Physics 129: Particle Physics
Lecture 2: Review of non-Relativistic Quantum Mechanics

Sept 3, 2020

- Suggested Reading:
  - Thomson Section 2.3
  - Selected topics in Bransden and Joachain (B& J) Quantum Mechanics 2^{nd} edition
- Today's lecture follows the format and notation from Thomson

  This is a brief review of terminology and results

  Please let me know if topics discussed are not familiar to you

Wave Mechanics

- Free particles described as wave packets
- Decomposed into Fourier integral of plane waves:

$$\psi(\vec{x}, t) = N \exp\left[i\left(\vec{p} \cdot \vec{x} - Et\right)\right]$$

  where we have used natural units ($\hbar = c = 1$)
- Here normalization N ensures for wave function (wf) $\psi(\vec{x}, t)$

$$\int \psi^*(\vec{x}, t)\psi(\vec{x}, t) d^3x = 1$$

- Physical observable A obtained by applying operator $\hat{A}$ to w.f.
- Eigenstates of $\hat{A}$ given by

$$\hat{A}\psi(\vec{x}, t) = a\psi(\vec{x}, t)$$

- Important operators:
  - Momentum $\hat{\mathbf{p}} = -i\nabla$
  - Energy $\hat{E} = i\frac{\partial}{\partial t}$
  - Angular Momentum $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$

  where **bf** indicates a 3-vector

Schrodinger Equation

- Hamiltonian operator gives total energy of the system
- For particle of mass m:

$$\begin{aligned} H &= KE + V \\ \hat{H} &= \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V} \end{aligned}$$

- Using operators from previous page:

$$\begin{aligned} \hat{E}\psi(\vec{x},t) &= \hat{H}\psi(\vec{x},t) \\ i\frac{\partial}{dt}\psi(\vec{x},t) &= -\frac{1}{2m}\nabla^2\psi(\vec{x},t) + \hat{V}\psi(\vec{x},t) \end{aligned}$$

Might look unfamiliar in natural units!

Probability Densities and Probability Current (I)

- Probability of observing a particle within volume d^3x around point $\vec{x}$ at time t is:

$$P(\vec{x},t)d^3x = |\psi(\vec{x},t)|^2 \, d^3x$$

(note: $d^3x \equiv dxdydz \equiv r^2 dr d\cos\theta d\phi$)

- Change in probability with time:

$$\begin{aligned}\frac{\partial}{\partial t}\int P(\vec{x},t)d^3x &= \frac{\partial}{\partial t}\int \psi^*(\vec{x},t)\psi(\vec{x},t)d^3x \\ &= \int \left[\frac{\partial\psi^*(\vec{x},t)}{\partial t}\psi(\vec{x},t) + \psi^*(\vec{x},t)\frac{\partial\psi(\vec{x},t)}{\partial t}\right] d^3x\end{aligned}$$

- But Schrodinger Eq says

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\vec{x},t) &= \left[-\frac{1}{2m}\nabla^2 + V(\vec{x},t)\right]\psi(\vec{x},t) \\ -i\frac{\partial}{\partial t}\psi^*(\vec{x},t) &= \left[-\frac{1}{2m}\nabla^2 + V(\vec{x},t)\right]\psi^*(\vec{x},t)\end{aligned}$$

- So, the expression above becomes

$$\begin{aligned}\frac{\partial}{\partial t}\int P(\vec{x},t)d^3x &= \frac{i}{2m}\int \left[\psi^*(\vec{x},t)(\nabla^2\psi(\vec{x},t)) - (\nabla^2\psi^*(\vec{x},t))\psi(\vec{x},t)\right] d^3x \\ &= \frac{i}{2m}\int \vec{\nabla}\cdot\left[\psi^*(\vec{x},t)(\vec{\nabla}\psi(\vec{x},t)) - (\vec{\nabla}\psi^*(\vec{x},t))\psi(\vec{x},t)\right] d^3x \\ &= -\int \vec{\nabla}\cdot\vec{j}(\vec{x},t)d\vec{x}\end{aligned}$$

where $j \equiv \frac{i}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$

Probability Densities and Probability Current (II)

