Physics 129: Particle Physics
Lecture 2: Relativity Review

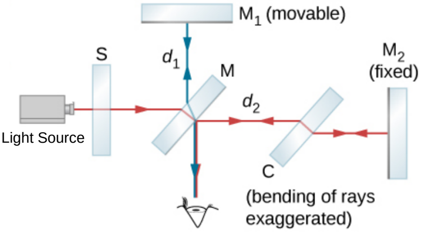
Fall 2020

Sept 1, 2020

- Suggested Reading:
  - Griffiths Chaper 3
  - Thomson Section 2.2

  Today's lecture follows the strategy and notation from Griffiths

Michaelson-Morley Experiment (1887)



- In 19^{th} century, scientists thought light propagated throug a medium that extended through all space. Ths was called the aether
- If light works like other waves, its speed is measured relative to the medium. If we move relative to the medium, the speed of the wave changes
- Michaelson and Morley tried to prove this was the case by measuring the change in the speed of light relative to the motion of the earth due to rotation about its axis
- They used an "interferometer" to measure the change in phase associated with a change in the speed of light
- Much to their surprise, they observed NO change in speed
- They found that the speed of like appeared to be independent of the speed of the observer!
- This unexpected fact is also predicted using Maxwell's Equations where

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

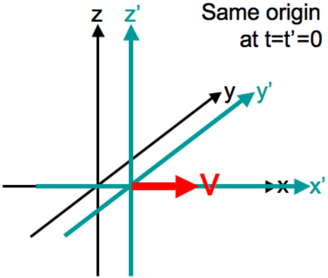
Einstein's Proposal

- Assume the observations of Michaelson-Morely and the predictions of Maxwell's Equations are correct
- Take this as the starting point and propose two *axioms*:
  1. The speed of light in empty space has the same value c in any inertial frame
  2. The laws of physics are the same in all inertial frames

  An inertial frame is a coordinate system where there is no acceleration
  - When there is acceleration, we "feel" fake forces
  - If there is no acceleration we don't feel the motion (since $F = ma$, no a means no F)

What we mean by frames of reference

- A frame of reference means a coordinate system and one or more "observers" that are part of the coordinate system
- Example: Two frames of reference could be someone on the platform in a train station and someone else on the train that is moving with constant velocity v relative to the platform. Each person has an x, y, z coordinate system and a clock to measure time.
- Let's setup of coordinate systems so that Frame F' is moving with velocity v in the x-direction with respect to frame F and that at time $t = 0$ the two coordinate systems have their origins at the same place



Galilean and Lorentz Tranformations

We want to relate the coordinate systems in the F and F' frames. For the case where F' is moving in the x-direction with velocity v

Galilean Transformation

- This describes what we would expect for Newtonian Physics:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

- x' and x move away from each other with time but the other coordinates stay consistent

Lorentz Transformation

- This is what Einstein found:

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt) \\ y' &= y \\ z' &= z \\ ct' &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(ct - \frac{vx}{c}\right) \end{aligned}$$

- y' and z' stay consistent but x and t move away from each other
- We'll see how Einstein reached this conclusion in the next slides
- Note: if $v/c << 1$ then the Lorentz Transformation approaches the Galilean Transformation

Nomenclature

- To save typing, physicists define two variables:

$$\begin{aligned} \beta &\equiv \frac{v}{c} \\ \gamma &\equiv \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

- Since $v \leq c$, $\beta \leq 1$ and $\gamma \geq 1$
- Using these definitions, the Lorentz Tranformation becomes:

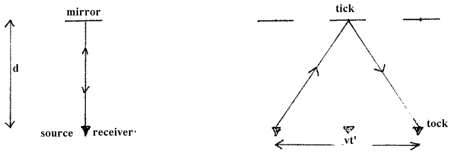
$$\begin{aligned} x' &= \gamma\,(x - \beta ct) \\ y' &= y \\ z' &= z \\ ct' &= \gamma\,(ct - \beta x) \end{aligned}$$

Note: Thomson writes boosts along the z axis rather than x axis. I'll use x here (as does Griffiths). Necessary changes to all formulae between the conventions are straightforward

