Problem Set 9 problems

Question 1: Deep Inelastic Scattering Kinematics (50 points)

Learning objectives

In this question you will:

- Review the kinematic variables used to describe Deep Inelastic Scattering
- Relate the variable y to the scattering angle in the parton-lepton center-or-mass frame
- Learn how to express the structure function F_2 in terms of the scaling variables x and y

Consider the process $ep \to e + X$. We are going to derive the expression that gives the cross section in terms of the structure functions and the dimensionless variables x and y:

$$x \equiv \frac{Q^2}{2P \cdot q}$$
$$y \equiv \frac{P \cdot q}{P \cdot k}$$

where P is the initial four-momentum of the proton, k is the initial four-momentum of the electron, k' is the final four-momentumof the electron, $q \equiv k - k'$ and $Q^2 = -q^2$ (see page 3 of the lecture 17 notes)

For this problem, instead of working in either the lab frame of the proton-lepton center of mass frame, we will work in the center of mass frame of the hard scatttering. That means we will work in the frame where the sum of the three momentum of the parton and the quark is zero::

$$\vec{p}_q^{\,*} + \vec{p}_e^{\,*} = 0$$

where we are using the superscript * to remind us that we are in the parton-lepton center of mass frame.

We will take our coordinates so that the inital momentum of the electron is in the z-direction and the final electron makes an angle θ^* with respect to the z-axis. We will also assume that all masses can be neglected so that $|\vec{p}_e^{\,*}| = |\vec{p}_q^{\,*}| \equiv E^*$.

1a.

From the definition of y, show that

$$y = \sin^2\left(\frac{\theta^*}{2}\right)$$

Solution

In the parton-electron center-of-mass frame with all masses set to 0:

$$\begin{aligned} k &= (E, 0, 0, E) \\ k' &= (E, E\sin\theta, 0, E\cos\theta) \\ xp &= (E, 0, 0, -E) \end{aligned}$$

Therefore

$$1 - y = 1 - \frac{p \cdot q}{p \cdot k} = \frac{p \cdot (k - q)}{p \cdot q}$$

Since $q = k - k'$ this becomes

$$\begin{aligned} 1 - y &= \frac{p \cdot \left(k - (k - k')\right)}{p\dot{q}} \\ &= \frac{p \cdot k'}{p \cdot k} = \frac{xp \cdot k'}{xp \cdot k} \\ &= \frac{E^2(1 + \cos\theta)}{2E^2} \\ &= \frac{1}{2}(1 + \cos\theta) \end{aligned}$$

1b.

It is possible to derive the cross section for the e-parton elastic scattering in the paton-lepton center of mass frame starting from Thomson eq 6.72, but you will not be asked to do that calculation. We will just quote the solution:

$$\frac{d\sigma^{eq}}{d\Omega} = e_q^2 \frac{\alpha^2}{8p^{*2}\sin^4(\theta^*/2)}\left[1 + \cos^4(\theta^*/2)\right]$$

where e_q is the charge of the quark or antiquark in units of e (eg $e_q = 2/3$ for up quarks) and p^* is the incoming momentum of the e in the center-of-mass frame. Starting with this expression, perform a change of variables to find the differential cross section $d\sigma^{eq}/dy$.

Solution

Starting with the given formula:

$$\frac{d\sigma^{eq}}{d\Omega} = e_q^2 \frac{\alpha^2}{8p^2\sin^4(\theta^*/2)}\left[1+\cos^4(\theta^*/2)\right]$$

And noting that $d\Omega = 2\pi\sin\theta d\theta$ and $y = \frac{1}{2}(1-cos\theta)$ we find

$$\begin{aligned} dy &= \frac{1}{2}\sin\theta d\theta \\ \Rightarrow \quad d\Omega &= 4\pi dy \end{aligned}$$

Using the double angle formula

$$y = \frac{1}{2}\left(1-(1-2\sin^2\frac{2\theta}{2})\right) = \sin^2\frac{\theta}{2}$$

so

$$\frac{d\sigma^{eq}}{dy} = e_q^2\frac{4\pi\alpha^2}{8p^2y^2}\left[1+(1-y)^2\right]$$

1c.

Use the fact that the deep inelastic ep cross section can be calculated as an incoherent sum over e-parton scattering cross sections to turn your expression from part (b) into an expression for the ep scattering cross section $d^2\sigma^{ep}/dxdy$ in terms of a sum over $f_i(x)$, where $f_i(x)$ is the parton distribution function for parton species i. In other words, $f_i(x)$ is defined as the probability of finding a parton of species i that carries a fraction of the proton's momentum between that lies between x and $x+dx$.