- From previous page:

$$\frac{\partial}{\partial t}\int P(\vec{x},t)d^3x = -\int \vec{\nabla}\cdot\vec{j}(\vec{x},t)d^3x$$

where $j \equiv \frac{i}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$

- This leads to the equation

$$\begin{aligned} \frac{\partial}{\partial t}\int P(\vec{x},t) + \int \vec{\nabla}\cdot\vec{j}(\vec{x},t)d^3x &= 0 \\ \frac{\partial}{\partial t}P(\vec{x},t) + \vec{\nabla}\cdot\vec{j}(\vec{x},t) &= 0 \end{aligned}$$

- This is called the continuity equation and if we multiply by charge is familiar from E&M
- $\vec{j}(\vec{x},t)$ can be interpreted as a probability current density
- We'll use the continuity eq in week 4 to describe scattering of particles and in week 7 when we discuss the Dirac Eq

bra-ket notation

- Write wf in more compact form

$$\psi(\vec{x}, t) \Rightarrow |\psi\rangle$$

- Hermitian conjugate becomes

$$\psi^\dagger \equiv (\psi^*)^T \Rightarrow \langle\psi|$$

- Expectation value of operator $\hat{A}$:

$$\left\langle \hat{A} \right\rangle \equiv \langle\psi| \hat{A} |\psi\rangle$$

- Taking derivation wrt time:

$$\frac{d\left\langle \hat{A} \right\rangle}{dt} = \left\langle \left[\hat{H}, \hat{A}\right] \right\rangle$$

where we use the chain rule and Schodinger eq to prove this

$\Rightarrow$ Operators that commute with the Hamiltonian are conserved!

Angular Momentum

- Write orbital angular momentum as **L**, spin as **S** and total angular momentum as **J**
- Commutation relations for orbital angular momentum can be derived from operator expression $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$
- Result:

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z \qquad \left[\hat{L}_y, \hat{L}_z\right] = i\hat{L}_x \qquad \left[\hat{L}_z, \hat{L}_x\right] = i\hat{L}_y$$

$$\equiv \left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k$$

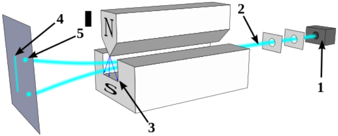
- All 3 components of $\hat{L}$ commute with $\hat{L}^2$, but since the 3 components don't commute with each other, can only have simulataneous eigenstates of $\hat{L}^2$ and one component, typically taken as $\hat{L}_z$

$$\left[\hat{L}^2, \hat{L}_z\right] = 0$$

- Eigenstates $|\ell, m_\ell\rangle$ have eigenvalues

$$\begin{aligned} \hat{L}^2 |\ell, m_\ell\rangle &= \ell(\ell+1)|\ell, m_\ell\rangle \\ \hat{L}_z |\ell, m_\ell\rangle &= m_\ell |\ell, m_\ell\rangle \end{aligned}$$

Spin



- Stern-Gerlach experiment: Silver atoms in a non-uniform B field

$$\vec{B} = B_z(z)\hat{z}$$
$$\Rightarrow \vec{F} = \nabla\left(\vec{\mu}\cdot\vec{B}\right) = \mu_z\frac{\partial B_z}{dz}$$

- Two lines with displacement proportional to B
- Two lines implies $S = \frac{1}{2}$
- Consistent with atom having a magnetic moment

$$\vec{\mu} = \frac{q}{2m}\vec{L} + g\frac{q}{2m}\vec{S}$$

- $\vec{S}$ is the intrinsic spin of the particle and is a fundamental quantum property.
- g depends on type of particle
- Spin-$\frac{1}{2}$ particles with no substructure have $g = 2$
  - QED give small, calculable corrections to g
- For a proton, $g = 5.58$ (indicating it has substructure)

Is spin a form of angular momentum?

