Physics 129: Particle Physics
Lecture 10: Quark Model and SU(3)

Sept 29, 2020

- Suggested Reading:
  - Thomson Sections 9.2-9.7
  - Griffiths 5.8-5.10
  - Perkins Sections 4.3-4.11

Reminder: Quiz #1 available now through midnight Wed

Review: Isospin

- Can classify hadrons with similar mass (and same spin & P) but different charge into multiplets
- Examples:

$$N \equiv \begin{pmatrix} p \\ n \end{pmatrix} \qquad \Pi \equiv \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

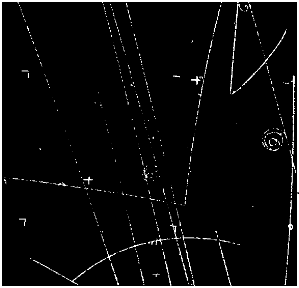
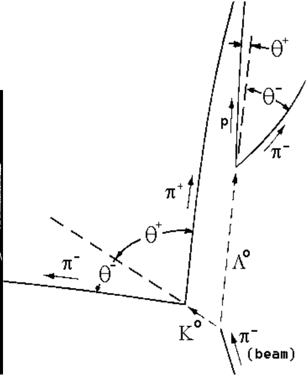
$$p = \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \qquad \pi^+ = |1, 1\rangle$$

$$n = \left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle \qquad \pi^0 = |1, 0\rangle$$

$$\pi^- = |1, -1\rangle$$

- Isospin has the same algebra as spin: SU(2)
  - Can confirm this by comparing decay or scattering rates for different members of the same isomultiplet
  - Rates related by normal Clebsh-Gordon coefficients

Review: Strangeness

- In 1950's a new class of hadrons seen
  - Produced in πp interaction via Strong Interaction
  - But travel measureable distance before decay, so decay is weak

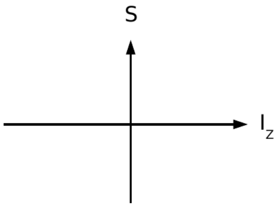
$\Rightarrow \exists$ conserved quantum number preventing the strong decay

Putting Strangeness and Isospin together

- Strange hadrons tend to be heavier than non-strange ones with the same spin and parity

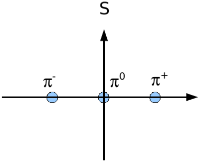
J^P	Name	Mass (MeV)
0^-	$\pi^\pm$	140
	$K^\pm$	494
1^-	$\rho^\pm$	775
	$K^{*\pm}$	892

- Associate strange particles with the isospin multiplets

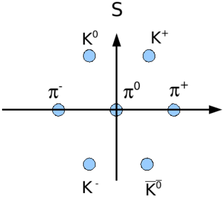


Adding Particles to the Axes: Pseudoscalar Mesons

- Pions have $S = 0$
- Three charge states $\Rightarrow I = 1$
- Draw the isotriplet:



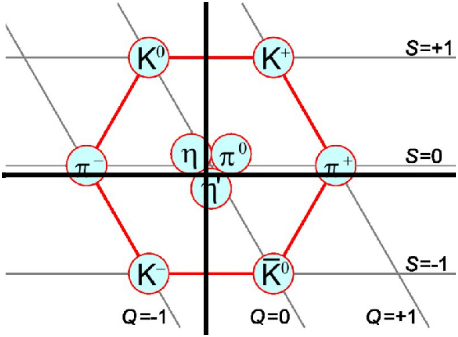
- From $\pi^- p \to \Lambda^0 K^0$ define K^0 to have $S = 1$
- If strangeness an additive quantum number, $\exists$ anti-K^0 with $S = -1$
- Also, K^+ and K^- must be particle-antiparticle pair: (eg from $\phi \to K^+ K^-$)



But this is not the whole story

There are 9 pseudoscalar mesons (not 7)!

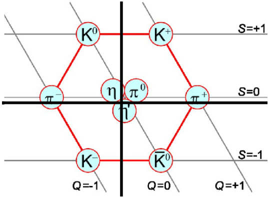
The Pseudoscalar Mesons



- Will try to explain this using group theory

Raising and Lowering Operators

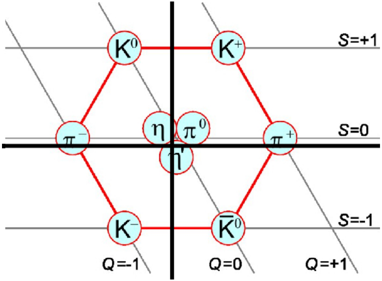
- In Quantum Mechanics, can start with state $|J, J_z = J\rangle$ and construct all other states of same J using lowering operator J_-
- Similarly with Isopsin, if start with $\pi^+ = |I = 1, I_z = 1\rangle$ and construct π^0 and π^- using lowering operator τ_-
- If we introduce strangeness, can we navigate among all the mesons in the same way?



