Modeling Complex Systems (CS/CSYS 302), Fall 2022 Assignment #1

Aviral Chawla[1], Mioto Takahashi[2], and William Thompson[3]

[1, 2, 3]University of Vermont Complex Systems and Data Science

October 5, 2022

1 Lotka-Volterra 4th Phase Plane

1.1 Perquisites

Taking the video lectures into account, we have the continuous dynamics of 2 competing species in the form of logistic growth models.

$$\begin{aligned} &\text{Lotka-Volterra} = \begin{cases} \dot{N}_1 & = r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1}\right) \\ \dot{N}_2 & = r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2}\right) \end{cases} \\ &\text{where} \begin{cases} a_{12} \equiv \alpha_2 \frac{r_1}{K_1} \\ a_{12} \equiv \alpha_1 \frac{r_2}{K_2} \end{cases} \end{aligned} \tag{1}$$

with the following variables,

- i: species ID
- r_i: population growth rate for species i
- N_i: population of species i
- K_i: population maximum capacity
- a_{ij}: interaction between species i and species j (predation and reproduction)

Nullclines for $\dot{N}_1 = 0$ and $\dot{N}_2 = 0$ are,

For $\dot{N}_1 = 0$,

- $N_1^* = 0$ (species 1 is extinct)
- $N_1^{**} = K_1 - \alpha_2 N_2$

For $\dot{N}_2 = 0$,

- $N_2^* = 0$ (species 2 is extinct)
- $N_2^{**} = K_2 - \alpha_1 N_1$

with $*$ and $**$ signifying 2 solutions. The same rule applies to the rest of the paper.

The uncoupled fixed points are the following.

$$N_1^{**} = \frac{K_1 - \alpha_2 K_2}{1 - \alpha_2 \alpha_1} \tag{2}$$

$$N_2^{**} = \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2} \tag{3}$$

1.2 Determining the Vector Space

The 4th phase space, succeeding the 3 phase spaces discussed in the lecture, has the following constraints.

$$\begin{aligned} K_1 &> \frac{K_2}{\alpha_1} \\ K_2 &> \frac{K_1}{\alpha_2} \end{aligned} \tag{4}$$

This system in a phase space should resemble Figure 1.

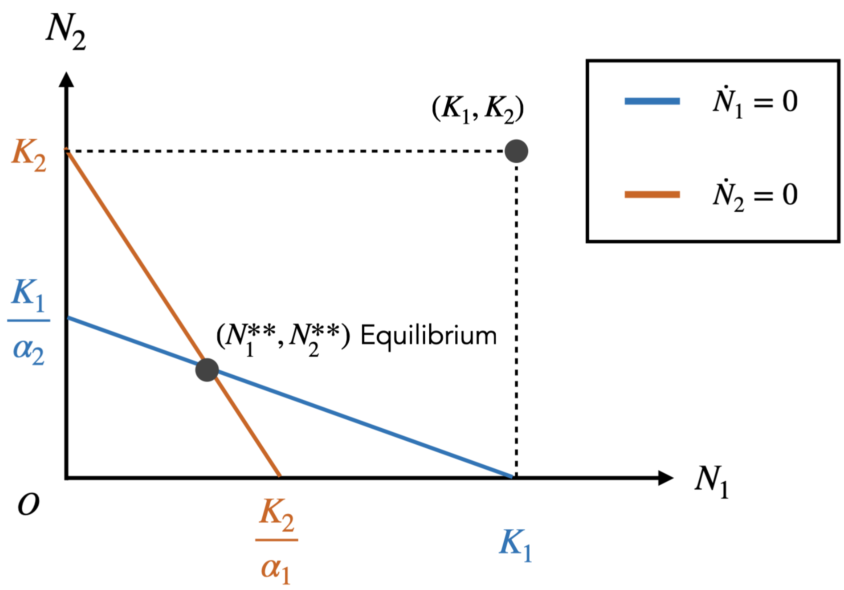

Figure 1: Nullclines for N_1 and N_2 and the equilibrium of the Lotka-Volterra system when Equation 4 is met.

