natural to write is as $\exp\left(-\frac{i}{\hbar}\vec{\sigma}\cdot\vec{n}\,\theta\right)$. This is the most general operation we can perform an a qubit.

Choosing $\theta = \pi/2$, we obtain now the specific important operations creating superposition states from energy eigenstates:

$$(\pi/2)_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, (\pi/2)_y = \frac{1}{\sqrt{2}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{14}$$

which resemble the so-called Hadamard operation, $\frac{2}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ used by Computer Scientists [2]. Finally, we also have the diagonal z-gate $(\pi/4)_z = \mathrm{diag}(e^{i\pi/4}e^{-i\pi/2})$, which applies a relative phase shift of $\pi/2$ between the two logical eigenstates.

Something to think about:

- What angle θ does one need to choose to perform the identity operation? What happens for $\theta = 2\pi$?
- Can one construct a rotation around the z-axis from the $(\pi/2)_x$ and $(\pi/2)_y$ gates?

5.5 Overview of the IBM quantum experience environment

For the time being see the separate document on bcourses entitled "TLS Lab Additional Information".

6 Experimentation

6.1 Getting started

In this lab you will study the dynamics of a single qubit and if you have time perform some basic experiments on one of IBM's multi-qubit machines. For this you will need to have python3 and preferably jupyter notebook installed. Specific packages for python we will need are qiskit (a frame work to control quantum computers, see Ref. [5] for a description), numpy and matplotlib. You should have installed these packages already before the lab and made yourself familiar with the IBM quantum experience ecosystem.

Before beginning experimentation, we first observe that we will be working with a single quantum system, i.e. when we measure it we will get only one piece of information, i.e. which eigenstate of the measurement operator we have projected the system to. By convention, we usually measure in the logical basis, $\{|0\rangle, |1\rangle\}$, which typically coincides with the energy eigenbasis. In most cases, we also choose the z-axis in this direction, i.e. the qubit Hamiltonian (without interactions) is proportional to $\hat{\sigma_z} = |0\rangle\langle 0| - |1\rangle\langle 1|$), i.e. we will ask whether the system is in the $|0\rangle$ or the $|1\rangle$ state. In order to find expectation values, we will need to repeat the same experiment multiple times, typically between 100 and 1000 times. Thus a typical experiment has the flow indicated in Fig. 4.

6.2 Getting familiar with the our qubit

The IBM quantum experiences provides access to a multitude of quantum processors and simulators (so-called backends). Most of them provide an additional layer of abstraction in form of quantum gates and do not allow for low-level programming. In this lab, however, we want to understand the physics and thus need direct hardware access, such as setting the frequency, phase and amplitude of the microwave pulses controlling the qubit. The hardware backend we will use for the majority of our work is called 'ibmq_armonk'. While this backend is capable of processing our pulses, it uses only a single Josephson junction qubit. Thus we will need to use a different backend in case you decide to explore multi-qubit operations. Since jobs

[2] The Hadamard operation has determinant -1 and can thus not be a result of a physical operation $\exp\left(-i\hat{H}t\right)$ with determinant 1.