▶ Need second lowering operator to navigate in the S direction

Extend group from SU(2) to SU(3)

Group Theory Interpretation of Meson Spectrum



- Particles with same spin, parity and charge conjugation symmetry described as multiplet
  - Different values of I_z and S
  - Will replace S with $Y = B + S$ where B is the baryon number
    - Reason for this will be clear soon

- Raising and lowering operators to navigate around the multiplet
- Gell Man and Zweig: Patterns of multiplets explained if all hadrons were made of quarks
  - Mesons: $q\bar{q}$ $\quad 3 \otimes \overline{3} = 1 \oplus 8$
  - Baryons: qqq $\quad 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$
- In those days, 3 flavors (extension to more flavors discussed later)
- Are the quarks real

Defining SU(3)

- SU(3): All unitary transformations on 3 component complex vectors without the overall phase rotation (U(1))

$$U^\dagger U = UU^\dagger = 1 \qquad \det U = 1$$

$$U = exp\left(i \sum_{a=1}^{8} \lambda_a \theta_a / 2 \right)$$

- The fundamental representation of SU(3) are 3×3 matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Commutation relations:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = i f_{abc} \frac{\lambda_c}{2}$$

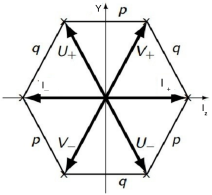
where $f_{123} = 1$, $f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$, $f_{156} = f_{367} = -\frac{1}{2}$ and $f_{458} = f_{678} = \sqrt{3}/2$.

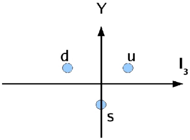
SU(3) Raising and Lowering Operators

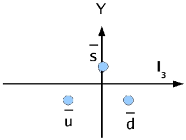
- SU(3) contains 3 SU(2) subgroups embedded in it

$$\begin{array}{llll} \text{Isospin}: & \lambda_1 & \lambda_2 & \lambda_3 \\ \text{U}-\text{spin}: & \lambda_6 & \lambda_7 & \sqrt{3}\lambda_8 - \lambda_3 \\ \text{V}-\text{spin}: & \lambda_4 & \lambda_5 & \sqrt{3}\lambda_8 + \lambda_3 \end{array}$$

- For each subgroup, can form raising and lowering operators
- Any two subgroups enough to navigate through multiplet







- Fundamental representation: A triplet
- Define group structure starting at one corner and using raising and lowering operators
- Define "highest weight state" ψ as state where both $I_+\psi = 0$ and $V_+\psi = 0$
- Quarks (u, d, s) have $p = 1,\ q = 0$ while antiquarks $(\overline{u}, \overline{d}, \overline{s})$ have $p = 0,\ q = 1$

$$\begin{array}{lcl} (V_-)^{p+1}\phi_{max} & = & 0 \\ (I_-)^{q+1}\phi_{max} & = & 0 \\ \text{structure}: & (p,q) & \end{array}$$

More on Quarks and SU(3)

- All the quarks have Baryon number $1/3$ (Baryons are qqq states)
- The quark quantum numbers are:

Quark	B	Q	I_3	S	Y
u	$1/3$	$2/3$	$1/2$	0	$1/3$
d	$1/3$	$-1/3$	$-1/2$	0	$1/3$
s	$1/3$	$-1/3$	0	-1	$-2/3$
$\overline{u}$	$-1/3$	$-2/3$	$-1/2$	0	$-1/3$
$\overline{d}$	$-1/3$	$1/3$	$1/2$	0	$-1/3$
$\overline{s}$	$-1/3$	$1/3$	0	1	$2/3$

- Notice from this table:

$$Q = I_3 + \frac{Y}{2}$$

- This general relationship holds for baryons and mesons as well as quarks and is called the Gell Mann-Nishijima Eq