In determining the vector space, or rather the vectors' orientation on arbitrary points in the phase plane, we need a sample of a point on the phase plane. For simplicity, we take (K_1, K_2).

Now we determine the orientation of the vector at (K_1, K_2).

$$\begin{aligned}\dot{N}_1 &= r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1}\right) \\ &= r_1 K_1 \left(\frac{K_1 - K_1 - \alpha_2 N_2}{K_1}\right) \\ &= -r_1 \alpha_2 K_2 < 0\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{N}_2 &= r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2}\right) \\ &= r_2 K_2 \left(\frac{K_2 - K_2 - \alpha_1 N_1}{K_2}\right) \\ &= -r_2 \alpha_1 K_1 < 0\end{aligned} \tag{6}$$

Based on Equations 5 and 6, we can determine that the vector at (K_1, K_2) is pointed toward the origin.

Given that there are 2 nullclines as specified in 1.1, the vector space would look like Figure 2.

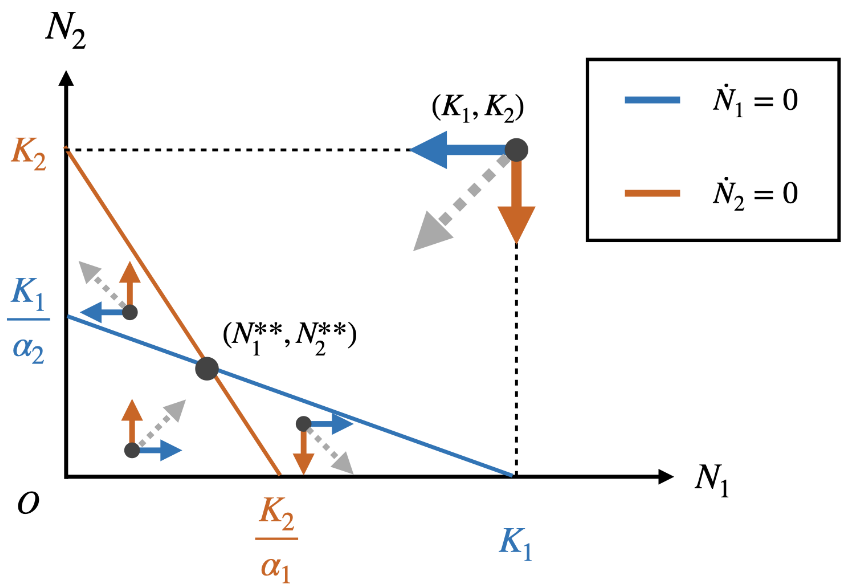

Figure 2: Vector space for the Lotka-Volterra model when Equation 4 is met. The blue and red arrows indicate the direction of the vector on the N_1 and N_2 axes per point, respectively. The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

1.3 Stability Analysis of the Equilibrium

Based on the previous section, we can draw the vectors around the equilibrium as Figure 3.

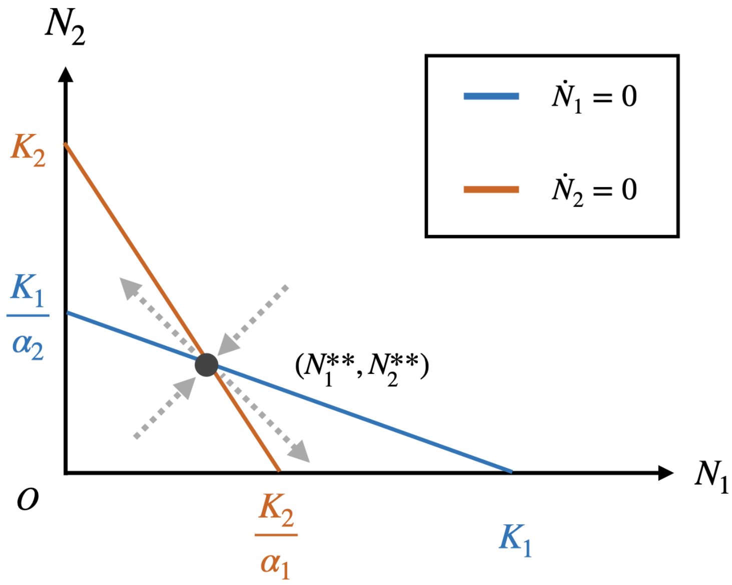

Figure 3: Vector space around the equilibrium in the Lotka-Volterra model when Equation 4 is met, derived from Figure 2 . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