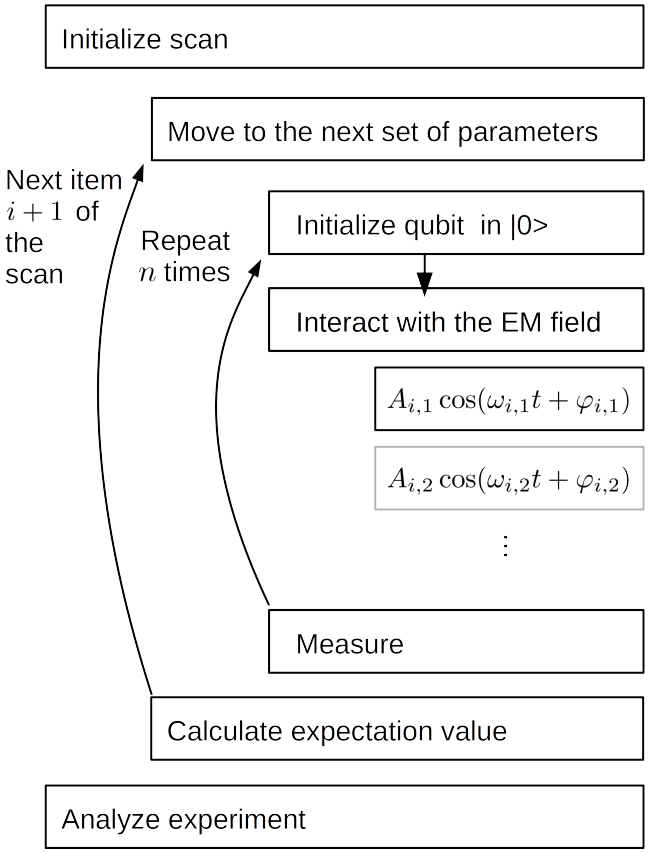

Figure 4: Flow of a typical quantum computing experiment.

submitted to a queue for the hardware backends may not be executed immediately (a typical wait time for completion of a job is two to five minutes for the ibmq_armonk backend), we recommend also using a simulated backend which simulates the behavior of qubits on our own computer. We recommend to use as the simulation backend 'open_pulse' provided by the 'Aer' package. The open_pulse backend can be programmed very similarly to the ibmq_armonk backend, but there are some important differences. We note that open_pulse does not yet support changing the frequency of the drive field during a single job (there are plans to add this feature in the near future). In addition, we also found no way to specify imperfections, i.e. decoherence. Still, the open_pulse backend allows us to test our programming fast, helps us to check our understanding and to make predictions for our experimental parts.

We will start now with programming our simulation backend. Goal will be study the dynamics of the qubit when irradiated with resonant microwaves at a constant amplitude for a given amount of time. For this, we will need to configure the backend.

To begin, we suggest to do part 1 of the tutorial provided by IBM: https://qiskit.org/textbook/ch-quantum-hardware/calibrating-qubits-openpulse.html. You can do this either within their online environment using a web-browser or use a copy of the notebook provided by the 111B lab here on bcourses. Make sure that you go through the tutorial step-by-step and that you understand what is done and how it is accomplished. Do not get hung up at details of the implementation yet, focus more on the concepts. Later when you will do your own experiments you can always go back and look up exactly how things are implemented.

One important part you will need to understand later is how to construct the actual program at the pulse level. For this it may be helpful to read Sec. II of https://arxiv.org/pdf/2004.06755.pdf.

6.3 Controlling the qubit

First we are going to verify some basic dynamics of the qubit. This understanding will be useful to construct the actual quantum gates, i.e. rotations around specific axes and rotation angles. As you may have noticed it takes a few minutes to receive the result from the armonk-backend, so we will first test with a simulation backend, called "pulse_simulator" from the "Aer" package. After testing and debugging our code, we will run it on the real machine. To make the test meaningful, we will need to write code which allows use to easily switch between simulation and execution on the real system. In the code snippet provided you can do this by modifying the flag `simulator` and running the corresponding cell. Note that you will need to execute the pulse sequence assembly as well as the actual submission of the pulse sequence after switching. The reason is that the measurement pulses for the simulator and the actual qubit differ, so the program for the backend needs to be reassembled and of course executed.

So lets get started! The probably simplest is to drive the qubit directly on resonance. In contrast to the IBM tutorial, we will use constant amplitude pulses and scan the time for which the drive is applied. The reason for this is that we are interested in following the time dynamics thereby providing a simpler physics interpretation[3]. The example code is called 'Getting-started-Rabi-flops.ipynb'. Before reviewing the code, you may run it. You should find that the qubit oscillates between states $|0\rangle$ and $|1\rangle$ sinusoidally as a function of the pulse length. Vary the amplitude of the pulse (variable `drive_amp`) and observe what changes.

After running a few simulations, run it now on the real system. Note that we can use the automatic calibration values for the qubit frequency, `qubit_lo_freq`, so we do not need to repeat the frequency calibration performed in the tutorial.

Questions you may ask yourself at this point:

- What is the value for the Rabi frequency for `drive_amp = 0.9`? What is the time required to perform a rotation around an axis in the x, y plane for a given drive amplitude? How do you expect that the Rabi frequency scales with the drive amplitude? Can you confirm this?

- It is interesting to observe how reproducible the qubit parameters, such as its frequency and Rabi frequency for a given drive strength are, i.e. log the value for a number of runs on the real machine to get a feel for how frequent recalibrations are necessary. Are they constant throughout the day, are they changing day by day?

