Physics 129: Particle Physics
Lecture 13: The Dirac Equation

Oct 8, 2020

- Suggested Reading:
  - Thomson Chapter 4
  - Griffiths 6.1-6.3,7.1-7.3
  - Robert Littlejohn's notes for Physics 221:
    - Klein-Gordon Eq:
      http://bohr.physics.berkeley.edu/classes/221/notes/kleing.pdf
    - Dirac Eq:
      http://bohr.physics.berkeley.edu/classes/221/notes/dirac.pdf

Today's class follows Littlejohn's notes closely with some additions of material from Thomson

Clarifications from last time: Calculation of density of states

- From last class:

$$\rho(E) = \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{E_i}$$

- Impose energy and momentum conservation:

$$\begin{aligned}
\rho(E) &= \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{E_i} \times \left[\int d^3 p_n \delta \left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i \right) \times \int dE \delta \left(E - \sum_{i=1}^{N} E_i \right) \right] \\
&= \frac{1}{(2\pi)^{3n}} \int \prod_{i=1}^{n} \frac{d^3 p_i}{E_i} \delta \left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i \right) \delta \left(E - \sum_{i=1}^{N} E_i \right)
\end{aligned}$$

So, the derivative of E removes the integral over E

Reminder: Why Dirac Equation?

- Lorentz invariant eq must be of same order in time and space dimensions
- Klein Gordon starts with $E^2 = p^2 + m^2$

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = 0$$

where

$$\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

  - Since $E = \pm\sqrt{p^2 + m^2}$ negative energy solutions exist
  - These negative energy solutions have negative probability densy states: impossible
  - Can kludge a fix if we multiply by charge and define as a current density
  - But, cannot introduce spin in a Lorentz invariant way

- Dirac eq first order time and space
  - Introduces spin from the start
  - We will see today:
    - Still has negative energy states
    - But probability density is always positive
    - Includes antiparticles in a "natural" way

Constructing the Dirac Eq (I)

- Dirac started familiar quantum mechanical expression

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

  and asked what form $\hat{H}$ should take
- Since first order derivative in time, $\hat{H}$ must be linear in $\frac{\partial}{\partial x_i}$
- He postulated (using natural units)

$$H = \vec{\alpha}\cdot\vec{p} + m$$

  where $\vec{\alpha}$ and β are coefficients to be determined

  Warning: β has nothing to do with our β from special relativity!
- Writing this eq explicitly using derivatives:

$$i\frac{\partial\psi}{\partial t} = -i\sum_{k=1}^{3}\alpha_k\frac{\partial\psi}{\partial x_k} + m\beta\psi$$

Constructing the Dirac Eq (II)

- From previous page:

$$i\frac{\partial\psi}{\partial t} = -i\sum_{k=1}^{3}\alpha_k\frac{\partial\psi}{\partial x_k} + m\beta\psi$$

- The coefficients $\vec{\alpha}$ cannot be ordinary numbers:
  - $\vec{\alpha}$ would specify some privileged direction in space, whic would violate rotational invariance
- Dirac assumed ψ was some sort of multicomponent object, a "spinor"

$$\psi = \begin{pmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_N \end{pmatrix}$$

  where N is to be determined
- The matrices α_k form a vector of matrices just as Pauli matrices σ_x, σ_y,σ_z do
- α_k and β are $N \times N$ matrices
- Since $\hat{H}$ is invariant under space and time transformations, these matrices must be independent of $\vec{x}$ and t
- Also, from form of Dirac eq, they must be Hermitian

Constructing the Dirac Eq (III)

- Dirac eq from previous page:

$$i\frac{\partial\psi}{\partial t} \quad = \quad -i\sum_{k=1}^{3}\alpha_k\frac{\partial\psi}{\partial x_k} + m\beta\psi$$

