Physics 129: Particle Physics
Lecture 11: Scattering Cross Sections

Oct 1, 2020

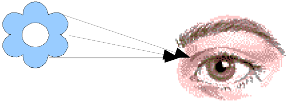
- Suggested Reading:
  - Thomson 3.4-3.5
  - Griffiths 6.5
  - Perkins Sections 2.10
  - http://www.tcm.phy.cam.ac.uk/ bds10/aqp/lec20-21_compressed.pdf
  - https://ocw.mit.edu/courses/physics/8-06-quantum-physics-iii-spring-2018/lecture-notes/MIT8_06S18ch7.pdf

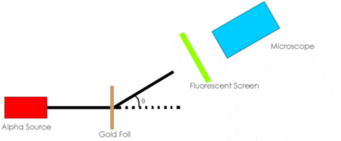
Reminder: Office hours at unusal time this week (see announcement)

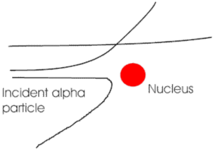
Overview amd Strategy

- We see objects via scattered light
- Scattering essential for understanding structure
- Rutherford: 1910
  - Shoot α at thin gold foil
  - Study recoil pattern
  - Large angle scatters provide evidence for small, hard nucleus within atom
- Particle physicists use same technique with higher momentum probes
  - 1968: Protons and neutrons are bound states of quarks

Studies rate and angular distribution of scattered particles tells us about the structure of the target(s) and about the nature of the interaction between the particles

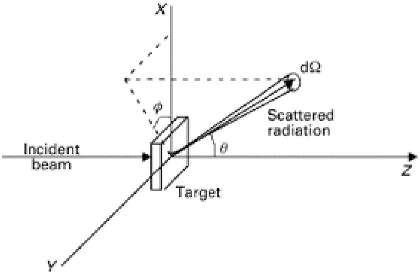






The Scattering Cross Section

- Beam of particles of type A incident on a target of type B
  - Target called the "scattering center"
- Beam characterized by number density $J = N/V$ and speed of particle v
- Differential cross section $d\sigma$ tells us number of particle per second scattering into solid angle $d\Omega$



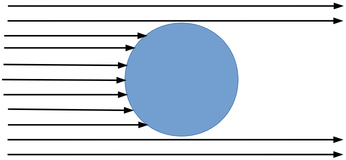
$$\frac{d\sigma}{d\Omega} = \frac{\text{\# particles scattering into d}\Omega\text{ per unit time}}{\text{\# particles per unit area of beam reaching target per unit time}}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

- σ is called the total scattering cross section
- Units of $\sigma = \frac{\#/\text{sec}}{\#\text{ per unit area per sec}}$ = area

Why the scattering cross sections has units of area

- Consider hard sphere of radius R in a beam of particles
- Any particle that hits the sphere will be scattered out of the beam
- Incident flux determined by number density J (# per unit volume) and speed v



$$\text{flux} \equiv \Phi = Jv \qquad \left(\frac{\#}{\mathrm{m}^3} \times \frac{\mathrm{m}}{s} = \frac{\#}{\mathrm{m}^2}\right)$$

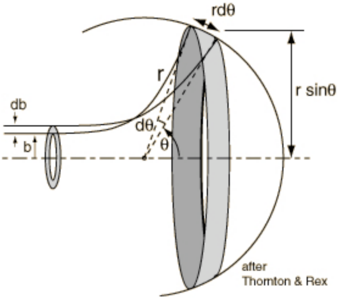
- Total cross section

$$\begin{aligned}\sigma &= \frac{\text{\# particles scattered per unit time}}{\text{\# particles per unit area of beam reaching target per unit time}} \\ &= \frac{JvA}{Jv} = A = \pi R^2\end{aligned}$$

- Where A is the cross sectional area of the sphere

The Classical View (I)

- Force beam particle sees depends distance from scattering center
- Particle further from the scatterer feel less force and are scattered less
- Characterize particles in beam by impact parameter b
  - b is $\perp$ distance from axis going through scattering center
  - Given value of b corresponds to a given scattering angle
  - If scattering center is cylindrically symmetric, cross section independent of ϕ



- The cross section is:

$$
\begin{aligned}
d\sigma &= \frac{JvA}{Jv} - \frac{(2\pi b db)Jv}{Jv} = -2\pi b\, db \\
\frac{d\sigma}{d\Omega} &= -\frac{2\pi b\, db}{2\pi d\cos\theta} = -\frac{b}{\sin\theta}\frac{db}{d\theta}
\end{aligned}
$$

where the $-$ sign comes from fact that larger b means smaller θ

The Classical View (II)