Einstein's Strategy

- Use one set of units
  - Distance measured in meters. Instead of measuring time in sec, multiply by c: ct is measured in meters.
- Define a measuriung scheme
  - Define a grid of equally spaces coordinates marked off with a ruler
  - Synchronize clocks by shooting light out from the origin at $t = 0$
  - At each grid point, observer uses her known position and the fact that the light left the origin at $t = 0$ to properly set her clock
- Figure out what one observer sees when she looks at things happening in the other observer's frame
  - See next page

Light Clocks: Derivation of the formula for time dilation (I)



- Consider a clock made of 2 mirrors separated bt a distance d. Light bounces back and forth between the mirrors. Since total round trip distance is $2d$, the time for a round trip is

$$t = \frac{2d}{c}$$

  this is one "tick" of the clock.
- But suppose our observer in frame F looks at a light clock in frame F'.
- The clock moves to the right while the light is traveling between the mirrors. Total pathlength is longer and so the time between ticks is longer.
  - The person in frame F says that a click of the clock in frame f' takes more time than a click in her own frame
  - The observer in frame F sats the clock in frame F' clock is running slow

Light Clocks: Derivation of the formula for time dilation (II)

- t is the time between ticks of the light clock on the train as measured by the observer on the platform. t' is the time between ticks of the light clock on the train as measured by the observer on the train
- The time that the observer on the train measures is: $ct' = 2d$
- The time that the observer on the platform measures is:

$$
\begin{aligned}
ct &= 2\sqrt{d^2 + (\frac{vt}{2})^2} \\
&= \sqrt{4d^2 + (vt)^2} \\
(ct)^2 &= 4d^2 + v^2 t^2 \\
\left(c^2 - v^2\right) t^2 &= 4d^2 \\
t^2 &= \frac{1}{c^2 - v^2} 4d^2 \\
&= \frac{1}{c^2 - v^2} \left(ct'\right)^2 \\
t &= \frac{1}{\sqrt{1 - v^2/c^2}}\, t' \\
t' &= \sqrt{1 - v^2/c^2}\, t = \frac{1}{\gamma} t
\end{aligned}
$$

- So, $t' < t$. The time the observer on the train measures is less than the time the observer on the platform measures

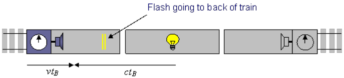
All clocks behave the same way

- Einstein's axioms say the laws of physics are the same in all inertial frames
- That means all clocks that give consistent measurements in a frame where the clocks are at rest must give consistent results in any another inertial frame
- The result we derived using a light clock holds for any kind of clock, including:
  - Radioactive decay (half-life)
  - Biological clock (the aging process)
  - A wrist watch
  - An atomic clock

Why we don't have a paradox?

- The observer on the platform says that the train's clocks are running slow. But we could make the same argument for the observer on the train.
  - The observer on the train thinks the platform is moving with speed v in the $-x$-direction
  - The observer on the train says the platform clocks are running slow
- How can both things be true?
- Clocks that are synchronized in one frame aren't synchronized in the other

The clocks are triggered when the flash of light from the central bulb reaches the attached photocells.



The train is moving to the right: the central bulb emits a flash of light. Seen from the ground, the part of the flash moving towards the rear travels at c, the rear travels at v to meet it.

- Lack of synchronization between clocks in train frame is why we don't have a paradox

Consequences of Lorentz Transformations

- Lorentz contraction
  - The observer who says an object is at rest measures the longest distance
  - An observer that sees the object moving says the length is shorter

$$L' = \frac{L}{\gamma}$$

- Time dilation
  - Clocks run slower in a moving frame
  - The shortest time is measured in the frame where the clock stays in one place
  - An observer that says the clock is moving says that the clock reads a time

$$t' = \gamma t$$

Four Vectors

- Define time-position 4-vector

$$x^{(0)} = ct \quad x^{(1)} = x \quad x^{(2)} = y \quad x^{(3)} = z$$

- Lorentz tranformation becomes:

$$\begin{aligned}
x^{(0)\prime} &= \gamma\left(x^{(0)} - \beta x^{(1)}\right) \\
x^{(1)\prime} &= \gamma\left(x^{(1)} - \beta x^{(2)}\right) \\
x^{(2)\prime} &= x^{(2)} \\
x^{(3)\prime} &= x^{(3)}
\end{aligned}$$

- Can write this as:

$$\begin{aligned}
x^{\mu\prime} &= \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} \\
\Lambda &\equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