- Apply $i\partial/\partial t$:

$$\begin{aligned}-\frac{\partial^2\psi}{\partial t^2} &= \hat{H}^2\psi \\ &= -\sum_{k,\ell}\alpha_k\frac{\partial^2\psi}{\partial x_k\partial x_\ell} - im\sum_k(\alpha_k\beta+\beta\alpha_k)\frac{\partial\psi}{\partial x_k} + m^2\beta^2\psi\end{aligned}$$

- Require $E^2 = p^2 + m^2$:

$$\frac{1}{2}\left(\alpha_k\alpha_\ell + \alpha_\ell\alpha_k\right) = \delta_{k\ell} \quad \alpha_k\beta + \beta\alpha_k = 0 \quad \beta^2 = 1$$

- Defining $\{a, b\} \equiv ab + ba$:

$$\{\alpha_k, \alpha_\ell\} = 2\delta_{k\ell} \quad \{\alpha_k, \beta\} = 0 \quad \{\beta, \beta\} = 2$$

Finding the Dirac Matrices $\vec{\alpha}$ and β (I)

- We know our matrices must be Hermian due to their presence in defn of $\hat{H}$
- Also $\alpha_k^2 = 1$ and $\beta^2 = 1$ so their eigenvalues are ± 1
- Starting with $\{\alpha_k, \beta\} = 0$, multiply on left by β and take the trace

$$\mathrm{tr}\left(\beta\alpha_{\mathrm{k}}\beta\right) + \mathrm{tr}\left(\alpha_{\mathrm{k}}\right) = 0$$

but we can also cycle the order of the matrics in the first term

$$\mathrm{tr}\left(\beta\alpha_{\mathrm{k}}\beta\right) = \mathrm{tr}\left(\beta^2\alpha_{\mathrm{k}}\right) = \mathrm{tr}\left(\alpha_{\mathrm{k}}\right)$$

so

$$\mathrm{tr}\left(\alpha_{\mathrm{k}}\right) = 0$$

similarly $\mathrm{tr}\left(\beta\right) = 0$

- Because the eigenvalues are ± 1 and the trace is the sum of the eigenvalues, trace=0 means the $N \times N$ matrices must have N even
- If $N = 2$ can always express any matrix as a sum over Pauli matrices
  - Not possible to meet conditions above

Smallest matrices where a solution is possible is $N = 4$

Finding the Dirac Matrices $\vec{\alpha}$ and β (II)

- Dirac showed (and you will prove on Homework 7) that the following matrices satisfy the required conditions:

$$\begin{aligned} \beta &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \vec{\alpha} &= \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \end{aligned}$$

- So, the matrix is partitioned into four 2×2 sub-matrices

Probability Density and probability current

- Taking Hermian conjugate, the column spinor ψ is mapped into a row spinor

$$\psi = \begin{pmatrix} \psi_1 \\ . \\ . \\ . \\ \psi_N \end{pmatrix} \Rightarrow \psi^\dagger = (\psi_0^*, \psi_1^*, \psi_2^*, \psi_3^*)$$

- So the Dirac eq maps

$$i\frac{\partial \psi}{\partial t} = -i\sum_{k=1}^{3} \alpha_k \frac{\partial \psi}{\partial x_k} + m\beta\psi \quad \Rightarrow \quad -i\frac{\partial \psi^\dagger}{\partial t} = +i\sum_{k=1}^{3} \alpha_k \frac{\partial \psi^\dagger}{\partial x_k} + m\beta\psi^\dagger$$

- Mulitply first eq by $\psi^\dagger$ on left and second by ψ on right and then subtract to get the continuity eq

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

where $\rho = \psi^\dagger \psi$ and $\vec{J} = \psi^\dagger \vec{\alpha} \psi$

- But note: $\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2$ is always positive, so we don't have the Klein-Gordon problem of negative proability densities

Spin and the Dirac Eq

- How do we write the spin operator so we can apply it to our spinors?
- The operator must satisfy the normal commutation relations for angular momentum
- Since we know $\frac{1}{2}\sigma_i$ satisfy this algebra, it is easy to construct a 4×4 matrix with the right commuation relations:

$$\hat{\vec{S}} = \frac{1}{2}\hat{\vec{\Sigma}} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