Combining SU(3) states: 2 quarks

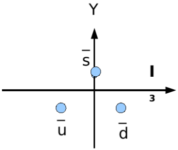
- Combining two SU(3) objects gives $3 \times 3 = 9$ possible states

$$
\begin{array}{cc}
uu & \\
\frac{1}{\sqrt{2}}(ud + du) & \frac{1}{\sqrt{2}}(ud - du) \\
dd & \\
\frac{1}{\sqrt{2}}(us + su) & \frac{1}{\sqrt{2}}(us - su) \\
ss & \\
\frac{1}{\sqrt{2}}(ds + sd) & \frac{1}{\sqrt{2}}(ds - sd) \\
\mathbf{6} & \overline{\mathbf{3}} \\
\mathrm{symmetric} & \mathrm{anti-symmetric}
\end{array}
$$

$$3 \otimes 3 \quad = \quad 6 \oplus \overline{3}$$

- Triplet is a $\overline{3}$ (not a 3):

$\frac{1}{\sqrt{2}}(ud - du)$ has $I = 0$, $I_z = 0$, and $Y = 2/3$
$\frac{1}{\sqrt{2}}(us - su)$ has $I = \frac{1}{2}$, $I_z = \frac{1}{2}$ and $Y = -1/3$
$\frac{1}{\sqrt{2}}(ds - sd)$ has $I = \frac{1}{2}$, $I_z = -\frac{1}{2}$ and $Y = -1/3$



Adding a 3^{rd} quark

- $3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \overline{3}) = (10_s \oplus 8_{M,S}) \oplus (8_{M,A} \oplus 1)$
- Start with the fully symmetric combination with the **6**:

$$\begin{array}{cl} uuu & \text{3 such states} \\ \frac{1}{\sqrt{3}}(ddu + udd + dud) & \text{6 such states} \\ \frac{1}{\sqrt{6}}(dsu + uds + sud + sdu + dus + usd) & \text{1 such state} \end{array}$$

- Now, the mixed symmetry combination with the **6**:

$$\frac{1}{\sqrt{6}}\left[(ud + du)u - 2uud\right] \quad \text{8 such states}$$

- Now the $\overline{3}$:

$$\frac{1}{\sqrt{6}}\left[(ud - du)s + (usd - dsu) + (du - ud)s\right] \quad \text{8 such states}$$

- One totally antisymmetric state

$$\epsilon_{ijk} q_i q_j q_k$$

Some Comments on what we just did

- Strategy same as how we constructed states of angular momentum in coupled basia
  - Two quarks play same role as two spins in uncoupled basis
  - Combinations have definite symmetry under interchange
    - Quarks are femions and if identical will need to obey Fermi statistics
- Size and symmetry properties of multiplets determined by SU(3) symmetry
  - But can be seen by inspection for the simple cases
- Group theory cannot tell us that all physically observed baryons are qqq combinations but it can tell us which multiplets can exist for such combinations
- Not all possible multiplets will exist (in some cases no bound state exists)
- But once we have observed one hadron in a multiplet, all the others must exist
  - Can construct all the hadrons from the one observed using raising and lowering operators

What about the mesons?

- Mesons are $q\overline{q}$ pairs
- We need to know represenation of the anti-quarks
- Strangeness and baryon number easy
  - Antiparticle has opposite strangeness from particle
  - Antiparticle has opposite baryon number from particle
- Peculiar sign convention for isospin (phases in Condon Shortly convention)
- If we write

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

  the convention for antiquarks is

$$\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix}$$

- With this definition, quarks and anti-quarks transform the same way under isospin and physical predictions are invarient under simultaneous tranformations of the form $u \Leftrightarrow d$ and $\overline{u} \Leftrightarrow \overline{d}$

Constructing the Mesons ($q\overline{q}$) states

- Start with $\pi^+ = u\,\overline{d}$
- Using:

$$\begin{aligned} I_-\,|\overline{u}\rangle &= -\,|\overline{d}\rangle \\ I_-\,|\overline{d}\rangle &= +\,|\overline{u}\rangle \end{aligned}$$

We find:

$$\begin{aligned} I_-\,|u\overline{d}\rangle &= \frac{1}{\sqrt{2}}\left(|d\overline{d}\rangle - |u\overline{u}\rangle\right) \\ &= |I=1\ I_3=0\rangle \\ \pi^0 &= \frac{1}{\sqrt{2}}\left(|d\overline{d}\rangle - |u\overline{u}\rangle\right) \end{aligned}$$

