Physics 129: Particle Physics

Lecture 14: From the Dirac eq to Feynman Diagrams

Oct 13, 2020

- Suggested Reading:
  - Thomson 4.6-4.9, 5.1-5.4
  - Griffiths 6.3,7.1-7.5

  Reminder: Quiz #2 next week. Focus: Lectures 7-13 and HW 4-6 (LIPS but no Dirac Eq for this quiz)

Review: The Dirac Equation

- Dirac started familiar quantum mechanical expression

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

  and asked what form $\hat{H}$ should take

- Since first order derivative in time, $\hat{H}$ must be linear in $\frac{\partial}{\partial x_i}$
- He postulated (using natural units)

$$H = \vec{\alpha}\cdot\vec{p} + m\beta$$

  where $\vec{\alpha}$ and β are coefficients to be determined

- Solutions not possible for α and β as numbers: must be matrices
- No 2×2 matrix solutions
- Can solve with 4×4 matrices applied to 4-column spinor

Review: Express $\vec{\alpha}$ and β as γ matrices

- γ-matrices defined:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

- Or writing out all components

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Note: Today we'll also use one other marix:

$$\gamma^5 = i\gamma^0\gamma^0\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Review: General Solution to Dirac Eq for Free Particle

- Can Lorentz boost rest frame wf to get general form. Won't do math here
- Solutions are:

$$\psi = u(E, \vec{p}) e^{i(\vec{p}\vec{x} - Et)}$$

with

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

- Note that the boost mixes the top two components with the bottom two
- For u_1 and u_2, $E = \sqrt{p^2 + m^2}$ while for u_3 and u_4, $E = -\sqrt{p^2 + m^2}$
- Also, we have to understand how to handle the negative energy solutions

Review: Reinterpreting the negative energy states

- Interpretation by Stuckelberg and Feyname
- $E < 0$ states are negative energy particles propagating backwards in time
- Reinterpret them as positive energy antiparticles with opposite charge propagating forward in time

$$E \Rightarrow -E \quad t \Rightarrow -t$$

- Redefine

$$\begin{aligned} v_1(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} &= u_4(-E,-\vec{p})e^{+i(-\vec{p}\cdot\vec{x}-(-E)t)} \\ v_2(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} &= u_3(-E,-\vec{p})e^{+i(-\vec{p}\cdot\vec{x}-(-E)t)} \end{aligned}$$

- We can rewrite our wf using this new notation and can also calculate the normalization N
- Final results on the next page

Review: Solutions to the Dirac Equation

- Normalized solutions to the Dirac Eq

$$\psi = u(E,\vec{p}) = e^{+i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu - m)\, u = 0$$

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- Antiparticle solutions:

$$\psi = v(E,\vec{p}) e^{-i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu + m)\, v = 0$$

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Some Comments

- Spinors have 4 components but they are NOT four-vectors
  - They play the same role as the spin matrices do in non-relativistic QM
  - As we saw last time, a spinor has total spin-$\frac{1}{2}$ evem though it has 4 components
- We also saw last time that

$$\begin{aligned} j^{\mu} &= \left(\rho, \vec{j}\right) = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi \\ &= \overline{\psi}\gamma^{\mu}\psi \end{aligned}$$

where we have defined

$$\overline{\psi} \equiv \psi^{\dagger}\gamma^{0}$$

  - $\overline{\psi}$ is the definition of the adjoint spinor that allows us to write the continuity equation compactly:

$$\partial_{\mu}j^{\mu} = 0$$

This means $\overline{\psi}$ plays the role in our theory that $\psi^{*}\psi$ played in NR QM

Bilinear Covariants

- Most general form for cross sections can be written in terms of Lorentz Invariant operators
  - For parity conserving interactions, cross sections must be scalars.
  - If interaction includes parity violation, pseudoscalars also possible
- Cross sections will be made from terms of form $\overline{\psi}\hat{O}\psi$
- Starting point: Obsserve from continuity eq that

$$\begin{aligned} j^\mu &= \left(\rho, \vec{j}\right) = \psi^\dagger \gamma^0 \gamma^\mu \psi \\ &= \overline{\psi}\gamma^\mu\psi \end{aligned}$$

