Introduction

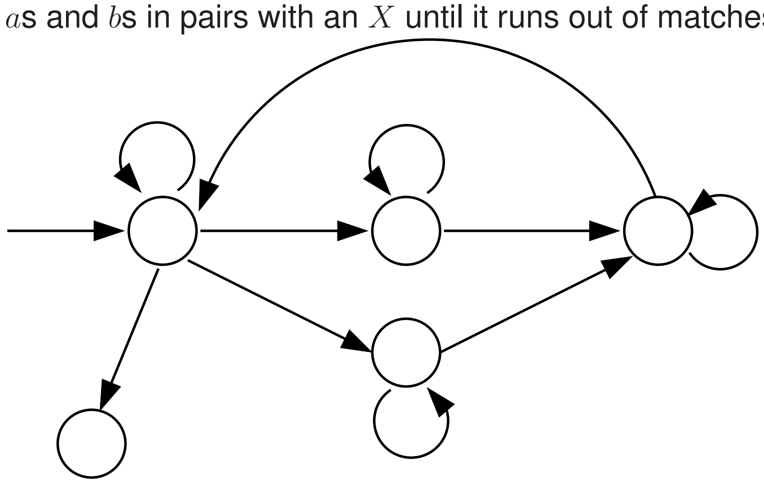
This is coursework assignment 2, due week 12, Thursday, 14 December 2017, 16:00 in the School Office of the School of Engineering and Informatics, Chichester I building, ground floor.

Please work on this alone and do not plagiarise. If your solution is suspected of plagiarism or collusion it will be reported and investigated. If you would like to know more about academic (mis)conduct, refer to `http://www.sussex.ac.uk/adqe/standards/academicmisconduct`.

A. Automata (18 marks)

Consider the language $\mathcal{B}$ over the alphabet $\{a, b\}$ that consists of all words that contain exactly the same number of as as bs (as and bs in any order), including the empty string ϵ. E.g., ϵ, ab, $abab$, $babaab$, $abab$, and so on would be in the laguage, but a, b, abb, bab, aab, aba and so on would not.

1. Design a pushdown automaton with a single state that accepts the language $\mathcal{B}$ [5 marks]
2. State whether the automaton is deterministic. [1 mark]
3. Show the protocol of the successful computation path for the input $abbbaa \in \mathcal{B}$ [4 marks]
4. Copy the diagram below and label it to become a Turing Machine that also accepts $\mathcal{B}$. Take care to mark (a) final state(s). (Hint: Use a strategy where the TM replaces as and bs in pairs with an X until it runs out of matches.) [8 marks]



B. Vectors and Matrices (30 marks)

1. The meaning of matrices

   (a) In lecture we learned to interpret what a matrix does by looking at its columns. According to this rule, describe in your own words what the matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ does geometrically. [2 marks]

   (b) Show that the Euclidean norm of any vector $\vec{x} \in \mathbb{R}^3$ does not change when A is applied to $\vec{x}$. [2 marks]

   (c) Calculate $A \cdot A$. Accordingly, what is A^{-1}? [2 marks]

   (d) Calculate the inverse of the matrix $B = \begin{pmatrix} 1 & x & 0 \\ -1 & -2 & 0 \\ 2 & 4 & 1 \end{pmatrix}$ [4 marks]

   (e) What value should x not take in order for B^{-1} to exist? [1 marks]

2. Proving things about matrices

   (a) Using summation notation for the components, prove that $(A \cdot B)\vec{x} = A(B \cdot \vec{x})$ for any matrix $A \in \mathcal{M}(m, n)$, $B \in \mathcal{M}(n, l)$ and vector $\vec{x} \in \mathbb{R}^l$. [4 marks]

   (b) Let $A, B \in \mathcal{M}(n, n)$ be two symmetric matrices, i.e. $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ for all $i, j = 1, \ldots, n$. Using summation notation, prove that $A \cdot B = (B \cdot A)^T$. [4 marks]

3. Norms and distances

   (a) Calculate the Manhatten distance and the Euclidean distance between the vectors $\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. [2 marks]

   (b) Prove that $||k \cdot \vec{x}||_2 = |k| \cdot ||\vec{x}||_2$ for any $k \in \mathbb{R}$ and any vector $\vec{x} \in \mathbb{R}^n$ [2 marks]

   (c) Show that for any $\vec{x} \in \mathbb{R}^n$, the identity $||\vec{x}||_2 = 0$ implies $\vec{x} = \vec{0}$, i.e. the vector with all 0 entries. [2 marks]

