Control of quantum two-level systems

How to control a qubit?

Does the answer depend on specific realization? → No, only its implementation

Qubit is **pseudo spin**

→ **General concept exists**
→ Independent of qubit realization
→ Methods from nuclear magnetic resonance (NMR)

Nuclear magnetic resonance

→ Method to explore the magnetism of nuclear spins
→ Important application → **Magnetic resonance imaging (MRI)** in medicine
→ MRI exploits the different nuclear magnetic signatures of different tissues

Brief MRI history

early suggestions → **H. Carr** (1950) and **V. Ivanov** (1960)

1972 → MRI imaging machine proposed by **R. Damadian** (SUNY)

1973 → 1st MRI image by **P. Lauterbur** (Urbana-Champaign)

Late 1970ies → Fast scanning technique proposed by **P. Mansfield** (Nottingham)

2003 Nobel Prize in Medicine → for P. Lauterbur and P. Mansfield

Overview: Important NMR techniques

Basic idea → Rotate spins by static or oscillating magnetic fields

Static fields parallel to quantization axis
→ **free precession**
→ changes φ on Bloch sphere
Oscillating fields perpendicular to quantization axis
→ **change population**
→ changes θ on Bloch sphere

Important protocols
→ Rabi → population oscillations
→ Relaxation measurement → T_1
→ Ramsey fringes → $T_2^\star$
→ Spin echo (Hanh echo) → $T_2^\star$ corrected for reversible dephasing

Hamiltonian of a quantum two-level system (TLS)

Arbitrary TLS → $\hat{H} = \frac{1}{2}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is Hermitian matrix

→ $H_{11}, H_{22} \in \mathbb{R}$ and $H_{21} = H_{12}^{\star}$

→ we choose $H_{11} > H_{22}$

Symmetrize → Subtract global energy offset $\frac{H_{11}+H_{22}}{4} \times \hat{1}$

$$\rightarrow \hat{H} = \frac{1}{2}\begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix} = \frac{\epsilon}{2}\hat{\sigma}_z + \frac{\tilde{\Delta}}{2}\hat{\sigma}_y + \frac{\Delta}{2}\hat{\sigma}_x$$

$$\epsilon \equiv \frac{H_{11} - H_{22}}{2} > 0 \qquad \Delta, \tilde{\Delta} \in \mathbb{R}$$

Natural or physical basis $\{|\varphi_+\rangle, |\varphi_-\rangle\}$

$$\rightarrow \hat{H}_0 = \frac{1}{2}\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} = \frac{\epsilon}{2}\hat{\sigma}_z$$

$$\rightarrow \hat{H}|\varphi_{\pm}\rangle = \pm\frac{\epsilon}{2}$$

$$\hat{H} = \frac{1}{2}\begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix}$$

Diagonalization of $\hat{H}$

Eigenvalues

$$\rightarrow \det\begin{pmatrix} \epsilon - 2E_\pm & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon - 2E_\pm \end{pmatrix} = 0$$

$$\rightarrow (\epsilon - 2E_\pm)(-\epsilon - 2E_\pm) - (\Delta - i\tilde{\Delta})(\Delta + i\tilde{\Delta}) = 0$$

$$\rightarrow E_\pm = \pm\frac{1}{2}\sqrt{\epsilon^2 + \Delta^2 + \tilde{\Delta}^2} \equiv \pm\frac{1}{2}\hbar\omega_\mathrm{q}$$

transition energy conveniently expressed in **frequency units**

Eigenvectors

$$\rightarrow |\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$

$$\rightarrow |\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\rightarrow \hat{H}|\Psi_\pm\rangle = \mathrm{E}_\pm|\Psi_\pm\rangle$$

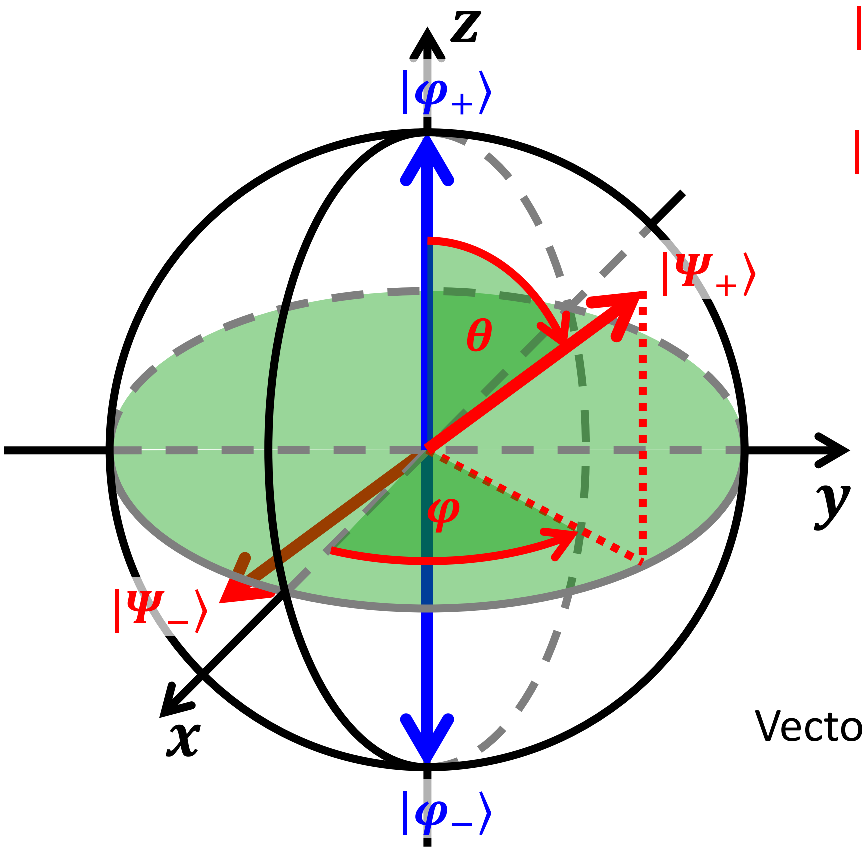
$$\tan\theta \equiv \frac{\sqrt{\Delta^2 + \tilde{\Delta}^2}}{\epsilon} \qquad \text{with } 0 \le \theta \le \pi$$

$$\tan\varphi \equiv \frac{\tilde{\Delta}}{\Delta} \qquad \text{with } 0 \le \varphi \le 2\pi$$

C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics Volume One*, Wiley-VCH

Basis $\{|\Psi_+\rangle, |\Psi_-\rangle\}$ → **energy eigenbasis** (because $\hat{H}$ has energy units)

Visualization on Bloch sphere



$$|\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$

$$|\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\tan\theta \equiv \frac{\sqrt{\Delta^2+\widetilde{\Delta}^2}}{\epsilon} \qquad \text{with } 0 \le \theta \le \pi$$

$$\tan\varphi \equiv \frac{\widetilde{\Delta}}{\Delta} \qquad \text{with } 0 \le \varphi \le 2\pi$$

Basis rotation on Bloch sphere!

Vectors $|\Psi_-\rangle$ and $|\Psi_+\rangle$ define new quantization axis

→ $\hat{H} = \frac{\sqrt{\epsilon^2+\Delta^2+\widetilde{\Delta}^2}}{2}\hat{\sigma}_z$ in basis $\{|\Psi_-\rangle, |\Psi_+\rangle\}$

→ $|\Psi_-\rangle$ and $|\Psi_+\rangle$ become new poles

Analogy to spin ½ in static magnetic field

Fictitious spin ½ in fictitious magnetic field $\boldsymbol{B}$

$$\rightarrow \widehat{H}_{\uparrow} = -\gamma\hbar\, \boldsymbol{B}\cdot\boldsymbol{S} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

→ γ is the gyromagnetic ratio

→ $B = \left(B_x, B_y, B_z\right)^T$ is the magnetic field vector

Fictitous spin in fictitious $\boldsymbol{B}$-field	→	**quantum TLS**
$\vert\uparrow\rangle$	→	$\vert\varphi_+\rangle$
$\vert\downarrow\rangle$	→	$\vert\varphi_-\rangle$
$\vert\uparrow\rangle_{\boldsymbol{u}}$	→	$\vert\Psi_+\rangle$
$\vert\downarrow\rangle_{\boldsymbol{u}}$	→	$\vert\Psi_-\rangle$

($\boldsymbol{u}$ denotes the quantization axis along which $\widehat{H}_{\uparrow}$ is diagonal)

$\hbar\vert\gamma\vert\vert\boldsymbol{B}\vert$	→	$\mathrm{E}_+ - \mathrm{E}_- = \hbar\omega_q$
Polar angles of $\boldsymbol{B}$	→	θ, φ
$-\gamma\hbar B_z$	→	ϵ
$-\gamma\hbar B_x$	→	Δ
$-\gamma\hbar B_y$	→	$\widetilde{\Delta}$

Orientation with respect to the quantization axis depends on $\epsilon, \Delta, \widetilde{\Delta}$

Any quantum TLS has a **“built-in” static magnetic field**

↓

NMR situation!