- To answer this, we need to know:
  1. Does $\vec{S}$ satisfy the same commutation relation as orbital angular momentum:
  $$[S_i, S_j] = i\epsilon_{ijk}S_k$$
  2. If a system has orbital angular momentum $\vec{L}$ and spin $\vec{S}$, can we add these together to form a total angular momentum $\vec{J} = \vec{L} + \vec{S}$ that satisfies the same commutation relation:
  $$[J_i, J_j] = i\epsilon_{ijk}J_k$$
- Experiment demonstrates answer is YES to both questions
- Spin is another form of angular momentum
- It is *total* angular momentum that is conserved for a closed system

Pauli Matrices

- We can write
$$\vec{S} = \frac{1}{2}\vec{\sigma}$$
Where $\vec{\sigma}$ is a set of three 2×2 matrices called the Pauli matrices:
$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

- With two eigenstates
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
that correspond to $|\uparrow\rangle$ and $|\downarrow\rangle$

- In simular way, for a state with angular momentum ℓ we can define threee $(2\ell + 1) \times (2\ell + 1)$ matrices and ℓ column vectors

Addition of Angular Momentum (I)

- Two angular momenta J_1 and J_2 can be combined

$$\begin{aligned} J_{tot}^2 &= (J_1 + J_2)^2 \\ &= J_1^2 + J_2^2 + 2J_1 \cdot J_2 \\ &= J_1^2 + J_2^2 + J_{1+}J_{2-} + J_{1-}J_{2+} + J_{1z}J_{2z} \end{aligned}$$

where J_+ and J_- are the raising and lowering ladder operators

$$\begin{aligned} J_+ &= J_x + iJ_y \\ J_- &= J_x - iJ_y \\ J + |j, m_j\rangle &= \sqrt{j(j+1) - m_j(m_j+1)}\, |j, m_{j+1}\rangle \\ J - |j, m_j\rangle &= \sqrt{j(j+1) - m_j(m_j-1)}\, |j, m_{j-1}\rangle \end{aligned}$$

- j_{tot} runs from $j_1 + j_2$ to $|j_1 - j_2|$

Addition of Angular Momentum (II)

- Let's look at two spin-$\frac{1}{2}$ particles as an example
- Uncoupled basis: $|S_1\rangle = \left|\frac{1}{2} m_1\right\rangle$, $|S_2\rangle = \left|\frac{1}{2} m_2\right\rangle$ where $m = \pm\frac{1}{2}$
- Coupled basis can have $S = 1$ or $S = 1$
- Construct a "stretch state" where there is only one option:

$$S_{tot,Z}\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle = (S_{1z} + S_{2z})\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle = +1\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle$$

- Apply lowering operator:

$$\begin{aligned}
S_{tot,-}\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle &= (S_{1-} + S_{2-})\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle \\
S_{tot,-}\left|s = 1, m_s = 1\right\rangle &= (S_{1-} + S_{2-})\left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle \\
\sqrt{1(2) - 1(0)}\left|s = 1, m_s = 0\right\rangle &= \sqrt{\frac{3}{4} + \frac{1}{4}}\left(\left|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle + \left|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}\right\rangle\right) \\
\left|s = 1, m_s = 0\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle + \left|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}\right\rangle\right)
\end{aligned}$$

- Apply again:

$$\left|s = 1, m_s = -1\right\rangle = \left|m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle$$

- Find one remaining state using orthogonality:

$$\left|s = 0, m_s = 0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle - \left|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}\right\rangle\right)$$

Hamiltonians that change with time

- There are many problems where $\hat{H}$ depends on time, eg:
  - Turn on an E field at time $t = 0$
  - A particle decays at time $t = 0$
  - A charged particle is in a region with an EM wave
- We'll study problems of the form

$$\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$$

  (eg, H_0 might be the kinetic energy term)
- If we start at time $t = t_0$ in a known state $|\psi(t = t_0)\rangle$ how does the w.f. change for later times?
- Several techniques to approach such problems:
  1. Time dependent Perturbation Theory.:

$$\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$$

     with λ small
  2. Sudden approximation: $\hat{H}'(t)$ turns on abruptly and is constant after turn-on
  3. Adiabatic approximation: $\hat{H}'(t)$ changes very slowly with time