Doing this again: $\pi^- = d\,\overline{u}$

- Now add strange quarks: 4 combinations

$$\begin{array}{cccc} u\overline{s} & d\overline{s} & \overline{u}s & \overline{d}s \\ K^+ & K^0 & K^- & \overline{K^0} \end{array}$$

- One missing combination we can construct using raising or lowering operators

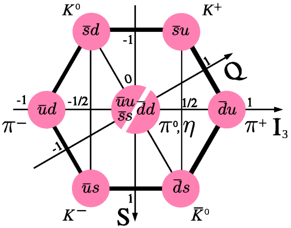
$$(d\overline{d} + u\overline{u} - 2s\overline{s})/\sqrt{6} \quad \equiv \eta'$$

These 8 states are called an octet

- One additional independent combination: the singlet state

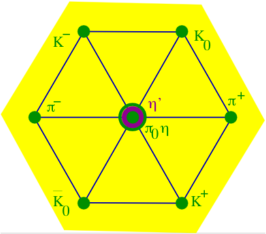
$$(u\overline{u} + d\overline{d} + s\overline{s})/\sqrt{6}$$

Pseudoscalar Mesons ($J^P = 0^-$)



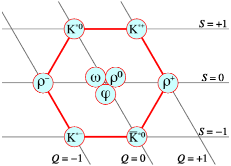
I	I_3	S	Meson	Combo	Example Decay	Mass (MeV)
1	1	0	π^+	$u\bar{d}$	$\mu^+\nu$	140
1	0	0	π^0	$\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$\gamma\gamma$	135
1	-1	0	π^-	$d\bar{u}$	$\mu^-\overline{\nu}$	140
$\frac{1}{2}$	$\frac{1}{2}$	$+1$	K^+	$u\bar{s}$	$\mu^+\nu$	494
$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	K^0	$d\bar{s}$	$\pi\pi$	498
$\frac{1}{2}$	$\frac{1}{2}$	-1	K^-	$\bar{u}s$	$\mu^-\overline{\nu}$	494
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\overline{K^0}$	$\bar{d}s$	$\pi\pi$	498
0	0	0	η_8	$\frac{1}{\sqrt{6}}(d\bar{d} + u\bar{u} - 2s\bar{s})$	see next page	
0	0	0	η_0	$\frac{1}{\sqrt{3}}(d\bar{d} + u\bar{u} + s\bar{s})$	see next page	

Observations about the Pseudoscalar Mesons



- Mass of strange mesons larger than non-strange by about 170 MeV
  - Strange quark has a larger mass than up and down
  - Leads to SU(3) breaking in H_{int}
- η_8 and η_0 would be degenerate if SU(3) were perfect symmetry
  - Degenerate PT: The states can mix. Physical states are:
    - η: Mass=549 Decay: $\eta \to 2\gamma$
    - η': Mass=958 Decays:$\eta' \to \eta\pi\pi$ or 2γ

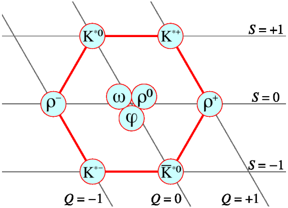
Vector Mesons ($J^P = 1^-$)



I	I_3	S	Meson	Combo	Decay	Mass (MeV)
1	1	0	ρ^+	$u\bar{d}$	$\pi^+\pi^0$	776
1	0	0	ρ^0	$\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$\pi^+\pi^-$	776
1	-1	0	ρ^-	$d\bar{u}$	$\pi^-\pi^0$	776
$\frac{1}{2}$	$\frac{1}{2}$	$+1$	K^{*+}	$u\bar{s}$	$K\pi$	892
$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	K^{*0}	$d\bar{s}$	$K\pi$	892
$\frac{1}{2}$	$\frac{1}{2}$	-1	K^{*-}	$\bar{u}s$	$\overline{K}\pi$	892
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\overline{K^{*0}}$	$\bar{d}s$	$\overline{K}\pi$	892
0	0	0	ω	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	783	3π
0	0	0	ϕ	$s\bar{s}$	1019	$K\overline{K}$

See comments on next page

Comments on Vector Mesons (1^-)