Note that you do not need to follow all suggestions. Do what you find interesting, but try to develop a complete understanding. Further, you do not need to present all data in your write-up. Only include the most important plots (as a guide your report should not exceed 20 pages, quality before quantity!)

Next we want to understand what happens if we drive the qubit non-resonantly. Switch back to the simulation mode and change the frequency of the applied microwave radiation, i.e. the value of `qubit_lo_freq` (the qubit local oscillator frequency) in the function call `assemble` from the estimated qubit frequency by a certain amount.

Question to ask yourself:

- What size should this detuning have. For making a guess try to think about the relevant scales of the dynamics. Hints: There are (at least) two ways of how to come up with an estimate. The first is rather abstract and is to inspect the relevant Hamiltonian. The second is to think about the spectrum of a time varying signal lasting for a given amount of time.

Before delving into more quantum information related concepts, we will study the spectrum of the qubit. Often the goal of a spectroscopic measurement is to determine the resonance condition (for instance an atomic transition frequency) as precisely as possible. For this an excitation pulse is sent to the system and its response is recorded. Goal of this part of the lab is to understand how the relevant pulse parameters (strength and length) influence how precisely the resonant frequency can be extracted. What are your expectations. Would you rather choose a large strength to get precise results or a long interrogation time?

[3]The reason the IBM tutorial scans the amplitude instead of the pulse time is that they would like to use a Gaussian shaped pulse to suppress off-resonant excitations thereby reducing imperfections slightly.

Write a program recording the qubit excitation probability as a function of the frequency. Try out various parameter combinations on the simulator before choosing (wisely) a few sets to run on the real system to demonstrate how to perform the spectroscopy best.

Question to ask yourself:

- Can you explain the shape of the resulting spectra?

6.4 Rotating the qubit around various axes and Ramsey spectroscopy

So far we have varied the frequency, strength, and length of the excitation pulse. However, there is another parameter hidden in the excitation pulse. This is the phase of the excitation. So how does the phase of the excitation affect results? Give it a try? How do we set the phase? By this time you may have realized that the drive amplitude determines the Rabi frequency. If you inspect the Hamiltonian in Eq. 23 you may notice that we can absorb the phase into a complex Rabi frequency. Thus, we will set the phase by adding a complex phase φ to the amplitude supplied to the drive pulse in the `assemble` statement, i.e. we call it with `drive_amp np.exp(1j*relative_phase*np.pi` instead of with `drive_amp`.

So what is this phase doing? Is it relevant at all? For this it may be helpful to use an analogy from a driven (classical) harmonic oscillator: consider driving a harmonic oscillator initially at rest (corresponding to the qubit being prepared in an energy eigenstate) resonantly either with phase 0 or phase π. Either drive will excite the oscillator by exactly the same amount. However imagine that we first apply the tone with phase 0 and then the second tone with its phase shifted by π. The second tone will deexcite the oscillator all the way to rest (think about a swing and pushing in sync or out of sync). From this observation, we conjecture that the effect of the phase may become relevant if we have two excitation pulses with a different (relative) phase.

In order to explore this, we will have to add a second pulse before the measurement pulse. Write a program and try it. We suggest that you set the length of the first pulse such that the system creates roughly an equal superposition of the $|0\rangle$ and $|1\rangle$ states (why?) and then scan the phase of the second pulse.

Questions to ask yourself (or your partner):

- What is the unitary operation a resonant drive with phase φ implements?
- Can you follow the dynamics on the Bloch sphere?

Consider now the case of two short $\pi/2$-pulses (they rotate the system half-way between the north and south pole of the Bloch sphere) in phase, however, detuned from the qubit frequency by some amount Δ and delayed from each other by some time $\tau \ll t_{\pi/2}$. Study the dynamics as a function of the detuning and delay, both on the simulator and the real system. Questions:

- Explain the dynamics in the Bloch sphere picture and demonstrate how it is useful for spectroscopy.
- What happens if the delay becomes very long (lets say 50-100 µs) on the real system. Can you explain your observations? Note that you will need to adjust your detuning. Alternatively, you can vary the phase for a constant delay and observe the resulting fringes.

Finally, we have nearly all understanding in place to perform an arbitrary unitary operation on our qubit. You may ask yourself: how would you implement a rotation around the x, y, and z-axis?