- From previous page:

$$\frac{d\sigma}{d\Omega} = -\frac{b}{\sin\theta}\frac{db}{d\theta}$$

- Total cross section obtained by integrating over b and angle
- Looking at this expression,

$$\sigma = \int_{d\Omega}\int_{b=0}^{b=\infty}\frac{d\sigma}{d\Omega}d\Omega db$$

  appears to give infinite cross section
- In practice, σ is finite for all real situations since
  - The force due to any scattering center usually falls off rapidly with distance
  - The beam has a finite size so b never really goes to ∞ (and hence θ doesn't go to 0)

The Classical View: Hard Sphere Scattering

- From previous page

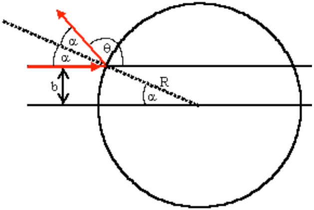
$$\frac{d\sigma}{d\Omega} = -\frac{b}{\sin\theta}\frac{db}{d\theta}$$

- For elastic scattering from a hard sphere

$$\begin{aligned} b(\theta) &= R\sin\alpha \\ &= R\sin\left(\frac{\pi-\theta}{2}\right) \\ &= = R\cos\left(\frac{\theta}{2}\right) \\ \frac{db}{d\theta} &= \frac{R}{2}\sin\left(\frac{\theta}{2}\right) \\ \frac{d\sigma(\theta)}{d\Omega} &= \frac{bR}{2\sin\theta}\sin\left(\frac{\theta}{2}\right) \end{aligned}$$

- But

$$b = R\cos\left(\frac{\theta}{2}\right)$$



- Putting together eq from left column:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{R^2}{2}\frac{cos(\theta/2)\sin(\theta/2)}{\sin\theta} \\ &= \frac{R^2}{4} \end{aligned}$$

- Total Cross Section

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega}d\Omega \\ &= \pi R^2 \end{aligned}$$

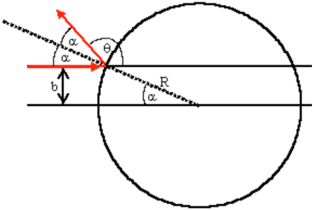
Rutherford Scattering in Classical Mechanics (I)

- Infinitely massive target $\Rightarrow$ elastic scattering conserves energy of incident particle
- Calculate change in momentum from change in angle and equate to change in momentum from impulse due to Coulomb force

$$\begin{aligned}\Delta p_z &= 2mV_0 \cos\left(\frac{\pi-\theta}{2}\right) \\ &= 2mv_0 \sin\frac{\theta}{2} \\ \Delta p_z &= \int F_z\,dt \\ &= \int \frac{Ze^2}{4\pi\epsilon_0 r^2}\cos\alpha\,dt \\ &= \int \frac{Ze^2}{4\pi\epsilon_0 r^2}\cos\alpha\,\frac{dt}{d\alpha}\,d\alpha\end{aligned}$$

- Conservation of angular momentum

$$\begin{aligned}mr^2\frac{d\alpha}{dt} &= mv_0 b \\ \frac{d\alpha}{dt} &= \frac{v_0 b}{r^2}\end{aligned}$$



- Continuing from left hand column:

$$\begin{aligned}\Rightarrow \Delta p_z &= \int \frac{Ze^2}{4\pi\epsilon_0 r^2}\cos\alpha\,\frac{dt}{d\alpha}\,d\alpha \\ &= \frac{Ze^2}{2\pi\epsilon_0 v_0 b}\cos\frac{\theta}{2}\end{aligned}$$

- Combining with expression from top of previous column

$$\begin{aligned}b &= \frac{Ze^2}{4\pi\epsilon_0 m v_0^2}\cot\frac{\theta}{2} \\ &= \frac{Ze^2}{8\pi\epsilon_0 E}\cot\frac{\theta}{2}\end{aligned}$$

Rutherford Scattering in Classical Mechanics (II)