Time Dependent Perturbation Theory: Two Equivalent Descriptions

- Writing $\hat{H}(t)$ factoring out λ to indicate that $\hat{H}'(t)$ is small::

$$\hat{H}(t) = \hat{H}_0 + \lambda \hat{H}'(t)$$

  If we start at time $t = t_0$ in a known state $|\psi(t = t_0)\rangle$ how does the w.f. change for later times?

- We can describe this problem in two ways (the math is the same although the words are different)
  1. Expand wf at t_0 in $\hat{H}_0$ basis. Since H' varies with time, eigenstates also vary with time. Calculate this time dependence allowing the coefficients of the expansion to vary with time as well.
  2. The $\hat{H}'$ term is changing the energy of our state as a function of time (pumping energy in or taking energy out of the original system). Adding or removing energy allows the particle to jump between eigenstates of the original $\hat{H}_0$, Such a jump is called a *transition*.

- In both descriptions we describe the particle in the original basis

  Note: As in time indep case, we'll have to worry about degeneracies.

Using description #1 from the previous page

- Time dependence of $\hat{H}_0$ can be written

$$\psi_n(\vec{x}, t) = \psi_n(\vec{x}) e^{-i\omega_n t}$$

where

$$\hat{H}_0 \psi_n = E_n^{(0)} \psi_n = \omega_n \phi_n$$

- Expand a general time dependent $\Psi(\vec{x}, t)$ in terms of this basis:

$$\Psi(\vec{x}, t) = \sum_n C_n(t) \psi_n(\vec{x}, t)$$

Note: we are assuming that $\hat{H}'(t)$ can be described using same basis as $\hat{H}_0$. If $H'(t)$ involves different degrees-of-freedom (eg spin) must sure that the ψ_n include an outer product with basus states of the new degrees of freedom

- We'll expand our time-dependent Ψ using this basis and then apply Schrodinger Equation
  - As with time-indep perturb theory, expand in powers of λ

Solution using time dependent Schrodinger Eq

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Psi(\vec{x},t) &= \hat{H}\Psi(\vec{x},t) \\
i\frac{\partial}{\partial t}\sum_n C_n(t)\psi_n(\vec{x},t) &= \left(\hat{H}_0 + \lambda\hat{H}'(t)\right)\sum_n C_n(t)\psi_n(\vec{x},t) \\
i\sum_n\left(\frac{d}{dt}C_n(t)\right)\psi_n(\vec{x},t) + C_n(t)\left(-i\omega_n\psi_n(\vec{x},t)\right. &= \sum_n\left(\omega_n C_n\psi_n(\vec{x},t) + \right. \\
&\quad \left.\lambda\hat{H}'(t)C_n(t)\psi_n(\vec{x},t)\right) \\
i\sum_n\left(\frac{d}{dt}C_n(t)\right)\psi_n(\vec{x},t) &= \sum_n\lambda\hat{H}'(t)C_n(t)\psi_n(\vec{x},t) \\
i\sum_n\left(\frac{d}{dt}C_n(t)\right)\psi_n(\vec{x})e^{-i\omega_n t} &= \sum_n\lambda\hat{H}'(t)C_n(t)\psi_n(\vec{x})e^{-i\omega_n t}
\end{aligned}
$$

Now take scalar product with $\langle\psi_m^*(\vec{x})|$:

$$
\begin{aligned}
i\sum_n\frac{d}{dt}C_n(t)\delta_{nm}e^{-i\omega_n t} &= \sum_n\lambda\left\langle\psi_m\right|H'(t)\left|\psi_n\right\rangle e^{-i\omega_n t}C_n(t) \\
i\frac{d}{dt}C_n(t)\delta_{nm}e^{-i\omega_n t} &= \lambda\sum_n\left\langle\psi_m\right|H'(t)\left|\psi_n\right\rangle e^{-i\omega_n t}C_n(t) \\
i\frac{d}{dt}C_n(t)\delta_{nm} &= \lambda\sum_n\left\langle\psi_m\right|H'(t)\left|\psi_n\right\rangle e^{-i(\omega_n-\omega_m)t}C_n(t)
\end{aligned}
$$