  - $\partial_\mu j^\mu = 0$ which is Lorentz invariant so $\overline{\psi}\gamma^\mu\psi$ is a 4-vector
  - $\overline{\psi}\psi$ is a scalar (won't prove here)
- We'd like to construct pseudoscalar and axial vector combinations as well, but to do so need to figure out how to apply parity operator to spinors
- We'll take that aside and come back to the remaining bilinear covariants after that

Parity and Spinors

- Under Parity

$$\psi \Rightarrow \psi' = \hat{P}\psi$$

- Using $\hat{P}^2 = 1$ this means $\hat{P}\psi' = \psi$
- Writing Dirac eq

$$\begin{aligned}
i\gamma^1 \frac{\partial\psi}{\partial x} + i\gamma^2 \frac{\partial\psi}{\partial y} + i\gamma^3 \frac{\partial\psi}{\partial z} - m\psi &= -i\gamma^0 \frac{\partial\psi}{\partial t} \\
i\gamma^1 \frac{\partial\psi'}{\partial x'} + i\gamma^2 \frac{\partial\psi'}{\partial y'} + i\gamma^3 \frac{\partial\psi'}{\partial z'} - m\psi &= -i\gamma^0 \frac{\partial\psi'}{\partial t'} \\
i\gamma^1 \hat{P} \frac{\partial\psi}{\partial x} + i\gamma^2 \hat{P} \frac{\partial\psi}{\partial y} + i\gamma^3 \hat{P} \frac{\partial\psi}{\partial z} - m\psi &= -i\gamma^0 \hat{P} \frac{\partial\psi}{\partial t}
\end{aligned}$$

- Multiply by γ^0 on left and express derivatives in primed system

$$\begin{aligned}
-i\gamma^0\gamma^1 \hat{P} \frac{\partial\psi}{\partial x'} - i\gamma^0\gamma^2 \hat{P} \frac{\partial\psi}{\partial y'} - i\gamma^0\gamma^3 \hat{P} \frac{\partial\psi}{\partial z'} - m\psi &= -i\gamma^0\gamma^0 \hat{P} \frac{\partial\psi}{\partial t'} \\
i\gamma^1\gamma^0 \hat{P} \frac{\partial\psi}{\partial x'} + i\gamma^2\gamma^0 \hat{P} \frac{\partial\psi}{\partial y'} + i\gamma^3\gamma^0 \hat{P} \frac{\partial\psi}{\partial z'} - m\psi &= -i\hat{P} \frac{\partial\psi}{\partial t'}
\end{aligned}$$

- For this to hold:

$$\gamma^0 \hat{p} - I$$

- This plus $\hat{P}^2 = 1$ means

$$\psi \Rightarrow \hat{P}\psi = \gamma^0\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \psi$$

Back to bilinear covariants

- We already saw that $\overline{\psi}\psi$ was a scalar
- How about $\overline{\psi}\gamma^5\psi$ with

$$i\gamma^0\gamma^0\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- Writng

$$\begin{aligned} \hat{P}\overline{\psi}\gamma^5\psi &= \psi^\dagger\gamma^0\gamma^0\gamma^5\gamma^0\psi \\ &= \psi^\dagger\gamma^5\gamma^0\psi \\ &= -\psi^\dagger\gamma^0\gamma^5\psi \\ &= -\overline{\psi}\gamma^5\psi \end{aligned}$$

- So, $\overline{\psi}\gamma^5\psi$ is a pseudoscalar
- Using similar arguments: $\overline{\psi}\gamma^\mu\gamma^5\psi$ is an axial vector
- One final combination: $\overline{\psi}\sigma^{\mu\nu}\psi$ with $\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^n u - \gamma^\nu\gamma^\mu\right)$ is an antisymmetric tensor

Spinors are NOT eigenstates of S_z

- Reminder:

$$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Looking at one of our solutions:

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

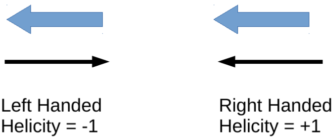
- We can see that it isn't an eigenstate of S_z
- Can we find an observable that is?
  - The hint: if $\vec{p}$ were along the z direction. we would be in an eigenstate of S_z
  - Same conclusion holds for the other 3 solutions
  - So, instead of taking S projection along arbitrary direction, we want the projection along the momentum

Helicity

- Spinors are not eigenstates of $\hat{S}_z$
- Component of particle's spin along its direction of flight is, however, a good quantum number: $\left[H, \hat{S} \cdot \hat{p}\right] = 0$
- Define components of a particle's spin along its direction of flight as **helicity**