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\mathrm{e}\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\mathrm{g}\rangle$$

Free precession

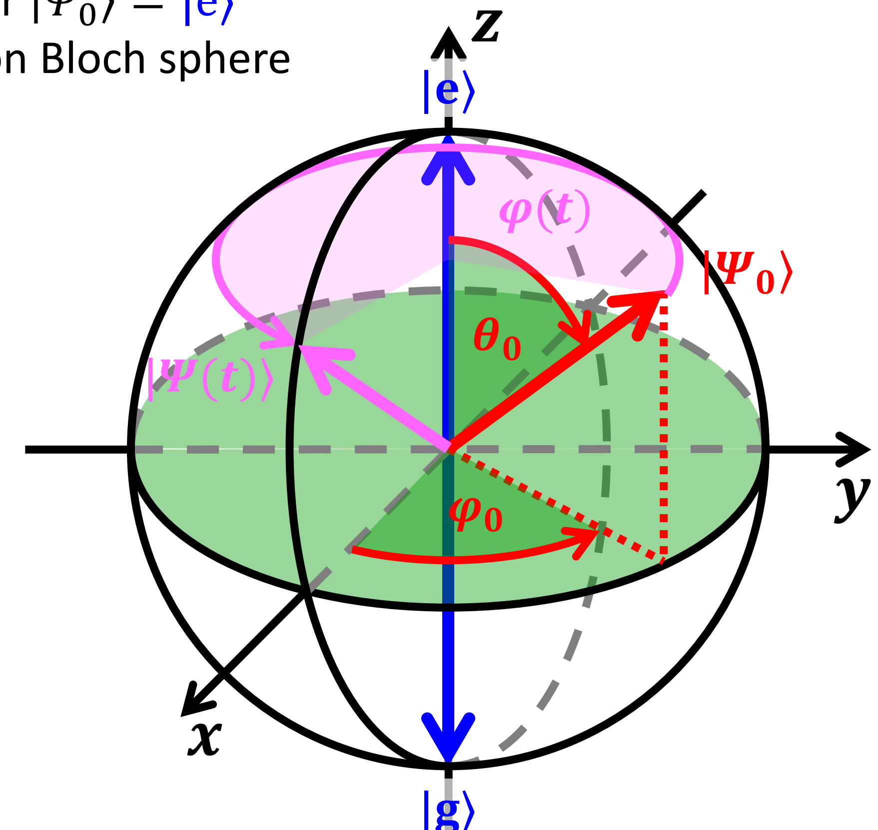
In the energy eigenbasis $\{|\mathrm{g}\rangle, |\mathrm{e}\rangle\}$, the qubit state vector $|\Psi_0\rangle$ is
- → parallel to built-in field for $|\Psi_0\rangle = |\mathrm{g}\rangle$
- → antiparallel to the built-in field for $|\Psi_0\rangle = |\mathrm{e}\rangle$
- → Built-in field points along z-axis on Bloch sphere

When qubit state vector $|\Psi_0\rangle$ is not parallel or antiparallel to built-in field

→ free evolution corresponds to **free precession about the z-axis**

→ Also called Larmor precession

→ In absence of decoherence, only $\varphi(t)$ evolves linearly with time t



$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\mathrm{e}\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\mathrm{g}\rangle$$

Larmor precession – formal calcualtion

Energy eigenbasis → $\widehat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z$ → fictitious field aligned along quantization axis

Time-independent problem → Develop into stationary states

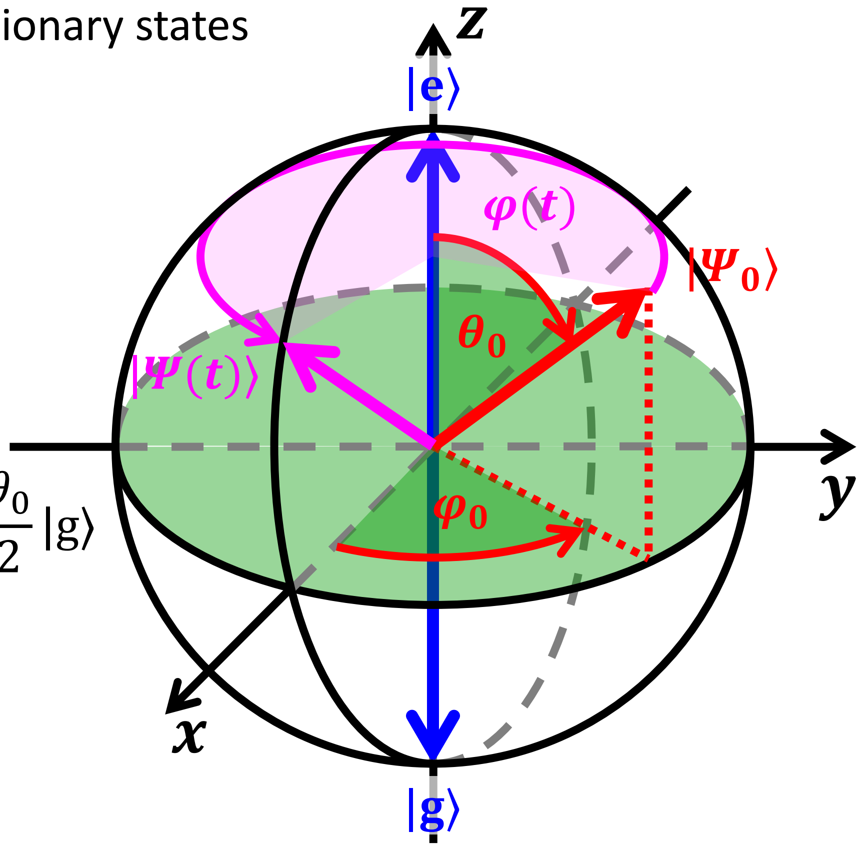
$$|\Psi(t)\rangle = \langle \mathrm{e}|\Psi_0\rangle e^{-i\omega_\mathrm{q}t/2}|\mathrm{e}\rangle + \langle \mathrm{g}|\Psi_0\rangle e^{i\omega_\mathrm{q}t/2}|\mathrm{g}\rangle$$

$$|\Psi_0\rangle = \cos\left(\frac{\theta_0}{2}\right)|\mathrm{e}\rangle + e^{i\varphi_0}\sin\left(\frac{\theta_0}{2}\right)|\mathrm{g}\rangle$$

$$|\Psi(t)\rangle = e^{-i\omega_\mathrm{q}t/2}\cos\frac{\theta_0}{2}|\mathrm{e}\rangle + e^{i\varphi_0}e^{i\omega_\mathrm{q}t/2}\sin\frac{\theta_0}{2}|\mathrm{g}\rangle$$

$$= \cos\frac{\theta_0}{2}|\mathrm{e}\rangle + e^{i(\varphi_0+\omega_\mathrm{q}t)}\sin\frac{\theta_0}{2}|\mathrm{g}\rangle$$

$\varphi(t)$

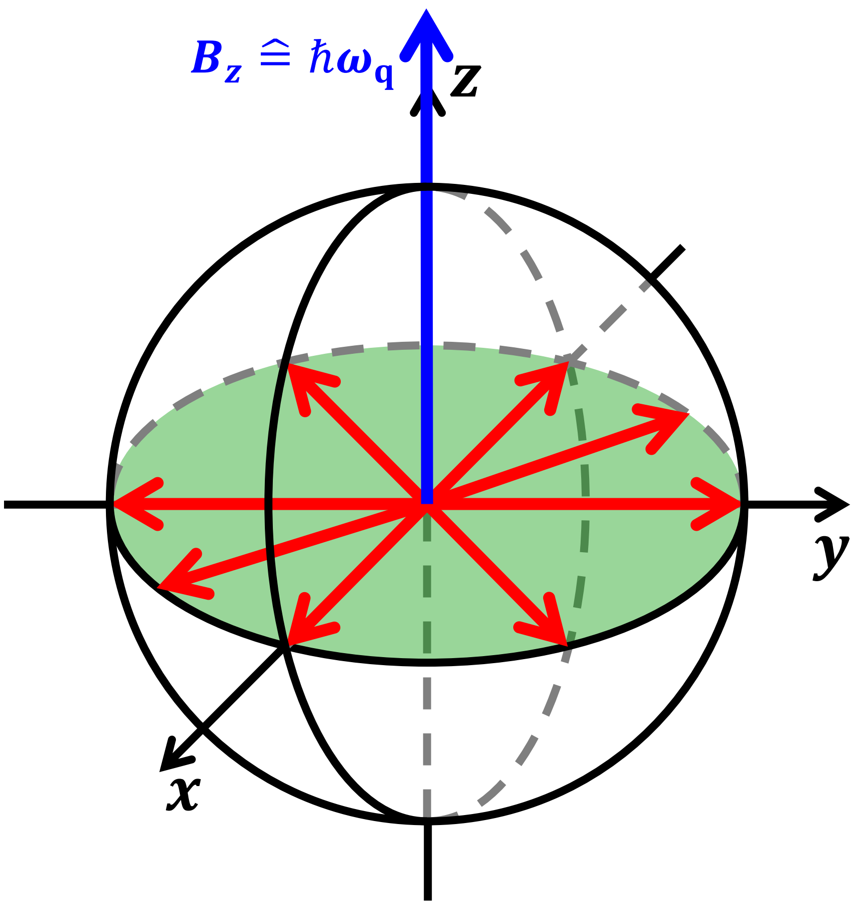


$$\widehat{H}_{\uparrow} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Rotating drive field

