

Modeling Complex Systems (CS/CSYS 302),  
Fall 2022  
Assignment #1

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# 1 Lotka-Volterra 4th Phase Plane

## 1.1 Perquisites

Taking the video lectures into account, we have the continuous dynamics of 2 competing species in the form of logistic growth models.

$$\begin{aligned} \text{Lotka-Volterra} &= \begin{cases} \dot{N}_1 &= r_1 N_1 \left( \frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\ \dot{N}_2 &= r_2 N_2 \left( \frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \end{cases} \\ \text{where } \begin{cases} a_{12} &\equiv \alpha_2 \frac{r_1}{K_1} \\ a_{21} &\equiv \alpha_1 \frac{r_2}{K_2} \end{cases} \end{aligned} \quad (1)$$

with the following variables,

- $i$ : species ID
- $r_i$ : population growth rate for species  $i$
- $N_i$ : population of species  $i$
- $K_i$ : population maximum capacity
- $a_{ij}$ : interaction between species  $i$  and species  $j$  (predation and reproduction)

Nullclines for  $\dot{N}_1 = 0$  and  $\dot{N}_2 = 0$  are,

For  $\dot{N}_1 = 0$ ,

- $N_1^* = 0$  (species 1 is extinct)
- $N_1^{**} = K_1 - \alpha_2 N_2$

For  $\dot{N}_2 = 0$ ,

- $N_2^* = 0$  (species 2 is extinct)
- $N_2^{**} = K_2 - \alpha_1 N_1$

with  $*$  and  $**$  signifying 2 solutions. The same rule applies to the rest of the paper.

The uncoupled fixed points are the following.

$$N_1^{**} = \frac{K_1 - \alpha_2 K_2}{1 - \alpha_2 \alpha_1} \quad (2)$$

$$N_2^{**} = \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2} \quad (3)$$

## 1.2 Determining the Vector Space

The 4th phase space, succeeding the 3 phase spaces discussed in the lecture, has the following constraints.

$$\begin{aligned} K_1 &> \frac{K_2}{\alpha_1} \\ K_2 &> \frac{K_1}{\alpha_2} \end{aligned} \tag{4}$$

This system in a phase space should resemble Figure ??.

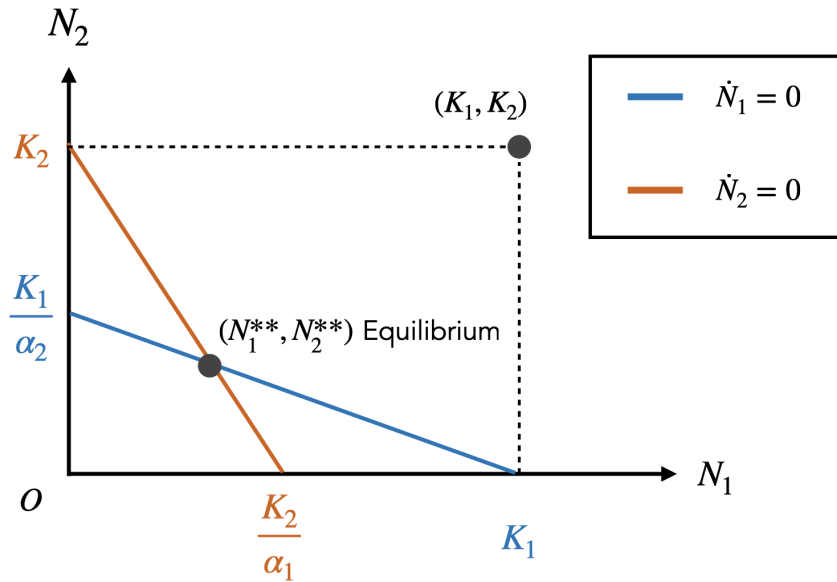


Figure 1: Nullclines for  $N_1$  and  $N_2$  and the equilibrium of the Lotka-Volterra system when Equation ?? is met.

In determining the vector space, or rather the vectors' orientation on arbitrary points in the phase plane, we need a sample of a point on the phase plane. For simplicity, we take  $(K_1, K_2)$ .

Now we determine the orientation of the vector at  $(K_1, K_2)$ .

$$\begin{aligned}
\dot{N}_1 &= r_1 N_1 \left( \frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\
&= r_1 K_1 \left( \frac{K_1 - K_1 - \alpha_2 N_2}{K_1} \right) \\
&= -r_1 \alpha_2 K_2 < 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\dot{N}_2 &= r_2 N_2 \left( \frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \\
&= r_2 K_2 \left( \frac{K_2 - K_2 - \alpha_1 N_1}{K_2} \right) \\
&= -r_2 \alpha_1 K_1 < 0
\end{aligned} \tag{6}$$

Based on Equations ?? and ??, we can determine that the vector at  $(K_1, K_2)$  is pointed toward the origin.

Given that there are 2 nullclines as specified in ??, the vector space would look like Figure ??.