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{S}|\ |\vec{p}|}$$

- Component of spin along any axis has 2 values for spin-$\frac{1}{2}$ particle: $\pm\frac{1}{2}$
  - Eigenvalues of helicity operator are : ± 1
- These are right- and left-handed helicity states



  - Warning: Helicity is not Lorentz invariant for paricles with mass
  - Can always boost to a frame where $\vec{p}$ changes sign

Helicity and Chirality

- For massless fermions, operator to project states of particular helicity are:

$$
\begin{aligned}
P_R &= \frac{1}{2}\left(1 + \frac{\sigma \cdot \mathbf{p}}{E}\right) \\
P_L &= \frac{1}{2}\left(1 - \frac{\sigma \cdot \mathbf{p}}{E}\right)
\end{aligned}
$$

- For massive fermions, need 4-component spinor and 4-component operator

$$P_{L,R} = \frac{1}{2}\left(1 \pm \gamma_5\right)$$

- Because direction of spin wrt momentum changes under boosts, this operator cannot represent helicity per se
- Instead, projects out state of polarization $P = \pm v/c$
  - In spite of this, everyone writes

$$\frac{1}{2}\left(1 - \gamma^5\right)\right) u \equiv u_L$$

$\frac{1}{2}\left(1 \pm \gamma^5\right)$ are called the chiral projection operators

- We'll talk more about chirality when we get to the weak interactions

From the Dirac Eq to Feynman Diagrams: Overview

- Dirac eq gives us free particle wf for spin-$\frac{1}{2}$ particles
- To calculate cross sections and decay rates, will use these wf together with time dep PT
  - Have already seen division into kinematics (LIPS) and dynamics (matrix elements)
  - Need procedure to calculate the matrix elements
  - Replace QM treatment using H_{int} with an interaction picture where fermion matter particles exchange spin-1 force carriers
    - Simplest theory: QED has one force carrier: the photon
    - Other forces have more more (gluons for QCD, W and Z for weak interactions)
    - Will need to introduce wf's for these
- Rules for calculating matrix elements pretty much cook-book but take time to learn and use
- You can find these calculations in Griffiths, but we will skip a lot of the math here
- Will motivate the results based on general principle such as Lorentz invariance

Perturbation Theory (I) Non-relativistic reminder

- Potential $\mathsf{V}(x,t)$ limited to finite spatial extent
- Assume $\mathsf{V}(x,t)$ small so PT works
- ϕ_n are solns to $H_0\phi_n = E_n\phi_n$
- For $H = H_0 + \mathsf{V}(x,t)$:

$$\begin{aligned}
(H_0 + \mathsf{V}(x,t))\,\psi &= -i\frac{\partial\psi}{\partial t}, \\
\psi &= \sum_n a_n(t)\phi_n(x)e^{-iE_nt}
\end{aligned}$$

therefore

$$i\sum_n \frac{da_n(t)}{dt}\phi_n(x)e^{-iE_nt} = \sum_n \mathsf{V}(x,t)a_n\phi_n(x)e^{-iE_nt}$$

- Multiply by ϕ^* and integrate:

$$\begin{aligned}
i\frac{da_f}{dt}e^{-iE_nt} &= -i\sum_n \int \mathsf{V}(x,t)a_n(t)\phi_f^*\phi_n e^{-iE_nt}d^3x \\
\frac{da_f}{dt} &= -i\sum_n \int a_n(t)\mathsf{V}(x,t)\phi_f^*\phi_n e^{-i(E_n-E_f)t}d^3x
\end{aligned}$$

Perturbation Theory (II)

- Integrate over time from $-\mathsf{T}/2$ to $\mathsf{T}/2$
- At time $t = -\mathsf{T}/2$ in state i:

$$a_i(-\mathsf{T}/2) = 1,$$
$$a_f(-\mathsf{T}/2) = 0, \quad \text{for} \;\; n \neq i$$

We find:

$$\begin{aligned}
\frac{da_f}{dt} &= -i \int a_i(t) \mathsf{V}(x,t) \phi_f^* \phi_n e^{i(E_f - E_n)t} d^3x \\
&= -i \int \phi_f^* \mathsf{V}(x,t) \phi_n e^{i(E_f - E_n)t} d^3x \\
T_{fi} \equiv a_f(\mathsf{T}/2) &= -i \int_{-\mathsf{T}/2}^{T/2} \int \phi_f^* \mathsf{V}(x,t) \phi_n e^{i(E_f - E_n)t} d^3x dt'
\end{aligned}$$