Consider the qubit state vector $|\Psi\rangle$ expressed in energy eigenbasis $\{|\mathrm{g}\rangle, |\mathrm{e}\rangle\}$

Apply a drive field with amplitude $\hbar\omega_\mathrm{d}$ **rotating about the z-axis** at frequency ω



$$\rightarrow \quad \widehat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix}$$

ωt	B_x	B_y
0	ω_d	0
$\pi/4$	$\omega_\mathrm{d}/\sqrt{2}$	$\omega_\mathrm{d}/\sqrt{2}$
$\pi/2$	0	ω_d
$3\pi/4$	$-\omega_\mathrm{d}/\sqrt{2}$	$\omega_\mathrm{d}/\sqrt{2}$
π	$-\omega_\mathrm{d}$	0
$5\pi/4$	$-\omega_\mathrm{d}/\sqrt{2}$	$-\omega_\mathrm{d}/\sqrt{2}$
$3\pi/2$	0	$-\omega_\mathrm{d}$
$7\pi/4$	$\omega_\mathrm{d}/\sqrt{2}$	$-\omega_\mathrm{d}/\sqrt{2}$

Driven quantum TLS

$$\hat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix} = \underbrace{\frac{\hbar\omega_\mathrm{q}}{2}\hat{\sigma}_z}_{\equiv \hat{H}_0} + \underbrace{\frac{\hbar\omega_\mathrm{d}}{2}\left(\hat{\sigma}_+ e^{+i\omega t} + \hat{\sigma}_- e^{-i\omega t}\right)}_{\equiv \hat{H}_\mathrm{d}}$$

Operators $\hat{\sigma}_- \equiv |\mathrm{e}\rangle\langle\mathrm{g}|$ and $\hat{\sigma}_+ \equiv |\mathrm{e}\rangle\langle\mathrm{g}|$ create or annihilate an excitation in the TLS

→ Here, our **physics convention is not very intuitive**, but consistent!

Matrix representation → $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Drive **rotating about arbitrary equatorial axis** on the Bloch sphere

→ $\hat{H}_\mathrm{d} = \frac{\hbar\omega_\mathrm{d}}{2}\left(\hat{\sigma}_+ e^{+i(\omega t+\varphi)} + \hat{\sigma}_- e^{-i(\omega t+\varphi)}\right)$

→ without loss of generality we choose $\varphi = 0$

$$\hat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix}$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

Time evolution of driven quantum TLS

Qubit state $|\Psi(t)\rangle = a_\mathrm{e}(t)|\mathrm{e}\rangle + a_\mathrm{g}(t)|\mathrm{g}\rangle$
obeys **Schrödinger equation**

$$i\frac{\mathrm{d}}{\mathrm{d}t}a_\mathrm{e}(t) = \frac{\omega_\mathrm{q}}{2}a_\mathrm{e}(t) + \frac{\omega_\mathrm{d}}{2}e^{-i\omega t}a_\mathrm{g}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}a_\mathrm{g}(t) = \frac{\omega_\mathrm{d}}{2}e^{i\omega t}a_\mathrm{e}(t) - \frac{\omega_\mathrm{q}}{2}a_\mathrm{g}(t)$$

Time-dependent → Difficult to solve → **Move to rotating frame!**

$$b_\mathrm{e}(t) \equiv e^{i\omega t/2}a_\mathrm{e}(t)$$

$$b_\mathrm{g}(t) \equiv e^{-i\omega t/2}a_\mathrm{g}(t)$$

→ Schrödinger equation **looses explicit time dependence**

$$i\frac{\mathrm{d}}{\mathrm{d}t}b_\mathrm{e}(t) = \frac{\omega_\mathrm{q}-\omega}{2}b_\mathrm{e}(t) + \frac{\omega_\mathrm{d}}{2}b_\mathrm{g}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}b_\mathrm{g}(t) = \frac{\omega_\mathrm{d}}{2}b_\mathrm{e}(t) - \frac{\omega_\mathrm{q}-\omega}{2}b_\mathrm{g}(t)$$

$$\hat{H}(t) = \frac{\hbar}{2}\begin{pmatrix} \omega_\mathrm{q} & \omega_\mathrm{d} e^{-i\omega t} \\ \omega_\mathrm{d} e^{i\omega t} & -\omega_\mathrm{q} \end{pmatrix}$$

Interpretation of the rotating frame

$$i\frac{\mathrm{d}}{\mathrm{d}t} b_\mathrm{e}(t) = \frac{\omega_\mathrm{q} - \omega}{2} b_\mathrm{e}(t) + \frac{\omega_\mathrm{d}}{2} b_\mathrm{g}(t)$$

$$i\frac{\mathrm{d}}{\mathrm{d}t} b_\mathrm{g}(t) = \frac{\omega_\mathrm{d}}{2} b_\mathrm{e}(t) - \frac{\omega_\mathrm{q} - \omega}{2} b_\mathrm{g}(t)$$

The **frame rotates at the angular speed $\boldsymbol{\omega}$** of the drive

→ Driving field appears at rest

→ Drive can be in resonance with Larmor precession frequency ω_q

→ Away from resonance $|\omega - \omega_\mathrm{q}| \gg 0$

→ red terms dominate → no $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ transitions induced by drive

→ Near resonance $\omega \approx \omega_\mathrm{q}$

→ blue terms dominate → $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ transitions induced by drive

Formal treatment → Effective Hamiltonian $\widetilde{H}$ describing the same dynamics as $\hat{H}(t)$

$$\widetilde{\mathrm{H}} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_\mathrm{d} \\ \omega_\mathrm{d} & \Delta\omega \end{pmatrix} \qquad i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{\Psi}(t)\rangle = \widetilde{H}|\widetilde{\Psi}(t)\rangle$$

with $\Delta\omega \equiv \omega - \omega_\mathrm{q}$ $\qquad |\widetilde{\Psi}(t)\rangle \equiv b_\mathrm{e}(t)|\mathrm{e}\rangle + b_\mathrm{g}(t)|\mathrm{g}\rangle$

$$|\Psi_+\rangle = +e^{-i\varphi/2}\cos\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\sin\frac{\theta}{2}|\varphi_-\rangle$$

$$|\Psi_-\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|\varphi_+\rangle + e^{+i\varphi/2}\cos\frac{\theta}{2}|\varphi_-\rangle$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

Dynamics of the effective Hamiltonian

$$\widetilde{\mathrm{H}} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_\mathrm{d} \\ \omega_\mathrm{d} & \Delta\omega \end{pmatrix}$$

→ **Diagonalize** → $\widetilde{\mathrm{H}} = \dfrac{-\hbar\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\hat{\sigma}_z$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

$$\tan\varphi = 0$$

→ New eigenstates →

$$|\widetilde{\Psi}_+\rangle = +\cos\frac{\theta}{2}|\mathrm{e}\rangle + \sin\frac{\theta}{2}|\mathrm{g}\rangle$$

$$|\widetilde{\Psi}_-\rangle = -\sin\frac{\theta}{2}|\mathrm{e}\rangle + \cos\frac{\theta}{2}|\mathrm{g}\rangle$$

Expand into stationary states

$$|\widetilde{\Psi}(t)\rangle = \langle\widetilde{\Psi}_-|\widetilde{\Psi}_0\rangle e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \langle\widetilde{\Psi}_+|\widetilde{\Psi}_0\rangle e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

Initial state $|\boldsymbol{\Psi}_0\rangle = |\mathrm{g}\rangle$ (energy ground state)

$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2}e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2}e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2} e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2} e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

Probability P_e to find TLS in $|\mathrm{e}\rangle$

$$|\widetilde{\Psi}_+\rangle = +\cos\frac{\theta}{2}|\mathrm{e}\rangle + \sin\frac{\theta}{2}|\mathrm{g}\rangle$$

$$|\widetilde{\Psi}_-\rangle = -\sin\frac{\theta}{2}|\mathrm{e}\rangle + \cos\frac{\theta}{2}|\mathrm{g}\rangle$$

$$P_\mathrm{e} \equiv |\langle \mathrm{e}|\Psi(t)\rangle|^2 = \left|\langle \mathrm{e}|\widetilde{\Psi}(t)\rangle\right|^2 =$$

$$= \left|\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}} - e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}\right)\right|^2$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

$$= \left|\sin\theta\ \sin\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)\right|^2$$

$$\boxed{P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2+\omega_\mathrm{d}^2}\sin^2\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)}$$

Driven Rabi oscillations
TLS population oscillates under transversal drive

..... Qubit Control: Rabi Oscillations

Rabi Oscillations on the Bloch sphere

$$|\widetilde{\Psi}(t)\rangle = \cos\frac{\theta}{2}\, e^{+\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_-\rangle + \sin\frac{\theta}{2}\, e^{-\frac{it}{2}\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}|\widetilde{\Psi}_+\rangle$$

$$\Delta\omega \equiv \omega - \omega_\mathrm{q}$$

$$\tan\theta = -\frac{\omega_\mathrm{d}}{\Delta\omega}$$

On resonance $\omega = \omega_\mathrm{q}$

→ Rotating frame cancels Larmor precession

→ State vector $|\widetilde{\Psi}(t)\rangle$ has no φ-evolution

→ $|\widetilde{\Psi}(t)\rangle = \cos\frac{\omega_\mathrm{d}t}{2}|\mathrm{g}\rangle + i\sin\frac{\omega_\mathrm{d}t}{2}|\mathrm{e}\rangle$