- Plugging expression for b from previous page into eq for cross section

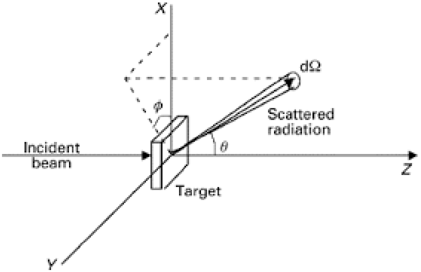
$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{Ze^2}{8\pi\epsilon_0 E}\right)^2 \cot\frac{\theta}{2}\frac{d}{d\theta}\cot frac\theta 2 \\
&= \left(\frac{Ze^2}{16\pi\epsilon_0 E\sin^2\frac{\theta}{2}}\right)^2
\end{aligned}
$$

- where above we used

$$\frac{d\cot\alpha}{d\alpha} = -\frac{1}{\sin^2\alpha}$$

- Note: The cross section diverges for $\theta = 0$. This is a feature of any potential that falls off like $1/r$ or slower. In practice, cannot tell if a particle at $\theta = 0$ scattered or not, so this is not a problem in practice

Scattering in Quantum Mechanics (I)



- Since particles don't have fixed trajectories, description in terms in impact parameter b not useful
- Instead:
  - At some location (we'll call it the origin) place a scatterer (taken here to be infinitely massive and at rest)
  - Prepare a beam of type A and fire it towards B
    - Beam has momentum distribution peaked at $\hbar\vec{k}$
    - Represent by a plane wave or wave packet (we'll use plane wave here)
  - Measure $\#$ of particles of type A that hit a detector subtending some angle $d\Omega$

Scattering in Quantum Mechanics (II)

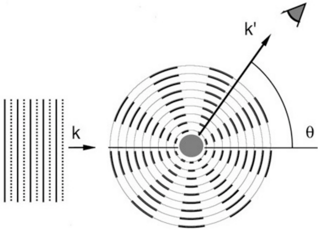
- In QM can only predict probabilities
- Goal is to calculate:

$$\text{Prob}\left(\left|\vec{k}_i\right\rangle \Rightarrow \left|\vec{k}_f\right\rangle\right)$$

- In QM, we don't talk about forces, but rather about potentials
- Describe our scatterer as providing a potential $V(\vec{r})$ that changes the wave function of the incoming particle
- Language and approach similar to what you are used to from classical wave phenomena as described in Physics 5 or 7

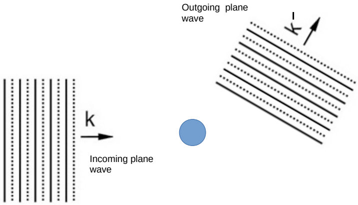
Two Scattering regimes

Low Energy Scattering



- Wavelength large compared to size of scatterer
- Incoming plane wave and outgoing wave largely spherical
  - Describe outgoing wave as sum over spherical harmonics
  - Dominated by small ℓ terms
- This is called the Partial Wave Expansion

High Energy Scattering



- Wavelength small compared to size of scatterer
- Incoming plane wave scatters
  - Describe outgoing wave as sum of plane waves
  - Use perturbation theory to calculate V_{if}
- This is called the Born Approximation

Review from Physics 137A: The continuity equation (I)

- Probability per observing at particle within volume $d\vec{r}$ around point $\vec{r}$ at time t is:

$$P(\vec{r},t)d\vec{r} = |\psi(\vec{r},t)|^2\, d\vec{r}$$

- Change in probability with time:

$$\begin{aligned}
\frac{\partial}{\partial t}P(\vec{r},t)d\vec{r} &= \frac{\partial}{\partial t}\int \psi^*(\vec{r},t)\psi(\vec{r},t)d\vec{r} \\
&= \int \left[\frac{\partial \psi^*(\vec{r},t)}{\partial t}\psi(\vec{r},t) + \psi^*(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t}\right] d\vec{r}
\end{aligned}$$

- But Schrodinger Eq says

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right]\psi(\vec{r},t) \\
-i\hbar\frac{\partial}{\partial t}\psi^*(\vec{r},t) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right]\psi^*(\vec{r},t)
\end{aligned}$$

- So, the expression above becomes

$$\begin{aligned}
\frac{\partial}{\partial t}P(\vec{r},t)d\vec{r} &= \frac{i\hbar}{2m}\int \left[\psi^*(\vec{r},t)(\nabla^2\psi(\vec{r},t)) - (\nabla^2\psi^*(\vec{r},t))\psi(\vec{r},t)\right] d\vec{r} \\
&= \frac{i\hbar}{2m}\int \vec{\nabla}\cdot\left[\psi^*(\vec{r},t)(\vec{\nabla}\psi(\vec{r},t)) - (\vec{\nabla}\psi^*(\vec{r},t))\psi(\vec{r},t)\right] d\vec{r} \\
&= -\int \vec{\nabla}\cdot\vec{j}(\vec{r},t)d\vec{r}
\end{aligned}$$

where $j \equiv \frac{i\hbar}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$