1.4 Stability Analysis of the Equilibrium

Question 2 presents an unstable state in which the population of organism 2 (N_2) dominates and reduces the population of organism $1(N_1)$ to local extinction. The proposed intervention involved reducing the populations of both N_1 and N_2 by a constant factor $\rho \in (0, 1)$ The question is, is this intervention effective and if so, is there a range of value of ρ in which the intervention is effective?

More formally the state of the system is represented by a vector

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

The system is an an region of the system that moves towards an equilibrium where $N_1 = 0$. The question is then: is there some scalar value ρ which results in the system moving towards an equilibirum where $N_1, N_2 > 0$

The result of the reduction reduction by rho is a vector **N'**

$$\mathbf{N'} = \begin{bmatrix} N_1' \\ N_2' \end{bmatrix}$$

Since **N** is multiplied by a scalar, **N'** will have the same direction as **N**, but a smaller magnitude.

From the initial wording of the question we know that $N1$ goes to 0. In order for an intervention to be successful the scaled **N'** must fall within a region such that the evolution will not cause the population of N_1 to go to 0. Fig 4 depicts this process.

This is not possible for the following reason. The derivative $\dot{N}_1$ constantly decreases going from positive to negative as it crossed the N_1 nullcline. An intervention can change the magnitude of **N** but not the direction so the resulting intervention will pull **N** the same distance or closer towards $N_1 = 0$. As a result no intervention and no value of rho can prevent the population of species 1 from collapsing.

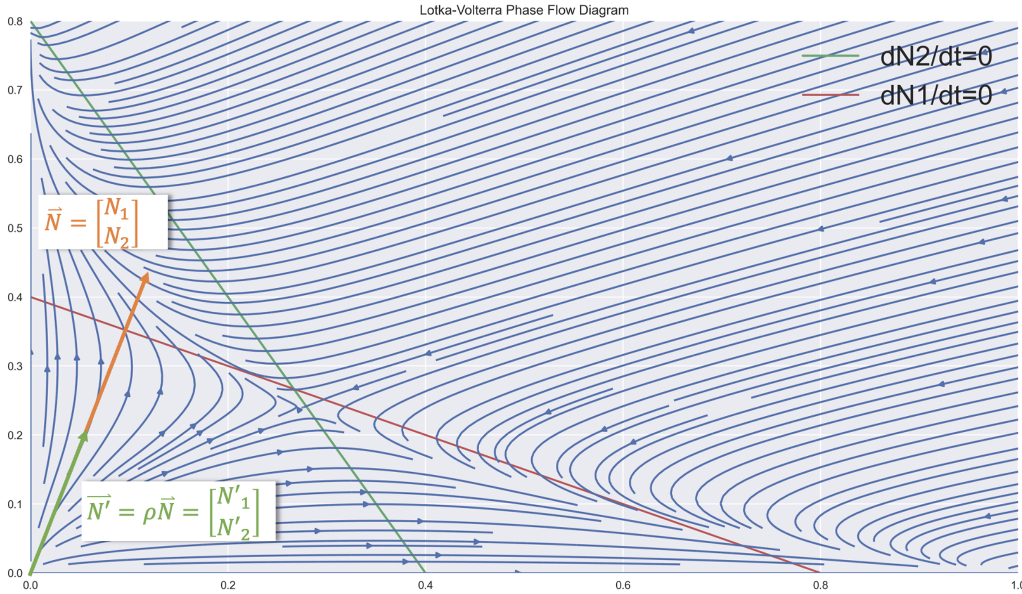

Figure 4: A phase flow diagram of the Lotka Volterra Model, the state of the system at any time is specified by a vector $\vec{N}$, the reduction by rho then corresponds to multiplying this vector by a scalar quantity ρ to a resulting vector $\vec{N}\prime$

.