→ Rotation about x-axis

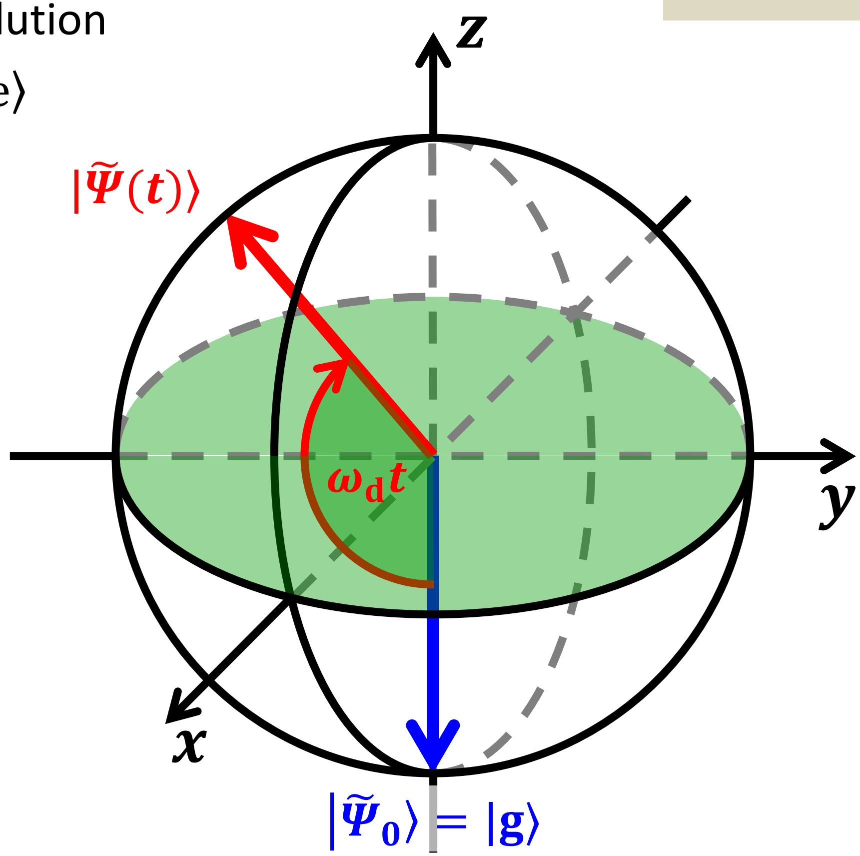
Finite detuning $|\Delta\omega| > 0$

→ Additional precession at $\Delta\omega$

→ Population oscillates faster

→ Reduced oscillation amplitude

$$\boxed{P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2+\omega_\mathrm{d}^2}\sin^2\left(\frac{t\sqrt{\Delta\omega^2+\omega_\mathrm{d}^2}}{2}\right)}$$

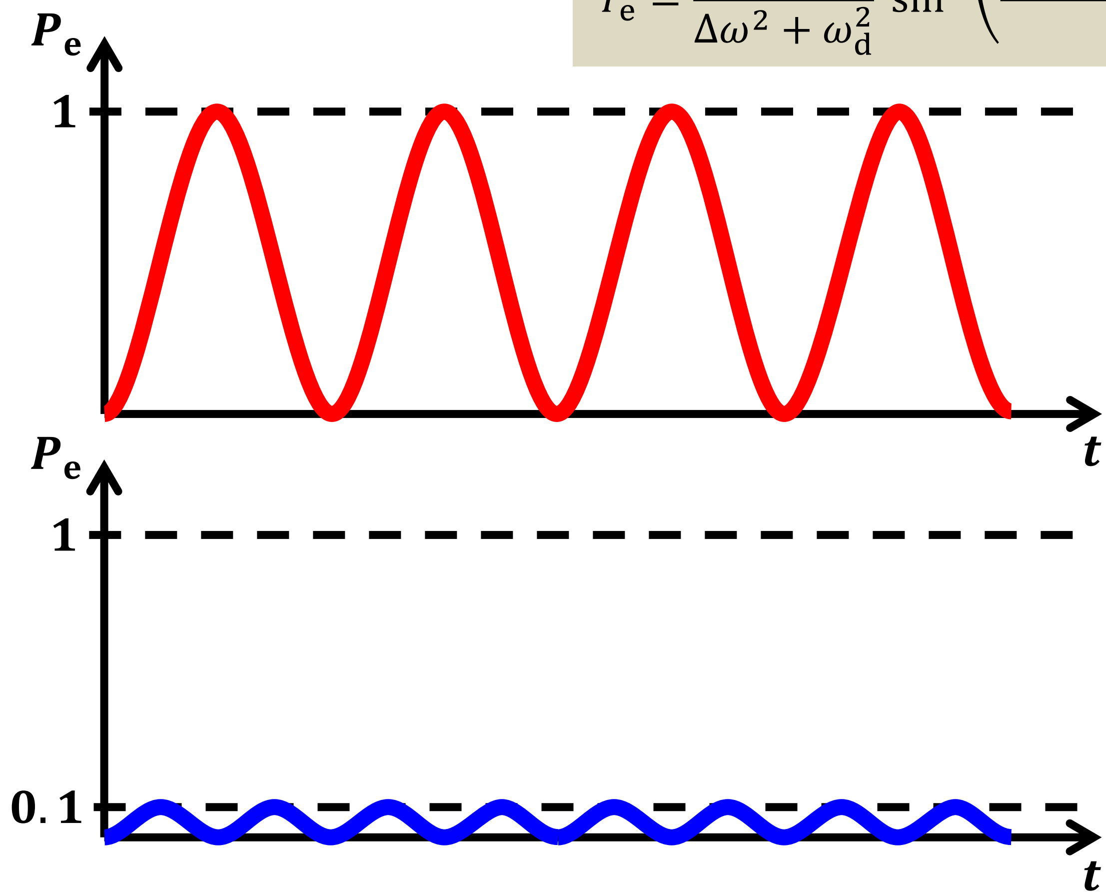


Rabi Oscillations – Graphical representation

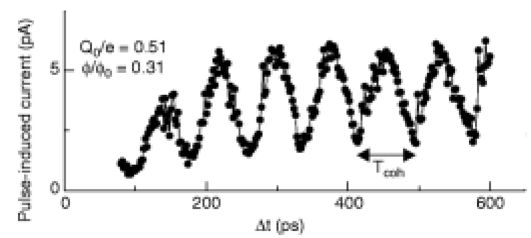
$$P_{\mathrm{e}} = \frac{\omega_{\mathrm{d}}^2}{\Delta\omega^2 + \omega_{\mathrm{d}}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_{\mathrm{d}}^2}}{2}\right)$$

On resonance $\omega = \omega_{\mathrm{q}}$

Detuning $|\Delta\omega| = 3\omega_{\mathrm{d}}$



First experimental demonstration with superconducting qubit
Y. Nakamura *et al.*, *Nature* **398**, 786 (1999)