6.5 Quantifying the fundamental noise properties of the qubit

In the previous section, you likely saw that noise limits the controllability of the qubit. In particular, if noise has a long time to disturb the quantum states, we expect that the actual qubit dynamics will deviate from the ideal one. Typically, we distinguish two types of noise sources. The first type of noise transfers population between the energy eigenstates. An example are noisy fields with spectral components near the qubit frequency) or so-called relaxation processes where the qubit looses energy to the environment. The second noise source affects the qubit frequency or the local oscillators. This disturbs the phase relation

between the driving field and the state of the qubit which is problematic as you have seen that the phase relation affects the dynamics if you do not start in an energy eigenstate. We can think of such phase noise (drifts) also as introducing some random detuning Δ. In quantum information, we borrow the language from NMR and call the former noise process affecting the energy of the system a T_1 process, where T_1 characterizes the time it takes for the population to change. For a spontaneous decay it would be the time constant in the exponential process. The second noise process refers to pure dephasing of the qubit and the associated time constant is T_φ. Measuring T_φ directly is difficult as the phase coherence is also affected by T_1 processes, i.e. if the qubit changes its energy eigenstate due to uncontrolled interactions it also looses its phase coherence. Thus, we introduce the T_2^* process which is a combination of T_1 and T_φ according to $1/T_2^* = 1/(2T_1) + 1/T_\varphi$.

Measure T_1 and T_2^* for the qubit of the real system. Extract T_φ.

Follow up question to think about:

- How many single-qubit operations can one hope to perform before the qubit has decohered?

Finally, we turn to the last single-qubit exercise. The previous estimates for the qubit phase coherence are somewhat pessimistic. In particular, it may be that the qubit frequency itself is stable, but that we just calibrated the qubit frequency wrongly or too long ago. "Long ago" in this context can really mean a few milliseconds ago. Indeed, in practice it turns out that the qubit frequency is actually quite stable on short times, but fluctuates on the second, minute or hour time scale. Since taking data involves many runs, the qubit frequency may have changed between the beginning and the end of data taking and wash out the interference fringes.

There is a trick to utilize the fact that the qubit frequency may be stable during a single experiment shot. Consider for instance an experiment where we create a superposition state $(|0\rangle + i\,|1\rangle)/\sqrt{2}$. In the rotating frame, the phase of the state will evolve as $(|0\rangle + ie^{i\delta\Delta t}\,|1\rangle)/\sqrt{2}$ where $\delta\Delta$ is a slowly fluctuating detuning Δ constant during one repetition (shot) of the experiment. What if we could switch the roles of $|0\rangle$ and $|1\rangle$ in the middle of the experimental sequence, i.e. at $t = T_{1/2}$? Then the state $(|0\rangle + ie^{i\delta\Delta T_{1/2}}\,|1\rangle)/\sqrt{2}$, would become $(|1\rangle + ie^{i\delta\Delta T_{1/2}}\,|0\rangle)/\sqrt{2}$ and after evolving for another time $t = T_{1/2}$ we would arrive at $(e^{i\delta\Delta T_{1/2}}\,|0\rangle + ie^{i\delta\Delta T_{1/2}}\,|1\rangle)/\sqrt{2} = e^{i\delta\Delta T_{1/2}}(|0\rangle + i\,|1\rangle)/\sqrt{2}$. Program a sequence implementing this dynamics and extract the so-called T_2 time of the qubit, i.e. the time constant the interference contrast decay in this spin-echo sequence.

6.6 Further ideas: single-qubit gate imperfections

First note that those are just suggestions on how you may (or may not) expand.

- One important aspect is the speed with which we can execute the operations. From the Fourier limit we expect that if the pulse is too short, we will can also excite other resonances, for instance, the higher states in our anharmonic qubit oscillator. Look whether you can find evidence for this. Can you mitigate it by using non-rectangular pulse shapes, such as Gaussian ones?

- Another important source for imperfections are amplitude fluctuations of the driving field. Can you find evidence and quantify these fluctuations. For this you may want to apply many pulses and observe the quality of the output states as compared to your expectations. How stable is the output? What does that mean for the number of gates we can run on this device?

6.7 Further ideas: Multi-qubit states

- You may use pretty much any of the other back-ends and use the IBM-quantum experience to create Bell states, GHZ states, the Deutsch-Josza or other algorithms.