Defining $\omega_{mn} \equiv \omega_m - \omega_n$ and $H'_{mn}(t) \equiv \left\langle\psi_m\right|H'(t)\left|\psi_n\right\rangle$:

$$
i\frac{d}{dt}C_m(t) = \lambda\sum_n H'_{mn}(t)e^{i\omega_{mn}t}C_n(t)
$$

Expanding in powers of λ

- From previous page:

$$i\frac{d}{dt}C_m(t) = \lambda \sum_n H'_{mn}(t)e^{i\omega_{mn}t}C_n(t)$$

- So far, solution is exact
- Now expand C_n in powers of λ:

$$\begin{aligned}
C_n &= C_n^{(0)} + \lambda C_n^{(1)}(t) + \lambda^2 C_n^{(2)}(t) + \ldots \\
\frac{d}{dt}C_m(t) &= \frac{\lambda}{i}\sum_n H'_{mn}(t)e^{i\omega_{mn}t}\left(C_n^{(0)} + \lambda C_n^{(1)}(t) + \ldots\right) \\
\Rightarrow \frac{d}{dt}\left(C_m^{(0)} + \lambda C_m^{(1)}(t) + \ldots\right) &= \frac{\lambda}{i}\sum_n H'_{mn}(t)e^{i\omega_{mn}t}\left(C_n^{(0)} + \lambda C_n^{(1)}(t) + \ldots\right)
\end{aligned}$$

- As for the time indep case, set equation for each power of λ to zero:

$$\frac{dC_m^{(0)}(t)}{dt} = 0 \qquad \Rightarrow C_m^{(0)}(t) = \text{constant}$$

$$\lambda\frac{dC_m^{(1)}(t)}{dt} = \frac{1}{i}\sum_n H'_{mn}(t)e^{i\omega_{mn}t}C_n^{(0)}(t)$$

- Using the same argument for $C_m^{(s)}$:

$$\frac{dC_m^{(s)}(t)}{dt} = \frac{1}{i}\sum_n H'_{mn}(t)e^{i\omega_{mn}t}C_n^{(s-1)}(t)$$

Solving the diff eq to first order in λ

$$\begin{aligned}
\Psi(\vec{x}, t) &= \sum_n C_n(t) \psi_n(\vec{x}, t) \\
\frac{dC_m^{(0)}(t)}{dt} = 0 & \quad \Rightarrow C_m^{(0)}(t) = \text{constant} \\
\frac{dC_m^{(1)}(t)}{dt} &= \frac{1}{i} \sum_n H'_{mn}(t) e^{i\omega_{mn} t} C_n^{(0)}(t)
\end{aligned}$$

- The constants $C_m^{(0)}$ together give the initial state of the system
- Just to simplify the math, let's assume we start in an eigenstate a so that:

$$C_m^{(0)}(t) = \delta_{ma}$$

- Integrating Eq above over time:

$$C_m^{(1)}(t) = \frac{1}{i} \int_{t_0}^{t} H'_{ma}(t) e^{i\omega_{ma} t'} dt' \quad (m \neq a)$$

This is our main result

Interpreting our result using description #2 language

- We start out in state $|a\rangle$ at $t = t_0$
- Time dependent perturbation adds or removes energy from the system, inducing transitions to other eigenstates of $\hat{H}_0$
- Can calculate transition probability to first order in λ:

$$P^{(1)}_{a\to m}(t) = \lambda^2 |C^{(1)}_m(t)|^2 = \lambda^2 \left| \int_{t_0}^{t} H'_{ma}(t) e^{i\omega_{ma}t'} dt' \right|^2$$

and coeff for remaining in state a

$$\begin{aligned} C_a(t) &= C_a^{(0)} + C_a^{(1)}(t) + ... \\ &\approx 1 + \frac{\lambda}{i} \int_{t_0}^{t} H'_{aa}(t) dt' \\ &\approx e^{-i\lambda \int_{t_0}^{t} H'_{aa}(t) dt'} \end{aligned}$$

where we have used the fact that λ is small to treat the middle eq as the beginning of a Taylor expansion.