- Or in covarient form

$$a_f = -i \int \phi_f^*(x) \mathsf{V}(x) \phi_i(x) d^4x$$

Same expression holds for relativisitic QM

Perturbation Theory (III)

- If V has no time dependence

$$\begin{aligned} T_{fi} &= -i\mathsf{V}_{fi}\int_{-\infty}^{\infty} e^{i(E_f - Ei)t}dt \\ &= -2\pi i\mathsf{V}_{fi}\delta(E_f - E_i) \end{aligned}$$

  Conservation of energy

- Transition rate

$$\begin{aligned} w_{fi} &= \lim_{T\to\infty} \frac{|T_{fi}|^2}{\mathsf{T}} \\ &= \lim_{T\to\infty} 2\pi\frac{|\mathsf{V}_{fi}|^2}{\mathsf{T}}\delta(E_f - E_i)\int_{-\mathsf{T}}^{\mathsf{T}} e^{i(E_f - E_i)t}dt \\ &= \lim_{T\to\infty} 2\pi\frac{|\mathsf{V}_{fi}|^2}{\mathsf{T}}\delta(E_f - E_i)\int_{-\mathsf{T}}^{\mathsf{T}} dt \\ &= 2\pi|\mathsf{V}_{fi}|^2\delta(E_f - E_i) \end{aligned}$$

- Must integrate over all possible final states for a given initial state
  - Introduce density of states $\mathcal{D}(E_f)$

Fermi Golden Rule

- The non-relativistic result holds for relativistic case as well

$$w_{fi} = 2\pi |\mathsf{V}_{fi}|^2 \mathcal{D}(E_i)$$

where w_{fi} is the transition rate, V_{fi} is the "matrix element" and $\mathcal{D}(E_i)$ is the density of states factor, also called the phase space factor

- To lowest order

$$\mathsf{V}_{fi} = \int d^3\, x \phi_f^*(x) \mathsf{V}(x) \phi_i(x)$$

- To next order

$$\mathsf{V}_{fi} \to \mathsf{V}_{fi} + \sum_{b \neq i} \mathsf{V}_{fn} \frac{1}{E_i - E_n} \mathsf{V}_{ni}$$

and so forth for higher orders

- Relativistic phase space factor (before including any spin factors):

$$\text{No of final states/particle} = \frac{V d^3 p}{(2\pi)^3 2E}$$

The volume V always cancels out when we properly normalize the single particle wave functions $N = 1/\sqrt{V}$

More comments on next order in Perturbation Theory

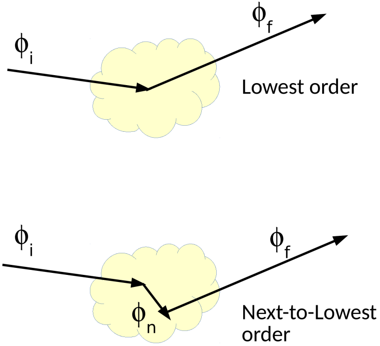
$$T_{fi} = T_{fi}^{lowest} - \sum_{n \neq i} \mathsf{V}_{fn} \mathsf{V}_{ni} \int_{-\infty}^{\infty} dt e^{i(E_f - E_n)t} \int_{-\infty}^{t} e^{i(E_n - E_i)t}$$

using
$$\int dt' e^{i(E_n - E_i - i\epsilon)} = \frac{i e^{i(E_n - E_i - i\epsilon)t}}{E_i - E_n - i\epsilon}$$

$$T_{fi} = T_{fi}^{lowest} - 2\pi i \sum_{n \neq i} \frac{\mathsf{V}_{fn} \mathsf{V}_{ni}}{E_i - E_n - i\epsilon} \delta(E_f - E_i)$$

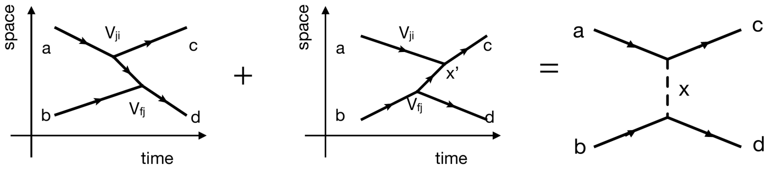
- Term in denominator is called the "'propagator factor"
- Intermediate states are virtual and don't have to conserve energy and momentum
- Overall δ-fn imposed energy conservation on result