7 Appendix A: solving for the time-dynamics of a two-level system interacting with an electromagnetic drive field

7.1 Overview of solving for the dynamics of a two-level system interacting with an electromagnetic field

In the main text, we discussed how we will first transform into the rotating frame to obtain a time independent Hamiltonian. We note that this method of transforming into the right frame is very useful in a variety of situations and that much more abstract frames can be used. We also will start with a circularly polarized field such that the solution is exact before then moving on to the practically more relevant case of linear polarization by performing the so-called rotating wave approximation.

7.2 Defining the Hamiltonian

For concreteness, we consider an electron spin (spin 1/2, magnetic moment $g\mu_{\mathrm{B}}$) interacting with a strong magnetic field $\vec{B_0} = (0, 0, B_z)$ in the z direction as well as with a much weaker magnetic field $\vec{B}_{\mathrm{rad}} = (B_1 \cos\omega_{\mathrm{rad}}t,\, B_1 \sin\omega_{\mathrm{rad}}t,\, 0)$ rotating in the x, y plane with speed ω_{rad}. We are interested in understanding the spin's time evolution and how it can be manipulated. Note that all results can be easily generalized for any qubit.

The first question we are asking ourselves is: What is the corresponding Hamiltonian and Schrödinger equation in its explicit matrix formulation?

For the energy of such a system we find:

$$
\begin{aligned}
H = & \quad \tfrac{\mu_B}{2}\vec{\sigma}\cdot\vec{B} && (15)\\
= & \quad \tfrac{\mu_B}{2}\left[\begin{pmatrix} 0 & B_1\cos(\omega_{\mathrm{rad}}t) \\ B_1\cos(\omega_{\mathrm{rad}}t) & 0 \end{pmatrix}\right. && (16)\\
& \quad + \begin{pmatrix} 0 & -iB_1\sin(\omega_{\mathrm{rad}}t) \\ iB_1\sin(\omega_{\mathrm{rad}}t) & 0 \end{pmatrix} && (17)\\
& \quad \left. + \begin{pmatrix} B_z & 0 \\ 0 & -B_z \end{pmatrix}\right] && (18)\\
= & \quad \tfrac{\mu_B}{2}\begin{pmatrix} B_z & B_1\cos(\omega_{\mathrm{rad}}t) - iB_1\sin(\omega_{\mathrm{rad}}t) \\ B_1\cos(\omega_{\mathrm{rad}}t) + iB_1\sin(\omega_{\mathrm{rad}}t) & -B_z \end{pmatrix}. && (19)
\end{aligned}
$$

To write the matrix form of the Schrodinger equation, we'll need to write the state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ as a two component spinor. Let us define

$$
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$
$$
|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

and hence the state $|\psi\rangle = a\,|1\rangle + b\,|0\rangle$ can be written as

$$
|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.
$$

Hence the Schrodinger equation is (note we work in units such that $\hbar = 1$)

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\psi\rangle =& H\,|\psi\rangle \\
i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} =& \frac{\mu_B}{2}\begin{pmatrix} B_z & B_1\cos(\omega_{\mathrm{rad}}t) - iB_1\sin(\omega_{\mathrm{rad}}t) \\ B_1\cos(\omega_{\mathrm{rad}}t) + iB_1\sin(\omega_{\mathrm{rad}}t) & -B_z \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.
\end{aligned}
$$

7.3 Rotating frames

At this point, we could solve the differential equation directly, however, to build intuition and to help with future more complicated scenarios, we will now use a trick and transform into a rotating frame such that the Hamiltonian becomes time-independent. The transformation is motivated by the observation that for classically evolving magnetic moment we could apply a basis transformation such that the rotating frame magnetic field becomes stationary. That is we would expect that the spin of an electron in the $(|0\rangle+|1\rangle)/\sqrt{2}$-state whose spin points in the direction of the rotating magnetic field $\vec{B}_{\text{rad}}$ at $t = 0$ would always remain aligned with $\vec{B}_{\text{rad}}$ simplifying the problem considerably.
First we are going to understand how the Schrödinger equation changes under a general basis transformation. This will be the key result which will allow us to apply this method to other situations. Consider a general basis transformation such that

$$|\Psi\rangle_{\text{rot}} = V\,|\Psi\rangle_{\text{lab}} \tag{20}$$

with V describing the basis transformation from the laboratory frame to the rotating frame and V^{-1} going from the rotating frame into the laboratory frame. In the new basis, we can rewrite the Schrödinger equation. The question will be how this Hamiltonian will look like in this basis?