- To avoid writing all the $\sum$'s, use "summation convention"
  - Greek letter indices run from 0-3
  - Repeated index with a subscript and a superscript are summed 0-3
- Using this notation, the Lorentz Transformation becomes

$$x^{\mu\prime} = \Lambda^{\mu}_{\nu} x^{\nu}$$

- This compact notation takes some getting used to but makes the equations much easier to write (and to read)
- Note: Changing direction of boost from x to an arbitrary direction only changes the explict form of Λ^{μ}_{ν}

Lorentz Invariant Quantities: Proper Time

- One combination of components remains the same in all frames:
  - Proper time $c\tau$ defined as:

$$\begin{aligned}(c\tau)^2 &\equiv (ct)^2 - (\vec{r})^2 \\ &= (x^{(0)})^2 - (x^{(1)})^2 - \\ &\quad (x^{(2)})^2 - (x^{(3)})^2\end{aligned}$$

- Would like to write this using "summation convention", but how do we handle the minus signs?
- Define the *metric* $g_{\mu\nu}$:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Then:

$$\begin{aligned}(c\tau)^2 &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}x^\mu x^\nu \\ &\equiv g_{\mu\nu}x^\mu x^\nu\end{aligned}$$

- Define covariant four-vector x_μ:

$$x_\mu = g_{\mu\nu}x^\nu$$

- Can contract contravarient x^μ with covariant x_μ:

$$(c\tau)^2 = x_\mu x^\mu$$

- Using the metric **g** puts the minus signs in the right place to calculate the invariant quantity

Lorentz Invariant Quantities: Scalar Products

- Just as the length of a 3-vector is invariant under rotations in space, the length of a 4-vector, calculated using the metric is invariant under Lorentz boosts
- This is true for any 4-vector
  - Four-vector is a 4-component object that transforms under boosts using a Lorentz transformation
- Suppose a^μ and b^μ are four vectors. Then

$$\begin{aligned} a_\mu b^\mu &= a^\mu b_\mu \\ &= a^{(0)}b^{(0)} - a^{(1)}b^{(1)} - a^{(2)}b^{(2)} - a^{(3)}b^{(3)} \\ &= a^{(0)}b^{(0)} - \vec{a}\cdot\vec{b} \end{aligned}$$

  is invariant under Lorenz boosts. This is the 4-vector scalar product
- More notation:
  - Write 4-vector products $a \cdot b$ as $a_\mu b^\mu$ and $a \cdot a$ as a^2
  - 3-vectors will always be written $\vec{a}$ and their squares as $(\vec{a})^2$
  - Explicit matrix form:

$$x^\mu = \begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix} \qquad x_\mu = \begin{pmatrix} x^{(0)} & -x^{(1)} & -x^{(2)} & -x^{(3)} \end{pmatrix}$$

Lorentz Invariant Quantities: Invariant Mass

- Energy and momentum form a 4-vector p^{μ}:

$$p^{(0)} = E, \quad p^{(1)} = p_x c, \quad p^{(2)} = p_y c, \quad p^{(3)} = p_z c$$

- From now on, use natural units so $c = 1$
- The Lorentz invariant

$$\begin{aligned} m^2 &= E^2 - (\vec{p})^2 \\ &= p_{\mu} p^{\mu} \\ &= p^2 \end{aligned}$$

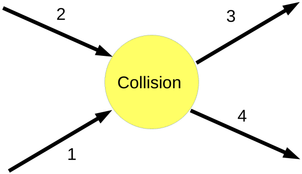
  is invariant mass-squared of a particle with momentum $\vec{p}$ and energy E
- In the limit where $|\vec{p}| = 0$ this reduces to the familiar

$$E = mc^2$$

  when we take $c = 1$

Conservation of Energy and Momentum

- Energy-momentum conservation in 2-body collisions: $1+2 \rightarrow 3+4$
  (also know as "two-to-two scattering")



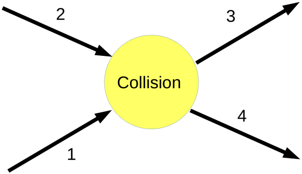
$$p_\mu^{(1)} + p_\mu^{(2)} = p_\mu^{(3)} + p_\mu^{(4)}$$

- Independent of type of interaction or type of particles, all 4 components of energy-momentum are conserved
- Therefore:

$$\left(p_\mu^{(1)} + p_\mu^{(2)}\right)^2 = \left(p_\mu^{(3)} + p_\mu^{(4)}\right)^2$$