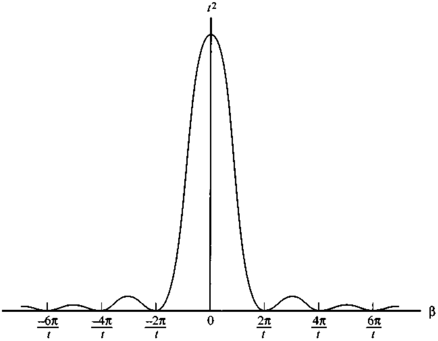
The simplest case: Turn on a constant H'

- Suppose $\hat{H}'$ has no time dependence other than being turned on at time $t = 0$
- Integrals from page 7 and 8 become straightforward

$$\begin{aligned}
C_a^{(1)} &= -i\lambda H'_{aa} t \\
\Rightarrow C_a(t) &\sim e^{-i\lambda H'_{aa} t} \\
C_m^{(1)} &= \frac{\lambda H'_{ma}}{\omega_{ma}} \left[1 - e^{i\omega_{ma}}\right] \qquad (m \neq a) \\
&= \frac{\lambda H'_{ma}}{\omega_{ma}} 2e^{i\omega_{ma}/2} \left[\frac{e^{-i\omega_{ma}/2} - e^{i\omega_{ma}/2}}{2}\right] \\
\Rightarrow P_{a\to m} &= \left|C_m^{(1)}\right|^2 \\
&= \frac{4\lambda^2 \left|H'_{ma}\right|^2}{\omega_{ma}^2} \sin^2\left(\omega_{ma} t/2\right)
\end{aligned}$$

What does this transition probability look like?

- For fixed time t plot $P_{a\to m}$ as a function of ω_{ma}



- Function sharply peaked about $\omega_{ma} = 0$ with height $\propto t^2$ and width $\approx 2\pi/t$
- Transitions to final states with energy within $2\pi/t$ strongly favored
- Result consistent with time-energy uncertainty principle

A continuous spectrum of final states

- Suppose we have a large number of closely spaced states
  - Eg, particle in a cube of length L per side where we let $L \to \infty$
- As when discussing Fermi energy, consider a continuous variable $\vec{k}$ that describes the momentum of the states
- Can count number of distinct states with energy between E_k and $E_k + dE_k$. This is the density of states $\mathcal{D}(E_k)$: $\mathcal{D}(E) = dn_{states}/dE_k$
- Our transition probability from Lecture 16 page 11 which was

$$P_{a\to m} = \sum_{m\neq a} \frac{4\lambda^2 \left|H'_{ma}\right|^2}{\omega_{ma}^2} \sin^2\left(\omega_{ma} t/2\right)$$

becomes

$$P_{a\to k} = \int dE_k \;\; \mathcal{D}(E_k) \frac{4\lambda^2 \left|H'_{ka}\right|^2}{\omega_{ka}^2} \sin^2\left(\omega_{ka} t/2\right)$$

Note:

1. B&J uses symbol ρ for what we call $\mathcal{D}(E)$
2. I have not put the limits on this integral. We'll discuss these limits on the next page

Integrating over the spectrum of final states

- Consider transitions from state $|a\rangle$ to a set of states in the range $E_k - \delta$ to $E_k + \delta$

$$P_{a\to k} = \int_{E_k-\delta}^{E_k+\delta} dE_k \;\; \mathcal{D}(E_k)\frac{4\lambda^2\left|H'_{ka}\right|^2}{\omega_{ka}^2}\sin^2\left(\omega_{ka}t/2\right)$$

