

Modeling Complex Systems (CS/CSYS 302),
Fall 2022
Assignment #1

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1 Lotka-Volterra 4th Phase Plane

1.1 Perquisites

Taking the video lectures into account, we have the continuous dynamics of 2 competing species in the form of logistic growth models.

$$\text{Lotka-Volterra} = \begin{cases} \dot{N}_1 &= r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\ \dot{N}_2 &= r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \end{cases} \quad (1)$$
$$\text{where } \begin{cases} a_{12} \equiv \alpha_2 \frac{r_1}{K_1} \\ a_{21} \equiv \alpha_1 \frac{r_2}{K_2} \end{cases}$$

with the following variables,

- i : species ID
- r_i : population growth rate for species i
- N_i : population of species i
- K_i : population maximum capacity
- a_{ij} : interaction between species i and species j (predation and reproduction)

Nullclines for $\dot{N}_1 = 0$ and $\dot{N}_2 = 0$ are,

For $\dot{N}_1 = 0$,

- $N_1^* = 0$ (species 1 is extinct)
- $N_1^{**} = K_1 - \alpha_2 N_2$

For $\dot{N}_2 = 0$,

- $N_2^* = 0$ (species 2 is extinct)
- $N_2^{**} = K_2 - \alpha_1 N_1$

with $*$ and $**$ signifying 2 solutions. The same rule applies to the rest of the paper.

The uncoupled fixed points are the following.

$$N_1^{**} = \frac{K_1 - \alpha_2 K_2}{1 - \alpha_2 \alpha_1} \quad (2)$$

$$N_2^{**} = \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2} \quad (3)$$

1.2 Determining the Vector Space

The 4th phase space, succeeding the 3 phase spaces discussed in the lecture, has the following constraints.

$$\begin{aligned} K_1 &> \frac{K_2}{\alpha_1} \\ K_2 &> \frac{K_1}{\alpha_2} \end{aligned} \tag{4}$$

This system in a phase space should resemble Figure 1.

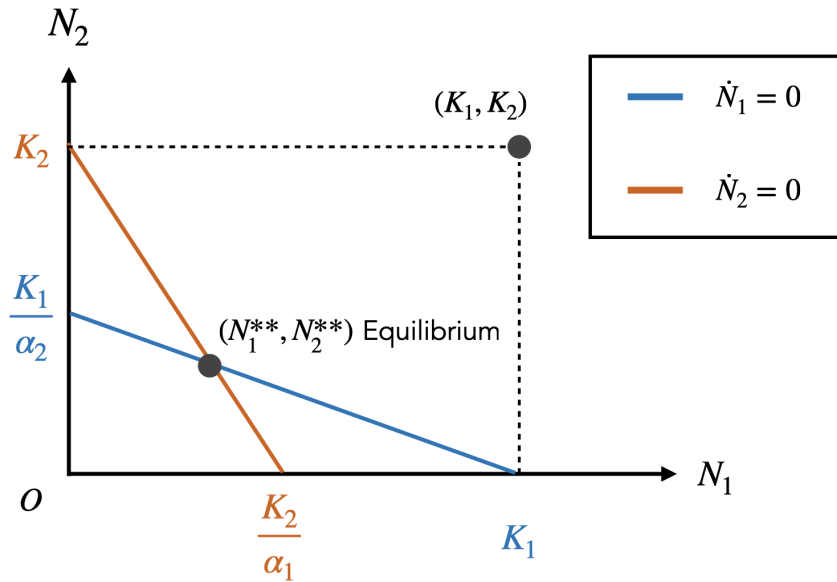


Figure 1: Nullclines for N_1 and N_2 and the equilibrium of the Lotka-Volterra system when Equation 4 is met.

In determining the vector space, or rather the vectors' orientation on arbitrary points in the phase plane, we need a sample of a point on the phase plane. For simplicity, we take (K_1, K_2) .

Now we determine the orientation of the vector at (K_1, K_2) .

$$\begin{aligned}
\dot{N}_1 &= r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\
&= r_1 K_1 \left(\frac{K_1 - K_1 - \alpha_2 N_2}{K_1} \right) \\
&= -r_1 \alpha_2 K_2 < 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\dot{N}_2 &= r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \\
&= r_2 K_2 \left(\frac{K_2 - K_2 - \alpha_1 N_1}{K_2} \right) \\
&= -r_2 \alpha_1 K_1 < 0
\end{aligned} \tag{6}$$

Based on Equations 5 and 6, we can determine that the vector at (K_1, K_2) is pointed toward the origin.

Given that there are 2 nullclines as specified in 1.1, the vector space would look like Figure 2.