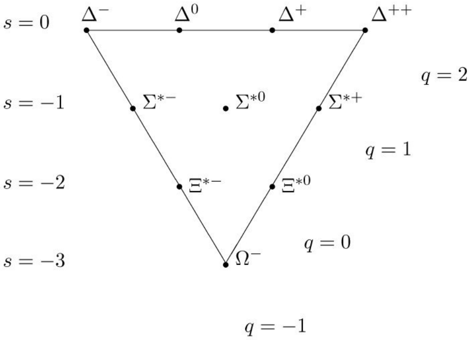


- Unlike the pseudoscalars which decay weakly, the vectors can decay strongly, eg

$$\begin{aligned} \rho^0 &\rightarrow \pi^+\pi^- \\ K^{*0} &\rightarrow K^+\pi^- \end{aligned}$$

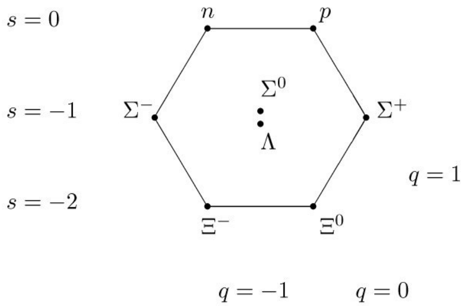
- Octet-nonet mixing is maximal in the case of the vector mesons
  - The ϕ is all $s\bar{s}$ while the ω is all $u\bar{u}$ and $d\bar{d}$
  - Related to mass difference between the s quark and the u or d quarks

Baryon Decouplet ($J = \frac{3}{2}^{-}$)



- Here is the Δ that we talked about earlier
- These are strongly decaying resonances since the can decay to the lighter octet baryons such as the p and n

Baryon Octet ($J = \frac{1}{2}^{-}$)



- Lightest baryons
- Decay weakly (except proton which is stable)

Comments on Antiparticles

- For mesons, particle and antiparticle are in the same multiplet
  - π^- is antiparticle of π^+
  - π^0 is its own antiparticle
  - The multiplet is called "self-charge conjugate"
- For baryons, the antiparticles are in different multiplets
  - $\mathbf{10} \Rightarrow \overline{\mathbf{10}}$
  - $\mathbf{8} \Rightarrow \overline{\mathbf{8}}$
  - $S \Rightarrow -S$; $I_z \Rightarrow -I_z$
    - Upper right corner moves to lower left
  - Baryon number $= 1 \Rightarrow$ Baryon number $= -1$

Fermi Statistics: Why Color

- Imposition of Fermi Statistics on Baryon States
  - $\Delta^{++} = uuu$, spin$=3/2$, s-wave: These are all symmetric under interchange
  - Need another degree of freedom to antisymmetrize
    - Must have at least 3 possible states, since we are antisymmetrizing 3 objects)
- We'll in 2 weeks, that the QCD Lagrangian is defined by $\mathrm{SU}(3)_{\mathrm{color}}$ interaction
  - Gluons are color octets
  - Observable hadrons are color singlets
  - The color singlet states are anti-symmetric under color exchange
    - This solves the Fermi statistics problem
    - All hadrons are color singlets
    - Thus quark wave function before adding color is symmetric

Backup slides with extra material

The following slides were not discussed in class but they will be of use for next week's homework

SU(3) Breaking and Mass Relations

- SU(3) symmetry: all members of multiplet have same mass
  - Mass depends on binding energy: cannot calculate, since perturbative calculations not possible for low energy QCD
- Reasons why the physical masses in a multiplet are different
  - Difference in quark masses
    - $m_d > m_u$ by a few MeV, m_s heavier by ~ 170 MeV
  - Coulomb energy difference associated with the electrical energy between pairs of quarks
    - Of order e^2/R_0. With $R_0 \sim 0.8$ fm, this ~ 2 MeV
  - Magnetic energy differences associated with the magnetic moments of the quarks (hyperfine interaction)

Vector-Pseudoscalar Meson Mass Differences

- ${m_\pi}^0 = 135$ MeV, ${m_\rho}^0 = 775$ MeV but SU(3) wave function is the same
- Only difference is the spin of the particles
- Both are $\ell = 0$ states of the $q\bar{q}$ pair
- Difference is spin: $S = 0$ or $S = 1$
- Consider magnetic dipole energy

$$U \propto \frac{1}{m_i m_j} \vec{S}_1 \cdot \vec{S}_2$$

- Meson mass is

$$m(q_1 q_2) = m_1 + m_2 + \frac{A}{m_i m_j} \vec{S}_1 \cdot \vec{S}_2$$

- Good agreement with data using

$$m_u, m_d = 0.307 \text{ GeV} \quad m_s = 0.490 \text{ GeV} \quad A = 0.06 \text{ GeV}$$