Review from Physics 137A: The continuity equation (II)

- From previous page:

$$\frac{\partial}{\partial t}P(\vec{r},t)d\vec{r} \quad = \quad -\int \vec{\nabla}\cdot\vec{j}(\vec{r},t)d\vec{r}$$

  where $j \equiv \frac{i\hbar}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$

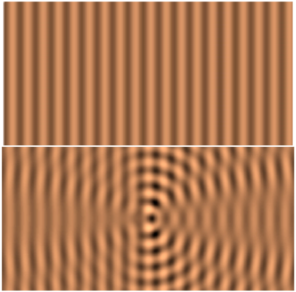
- This leads to the equation

$$\frac{\partial}{\partial t}P(\vec{r},t) + \int \vec{\nabla}\cdot\vec{j}(\vec{r},t) = 0$$

- This is called the continuity equation and if we multiply by charge is familiar from E&M
- $\vec{j}(\vec{r},t)$ can be interpreted as a probability current density
- We'll use the continuity eq later to interpret our scattering equation

Overall strategy for describing scattering using partial waves

- Idealized descrip of incoming beam
  - Assume single value of momentum rather than a wave packet
  - Plane wave $\psi_{in} = e^{i\vec{k}\cdot\vec{r}}$
  - Take z as incident beam direction
- Scattering center assumed to have limited spatial extent
  - $V(\vec{r})$ localized
  - Fall-off faster than $1/r$
- Only consider wave function far from scattering center: $V = 0$
- Outgoing wave propagates outward from scattering center
- Beam remains on for long time
  - Steady state soln



- Scattered wave probability stays normalized during propagation

$$\begin{aligned} P(R)dV &= (\psi^*\psi)\,\pi R^2 dr \\ \Rightarrow \psi_{scat}(r,t) &\propto \frac{e^{ikr}}{r} \end{aligned}$$

- In general, probability depends on angle

$$\psi_{scat} = f(k,\theta,\phi)\frac{e^{ikr}}{r}$$

Putting it all together

- Far from scattering center

$$\psi_k(\vec{r}) = \psi_{in}(\vec{r}) + \psi_{scat}(\vec{r})$$
$$= e^{ikz} + f(k,\theta,\phi)\frac{e^{ikr}}{r}$$

- Using the defn of prob current $\vec{j}$:

$$\vec{j} = \frac{i\hbar}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$$

- And noting

$$\nabla = \frac{\partial}{dr}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$$

- For large r, the radial component dominates

$$j_r = \frac{\hbar k}{m}\left|f(k,\theta,\phi)\right|^2$$

- Since j_r is prob of particle crossing unit area per unit time

$$N_{scat}\ \ d\Omega = \frac{\hbar k}{m}\left|f(k,\theta,\phi)\right|^2 d\Omega$$

- With our normalization, the flux Φ of incident particles is $\Phi = N/Vv$
- But $v = p/m = \hbar k/m$ so

$$\Phi = (\psi^*\psi)\hbar k/m$$

- The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\#\text{ scattering into d}\Omega\text{ per unit time}}{\#\text{ per unit area per unit time}}$$
$$= \left|f(k,\theta,\phi)\right|^2$$

- Our goal is to find $f(k,\theta,\phi)$

Method of partial waves (I)

- Consider problems with ϕ symmetry (eg central potential)
- ψ and f have no ϕ dependence: expand both ψ and f in Legendre polynomials

$$\begin{aligned}
\psi(r,\theta) &= \sum_{\ell=0}^{\infty} R_\ell(k,r) P_\ell(\cos\theta) \\
f(k,\theta) &= \sum_{\ell=0}^{\infty} f_\ell(k,r) P_\ell(\cos\theta)
\end{aligned}$$

- Each term is called a "partial wave"
- Any problem with ϕ symmetry can be expanded this way
- But method most useful for cases where only a first few terms important
- Schrodinger Eq for steady-state

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi$$

- Far from the scattering center

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

- so the Schrodinger eqation becomes

$$\begin{aligned}
\left[-\frac{\hbar^2}{2m}\nabla^2 - E + V(r)\right]\psi &= 0 \\
\left[\nabla^2 + k^2 - U(r)\right]\psi &= 0 \\
\text{with}\quad U(r) &\equiv \frac{2m}{\hbar^2}V(r)
\end{aligned}$$