- Assuming δ is small enough that $\mathcal{D}$ and H'_{ka} are approximately constant over the integral:

$$P_{a\to k} = \mathcal{D}(E_k)2\lambda^2\left|H'_{ka}\right|^2\int_{E_k-\delta}^{E_k+\delta} dE_k \;\; \frac{2\sin^2\left(\omega_{ka}t/2\right)}{\omega_{ka}^2}$$

- Also, $dE_k = d\left(E_k - E_a\right) = d\omega_{ka}$
  - Using the change of variables $x = \omega_{ka}t/2$ we find

$$\int_{-\infty}^{\infty}\frac{2\sin^2\left(\omega_{ka}t/2\right)}{\omega_{ka}^2}d\omega_{ka} = t\int_{-\infty}^{\infty}\frac{sin^2 x}{x^2}dx = \pi t$$

(look up the integral in a standard reference)

- If t is large, $F(t,\omega_{ka})$ is very peaked where energy is conserved. In this limit:

$$\begin{aligned} P_{a\to k} &= \mathcal{D}(E_k)2\pi\lambda^2\left|H'_{ka}\right|^2 t \\ \frac{d}{dt}\left(P_{a\to k}\right) &= \mathcal{D}(E_k)2\pi\lambda^2\left|H'_{ka}\right|^2 \end{aligned}$$

Fermi's Golden Rule

- The *transition rate* W_{ka} is the transition probability per unit time for going from state $|a\rangle$ to a state with energy in the range δ around E_k
- Our result from the previous page says

$$\frac{d}{dt}\left(P_{a\rightarrow k}\right) \equiv W_{ka} = 2\pi\lambda^2 \left|H'_{ka}\right|^2 \mathcal{D}(E_k)$$

- Fermi used this expression to calculate nuclear transition rates
  - He used this so often, he called it his "Golden Rule"
- Other calculations where Fermi's Golden Rule can be used:
  - Emission or absorption of photons by an atom
  - Transition rates for electrons in the conduction band of a semiconductor
  - Decays of elementary particles

Fermi's Golden Rule and Beta Decay

- β-decay is process by which one particle species can transform into another
- Examples:

$$\begin{aligned} n &\rightarrow pe^-\overline{\nu}_e \\ \mu^- &\rightarrow e^-\overline{\nu}_e\nu_\mu \end{aligned}$$

  One particle decays into three
- Occurs via "the weak interaction"
  - We cannot calculate H'_{ka} without studying some particle physics, but can understand the decay distributions under assumption that this term is approximately constant (and so can be factored out of all integrals)
- If H'_{ka} constant, transition probability depends only on determining $\mathcal{D}(E_k)$
  - ν are massless and e^- is very light, so particles are fully relativistic
  - $E = pc$ as we used for the the neutron star in homework 6 problem 5b.

Density of States for β-decay $p \to n e^- \overline{\nu}_e$

- Work in frame where neutron at rest before it decays
- Momentum conservation means that if we know the momentum of 2 of the 3 decay products, the momentum of the third is determined
- Calculate densithy of states in terms of e^- and ν_e momenta
- Also since $m_n \approx m_p$, the nuclear recoil is small. We'll ignore it
- Let E_f be the energy released in the decay
- The density of states is

$$d^2N = p_e^2 dp_e \; p_\nu^2 dp_\nu$$

For a massless neutrino (and ignoring small nuclear recoil)

$$p_\nu = (E_f - E_e); \quad dp_\nu = dE_f$$

Thus

$$\frac{dN}{dE_f} = p_e^2 (E_f - E_e)^2 dp_e$$

- Assuming that $|H'|^2$ is constant. Electron spectrum is

$$N(p_e) dp_e \propto p_e^2 (E_f - E_e)^2 dp_e$$

(You will work out this math on the next homework)

- Modification for non-zero neutrino mass

$$N(p_e) \propto p_e^2 (E_f - E_e)^2 \left[1 - \frac{m_\nu}{(E_f - E_e)}\right] dp_e$$