PSET 7 Problems

1) $F \to F'$ w/ velocity β wrt F, then $\frac{d^3p}{E}$ is Lorentz invar

i.e. show $\frac{dp_x}{E} = \frac{dp_x'}{E'}$

$\frac{dp_y}{E} = \frac{dp_y'}{E'}$

$\frac{dp_z}{E} = \frac{dp_z'}{E'}$

$p_x = \frac{2\pi n_x}{L}$, $p_y = \frac{2\pi n_y}{L}$, $p_z = \frac{2\pi n_z}{L}$

$\to$ discrete space, $\therefore d^3\vec{p} = dp_x\, dp_y\, d_z$

$= \left(\frac{2\pi}{a} 1\right)^3 = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$

$\to$ Lorentz boost: $p_x' = p_x$

$p_y' = p_y$

$p_z' = \gamma(p_z - \beta E)$ ☆

$E' = \gamma(E - \beta p_z)$

$\therefore d^3\vec{p}\,' = dp_x'\, dp_y'\, dp_z' = (dp_x)(dp_y)\left(dp_z \frac{dp_z'}{dp_z}\right)$

$= d^3\vec{p}\, \frac{dp_z'}{dp_z}$.

$$\frac{dp_z'}{dp_z} = \frac{d}{dp_z}\left(\gamma\left(p_z - \beta E\right)\right) = \gamma\left(1 - \beta \frac{\partial}{\partial p_z} E\right)$$

$$\frac{\partial E}{\partial p_z} = \frac{\partial}{\partial p_z}\left(\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\right)$$

$$= \frac{\cancel{2} p_z}{\cancel{2}\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}} = \frac{p_z}{E}$$

$$\therefore \frac{dp_z'}{dp_z} = \gamma\left(1 - \beta \frac{\partial E}{\partial p_z}\right) = \gamma\left(1 - \beta \frac{p_z}{E}\right)$$

$$= \frac{\gamma}{E}\left(E - \beta p_z\right)$$

$$= \frac{1}{E}\left(E'\right) = \frac{E'}{E}$$

$$\therefore d^3\vec{p}\,' = d^3\vec{p}\left(\frac{dp_z'}{dp_z}\right) = d^3\vec{p}\left(\frac{E'}{E}\right)$$

$$\therefore \boxed{\frac{d^3\vec{p}\,'}{E'} = \frac{d^3\vec{p}}{E}}$$

Q2) $(\gamma^\mu p_\mu - m)u = 0$

$$p_\mu = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

where $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}$$

$$u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$N_1 = N_2 = +\sqrt{p^2+m^2}$

$N_3 = N_4 = -\sqrt{p^2+m^2}$

show $\longrightarrow$ $E^2 = p^2 + m^2$ for all 4 u's.

$$\gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3$$

$$= E \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} - p_x \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} - p_y \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix} - p_z \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$= \longrightarrow$

$$\gamma^\mu p_\mu = \begin{pmatrix} E & 0 & -p_z & -p_x + ip_y \\ 0 & E & -p_x - ip_y & +p_z \\ +p_z & p_x - ip_y & -E & 0 \\ p_x + ip_y & -p_z & 0 & -E \end{pmatrix}$$

$$\gamma^\mu p_\mu - \mathbb{1} m = \begin{pmatrix} E-m & 0 & -p_z & -p_x + ip_y \\ 0 & E-m & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & -E-m & 0 \\ p_x + ip_y & -p_z & 0 & -E-m \end{pmatrix}$$

$$\downarrow$$

$$(\gamma^\mu p_\mu - \mathbb{1} m)\, u_1$$

$$= \begin{pmatrix} E-m & 0 & -p_z & -p_x + ip_y \\ 0 & E-m & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & -E-m & 0 \\ p_x + ip_y & -p_z & 0 & -E-m \end{pmatrix} N_1 \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ \dfrac{p_x + ip_y}{E+m} \end{pmatrix} = 0$$