- But for large r $U(r) \to 0$. Asymtopic solution:

$$\left[\nabla^2 + k^2\right]\psi = 0$$

Method of partial waves (II)

- Using our asymptotic solution and remembering what you derived for central potentials in Physics 137:

$$\begin{aligned}\left[\nabla^2 + k^2\right]\psi &= 0\\ \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(ell+1)}{r^2} + k^2\right]R_\ell(k,r) &= 0\end{aligned}$$

  where last expression comes from separation of variables

- As usual, solutions are spherical Bessel and Neumann functions
- Writing soln as linear combination and using asymptotic form

$$\begin{aligned}R_\ell(k,r) \quad \overrightarrow{r \to \infty} \quad & \frac{1}{kr}\left[B_\ell(k)\sin\left(kr - \ell\frac{\pi}{2}\right) - C_\ell(k)\cos\left(kr - \ell\frac{\pi}{2}\right)\right]\\ \overrightarrow{r \to \infty} \quad & A_\ell(k)\frac{1}{kr}\sin\left(kr - \ell\frac{\pi}{2} + \delta_\ell(k)\right)\end{aligned}$$

- Here $\delta_\ell(k)$ is called the phase shift
- As with one dimensional scattering problems, we need to match the asymptotic soln to the soln where $U(r) \neq 0$
- But first, we'll relate $\delta_\ell(k)$ to $f(k,\theta)$

Finding $f(k,\theta)$

- Summarizing what we have learned so far:

$$
\begin{aligned}
\psi(r,\theta) &= \sum_{\ell=0}^{\infty} R_\ell(k,r) P_\ell(\cos\theta) \\
f(k,\theta) &= \sum_{\ell=0}^{\infty} f_\ell(k,r) P_\ell(\cos\theta) \\
\psi(r,\theta) &\overrightarrow{r \rightarrow \infty} \quad e^{ikz} + f(k,\theta,\phi)\frac{e^{ikr}}{r}
\end{aligned}
$$

- and using

$$
e^{i\vec{k}\cdot\vec{r}} \quad \overrightarrow{r \rightarrow \infty} \quad \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta)
$$

(see B&J 7.77) we find:

$$
\begin{aligned}
R_\ell(k,r) &\overrightarrow{r \rightarrow \infty} \quad (2\ell+1)\frac{i^\ell}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right) + \frac{1}{r}e^{ikr} f_\ell(k) \\
\Rightarrow f_\ell(r) &= \frac{2\ell+1}{2ik}\left[e^{2i\delta_\ell(k)} - 1\right]
\end{aligned}
$$

- Using the expression for $f_\ell(k)$ from above:

$$
f(k,\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{2i\delta_\ell(k)} - 1\right] P_\ell(\cos\theta)
$$

The total cross section

- From page 6

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(k,\theta,\phi)|^2 \\
\sigma_{tot} &= \int |f(k,\theta,\phi)|^2 \, d\Omega \\
&= 2\pi \int_{-1}^{1} |f(k,\theta,\phi)|^2 \, d\cos\theta \\
&= 2\pi \int_{-1}^{1} \left(\sum_{\ell=0}^{\infty} f_\ell P_\ell(cos\theta) \right)^* \left(\sum_{\ell'=0}^{\infty} f_{\ell'} P_{\ell'}(cos\theta) \right) d\cos\theta
\end{aligned}
$$

- But orthogonality of the P_ℓ;s gives

$$
\int_{-1}^{1} P_\ell(\cos\theta)) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1} \delta_{\ell\ell'}
$$

giving:

$$
\begin{aligned}
\sigma_{tot} &= \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} |f_\ell(k)|^2 \\
&= \sum_{\ell=0}^{\infty} \frac{4\pi}{k^2} (2\ell+1) \sin^2 \delta_\ell \\
&= \sum_{\ell=0}^{\infty} \sigma_\ell
\end{aligned}
$$

The optical theorm

- From page 9

$$f(k,\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{2i\delta_\ell(k)}-1\right]P_\ell(\cos\theta)$$

- Looking at $\theta = 0$ and noting $P_\ell(1) = 1$:

$$\begin{aligned}
f(k,0) &= \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k) \\
&= \sum_{\ell=0}^{\infty}(2\ell+1)\frac{e^{i\delta_\ell(i)}}{k}\sin\delta_\ell \\
\mathrm{Im} f(k,0) &= \frac{k}{4\pi}\sigma_{tot} \\
\sigma_{tot} &= \frac{4\pi}{k}\mathrm{Im} f(k,\theta=0)
\end{aligned}$$