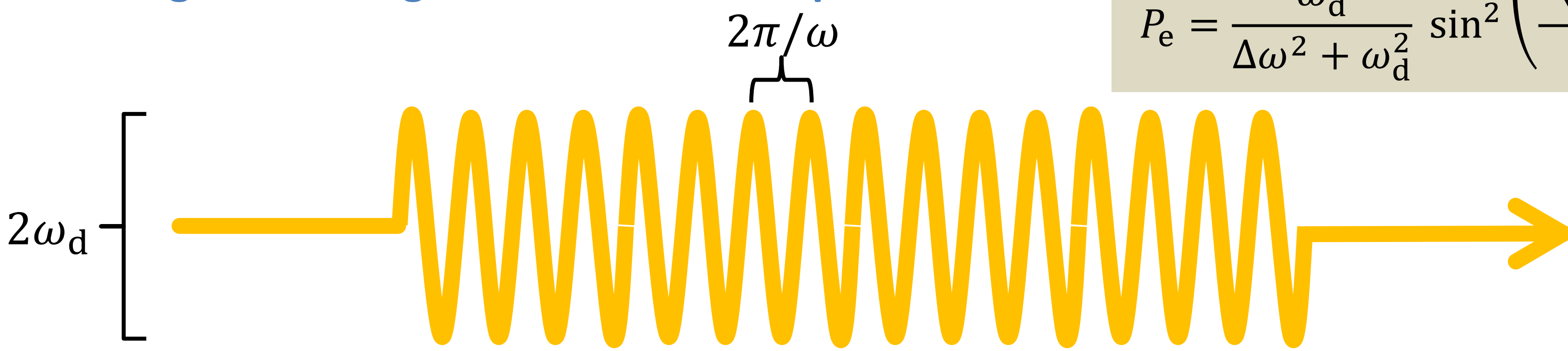


10.3 Control of quantum two-level systems

….. Qubit Control: Rabi Oscillations

Oscillating vs. rotating drive – Microwave pulses

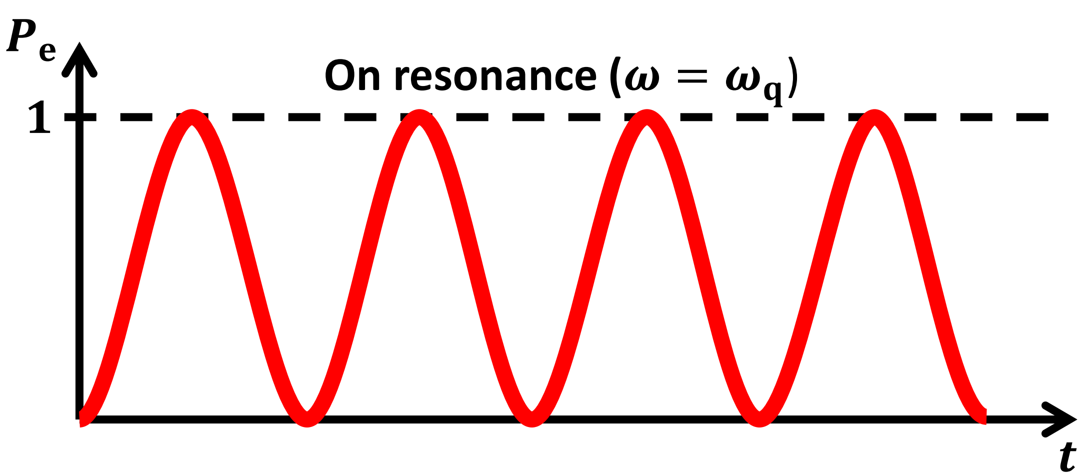
$$P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2 + \omega_\mathrm{d}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_\mathrm{d}^2}}{2}\right)$$



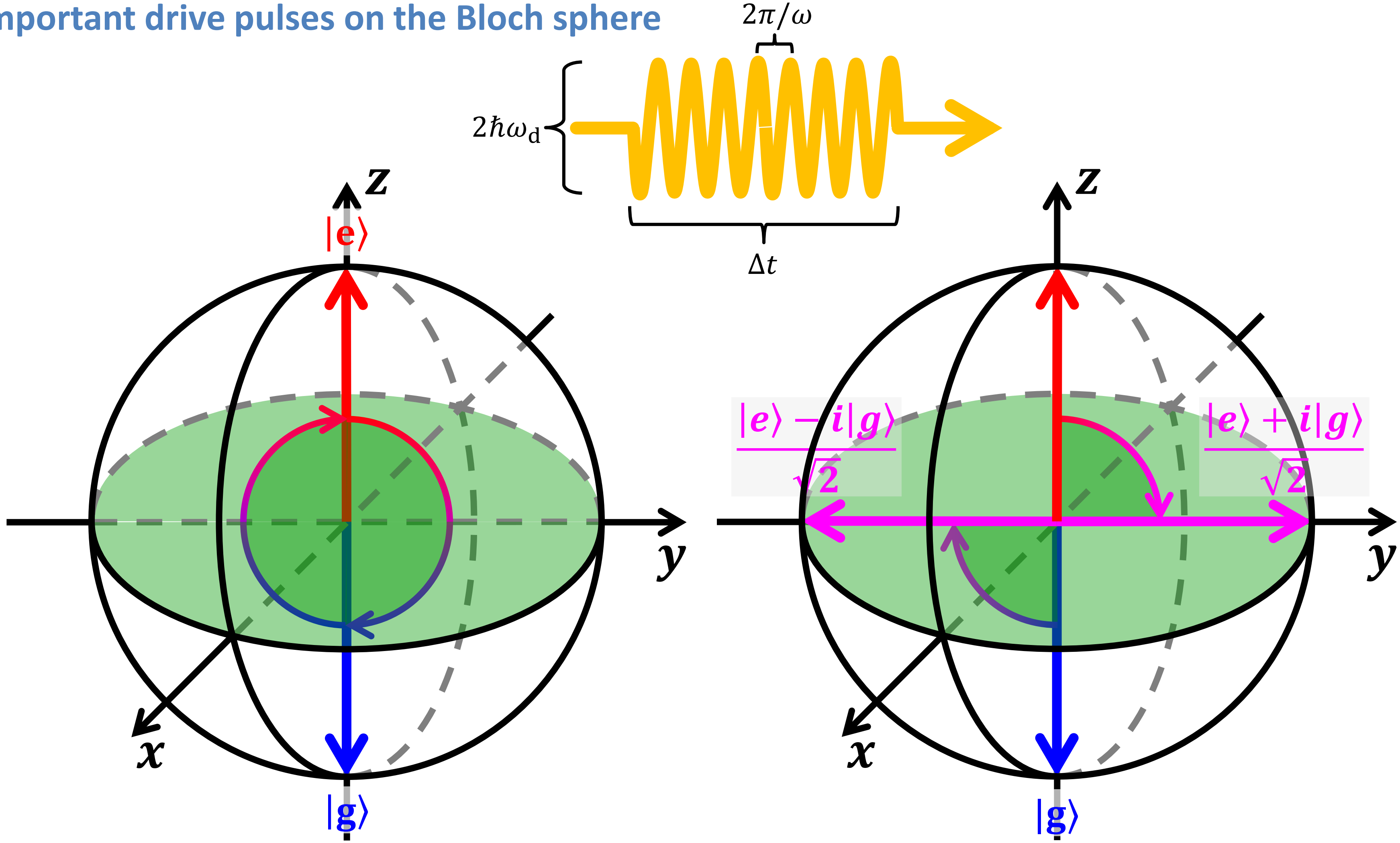
Oscillating drive $2\hbar\omega_\mathrm{d}\cos\omega t = \hbar\omega_\mathrm{d}\left(e^{+\mathrm{i}\omega t} + e^{-\mathrm{i}\omega t}\right)$

→ in frame rotating with $+\omega$ the $e^{-\mathrm{i}\omega t}$ -component rotates fast with -2ω

→ For $\omega_\mathrm{d} \ll \omega$ this fast contribtion averages out on the timescale of the slowly rotating component → **Rotating wave approximation**



Important drive pulses on the Bloch sphere



π-pulse ($\omega_d \Delta t = \pi$)
$|g\rangle \leftrightarrow |e\rangle$ flips, refocus phase evolution

$\pi/2$-pulse ($\omega_d \Delta t = \pi/2$)
Rotates into equatorial plane and back

Rabi Oscillations in presence of decoherence

$$P_\mathrm{e} = \frac{\omega_\mathrm{d}^2}{\Delta\omega^2 + \omega_\mathrm{d}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_\mathrm{d}^2}}{2}\right)$$

Effect of decoherence (qualitative approach)

→ Loss of coherent properties to environment at a constant rate $\Gamma_\mathrm{dec} = 2\pi/T_\mathrm{dec}$

→ "Change of coherence" (time derivative) proportional to "amount of coherence"

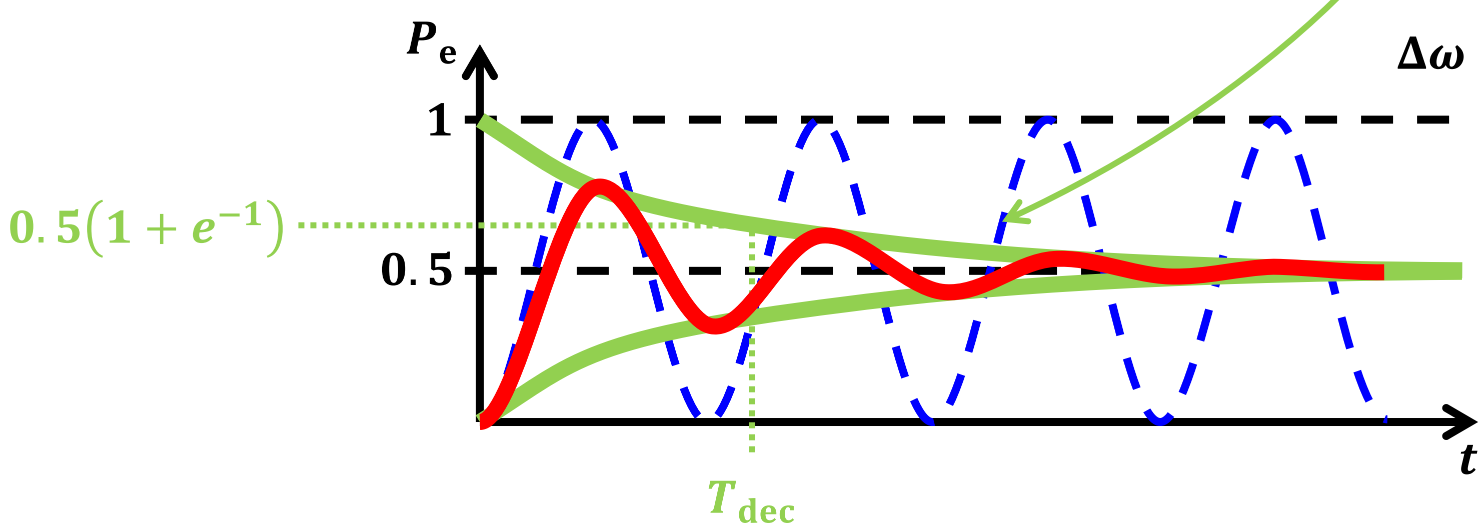
→ Exponential decay of coherent property with factor $e^{-\frac{\Gamma_\mathrm{dec} t}{2\pi}} = e^{-\frac{t}{T_\mathrm{dec}}}$