$$= N_2 \begin{pmatrix} (E-m) - \dfrac{p_z^2}{(E+m)} + \dfrac{(-p_x + ip_y)(p_x + ip_y)}{(E+m)} \\ \cancel{\dfrac{p_z(-p_x - ip_y)}{E+m} + \dfrac{p_z(p_x + ip_y)}{E+m}} \\ \cancel{p_z + 0 + p_z\left(-\dfrac{E+m}{E+m}\right)} \\ \cancel{p_x + ip_y - (p_x + ip_y)} \end{pmatrix} = \longrightarrow$$

$$(-p_x+ip_y)(p_x+ip_y) = -p_x^2 - \cancel{ip_xp_y} + \cancel{ip_yp_x} - p_y^2$$

$$\cancel{N_2}\begin{bmatrix} (E-m) - \frac{p_z^2}{E+m} + \frac{(-p_x^2 - p_y^2)}{E+m} \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\therefore (E-m) - \frac{(p_x^2+p_y^2+p_z^2)}{E+m} = 0.$$

$$E^2 - m^2 = |\vec{p}|^2$$

$$\therefore \boxed{E^2 = |\vec{p}|^2 + m^2} \checkmark$$

$\underline{u_2:}$

$$(\gamma^\mu p_\mu - \mathbb{1}m)u_2 = 0.$$

$$= \begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} \cancel{N_2} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = 0.$$

$$= \begin{pmatrix} \cancel{0+0-\frac{p_z(p_x-ip_y)}{E+m} - \frac{p_z(-p_x+ip_y)}{E+m}} \\ (E-m) + \frac{(p_x-ip_y)(-p_x-ip_y)}{E+m} - \frac{p_z^2}{E+m} \\ \cancel{(p_x-ip_y)} - \cancel{(p_x-ip_y)} \\ -\cancel{p_z} + \cancel{p_z} \end{pmatrix} \cancel{N_2} = 0 -$$

$$\rightarrow (E-m) + \frac{(-p_x^2 - p_y^2 - p_z^2)}{E+m} = 0.$$

$\rightarrow$ same as u_1 : $\underline{\underline{E^2 = |\vec{p}|^2 + m^2}}$ ✓

$\underline{u_3}$: $(\gamma^\mu p_\mu - \mathbb{1} m) u_3 = 0$

$$= \begin{pmatrix} (E-m) & 0 & -p_z & -p_x + i p_y \\ 0 & (E-m) & -p_x - i p_y & p_z \\ p_z & p_x - i p_y & -E-m & 0 \\ p_x + i p_y & -p_z & 0 & -E-m \end{pmatrix} \cancel{N_3} \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + i p_y}{E-m} \\ 1 \\ 0 \end{pmatrix} = 0,$$

$$= \begin{pmatrix} \cancel{p_z} - \cancel{p_z} \\ \cancel{p_x + i p_y} - \cancel{(p_x + i p_y)} \\ \frac{p_z^2}{E-m} + \frac{(p_x^2 + p_y^2)}{E-m} - (E+m) \\ \cancel{p_z(p_x + i p_y) - p_z(p_x + i p_y)} \end{pmatrix} = 0$$

$$\boxed{E^2 = |\vec{p}|^2 + m^2}$$ ✓

$\underline{u_4}$: $(\gamma^\mu p_\mu - \mathbb{1}m)\, u_4 = 0$.

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} \cancel{N_4} \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ +1 \end{pmatrix} = 0$$

$$= \begin{pmatrix} \cancel{p_x-ip_y} + (\cancel{-p_x+ip_y}) \\ \cancel{-p_z} + \cancel{p_z} \\ \cancel{p_z(p_x-ip_y)} - \cancel{p_z(p_x-ip_y)} \\ \frac{p_x^2+p_y^2+p_z^2}{E-m} - (E+m) \end{pmatrix} = 0.$$

$$\boxed{E^2 = |\vec{p}|^2 + m^2} \checkmark$$

↳ u_1, u_2, u_3, u_4 reproduce $E^2 = p^2 + m^2$

when inserted into the Dirac eq ✓

Q3:

consider = $\begin{cases} \mu = \nu = 0 & ① \\ \mu = \nu \neq 0 & ② \\ \mu \neq \nu & ③ \end{cases}$

show that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

① $\{\gamma^0, \gamma^0\} = \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\gamma^0\gamma^0.$