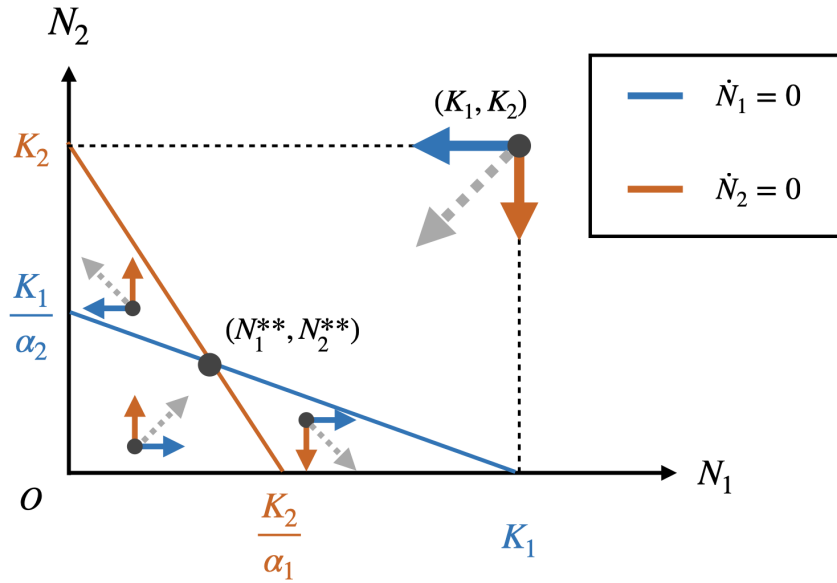


Figure 2: Vector space for the Lotka-Volterra model when Equation ?? is met. The blue and red arrows indicate the direction of the vector on the  $N_1$  and  $N_2$  axes per point, respectively. The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

### 1.3 Stability Analysis of the Equilibrium

Based on the previous section, we can draw the vectors around the equilibrium as Figure ??.

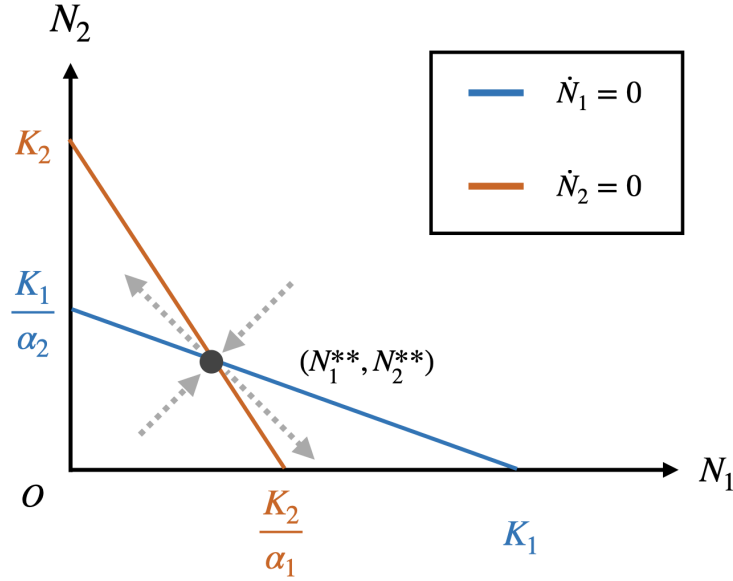


Figure 3: Vector space around the equilibrium in the Lotka-Volterra model when Equation ?? is met, derived from Figure ?? . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

## 1.4 Stability Analysis of the Equilibrium

Question 2 presents an unstable state in which the population of organism 2 ( $N_2$ ) dominates and reduces the population of organism 1 ( $N_1$ ) to local extinction. The proposed intervention involved reducing the populations of both  $N_1$  and  $N_2$  by a constant factor  $\rho \in (0,1)$ . The question is, is this intervention effective and if so, is there a range of value of  $\rho$  in which the intervention is effective?

More formally the state of the system is represented by a vector

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

The system is an an region of the system that moves towards an equilibrium where  $N_1 = 0$ . The question is then: is there some scalar value  $\rho$  which results in the system moving towards an equilibrium where  $N_1, N_2 > 0$

The result of the reduction reduction by rho is a vector  $\mathbf{N}'$

$$\mathbf{N}' = \begin{bmatrix} N'_1 \\ N'_2 \end{bmatrix}$$

Since  $\mathbf{N}$  is multiplied by a scalar,  $\mathbf{N}'$  will have the same direction as  $\mathbf{N}$ , but a smaller magnitude.

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

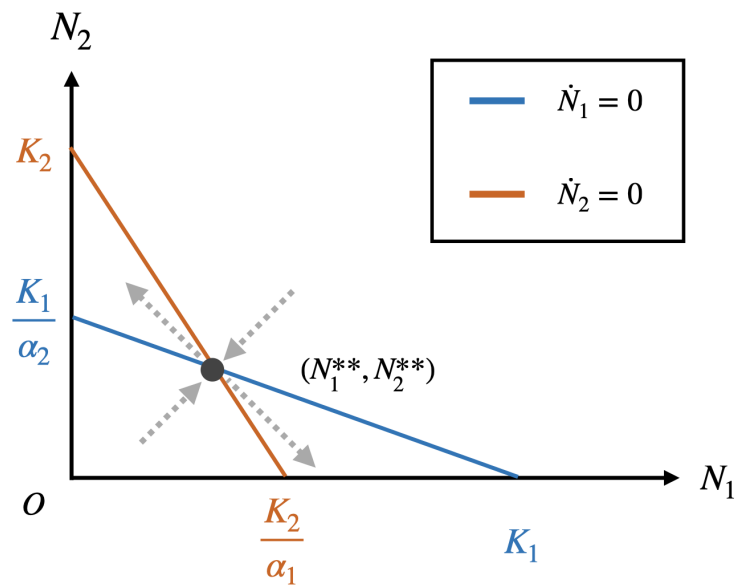


Figure 4: Vector space around the equilibrium in the Lotka-Volterra model when Equation ?? is met, derived from Figure ?? . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.