- Using this definition:

$$\hat{S}^2 = \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So, even though our spinors have 4 components, they are eigenstate of S^2 corresponding to spin-$\frac{1}{2}$

- Note: This means our usual definition of the magnetic moment still works

$$\vec{\mu} = \frac{g}{2m}\vec{S}$$

where $g = 2$ for a spin-$\frac{1}{2}$ Dirac particle

Covariant form of Dirac equation

- Take Dirac Eq (from page 6)

$$i\frac{\partial\psi}{\partial t} = -i\sum_{k=1}^{3}\alpha_k\frac{\partial\psi}{\partial x_k} + m\beta\psi$$

move all terms to one side and multiply by β:

$$\begin{aligned} 0 &= i\beta\frac{\partial\psi}{\partial t} + i\sum_{k=1}^{3}\alpha_k\beta\frac{\partial\psi}{\partial x_k} - \beta^2 m\psi \\ &= i\gamma^0\frac{\partial\psi}{\partial t} + +i\sum_{k=1}^{3}\gamma^k\frac{\partial\psi}{\partial x_k} - m\psi \\ 0 &= \left(\gamma^\mu\partial_\mu - m\right)\psi \end{aligned}$$

where

$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha_i$$

- This is the form we will use for the free particle Dirac eq from now on

Properties of γ Matices

- From previous page
$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha_i$$
- From pages 7 and 8:
$$\begin{aligned} \alpha_x^2 = \alpha_y^2 = \alpha_z^2 &= I \\ \alpha_j\beta + \beta\alpha_j &= 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j &= 0 \quad (j \neq k) \end{aligned}$$
- This translates to:
$$\begin{aligned} \left(\gamma^0\right)^2 = \beta^2 &= 1 \\ \left(\gamma^i\right)^2 = -\alpha_i\beta\beta\alpha_i &= -\alpha_i^2 = -1 \end{aligned}$$
- Full set of relations are:
$$\begin{aligned} \left(\gamma^0\right)^2 &= 1 \\ \left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 &= -1 \\ \gamma^0\gamma^i + \gamma^i\gamma^0 &= 0 \\ \gamma^i\gamma^j + \gamma^j\gamma^i &= 0 \quad (i \neq j) \end{aligned}$$
- Or more compactly: $\{\gamma^\mu\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

Pauli-Dirac Representation of the γ matrices

- γ-matrices defined:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

- Or writing out all components

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The Continuity Eq Revisited

- From page 9:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$
where $\rho = \psi^\dagger \psi$ and $\vec{J} = \psi^\dagger \vec{\alpha} \psi$
- But
$$\gamma^0 \equiv \beta, \quad \vec{\gamma} \equiv \beta \vec{\alpha}$$
so
$$\vec{\alpha} = \beta^2 \vec{\alpha} = \beta \vec{\gamma} = \gamma^0 \vec{\gamma}$$
- Can write four-vector current
$$\begin{aligned} j^\mu &= \left(\rho, \vec{j}\right) = \psi^\dagger \gamma^0 \gamma^\mu \psi \\ &= \overline{\psi} \gamma^\mu \psi \end{aligned}$$
where we have defined
$$\overline{\psi} \equiv \psi^\dagger \gamma^0$$
- $\overline{\psi}$ is the definition of the adjoint spinor that allows us to write the continuity equation compactly:
$$\partial_\mu j^\mu = 0$$
- We'll use $\overline{\psi}$ rather than $\psi^\dagger$ from now on