$= 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = 2\mathbb{1}$

$= 2g^{\mu\nu}$ ✓

$g^{\mu\nu}(\mu = 0, \nu = 0) = 1.$ ✓

② $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu, \quad \mu = \nu = [1, 2, 3]$

from relations

i) $\gamma^1\gamma^1 + \gamma^1\gamma^1 = 2\gamma^1\gamma^1 = 2(-1) = 2g(\mu=1, \nu=1)$

ii) $\gamma^2\gamma^2 + \gamma^2\gamma^2 = 2\gamma^2\gamma^2 = 2(-1) = 2g(\mu=2, \nu=2)$

iii) $\gamma^3\gamma^3 + \gamma^3\gamma^3 = 2\gamma^3\gamma^3 = 2(-1) = 2g(\mu=3, \nu=3)$

$g^{\mu\nu}(\mu=1, \nu=1) = g^{\mu\nu}(2,2) = g^{\mu\nu}(3,3) = -1$

✓

→

③ $\mu \neq \nu$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}$$

from relations: $\gamma^{i}\gamma^{j} + \gamma^{j}\gamma^{i} = 0$ for $i \neq j$

$$\therefore \underline{\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 0 = 2g \ (\mu \neq \nu)}$$

$\hookrightarrow g^{\mu\nu}(\mu \neq \nu) = 0.$

$$\therefore \boxed{g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

✓ recovered entirely of $g^{\mu\nu}$

Q4: a) show that $u_1^{\dagger}u_2 = 0$ and $v_1^{\dagger}v_2 = 0$

from Lec13 slide 18:

$$u_1 = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$v_1 = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

4a), cont.

$$u_1^\dagger = \sqrt{E+m}\begin{pmatrix}1 & 0 & \frac{p_z}{E+m} & \frac{p_x - ip_y}{E+m}\end{pmatrix}$$

$$u_1^\dagger u_2 = (E+m)\begin{pmatrix}1 & 0 & \frac{p_z}{E+m} & \frac{p_x - ip_y}{E+m}\end{pmatrix} \cdot \begin{pmatrix}0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m}\end{pmatrix}$$

$$= (E+m)\left[0 + 0 + \frac{\cancel{p_z(p_x - ip_y) - p_z(p_x - ip_y)}}{(E+m)^2}\right]$$

$$= (E+m)(0) = 0.$$

$$\boxed{u_1^\dagger u_2 = 0 \quad \therefore u_1 \text{ orthogonal to } u_2}$$

$$v_1^\dagger v_2 = (E+m)\begin{pmatrix}\frac{p_x + ip_y}{E+m} & \frac{-p_z}{E+m} & 0 & 1\end{pmatrix}\begin{pmatrix}p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0\end{pmatrix}$$

$$= (E+m)\left[\frac{\cancel{(p_x+ip_y)p_z - p_z(p_x+ip_y)}}{(E+m)^2} + 0 + 0\right]$$

$$= \boxed{0 = v_1^\dagger v_2}$$ ✓ $\therefore v_1$ and v_2 are orthogonal

$\longrightarrow$

4b) show u_1 and v_1 are not orthogonal

$$u_1^\dagger v_1 \overset{?}{\neq} 0$$

$$u_1^\dagger v_1 = (E+m)\begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x - ip_y}{E+m} \end{pmatrix} \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

$$= (E+m)\left[\frac{p_x - ip_y}{E+m} + 0 + 0 + \frac{p_x - ip_y}{E+m} \right]$$

$$= \boxed{2(p_x - ip_y) = u_1^\dagger v_1}$$

$\boxed{u_1^\dagger v_1 \neq 0 \quad \therefore u_1 \text{ is not orthogonal to } v_1}$

↳ (unless $p_x = p_y = 0$)