→ Argument holds well for population decay (energy relaxation, T_1)

→ Loss of phase coherence more diverse depending on environment (exponential, Gaussian, or power law)

→ Experimental timescales range from few ns to 100 µs

Decay envelope



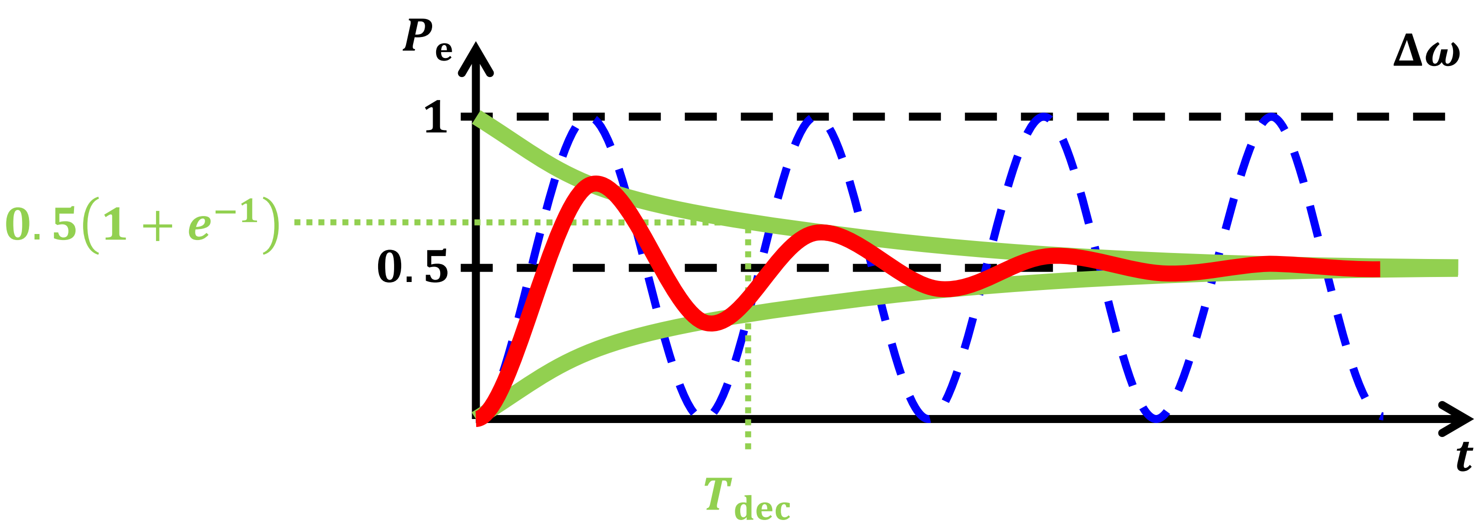
$$P_{\mathrm{e}} = \frac{\omega_{\mathrm{d}}^2}{\Delta\omega^2 + \omega_{\mathrm{d}}^2} \sin^2\left(\frac{t\sqrt{\Delta\omega^2 + \omega_{\mathrm{d}}^2}}{2}\right)$$

Rabi decay time

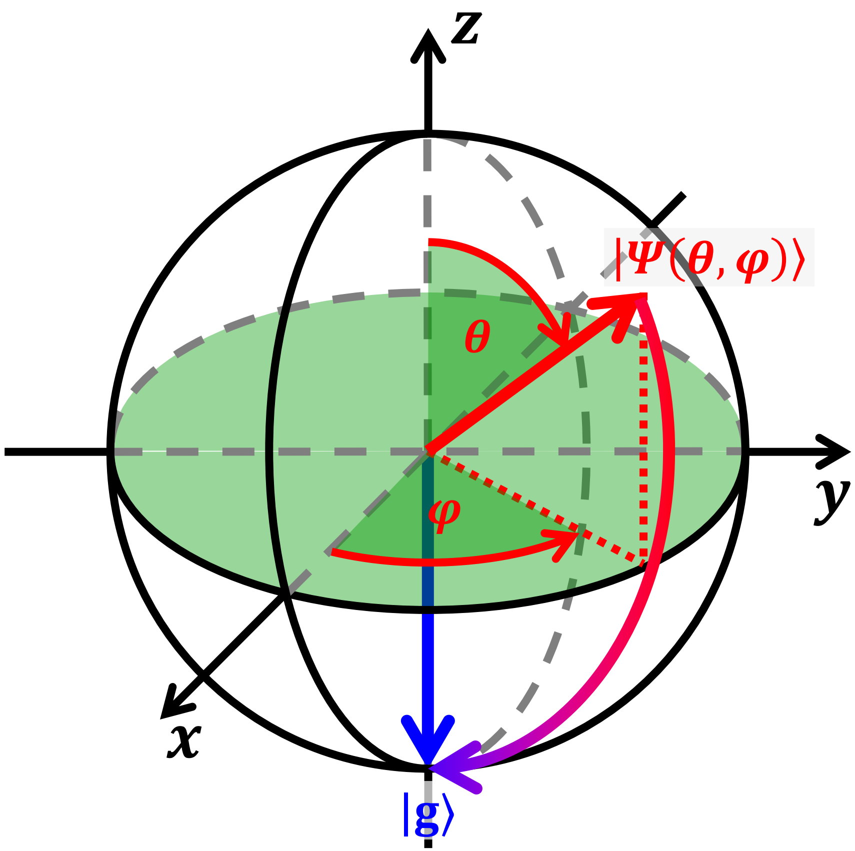
Complicated interplay between T_1, T_2^*, and the drive

→ At long times, small oscillations persist
→ Nevertheless useful **order-of-magnitude check** for T_{dec}
→ Important tool for single-qubit gates

→ To determine T_1, T_2^*, T_2 correctly, more sophisticated protocols are required
→ **energy relaxation measurements, Ramsey fringes, spin echo**



Energy relaxation on the Bloch sphere



Environment induces energy loss

→ State vector collapses to $|g\rangle$
→ Implies also loss of phase information
→ Intrinsically irreversible
→ T_1-time → rate $\Gamma_1 = \frac{2\pi}{T_1}$

Golden Rule argument

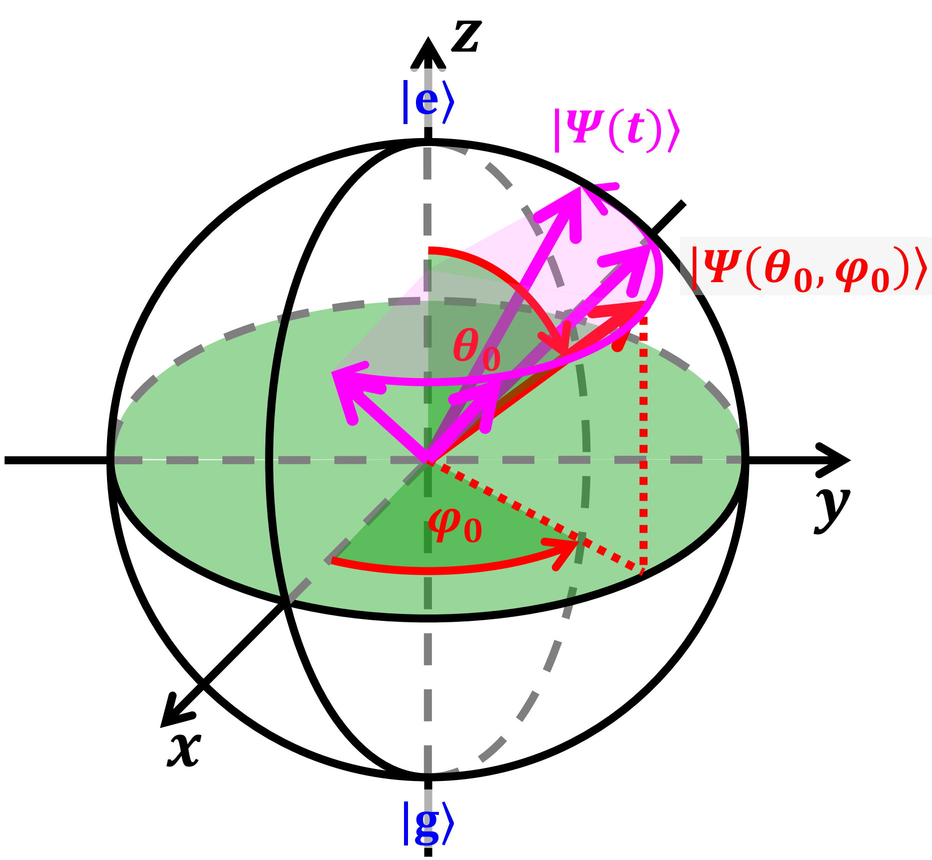
→ $\Gamma_1 \propto S(\omega_q)$
→ $S(\omega)$ is noise spectral density
→ High frequency noise
→ Intuition: Noise induces transitions

Quantum jumps

→ Single-shot, quantum nondemotiton measurement yields a discrete jump to $|g\rangle$ at a random time
→ Probability is equal for each point of time
→ Exponential decay with $e^{-\frac{t}{T_1}}$