Some comments on quark masses

- Mass values from previous page not universal
- Quarks are bound in hadrons: the quark “mass” needed in these calculations affected by binding energy
- Baryons and mesons have different wave functions and hence different binding energies
  - Thus slightly different effective masses
- Above masses called “constituent mass”
- Completely different from “current mass” that is used in QFT Lagrangian
  - Current masses for quarks are a few MeV

Baryon Masses

- Similar idea as for mesons

$$m(q_1q_2q_3) = m_1 + m_2 + m_3 + A' \left(\frac{\left\langle \vec{S}_1 \cdot \vec{S}_2 \right\rangle}{m_1 m_2} \frac{\left\langle \vec{S}_1 \cdot \vec{S}_3 \right\rangle}{m_1 m_3} \frac{\left\langle \vec{S}_2 \cdot \vec{S}_3 \right\rangle}{m_2 m_3} \right)$$

- Good agreement with data using

$$m_u, m_d = 0.365 \text{ GeV} \quad m_s = 0.540 \text{ GeV} \quad A = 0.026 \text{ GeV}$$

- Notice these values aren't the same as for the mesons.

Baryon Magnetic Moments (I)

- Baryons are not Dirac particles
  - They have structure (quark bound states)
- Magnetic moment differs from that of pointlike fermions
  - $\mu_{proton} = 2.792\mu_N$ and $\mu_{neutron} = -1.913\mu_N$
  - where μ_N is the nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p}$$

- Quarks are Dirac particles so for a given spin-state we can calculate the magnetic moment. Thus for spin up quarks

$$\mu_u = \langle u \uparrow | \hat{\mu}_z | u \uparrow \rangle = \frac{2}{3} \frac{e\hbar}{m_u}$$
$$\mu_d = \langle d \uparrow | \hat{\mu}_z | d \uparrow \rangle = \frac{-1}{3} \frac{e\hbar}{m_u}$$

- Hadron magnetic moments can be built from the quarks

Baryon Magnetic Moments (II)

- Combine the 3 quarks by first combining 2 and then adding the third
- Since color state asymmetric quark wave function without color is symmetric
- Let's see how this works for the protom
  - Isopsin of the first two quarks
    - $I = 0$: Antisymmetric state. Two quarks must be different $\psi_0 = \frac{1}{\sqrt{2}}(ud - du)$
    - $I = 1$: Symmetric state ψ_1: uu, $\frac{1}{\sqrt{2}}(ud - du)$, dd
  - Combine with $3rd$ quark and require $I = \frac{1}{2}$, $I_z = \frac{1}{2}$
    - $I = 0$: Third quark must be a u
    - $I = 1$: Use Clebsch-Gordon coeff. 3-quark state is $\sqrt{2/3}d(uu) - \sqrt{1/3}u(\frac{1}{\sqrt{2}}(ud - du))$,
  - Now add spin which must be antisymmetric for ψ_0 and symmetric for ψ_1
  - Final result for spin up proton:

$$\begin{aligned}\Phi \quad = \quad & \sqrt{1/8}\,[4/3uud \uparrow\uparrow\downarrow -2/3uud \uparrow\downarrow\uparrow -2/3uud \downarrow\uparrow\uparrow -2/3udu \uparrow\uparrow\downarrow +4/3udu \uparrow\downarrow\uparrow \\ & -2/3udu \downarrow\uparrow\uparrow -2/3duu \uparrow\uparrow\downarrow -2/3duu \uparrow\downarrow\uparrow +4/3duu \downarrow\uparrow\uparrow]\end{aligned}$$

Baryon Magnetic Moments (III)

- Combining terms from previous page:

$$|p\uparrow\rangle = \frac{1}{\sqrt{6}}\left(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\downarrow d\uparrow\right)$$

- This gived

$$\begin{aligned}\mu_p &= \frac{4}{6}\left(\mu_u + \mu_u - \mu_d\right) + \frac{1}{6}\left(\mu_u - \mu_u + \mu_d\right) + \frac{1}{6}\left(-\mu_u + \mu_u + \mu_d\right) \\ &= \frac{4}{3}\mu_u - \frac{1}{3}\mu_d\end{aligned}$$

- Similarly

$$\mu_n = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u$$

- Predictions reasonably reproduce data values