- This is called the optical theorem
- Optical theorm comes from conservation of probability
  - Scattering cross section related to loss of flux out of the beam

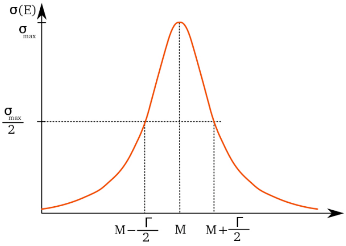
Resonances

- Partial wave expansion:

$$\sigma_{tot} = \sum_\ell \sigma_\ell$$
$$\sigma_\ell = \frac{4\pi}{k^2}\frac{1}{1+\cot^2\delta_\ell(k)}$$

- Maximum occurs when $\cot\delta_\ell$ vanishes
- If this happens when δ_ℓ rapidly passes through a multiple of $\pi/2$, see a large increase in cross section over a small variation of E around a specific value E_R
- This is called a resonance

$$\cot\delta_\ell(k) = \frac{E_r - E}{\Gamma/2}$$
$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\Gamma^2/4}{(E-E_r)^2+\Gamma^2/4}$$



This shape called a Breit-Wigner or Lorentzian

Green's Functions and the integral eq for scattering (I)

- Derivation of expression for Born approx depends on use of "Green's functions"
- Some of you may have seen this in math classes or in E&M
- Derivation of Green's function expressions go beyond what we can do here
- Will qualitatively explain method and quote result
- Consider Schrodinger Eq

$$\begin{aligned} \left(\nabla^2 + k^2\right)\psi(\vec{r}) &= U(\vec{r})\psi(\vec{r}) \\ \text{with} \quad U(r) &\equiv 2mV(r)/\hbar^2 \end{aligned}$$

- Integral solutuion can be obtained as

$$\psi(\vec{r}) = \phi(\vec{r}) + \int G_0(k, \vec{r}, \vec{r}')U(\vec{r}')\psi(\vec{r}')d\vec{r}'$$

where $\phi(\vec{r})$ is the soln to the case where $U(r) = 0$ and G_0 satisfies

$$\left(\nabla^2 + k^2\right)G_0(k, \vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')$$

- Won't solve this here, but for

$$\begin{aligned} \psi_k(\vec{r}) &= e^{ik\cdot r} \\ G_0(\vec{r}, \vec{r}') &= -\frac{1}{4\pi}\frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \end{aligned}$$

Born Approx: Green's Functions and the integral eq for scattering (II)

- Solution for $\psi_k(\vec{r})$ is

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int d^3r' \frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(\vec{r}')\psi_k(\vec{r}')$$

- Solve this expression as an expansion where

$$\begin{aligned}
\psi_0(\vec{r}) &= \phi_k(\vec{r}) = e^{ik\cdot r} \\
\psi_1)\vec{r}) &= \phi_k(\vec{r}) + \int G_0(k,\vec{r},\vec{r}')U(\vec{r})\psi_0(\vec{r}) \\
&\dots \\
\psi_n)\vec{r}) &= \phi_k(\vec{r}) + \int G_0(k,\vec{r},\vec{r}')U(\vec{r})\psi_{n-1}(\vec{r})
\end{aligned}$$

- This is called the Born series
- If we only solve to ψ_1 this is called the first Born Approximation

First Born Approximation

- Using results from previous page

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int d^3r'\,\frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(\vec{r}')\psi_k(\vec{r}')$$

- Taking case of large r so $|\vec{r}-\vec{r}'| \approx r - \hat{e}_r \cdot r'$, we find

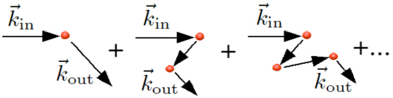
$$\begin{aligned} \frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} &\approx \frac{e^{ikr}}{r}e^{-i\vec{k}'\cdot r'} \\ \text{with} \quad \vec{k}' &\equiv k\hat{e}_r \end{aligned}$$

- That means

$$\begin{aligned} \psi_k(\vec{r}) &= e^{i\vec{k}\cdot r} + f(\theta,\phi)\frac{e^{ikr}}{r} \\ \text{with} \quad f(\theta,\phi) &\approx -\frac{1}{4\pi}\int d^3r'\,e^{i\vec{k}'\cdot r'}U(\vec{r}')\psi_k(\vec{r}') \\ &= -\frac{1}{4\pi}\langle\psi_{k'}|\,U\,|\psi_k\rangle \end{aligned}$$