….. Qubit Control: Rabi Oscillations

Dephasing on the Bloch sphere



Environment induces random phase changes

→ $\Gamma_2 \propto S(\omega \to 0)$, $T_2 = \frac{2\pi}{\Gamma_2}$

→ Low-frequency noise is detuned

→ **No energy transfer**

→ $1/f$-noise → $S(\omega) \propto \frac{1}{\omega}$

→ Example: Two-level fluctuator bath

→ To some extent reversible

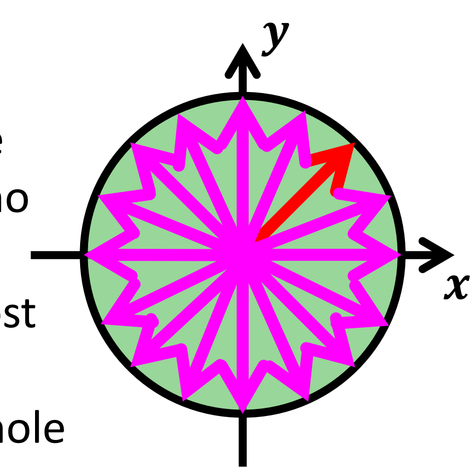
→ Decay laws $e^{-\frac{t}{T_2}}$, $e^{-\left(\frac{t}{T_2}\right)^2}$, $\left(\frac{t}{T_2}\right)^\beta$

Visualization

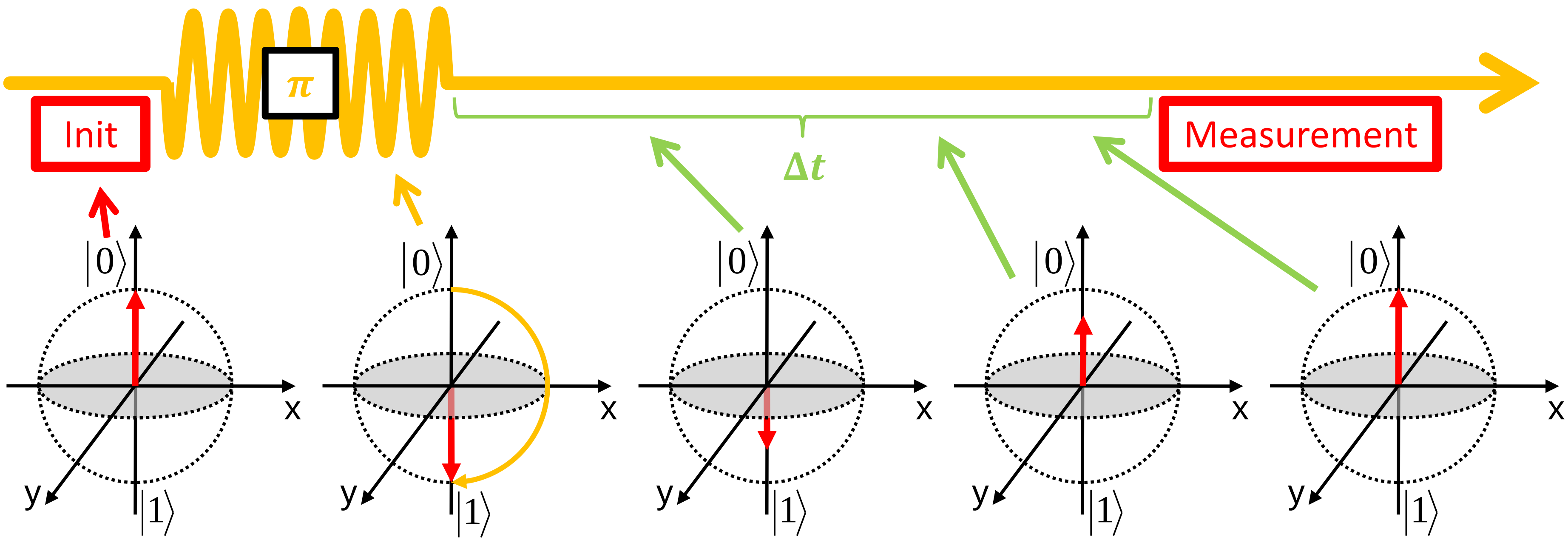
→ Phase φ becomes more and more unknown with time

→ Classical probility, no superposition!

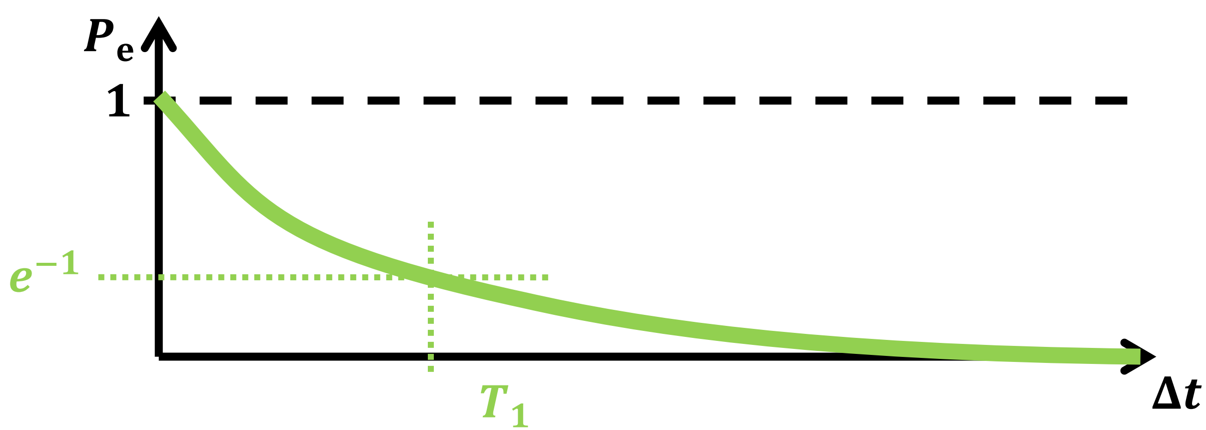
→ Phase coherence lost when arrows are distributed over whole equatorial plane