We will start from the Schrodinger equation in the lab frame, $i\frac{\partial}{\partial t}|\psi\rangle_{\text{lab}} = H_{\text{lab}}|\psi\rangle_{\text{lab}}$, and substitute in $|\psi\rangle_{\text{lab}} = V^{-1}|\psi\rangle_{\text{rot}}$. We then reorganize the terms until it looks like the Schrodinger equation again, but with a different Hamiltonian.

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\psi\rangle_{\text{lab}} &= H_{\text{lab}}|\psi\rangle_{\text{lab}} \\
i\hbar\frac{\partial}{\partial t}\left(V^{-1}|\psi\rangle_{\text{rot}}\right) &= H_{\text{lab}}V^{-1}|\psi\rangle_{\text{rot}} \\
i\hbar\left(\frac{\partial}{\partial t}V^{-1}\right)|\psi\rangle_{\text{rot}} + i\hbar V^{-1}\frac{\partial}{\partial t}|\psi\rangle_{\text{rot}} &= H_{\text{lab}}V^{-1}|\psi\rangle_{\text{rot}} \\
i\hbar\frac{\partial}{\partial t}|\psi\rangle_{\text{rot}} &= VH_{\text{lab}}V^{-1}|\psi\rangle_{\text{rot}} - i\hbar V\left(\frac{\partial}{\partial t}V^{-1}\right)|\psi\rangle_{\text{rot}} \\
i\hbar\frac{\partial}{\partial t}|\psi\rangle_{\text{rot}} &= \left(VH_{\text{lab}}V^{-1} - i\hbar V\frac{\partial}{\partial t}V^{-1}\right)|\psi\rangle_{\text{rot}}\ .
\end{aligned}$$

This is the Schrodinger equation with Hamiltonian

$$H_{\text{rot}} \equiv VH_{\text{lab}}V^{-1} - i\hbar V\frac{\partial}{\partial t}V^{-1}\,. \tag{21}$$

We note that this is a rather general result used quite often in quantum optics, atomic physics and quantum information.

7.4 Finding the right frame of reference

Our next goal is now to find V such that the Hamiltonian in Eq. 15 becomes time-independent. From our intuition, we know that we will need to transform into a frame rotating with angular velocity ω_{rad} such that the excitation field looks constant. But what does mean for the unitary basis transformation V? To get an idea, we observe that a superposition state $(|0\rangle + |1\rangle)/\sqrt{2}$ in a constant magnetic field B_z will time evolve as $|\Psi(t)\rangle_{\text{lab}} = (e^{-i\frac{\mu B_z}{2}t}|0\rangle_{\text{lab}} + e^{i\frac{\mu B_z}{2}t}|1\rangle_{\text{lab}})/\sqrt{2}$ in the laboratory frame. However, in the rotating frame we would expect no time evolution, i.e. $|\Psi(t)\rangle_{\text{rot}} = (|0\rangle_{\text{rot}} + |1\rangle_{\text{rot}})/\sqrt{2}$. A basis transformation of the form $V' = \text{diag}(e^{i\frac{\mu B_z}{2}t}, e^{-i\frac{\mu B_z}{2}t})$ will accomplish the desired transformation such that $|\psi\rangle_{\text{rot}}$ becomes time independent. In other words V' transforms into the frame co-rotating with the Larmor precession frequency $\omega_{\text{L}} = \mu B_z/\hbar$ (we inserted $\hbar$ for clarity). Thus, we postulate that the transformation

$$V = \begin{pmatrix} e^{i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \tag{22}$$

will do the job.

7.5 Solving the Schrödinger equation

Thus, we find for the Hamiltonian Eq. 15 in the frame rotating with the excitation field:

$$
\begin{aligned}
H_{\text{rot}} &= V H_{\text{lab}} V^{-1} - iV\tfrac{\partial}{\partial t}V^{-1} \\
&= \begin{pmatrix} e^{i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \begin{pmatrix} \frac{\omega_0}{2} & \mu B_1\cos(\omega_{\text{rad}}t) - i\mu B_1\sin(\omega_{\text{rad}}t) \\ \mu B_1\cos(\omega_{\text{rad}}t) + i\mu B_1\sin(\omega_{\text{rad}}t) & -\frac{\omega_0}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \\
&\quad - i\begin{pmatrix} e^{i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \begin{pmatrix} (-i\frac{\omega_{\text{rad}}}{2})e^{-i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & (i\frac{\omega_{\text{rad}}}{2})e^{i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\omega_0}{2} & e^{i\omega_{\text{rad}}t}\left(\mu B_1\cos(\omega_{\text{rad}}t) - i\mu B_1\sin(\omega_{\text{rad}}t)\right) \\ e^{-i\omega_{\text{rad}}t}\left(\mu B_1\cos(\omega_{\text{rad}}t) + i\mu B_1\sin(\omega_{\text{rad}}t)\right) & -\frac{\omega_0}{2} \end{pmatrix} - \begin{pmatrix} \frac{\omega_{\text{rad}}}{2} & 0 \\ 0 & \frac{-\omega_{\text{rad}}}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\Delta}{2} & e^{i\omega_{\text{rad}}t}\left(\mu B_1\cos(\omega_{\text{rad}}t) - i\mu B_1\sin(\omega_{\text{rad}}t)\right) \\ e^{-i\omega_{\text{rad}}t}\left(\mu B_1\cos(\omega_{\text{rad}}t) + i\mu B_1\sin(\omega_{\text{rad}}t)\right) & \frac{\Delta}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\Delta}{2} & \mu B_1 \\ \mu B_1 & \frac{\Delta}{2} \end{pmatrix} .
\end{aligned}
$$

In this frame, the $x-y$ magnetic field stays fixed along the $x-axis$ (we see this from the fact that the off diagonal terms are fixed at μB. However, it is like there's a small magnetic field pointing in the $-z$ direction that rotates states around the $z-axis$ (in a left-handed manner) with frequency Δ, i.e. the detuning of the excitation from resonance. H_{rot} is time-independent, which is a very nice feature!

Experimentally, it is simpler to apply a linearly polarized field $\vec{B}_{\text{rad}} = (B_1\cos\omega_{\text{rad}}t, 0, 0)$ rather than the rotating (circularily polarized) field assumed above.

The Hamiltonian in the rotating frame is now given as:

$$
\begin{aligned}
H_{\text{rot}} =& V H_{\text{lab}} V^{-1} - iV\frac{\partial}{\partial t}V^{-1} \\
&= \begin{pmatrix} e^{i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \begin{pmatrix} \frac{\omega_0}{2} & \mu B_1\cos(\omega_{\text{rad}}t) \\ \mu B_1\cos(\omega_{\text{rad}}t) & -\frac{\omega_0}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \\
&\quad - i\begin{pmatrix} e^{i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \begin{pmatrix} (-i\frac{\omega_{\text{rad}}}{2})e^{-i\frac{\omega_{\text{rad}}}{2}t} & 0 \\ 0 & (i\frac{\omega_{\text{rad}}}{2})e^{i\frac{\omega_{\text{rad}}}{2}t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\omega_0}{2} & e^{i\omega_{\text{rad}}t}\mu B_1\cos(\omega_{\text{rad}}t) \\ e^{-i\omega_{\text{rad}}t}\mu B_1\cos(\omega_{\text{rad}}t) & -\frac{\omega_0}{2} \end{pmatrix} - \begin{pmatrix} \frac{\omega_{\text{rad}}}{2} & 0 \\ 0 & \frac{-\omega_{\text{rad}}}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\Delta}{2} & \frac{\mu B_1}{2}e^{i\omega_{\text{rad}}t}\left(e^{i\omega_{\text{rad}}t} + e^{-i\omega_{\text{rad}}t}\right) \\ \frac{\mu B_1}{2}e^{-i\omega_{\text{rad}}t}\left(e^{i\omega_{\text{rad}}t} + e^{-i\omega_{\text{rad}}t}\right) & \frac{\Delta}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\Delta}{2} & \frac{\mu B_1}{2}(1 + e^{2i\omega_{\text{rad}}t}) \\ \frac{\mu B_1}{2}(1 + e^{-2i\omega_{\text{rad}}t}) & \frac{\Delta}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\Delta}{2} & \frac{\mu B_1}{2} \\ \frac{\mu B_1}{2} & \frac{\Delta}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{\mu B_1}{2}e^{2i\omega_{\text{rad}}t} \\ \frac{\mu B_1}{2}e^{-2i\omega_{\text{rad}}t} & 0 \end{pmatrix} .
\end{aligned}
$$

