LP problems

- A standard linear programming (LP) problem is:

$$\begin{aligned} &\min_z cz \\ \text{s.t. } &Az \geq b \\ &z \geq 0 \end{aligned} \tag{1}$$

  - z is the vector of decision variables, the *solutions*.
  - c is called objective coefficients.
  - Problem (1) is to determine an optimal value of z (searching for the best z), denoted as z^*, such that cz^* is the minimum.
  - $Az \geq b$ represent the *constraints* in a matrix form. Each row of $Az \geq b$ is a *constraint*.
  - b is called the right-hand-side (rhs) vector.
  - A is the left-hand-side (lhs) matrix.
  - z can have upper bounds and lower bounds. Here the lower bounds are 0 and upper bounds are ∞.

- Gurobi (a *solver*) solves Problem (1). Above list the inputs for Gurobi.

- (Note) Gurobi does not recommend dense matrix form shown above. Check the comments in the example dense_c.c. Gurobi does not store A in a dense matrix, but in a sparse matrix (CSR format) which ignores 0's.

- **Task 1** Write a function: given any size of matrices A, b and c, implement mip1_c.c with x, y and z are continuous and non-negative (as in dense_c.c).

- **Task 2** Read data from a csv file and store as c array (matrix).

DEA problems

- $X = [x_r] = [x_1\ x_2\ \cdots\ x_n]$ where x_r is a $p \times 1$ vector.

- $Y = [y_r] = [y_1\ y_2\ \cdots\ y_n]$ where y_r is a $q \times 1$ vector.

- Data envelopment analysis (DEA) problem:

$$\begin{aligned} &\min_{\theta,\lambda} \theta \\ \text{s.t. } &X\lambda \leq x_r\theta \\ &Y\lambda \geq y_r \\ &e\lambda = 1 \\ &\lambda \geq 0 \end{aligned} \tag{2}$$

  where is a $\lambda = [\lambda_1\ \lambda_2\ \cdots\ \lambda_n]$ is a $1 \times n$ vector (or $n \times 1$ vector). x_r and y_r are the rth column of X and Y, respectively.
  It is equivalent to:

$$\begin{aligned} &\min_{\theta,\lambda} \theta \\ \text{s.t. } &-X\lambda \geq -x_r\theta \\ &Y\lambda \geq y_r \\ &e\lambda = 1 \\ &\lambda \geq 0 \end{aligned} \tag{3}$$

It can be rearranged as:

$$
\begin{aligned}
&\min_{\theta,\lambda} \theta \\
\text{s.t. } & x_r\theta - X\lambda \geq \mathbf{0} \\
& \mathbf{0}\theta + Y\lambda \geq y_r \\
& 0\theta + e\lambda = 1 \\
& \lambda \geq 0, \theta \geq 0
\end{aligned}
\tag{4}
$$

Solving Problem (4)

- I want to solve Problem (2) (equivalently, Problem (4)), which is an instance of Problem (1). Comparing Problems (1) and (4), we have

$$A = \begin{bmatrix} x_r & -X \\ \mathbf{0} & Y \end{bmatrix}$$

$$b = \begin{bmatrix} \mathbf{0} \\ y_r \end{bmatrix}$$

  x_r and y_r are the rth column of X and Y respectively. Thus here A is a $(p+q) \times (1+n)$ matrix, and b is a $(p+q) \times 1$ vector. $\mathbf{0}$ is a $p \times 1$ vector with zeros.

$$c = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}$$

  c is a $(1+n) \times 1$ (or $1 \times (1+n)$) vector.

$$z = \begin{bmatrix} \theta & \lambda \end{bmatrix}$$

  z is a $(1+n) \times 1$ (or $1 \times (1+n)$) vector.

- X and Y are read from a csv file, in which row r represent column $[x_r, y_r]$. In the csv file, the first few columns associates with X, and follow by Y. For example, in file 5-2-25k_10.csv, the first two columns are X^T (transpose of X) and the last three column are Y^T.

- $0\theta + e\lambda = 1$ in (4) is an an *equality constraint*, or *constraint with equality*. It is that you replace $\leq$ or $\geq$ by $=$. Check the manual for how to set it up. It means that the coefficient matrix (vector) is a $1 \times (1+n)$ vector ,$[0, e]$, where $e = [1, 1, 1 \cdots, 1]$ is a $1 \times n$ vector with values 1.

- **Task:** We need to do the following.

Algorithm 1: Naive DEA

Input: X, Y
Output: optimal objective values
for $r = 1, \cdots, n$ **do**
  populates A, b, c defined above;
  $\theta_r^* \longleftarrow$solve Problem (4) ; `// get optimal value`
write θ_r^*'s to a csv.

- Each iteration of Algorithm (1) computes an LP problem.

- Please record the total computation time, but ignore the IO time.