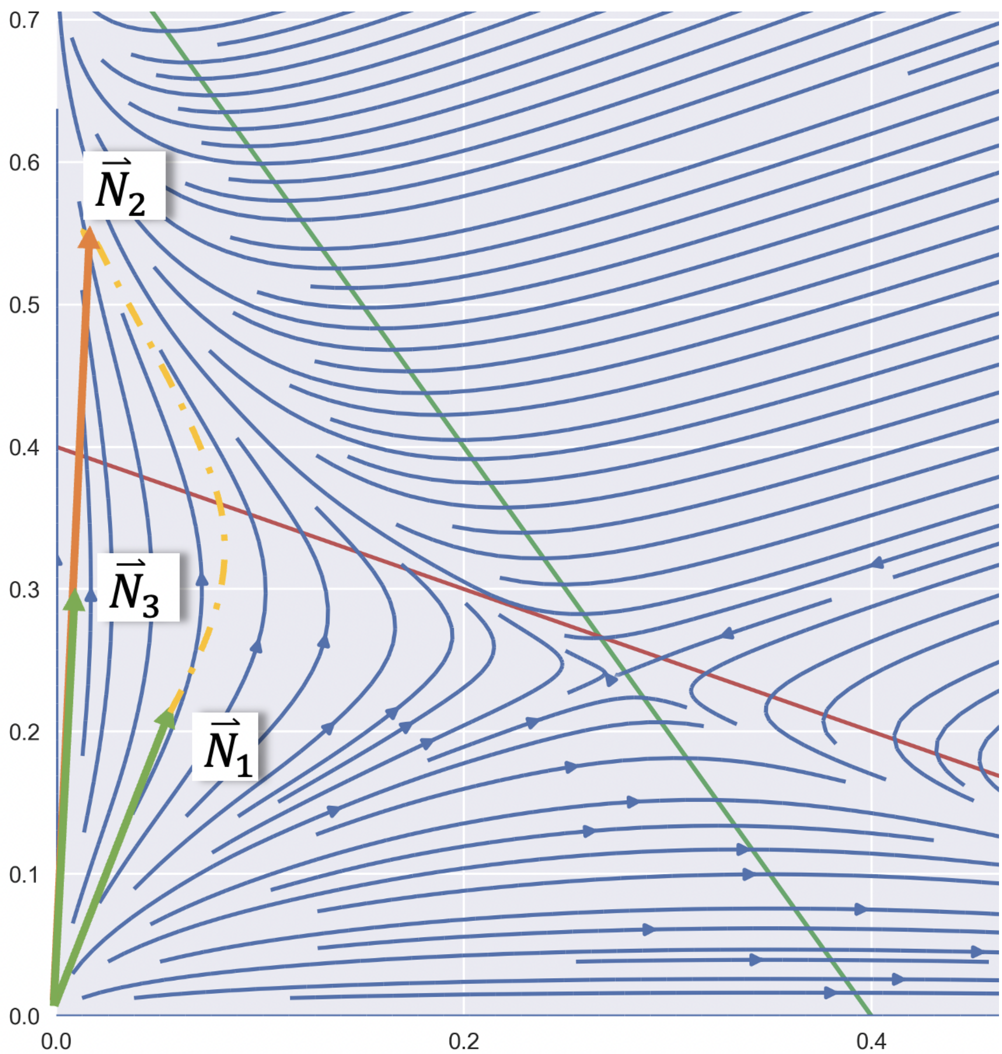

Figure 5: The result of an intervention by reducing the population by ρ. The state begins at $\vec{N}_1$ and evolves forward in time. The system evolves in time forward crossing the nullcline to into the region on the top right.

.