   (d) You have a fishing rod of length 2 and need to ship it in a box which sides are not longer than 1. In spaces $\mathbb{R}^n$ of at least which dimensionality n will you be able to fit the rod into the box without bending or breaking at the inter-dimensional post office (which works with Euclidean distances)?
   Find the smallest possible dimension n as any additional dimensions cost extra. Show clearly explained workings to support your answer. [5 marks]

C. Calculus (30 marks)

1. Write down the following limits or "doesn't converge" if it does not have a limit. (No workings or proof needed for this sub-problem, you can use the synbols $= \infty$ and $= -\infty$ for diverging series.):

   (a) $\lim\limits_{n\to\infty} \left(\frac{1}{n^a}\right)$ for $a > 0$ [1 mark]

   (b) $\lim\limits_{x\to\infty} \left(1 + \frac{\sin(x)}{x}\right)$ [1 mark]

   (c) $\lim\limits_{x\to\infty} \cos\left(\frac{1}{x}\right)$ [1 mark]

   (d) $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)$ [1 mark]

   (e) $\lim\limits_{n\to\infty} (-2)^n$ [1 mark]

   (f) $\lim\limits_{x\to 0,\, x<0} \exp\left(\frac{1}{x}\right)$ [1 mark]

   (g) $\lim\limits_{x\to 0,\, x>0} \exp\left(\frac{1}{x}\right)$ [1 mark]

2. Prove formally, i.e. using "epsilontic", that if $\lim\limits_{n\to\infty} x_n = x$ then $\lim\limits_{n\to\infty} ax_n + b = ax + b$. [4 marks]

3. In the lecture I told you that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$. The same is not true for x_n/y_n. Give an example of two sequences x_n and y_n where for $n \to \infty$, x_n and y_n converge, but x_n/y_n does not. Write down the limits for x_n and y_n. [2 marks]

4. A function is said to be continuous at x_0 if for all $\delta > 0$ there is an $\epsilon > 0$ such that $\forall x \in \mathbb{R}$, $|x - x_0| < \epsilon \;\Rightarrow\; |f(x) - f(x_0)| < \delta$. Show formally that this implies that the series $y_n = f(x_0 + \frac{1}{n})$ converges to $f(x_0)$, i.e. $\lim\limits_{n\to\infty} y_n = f(x_0)$ if f is a continuous function. [5 marks]

5. Calculate the following derivatives using the rules for derivatives that we have discussed in lecture:

   (a) $\frac{d}{dx} 2x^4$ [2 marks]

   (b) $\frac{d}{dx} \exp\left(\sqrt{x^3}\right) = \frac{d}{dx} e^{\sqrt{x^3}}$ [2 marks]

   (c) $\frac{d}{dx}\left(\frac{1}{1 + 2\cos(x^3)}\right)$ [2 marks]

6. Calculate the following integrals using the "inverse derivatives" we discussed in lecture:

   (a) $\int_0^2 x^3 - x\,dx$ [2 marks]

   (b) $\int_0^{\pi/2} \cos(x)\,dx$ [2 marks]

   (c) $\int_a^b \frac{1}{x}\,dx$ [2 marks]

D. Probability (22 marks)

1. Three Dice
   Consider a game of dice with three 6-sided, fair dice, i.e. $P(x_i = j) = \frac{1}{6}$ for $i = 1, 2, 3$ and $1 \leq j \leq 6$.

   (a) What is the probability for all of the three dice to show the same number? (Please show detailed workings) [4 marks]

   (b) What is the probability for at least one of the dice to show the number 6? (Please show detailed workings) [4 marks]

   (c) Show formally whether the two events "all three dice the same" and "at least one dice shows a 6" are independent. [3 marks]

2. Prof Nowotny's fried potatoes
   Prof Nowotny likes to cook. He cuts his potatoes into thin slices and fries them in a pan. He puts them in and occasionally tosses them. At each toss the potato slices fall independently of each other and with equal probability onto the same side or onto the other side. (Please show detailed workings with your answers to each part question)

   (a) For n slices, what is the probability that they all fall back onto the same side in a single toss? [2 marks]

   (b) For a single slice, how often does he have to toss it to get both sides fried with probability > 85%? [4 marks]

   (c) With n slices in the pan, how often does he have to toss such that all sides are fried (i.e. 100% of the slices are fried on both sides) with probability > 85%? Find the formula for general n and then try it for $n = 10$ and $n = 100$. [4 marks]

   (d) How often would he have to toss to have a probability of 100% that more than 85% of the sides are fried? Explain in your own words. [1 marks]