- Notice that our outgoing wave is a plane wave
- Although the intermediate steps are tough, our final result has a simple form

A physical interpretation of the Born series



- Iterative integral eq from previous page:

$$\begin{aligned}
\psi^{(0)}(\vec{r}) &= \phi_k(\vec{r}) = e^{ik\cdot r} \\
\psi^{(1)})\vec{r}) &= \phi_k(\vec{r}) + \int G_0(k,\vec{r},\vec{r}')U(\vec{r})\psi^{(0)}(\vec{r}) \\
&\ldots \\
\psi^{(n)})\vec{r}) &= \phi_k(\vec{r}) + \\
&\quad \int G_0(k,\vec{r},\vec{r}')U(\vec{r})\psi^{(n-1)}(\vec{r})
\end{aligned}$$

- Writing this equation is a more compact form:

$$\begin{aligned}
|\psi_k\rangle &= |\phi_k\rangle + \hat{G}_0\hat{U}\,|\phi_k\rangle + \hat{G}_0\hat{U}\hat{G}_0\hat{U}\,|\phi_k\rangle + \ldots \\
&= \sum_{n=0}^{\infty}\left(\hat{G}_0\hat{U}\right)^n|\phi_k\rangle
\end{aligned}$$

- Using

$$f(\theta,\phi) = -\frac{1}{4\pi}\left\langle \phi_k' \right| U\,|\psi_k\rangle$$

- We find

$$\begin{aligned}
f(\theta,\phi) &= -\frac{1}{4\pi}\left\langle \phi_k' \right| (U_U G_0 U + \\
&\quad UG_0UG_0U + \ldots)\,|\phi_k\rangle
\end{aligned}$$

- Series represents a sequence of multiple scattered events
- First Born Approximation corresponds to a single scatter

First Born Approximation (I)

- Using results from pg 25:

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int d^3r' \frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} U(\vec{r}')\psi_k(\vec{r}')$$

- Taking case of large r so $|\vec{r}-\vec{r}'| \approx r - \hat{e}_r \cdot r'$, we find

$$\frac{e^{i|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{e^{ikr}}{r} e^{-i\vec{k}'\cdot r'}$$

$$\text{with} \quad \vec{k}' \equiv k\hat{e}_r$$

- That means

$$\begin{aligned}
\psi_k(\vec{r}) &= e^{i\vec{k}\cdot r} + f(\theta,\phi)\frac{e^{ikr}}{r} \\
\text{with} \quad f(\theta,\phi) &\approx -\frac{1}{4\pi}\int d^3r' e^{i\vec{k}\cdot r'} U(\vec{r}')\psi_k(\vec{r}') \\
f_{Born}(\theta,\phi) &= -\frac{1}{4\pi}\left\langle \phi_{k'} \right| U \left|\psi_k\right\rangle
\end{aligned}$$

- Taking $\phi_k = e^{i\vec{k}\cdot\vec{r}}$ and $V(r) = \frac{\hbar^2}{2m}U(r)$:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(\theta,\phi)|^2 \\
&= \frac{m^2}{(2\pi)^2\hbar^4}\left|\left\langle \phi_k' \right| V \left|\psi_k\right\rangle\right|^2
\end{aligned}$$

- $\left\langle \phi_k' \right| V \left|\psi_k\right\rangle \equiv T_{kk'}$ sometimes called the transition matrix element

First Born Approximation (I)

- Writing expression from previous page with plane wave form of ϕ_k and ψ'_k:

$$\begin{aligned} f_{Born}(\theta,\phi) &= -\frac{1}{4\pi}\langle \phi_{k'}|\, U\,|\psi_k\rangle \\ &= -\frac{1}{4\pi}\int d^3r'\, e^{i\vec{k}\cdot\vec{r}'}U(r')e^{\vec{k}'\cdot\vec{r}'} \\ &= -\frac{1}{4\pi}\int d^3r'\, e^{i\vec{\Delta}\cdot\vec{r}'}U(r') \\ \text{where} \qquad & \Delta \equiv \vec{k}-\vec{k}' \end{aligned}$$

- But we are talking about elastic scattering

$$\begin{aligned} |\vec{k}| &= |\vec{k}'| \\ |\vec{\Delta}|^2 &= \left(\vec{k}-\vec{k}'\right)^2 \\ &= 2k^2\,(1-\cos\theta)^2 \\ &= 4k^2\sin^2\frac{\theta}{2} \end{aligned}$$