We consider $\omega_{\text{rad}} \approx \omega_0$, such that Δ is very small as compared to ω_0. Given that $|\vec{B}_1| \ll |\vec{B}_0|$, then the terms $e^{2i\omega_{\text{rad}}t}$ are oscillating much faster than anything else in the problem. In the last line, we wrote these terms as a separate matrix contribution to the Hamiltonian. We can convince ourselves that this term can be dropped by considering the time-averaged effect of this Hamiltonian, averaged over a time $t = \frac{1}{2\omega_{\text{rad}}}$. This is a very small time compared to the frequency of rotation about this axis, μB_1. Hence in one period $T = 2\omega_{\text{rad}}$, states will rotate slightly in one direction, and then slightly back in exactly the opposite direction a small time later. A more careful demonstration of this fact could come from integrating the effect of this part of the Hamiltonian over one period $T = 2\omega_{\text{rad}}$. For now, we will drop these terms to give ourselves the

Hamiltonian

$$H_{\text{rot}} \approx \begin{pmatrix} -\frac{\Delta}{2} & \frac{\mu B_1}{2} \\ \frac{\mu B_1}{2} & \frac{\Delta}{2} \end{pmatrix} .$$

Except a factor of 2 in tn the off-diagonals, this is the same time-independent Hamiltonian we had from a circularly-polarized field in the previous part!

We now need to understand one last bit, i.e. the significance of the phase of the field. For this, we need to repeat the derivation with $\vec{B}_{\text{rad}} = (B_1 \cos(\omega_{\text{rad}} t + \varphi), 0, 0)$. Using $\cos(\omega_{\text{rad}} t + \varphi) = e^{-i\varphi} \frac{e^{i(\omega_{\text{rad}} t + 2\varphi)} + e^{-i\omega_{\text{rad}} t}}{2}$, we find the Hamiltonian in the rotating frame is given by:

$$H_{\text{rot}} = \begin{pmatrix} -\frac{\Delta}{2} & \frac{\hbar\mu B_1}{2} e^{-i\varphi} \\ \frac{\hbar\mu B_1}{2} e^{i\varphi} & \frac{\Delta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\Delta}{2} & \frac{\Omega}{2} e^{-i\varphi} \\ \frac{\Omega}{2} e^{i\varphi} & \frac{\Delta}{2} \end{pmatrix} , \tag{23}$$

where we have introduced the so-called Rabi frequency $\Omega = \mu B_1/\hbar$, i.e. the rotation speed of the magnetization in the rotating frame This is the Hamiltonian of a spin-1/2 system with magnetic moment μ in a magnetic field given by $(\mu B_1 \cos\varphi, \mu B_1 \sin\varphi, \Delta)$ We also see that this can be written as

$$H_{\text{rot}} = \Omega(\cos\varphi S_x + \sin\varphi S_y) + \Delta S_z.$$

. So the time evolution is a rotation about the axis $(\cos\varphi, \sin\varphi, \Delta/\mu B_1)$ with angular velocity $\Omega_{\text{eff}} = \sqrt{\Omega^2 + \Delta^2} t$. Ω_{eff} is sometimes also called the effective Rabi frequency. This shows how we can rotate our two-level system around any axis by any desired amount.

7.6 More about the phase

Because of time-translation symmetry of physics, the phase of the field needs to be defined with respect to something, i.e. the phase of an oscillation requires us to pick a specific point in time. This something to which the phase of the drive is referenced is the phase of the qubit at the starting point of the experiment. However, the quibt phase is only meaningful in if the qubit starts in superposition of two energy eigenstates. When we think about a typical experiment starting in an energy eigenstate, it does not matter with what phase you excite the system, i.e. it does not matter when you do the experiment for a given trajectory of the driving field. The result (in terms of a measurement in the eigenbasis) will be independent with which phase you excite the qubit. This makes a lot of sense since you can shift the phase of an oscillator by just starting the experiment some time later. So what is the role of the phase of the driving field? Experimentally, the canonical method to create the superposition necessary to give the driving fields phase some meaning is to excite the qubit from an energy eigenstate with another radiation pulse, presumably resonant with the qubit frequency. Thus, the phase of the following pulse can be measured relative to the phase of the first pulse generating the superposition in the first place. You will see this principle in action when you do Ramsey spectroscopy.

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