The physical interpretation



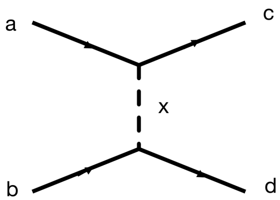
- In NR QM, top picture is lowest order PT calc of interactionof particle with fixed potential
- In field theory, no fixed potentials, particles interact by exchanging force carriers
- Bottom diagram is lowest order example of an exchange

Replacing time-ordered PT with Feynman Diagrams



- In QM would need to consider two cases
  - a emits particle that is absorbed by b
  - b emits particle that is absorbed by a
- But order of these events not independent of reference frame!
- Turns out that sum of both possibilities is Lorentz invariant (see Thomson 5.3)
- Draw both as a single "Feynman diagram"

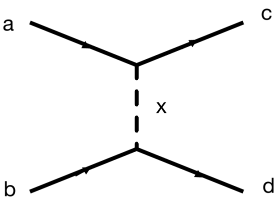
Feynman Diagrams



- Two conventions for drawing Feynman Diagrams
  - Time goes left-to-right
  - Time goes up to down

  We'll mainly use the first convention
- Incoming and outgoing particles are physical states we can observe
  - With left-to-right convention, incoming particles on lhs; outgoing on rhs
  - Will often draw arrows, but as we'll see on Thurs, due to Stuckelberg and Feyname, antiparticles will have arrows reversed
- Everything between incoming and outgoing is how the interaction happens
- The exchanged particle (labeled X) is "virtual."
- Direction of arrow is arbitrary for these virtual particles

Pieces of the Matrix Element Calculation

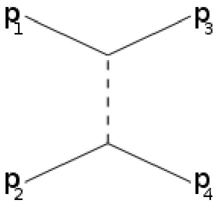


- Incoming and outgoing particles have standard wf's
- At each vertex
  - need a constant g that tells us the strength of the interaction
  - Energy and momentum conserved at all vertices
  - The exchanged particle (labeled X) is "virtual". For this particle $E^2 - p^2 \neq m^2$ ("off-shell"')
- For each vertex, there is "propagator" term $\propto \frac{1}{q^2 - m_X^2}$. This term comes from the denominator of the eq on page 16
  - We'll see next time that for exchange of photon (spin 1), will need an additioanl $ig^{\mu\mu}$ for the photon propagator
- Remember, we'll have to square matrix element to get decay rate or scattering cross section

Cross Sections and Lorentz Invariants

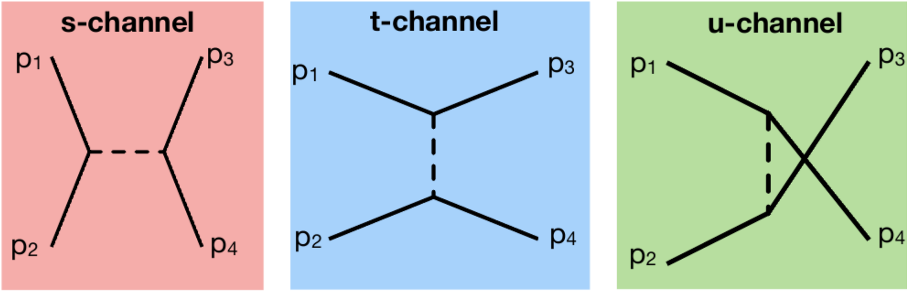
- Cross sections are easy to estimate at high energies, where we can ignore masses of scattered particles
- For $p_1 + p_2 \to p_3 + p_3$ the Mandelstam variables are

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_3 - p_1)^2 = (p_4 - p_2)^2 \\ u &= (p_3 - p_2)^2 = (p_4 + p_1)^2 \end{aligned}$$



- In all cases $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

s,t,u Exchange



- Interactions occur via exchange of vector bosons or of fermions item Interactions can be:
  - annihilation
  - absorption
  - emission
- Describe the interactions in terms of the Mandelstam variables
  - Propagator terms from page 23 can be written in terms of s, t and u