Solution

Summing over the individual partons gives:

$$\begin{aligned} \frac{d\sigma^{eq}}{dy} &= \frac{4\pi\alpha^2}{8p^2y^2}\left[1+(1-y)^2\right]\sum_i e_i^2 f_i(x)dx \\ \frac{d\sigma^{eq}}{dydx} &= \frac{4\pi\alpha^2}{8p^2y^2}\left[1+(1-y)^2\right]\sum_i e_i^2 f_i(x) \end{aligned}$$

d.

Use the definition of $F_2(x)$:

$$F_2(x) \equiv \sum_i e_i^2 x f_i(x)$$

to rewrite $d^2\sigma^{ep}/dxdy$ in terms of $F_2(x)$ instead of the sum over the $f_i(x)$.

Solution

Using $F_2(x) = \sum_i e_i^2 f_i(x)$ we find

$$\frac{d\sigma^{ep}}{dydx} = \frac{4\pi\alpha^2}{8p^2y^2}\left[1+(1-y)^2\right]F_2(x)$$

Question 2: Hadron Hadron Collisions (50 points)

Learning objectives

In this question you will:

- Learn that the parton distribution functions measured in Deep Inelastic Scattering can be used in other processes
- Apply this to the case of proton-proton collisions

2a.

In class this week we learned that Deep Inelastic Scattering cross sections can be used to determine the distribution of partons inside the proton. The parton distribution functions measured this way can be used in other hadron scattering processes. For example, consider the process

$$pp \to \mu^+\mu^- X$$

The cross section for this process can be calculated by convoluting the cross section

$$\sigma(q\bar{q} \to \mu^+\mu^-)$$

with the appropriate probability distribution functions for finding quark q in one proton and and anti-quark $\bar{q}$ in the other proton. A good picture of this process (taken from Wikpedia) is shown below. The q amd $\bar{q}$ annihilate into a virtual photon that then turns into the lepton pair.

Drawing

The cross section for the reaction $pp \to \mu^+\mu - +X$ is given by

$$\frac{d\sigma(pp \to \mu^+\mu^- X)}{d\hat{s}} = \sum_{ij} f_i^{(q)}(x_1) f_j^{(\bar{q})}(x_2) \frac{d\sigma(q_i\bar{q}_j \to \mu^+\mu^- X)}{d\hat{s}}$$

where $\hat{s}$ is the square of the $\mu^+\mu^-$ invariant mass, $f_i^{(q)}(x_1)$ is the parton distribution function for quarks of species i to carry a fraction x_1 of the proton's momentum and $f_y^{(\bar{q})}(x_2)$ is the parton distribution function for anti-quarks of species i to carry a fraction x_2 of the proton's momentum. (this is the same $f_i(x)$ as in the previous problem) Note that for the process above $i = j$ since the quark and anti-quark must be of the same species. Later this semester we will see that W-boson production occurs with a similar diagram but in that case $i \neq j$

2a.

Prove that the invariant mass squared of the $\mu^+\mu^-$ system

$$\hat{s} = x_1 x_2 s$$

where s is the center-of-mass energy squared of the proton-proton collision.

Note: As usual in the parton model, we will assume that the masses of the quarks can be neglected and that their momentum perpendicular to the proton direction of motion is negligable.

Solution

The invariant mass squared of the hard scattering system is

$$\begin{aligned} \hat{s} &= (x_i p_a + x_j p_b)^2 \\ &= x_i^2 p_a^2 + 2x_i x_j p_a \cdot p_b + x_j^2 p_b^2 \\ &\simeq 2x_i x_j p_a \cdot p_b \\ &\simeq x_j x_j s = \tau s \end{aligned}$$

Where we have used $s = p_a^+ p_b^2 + 2p_a \cdot p_b \simeq 2p_a \cdot p_b$

2b.

Prove that the longitudinal momentum of the $\mu^+\mu^-$ system (ie the momentum along the beamline) is

$$p_{||} = (x_1 - x_2)\frac{\sqrt{s}}{2}$$

where our sign convention is that parton 1 comes from the left and parton 2 comes from the right.

Solution

The longitudinal momentum is given by

$$p_{\|a} = \frac{x_a\sqrt{s}}{2}, \quad p_{\|b} = \frac{x_b\sqrt{s}}{2}$$

So the longitudinal momentum

$$p_{\|} = p_{\|a} - p_{\|\ b} = (x_a - x_b)\frac{\sqrt{s}}{2} = x\frac{\sqrt{s}}{2}$$