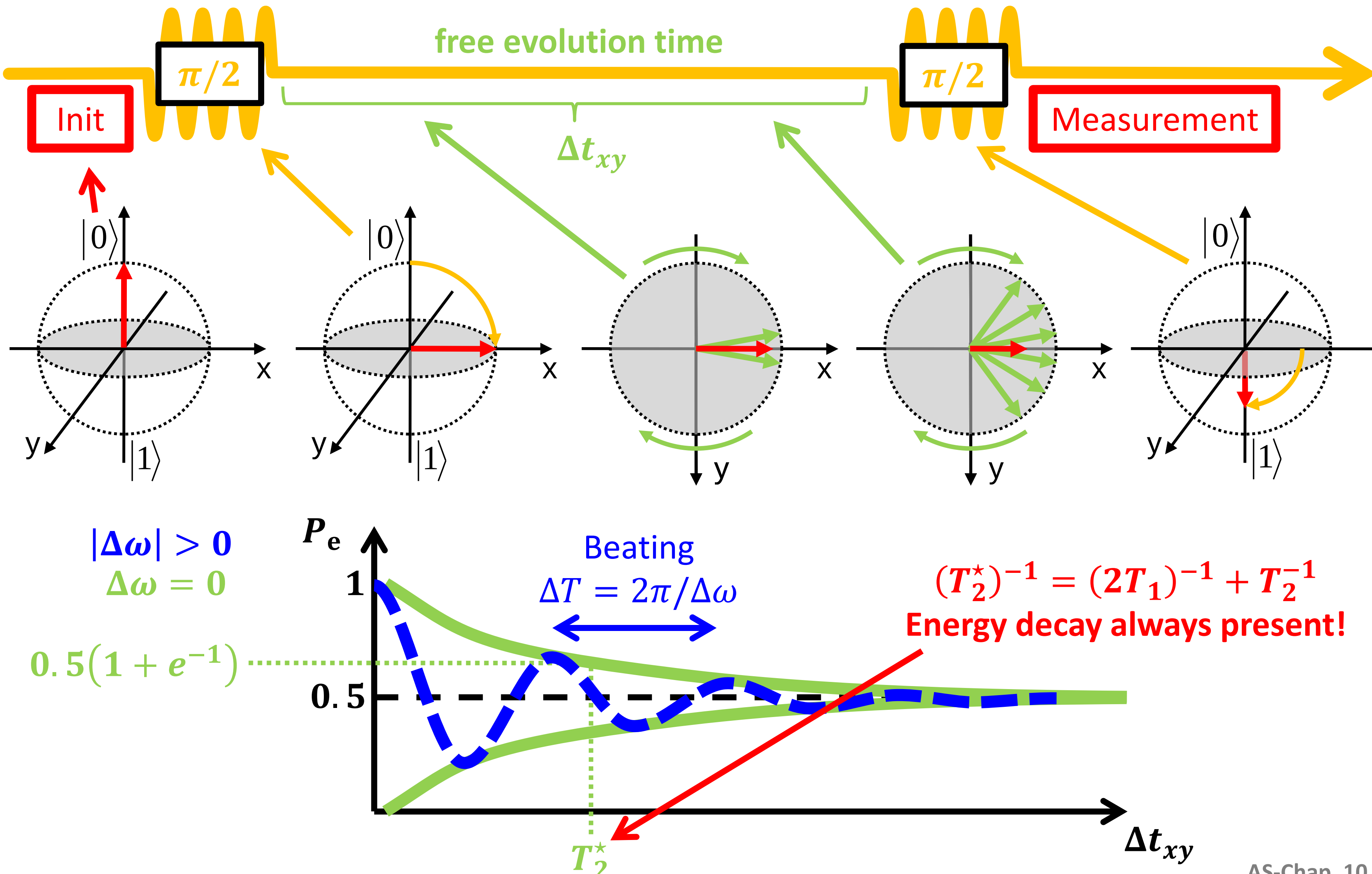
Qubit dynamics – Relaxation



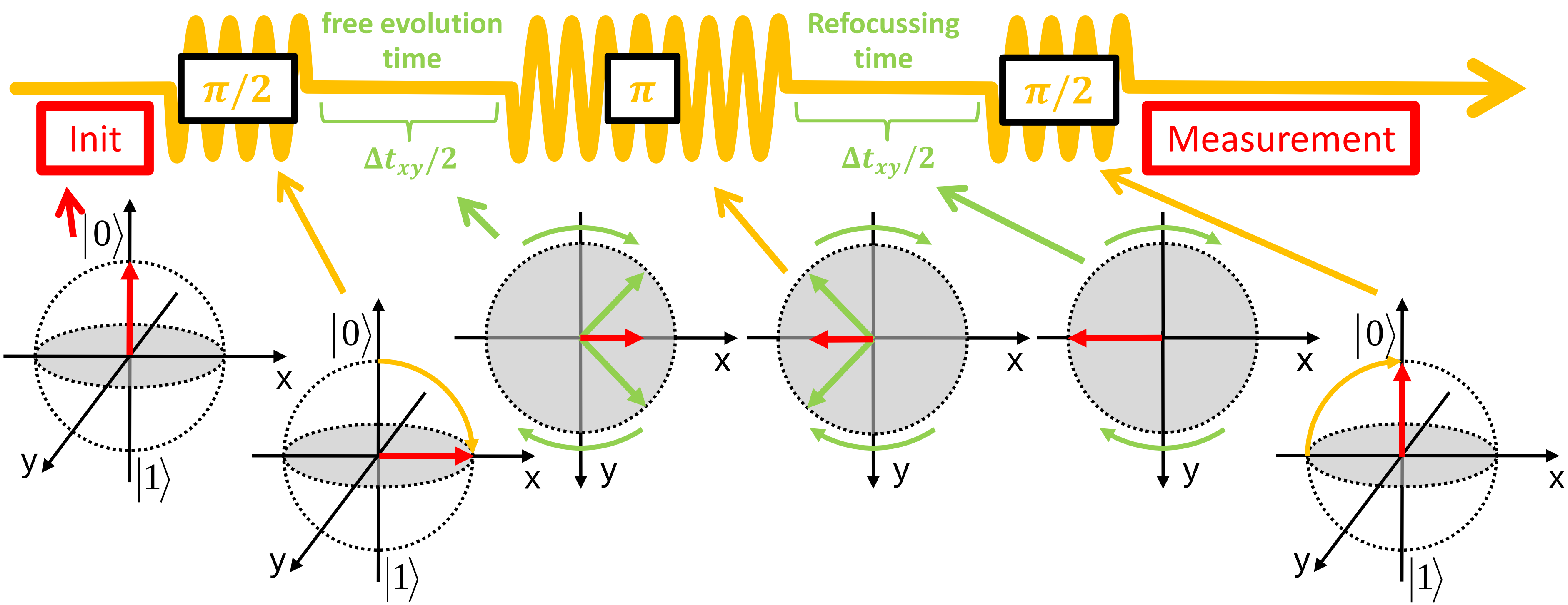
Rotating frame & no detuning ($\Delta\omega = \omega - \omega_q = 0$) → no xy-evolution



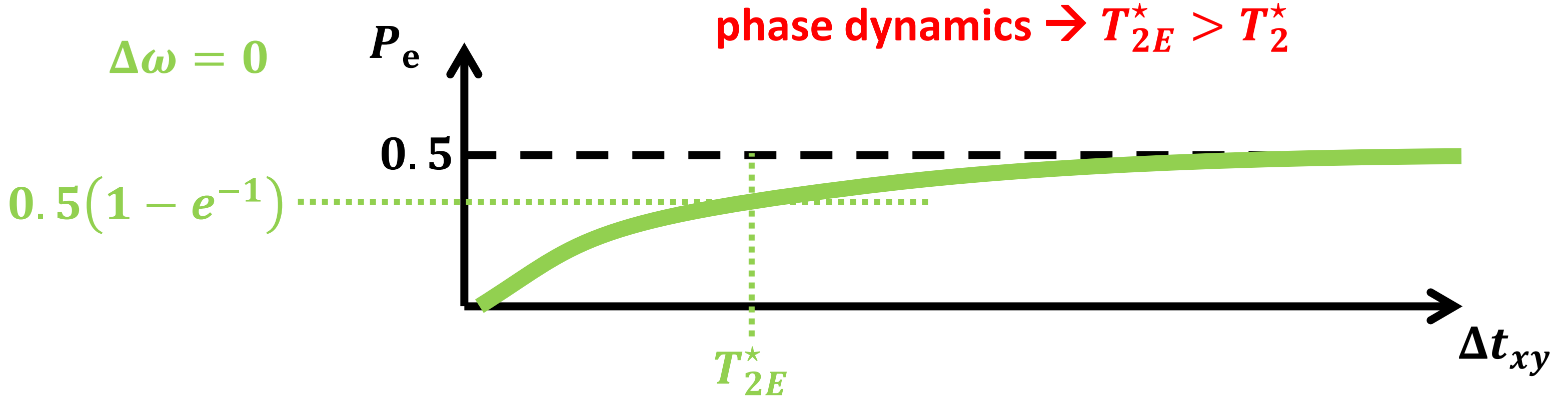
Qubit dynamics – Ramsey fringes ($T_2^\star$)



Qubit dynamics – Spin echo ($T_{2E}^{\star}$)



Refocussing pulse reverses low-frequency phase dynamics → $T_{2E}^{\star} > T_2^{\star}$



Ramsey vs. spin echo sequence

Spin echo cancels the effect of low-frequency noise in the environment

→ **Pulse sequences act as filters!**

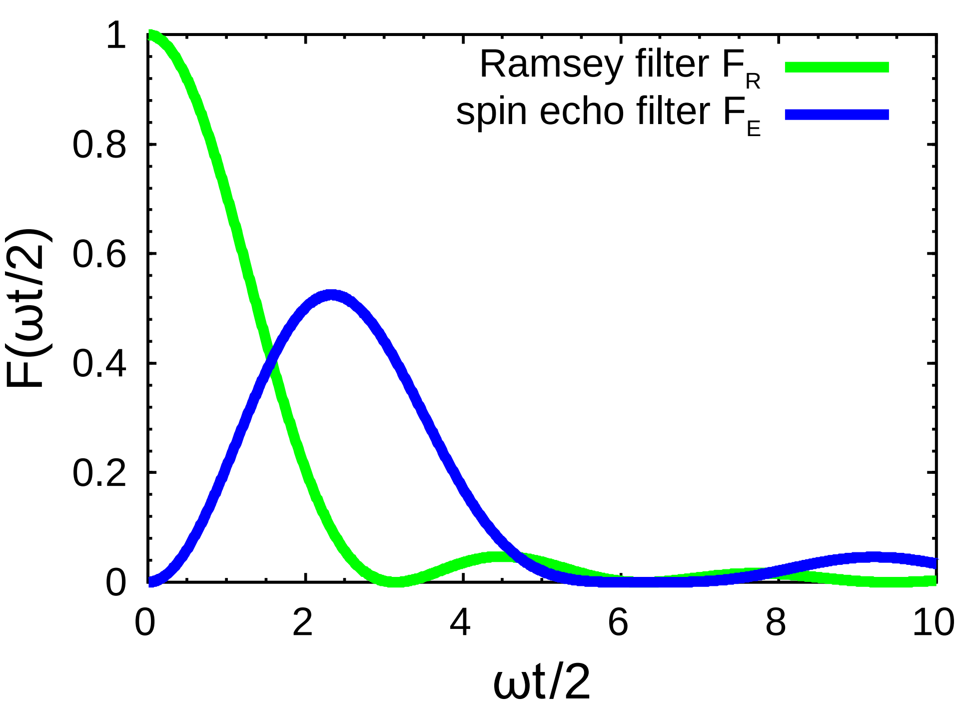
→ Environment described by noise spectral density $S(\omega)$

Decay envelope $\propto e^{-t^2 \int_{-\infty}^{+\infty} S(\omega) F_{\mathrm{R,E}}\left(\frac{\omega t}{2}\right) d\omega}$

$$F_{\mathrm{R}}(\omega t) = \frac{\sin^2 \frac{\omega t}{2}}{\left(\frac{\omega t}{2}\right)^2}$$

$$F_{\mathrm{E}}(\omega t) = \frac{\sin^4 \frac{\omega t}{4}}{\left(\frac{\omega t}{4}\right)^2}$$

Sequence length t is important!



Spin echo sequence

→ Filters low-frequency noise for $\omega t \to 0$

→ $\omega t \approx 2$ → Noise field fluctuates synchronously with π-pulse → No effect