Solving the free particle Dirac Eq

- Reminder: Dirac Eq

$$(\gamma^\mu \partial_\mu - m)\,\psi = 0$$

- Here ψ is a four component spinor. For plane wave solutions

$$\psi = u(E, \vec{p})e^{i(\vec{p}\cdot x - Et)}$$

where $u(E, \vec{p})$ is also a four component spinor

- Start with the special case of a particle at rest. Here $\vec{p} = 0$ and our wf is $\psi = u(E, 0)e^{-Et}$. The Dirac eq reduces to

$$E\gamma^0 u = mu$$

or in matrix form

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

- ϕ_1 and ϕ_2 have positive energy and ϕ_1 and ϕ_2 have negative energy
- If we define N as a normalization factor

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt},\ \psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt},\ \psi_3 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-+mt},\ \psi_4 = N \begin{pmatrix} 9 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-+m}$$

General Solution to Dirac Eq for Free Particle

- Can Lorentz boost rest frame wf to get general form. Won't do math here
- Solutions are:

$$\psi u(E,\vec{p})e^{i(\vec{p}\vec{x}-Et)}$$

  with

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

- Note that the boost mixes the top two components with the bottom two
- For u_1 and u_2, $E = \sqrt{p^2+m^2}$ while for u_3 and u_4, $E = -\sqrt{p^2+m^2}$
- Also, we have to understand how to handle the negative energy solutions

Reinterpreting the negative energy states

- Interpretation by Stuckelberg and Feyname
- $E < 0$ states are negative energy particles propagating backwards in time
- Reinterpret them as positive energy antiparticles with opposite charge propagating forward in time

$$E \Rightarrow -E \quad t \Rightarrow -t$$

- Redefine

$$\begin{aligned} v_1(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} &= u_4(-E,-\vec{p})e^{+i(-\vec{p}\cdot\vec{x}-(-E)t)} \\ v_2(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)} &= u_3(-E,-\vec{p})e^{+i(-\vec{p}\cdot\vec{x}-(-E)t)} \end{aligned}$$

- We can rewrite our wf using this new notation and can also calculate the normalization N
- Final results on the next page

Solutions to the Dirac Equation

- Normalized solutions to the Dirac Eq

$$\psi = u(E,\vec{p}) = e^{+i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu - m)\, u = 0$$

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- Antiparticle solutions:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu + m)\, v = 0$$

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Spinors are NOT eigenstates of S_z

- Reminder:

$$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Looking at one of our solutions:

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

- We can see that it isn't an eigenstate of S_z
- Can we find an observable that is?
  - The hint: if $\vec{p}$ were along the z direction. we would be in an eigenstate of S_z
  - Same conclusion holds for the other 3 solutions
  - So, instead of taking S projection along arbitrary direction, we want the projection along the momentum

Helicity

- Spinors are not eigenstates of $\hat{S}_z$
- Component of particle's spin along its direction of flight is, however, a good quantum number: $\left[H, \hat{S} \cdot \hat{p}\right] = 0$
- Define components of a particle's spin along its direction of flight as **helicity**

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{S}|; |\vec{p}|}$$

- Component of spin along any axis has 2 values for spin-$\frac{1}{2}$ particle: $\pm\frac{1}{2}$
  - Eigenvalues of helicity operator are : ± 1
- These are right- and left-handed helicity states

Left Handed
Helicity = -1

Right Handed
Helicity = +1

Helicity and Chirality

- For massless fermions, operator to project states of particular helicity are:

$$P_R = \frac{1}{2}\left(1 + \frac{\sigma \cdot \mathbf{p}}{E}\right)$$
$$P_L = \frac{1}{2}\left(1 - \frac{\sigma \cdot \mathbf{p}}{E}\right)$$

- For massive fermions, need 4-component spinor and 4-component operator

$$P_{L,R} = \frac{1}{2}\left(1 \pm \gamma_5\right)$$

- Because direction of spin wrt momentum changes under boosts, this operator cannot represent helicity per se
- Instead, projects out state of polarization $P = \pm v/c$
    - In spite of this, everyone writes

$$\frac{1}{2}\left(1 - \gamma^5\right)\right) u \equiv u_L$$

$\frac{1}{2}\left(1 \pm \gamma^5\right)$ are called the chiral projection operators