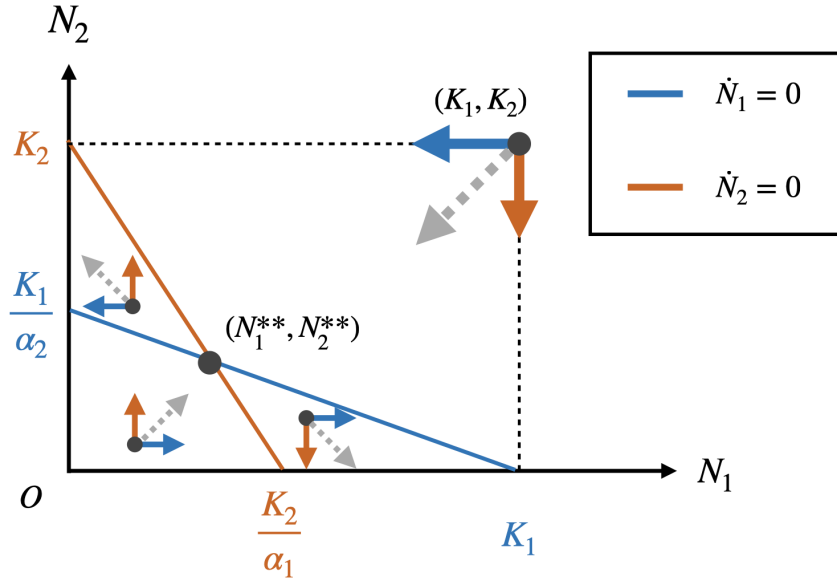


Figure 2: Vector space for the Lotka-Volterra model when Equation 4 is met. The blue and red arrows indicate the direction of the vector on the N_1 and N_2 axes per point, respectively. The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

1.3 Stability Analysis of the Equilibrium

Based on the previous section, we can draw the vectors around the equilibrium as Figure 4.

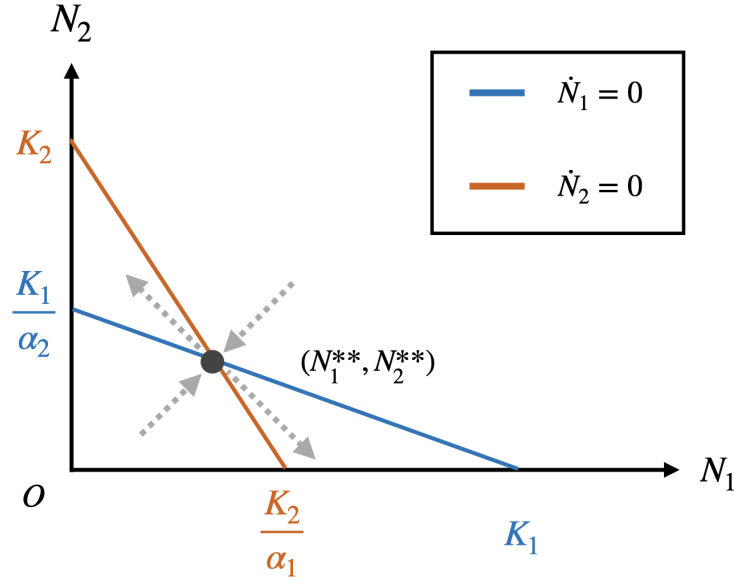


Figure 3: Vector space around the equilibrium in the Lotka-Volterra model when Equation 4 is met, derived from Figure 2 . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

1.4 Stability Analysis of the Equilibrium

Question 2 presents an unstable state in which the population of organism 2 (N_2) dominates and reduces the population of organism 1 (N_1) to local extinction. The proposed intervention involved reducing the populations of both N_1 and N_2 by a constant factor $\rho \in (0,1)$. The question is, is this intervention effective and if so, is there a range of value of ρ in which the intervention is effective?

More formally the state of the system is represented by a vector

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

The system is an an region of the system that moves towards an equilibrium where $N_1 = 0$. The question is then: is there some scalar value ρ which results in the system moving towards an equilibrium where $N_1, N_2 > 0$

The result of the reduction reduction by rho is a vector \mathbf{N}'

$$\mathbf{N}' = \begin{bmatrix} N'_1 \\ N'_2 \end{bmatrix}$$

Since \mathbf{N} is multiplied by a scalar, \mathbf{N}' will have the same direction as \mathbf{N} , but a smaller magnitude.

From the initial wording of the question we know that N_1 goes to 0. The region of the phase space for which this is the case is the region in which

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

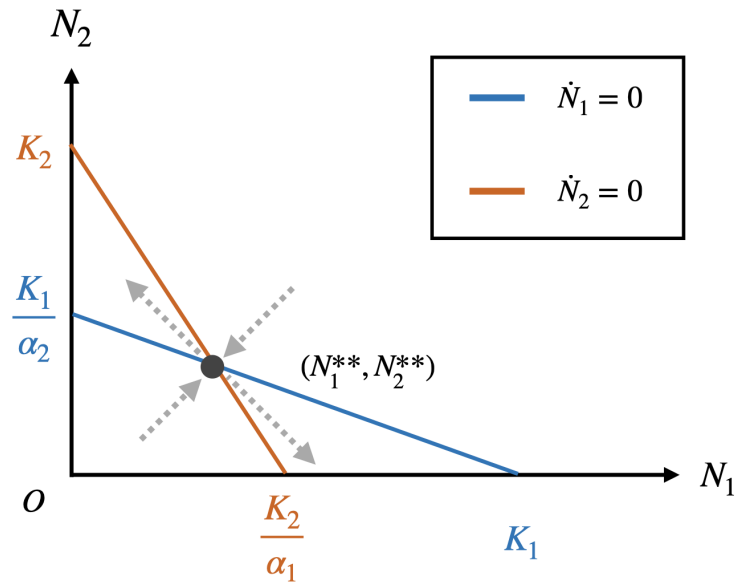


Figure 4: Vector space around the equilibrium in the Lotka-Volterra model when Equation 4 is met, derived from Figure 2 . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.