- For central potentials

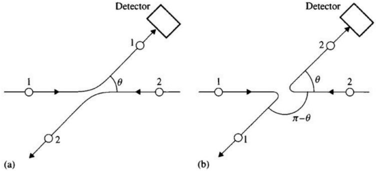
$$\begin{aligned} f_{Born}(\theta,\phi) &= -\frac{1}{4\pi}\int d^3r'\, e^{i\vec{\Delta}\cdot\vec{r}'}U(r') \\ &= -\frac{1}{4\pi}\int_0^\infty dr'\, r'^2 U(r')\int_0^\pi \sin\alpha \int_0^{2\pi} d\beta e^{i\Delta r\cos\alpha} \\ &= -\frac{1}{2}\int_0^\infty dr'\, r'^2 U(r')\int_{-1}^{1} d\cos\alpha e^{i\Delta r\cos\alpha} \\ &= -\frac{1}{\Delta}\int_0^\infty dr'\, r'\sin(\Delta r')U(r') \end{aligned}$$

Scattering of Identical Particles: the Center of Mass Frame

- Until now, consider scattering due to fixed potential (eg infinitely heavy target)
- Next topic: identical particles
  - Beam and target particles have same mass
  - Infinite mass approx not sensible
- Two particle system characterized by motion of center-of-mass (cm) and relative motion of particles with respect to each other in cm
  - Easiest to work in center-of-mass frame
  - Will discuss interaction between particles as function of separation r
  - Central force problem using reduced mass

$$\begin{aligned} \mu &= \frac{m_A m_B}{m_A + m + b} \\ &= m/2 \quad \text{if } m_A = m_B = m \end{aligned}$$

Collisions of Identical Particles in CM)



- Outgoing particles back-to-back
- Cannot distinguish particles: Cannot distinguish cases (a) and (b)
- Must write wave function in properly symmetrized/antisymmetrized form

$$\psi(\vec{r}) \quad = \quad e^{ikz} \pm e^{-ikz} + [f(\theta,\phi) \pm f(\pi-\theta,\phi+\pi)]\,\frac{e^{ikr}}{r}$$

where the $+$ sign is for bosons or for fermions with a symmetric spacial wf and the $-$ sign is for fermions with an antisymmetric spacial wf

$$\frac{d\sigma}{d\Omega} \quad = \quad |f(\theta,\phi) \pm f(\pi-\theta,\phi+\pi)|^2$$

- If particles are distinguishable cross section for seeing either of the two is:

$$\frac{d\sigma}{d\Omega} \quad = \quad |f(\theta,\phi)|^2 + |f(\pi-\theta,\phi+\pi)|^2$$

Collisions of Identical Bosons in CM

- Continuing from result of previous page:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= |f(\theta,\phi)+f(\pi-\theta,\phi+\pi)|^2 \\ &= |f(\theta,\phi)|^2+|f(\pi-\theta,\phi+\pi)|^2+2\mathrm{Re}\left[f(\theta,\phi)f^*(\pi-\theta,\phi+\pi)\right]\end{aligned}$$

- Going to the case with azimuthal symmetry and using the partial wave expansion, for bosons:
  - Expand in Legendre polynomials, but

$$P_\ell(\cos(\pi-\theta)=(-1)^\ell P_e ll(\cos\theta)$$

  - Boson scattering only contains even terms in ℓ
  - At $\theta=\pi/2$:

$$\frac{d\sigma(\pi/2)}{d\Omega}=4f(\pi/2)$$

    Four times as big as distinguishable particles

Collisions of Identical Fermions in CM

- Continuing from result of previous page:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left|f(\theta,\phi) \pm f(\pi-\theta,\phi+\pi)\right|^2 \\ &= \left|f(\theta,\phi)\right|^2 + \left|f(\pi-\theta,\phi+\pi)\right|^2 \pm 2\mathrm{Re}\left[f(\theta,\phi)f^*(\pi-\theta,\phi+\pi)\right]\end{aligned}$$

  where the $+$ sign is for fermions with a symmetric spacial wf and the $-$ sign is for fermions with an antisymmetric spacial wf

- Two spin-$\frac{1}{2}$ fermions can have
  - $S=0$: spin anti-symmetric, spatial symmetric
  - $S=1$: spin symmetric, spatial anti-symmetric
- There are 3 $S=1$ states ($m_s=-0,\pm1$):

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{d\sigma_s}{d\Omega} + \frac{3}{4}\frac{d\sigma_t}{d\Omega}$$

- Triplet state has only odd terms in ℓ and singlet has only even terms in ℓ