- Note: This only holds for particles that appear in the initial or final state
  - Virtual intermediate particles can be "off-shell" which means that the virtual exchanged particle can have an unphysical mass

The Mandelstam Variables



- For $p_1 + p_2 \to p_3 + p_4$ we define the "Mandelstam variables":

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned}$$

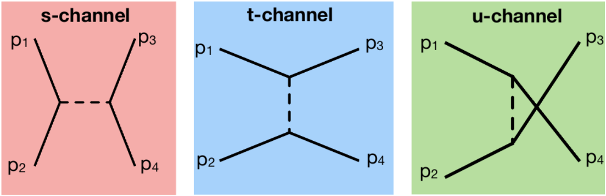
- You will prove on Problem Set #2 that:

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

- Thus, only 2 of the 3 variables are independent

Physical Interpretation of the Mandelstam Variables

Time runs left-to-right

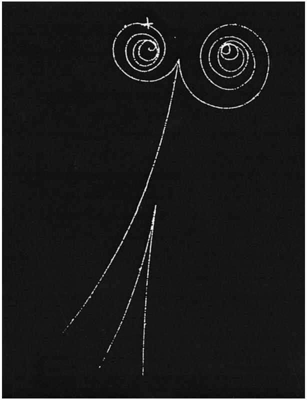


- Feynman diagrams show interactions as exchanges of virtual particles
- These exchanges allow "4-momentum transfer" q between particles
  - $q_\mu q^\mu = q^2$ is Lorentz invariant
- For 2-to-2 scattering, three possible exchanges
  - s-channel: initial particle annihilate
    - s is the square of the center-of-mass energy of the collision
  - t-channel: scattering via exchange of a particle
    - t is the square of the 4-momentum transfered between particles 1 and 2
  - u-channel: scattering via exchange where final particles exchanged
    - important for symmetrization with identical particles

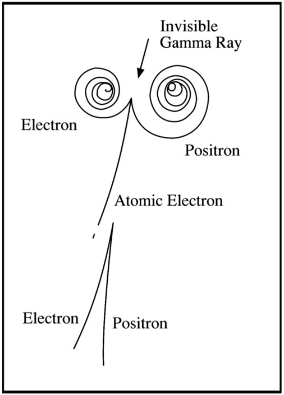
You will explore the properties of these variables on Problem Set #2

Energy transformed into Mass: Pair-Production

Bubble chamber picture

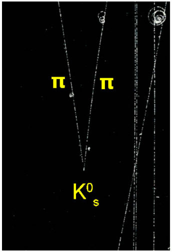


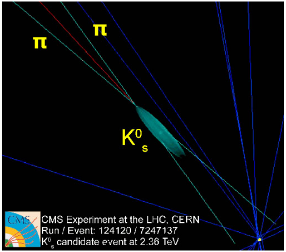
Annotated explanation



- Bubble chambers sensitive to charged particles
- Magnetic field allows determination of momentum and sign of charge
- Lower e^+e^- accompanied by nucleus with momentum too low to detect
    - γ cannot convert unless other matter present to allow conservation of energy-momentum

Mass Transformed into Energy: Particle Decay





- Massive particles can decay conserving energy and momentum as long as sum of mass of outgoing particles less than mass of initial particle
- Calculation most straightforward in center-of-mass frame
- Example: $K^0{}_S \to \pi^+\pi^-$

$$\vec{p}_{init} = 0 \quad \Rightarrow \quad \vec{p}_{final} = \vec{p}_{\pi^+} + \vec{p}_{\pi^-} = 0$$

$$\vec{p}_{\pi^+} = -\vec{p}_{\pi^-}$$

$$E_{init} = m_{K^0{}_S} \quad \Rightarrow \quad E_{final} = 2\sqrt{m_\pi{}^2 + p_\pi^2} = m_{K^0{}_S}$$

$$4\left(m_\pi{}^2 + p_\pi^2\right) = \left(m_{K^0{}_S}\right)^2$$

$$p_\pi = \sqrt{\frac{\left(m_{K^0{}_S}\right)^2}{4} - m_\pi{}^2}$$

- Fixed magnitude for p_π
- Angular distribution uniform in $d\cos\theta$ and $d\phi$