Modeling Complex Systems (CS/CSYS 302), Fall 2022 Assignment #1

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1 Lotka-Volterra 4th Phase Plane

1.1 Perquisites

Taking the video lectures into account, we have the continuous dynamics of 2 competing species in the form of logistic growth models.

$$\text{Lotka-Volterra} = \begin{cases} \dot{N}_1 & = r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\ \dot{N}_2 & = r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \end{cases}$$
$$\text{where} \begin{cases} a_{12} \equiv \alpha_2 \frac{r_1}{K_1} \\ a_{12} \equiv \alpha_1 \frac{r_2}{K_2} \end{cases} \tag{1}$$

with the following variables,

- i: species ID
- r_i: population growth rate for species i
- N_i: population of species i
- K_i: population maximum capacity
- a_{ij}: interaction between species i and species j (predation and reproduction)

Nullclines for $\dot{N}_1 = 0$ and $\dot{N}_2 = 0$ are,

For $\dot{N}_1 = 0$,

- $N_1^* = 0$ (species 1 is extinct)
- $N_1^{**} = K_1 - \alpha_2 N_2$

For $\dot{N}_2 = 0$,

- $N_2^* = 0$ (species 2 is extinct)
- $N_2^{**} = K_2 - \alpha_1 N_1$

with $*$ and $**$ signifying 2 solutions. The same rule applies to the rest of the paper.

The uncoupled fixed points are the following.

$$N_1^{**} = \frac{K_1 - \alpha_2 K_2}{1 - \alpha_2 \alpha_1} \tag{2}$$

$$N_2^{**} = \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2} \tag{3}$$

1.2 Determining the Vector Space

The 4th phase space, succeeding the 3 phase spaces discussed in the lecture, has the following constraints.

$$
\begin{aligned}
K_1 &> \frac{K_2}{\alpha_1} \\
K_2 &> \frac{K_1}{\alpha_2}
\end{aligned}
\tag{4}
$$

This system in a phase space should resemble Figure 1.

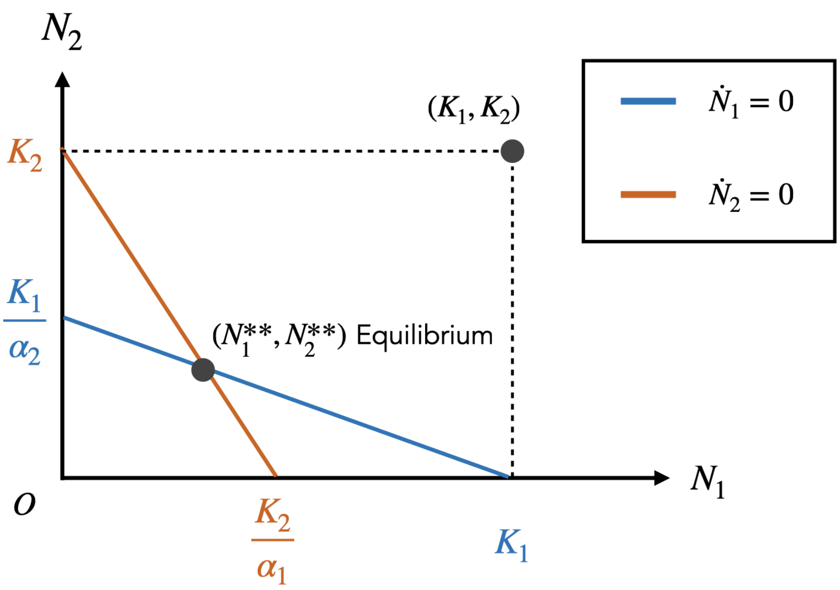

Figure 1: Nullclines for N_1 and N_2 and the equilibrium of the Lotka-Volterra system when Equation 4 is met.

In determining the vector space, or rather the vectors' orientation on arbitrary points in the phase plane, we need a sample of a point on the phase plane. For simplicity, we take (K_1, K_2).

Now we determine the orientation of the vector at (K_1, K_2).

$$
\begin{aligned}
\dot{N}_1 &= r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_2 N_2}{K_1} \right) \\
&= r_1 K_1 \left(\frac{K_1 - K_1 - \alpha_2 N_2}{K_1} \right) \\
&= -r_1 \alpha_2 K_2 < 0
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\dot{N}_2 &= r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_1 N_1}{K_2} \right) \\
&= r_2 K_2 \left(\frac{K_2 - K_2 - \alpha_1 N_1}{K_2} \right) \\
&= -r_2 \alpha_1 K_1 < 0
\end{aligned} \tag{6}
$$

Based on Equations 5 and 6, we can determine that the vector at (K_1, K_2) is pointed toward the origin.

Given that there are 2 nullclines as specified in 1.1, the vector space would look like Figure 2.

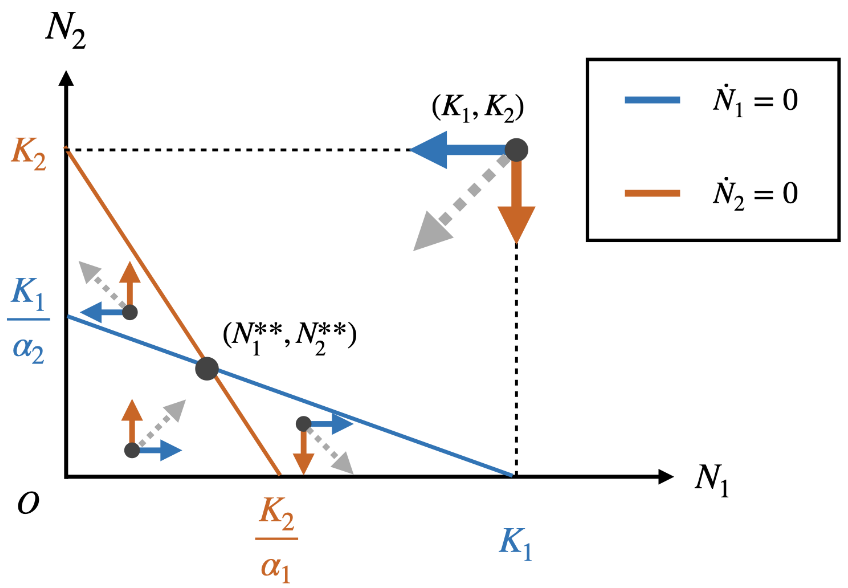

Figure 2: Vector space for the Lotka-Volterra model when Equation 4 is met. The blue and red arrows indicate the direction of the vector on the N_1 and N_2 axes per point, respectively. The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

1.3 Stability Analysis of the Equilibrium

Based on the previous section, we can draw the vectors around the equilibrium as Figure 3.

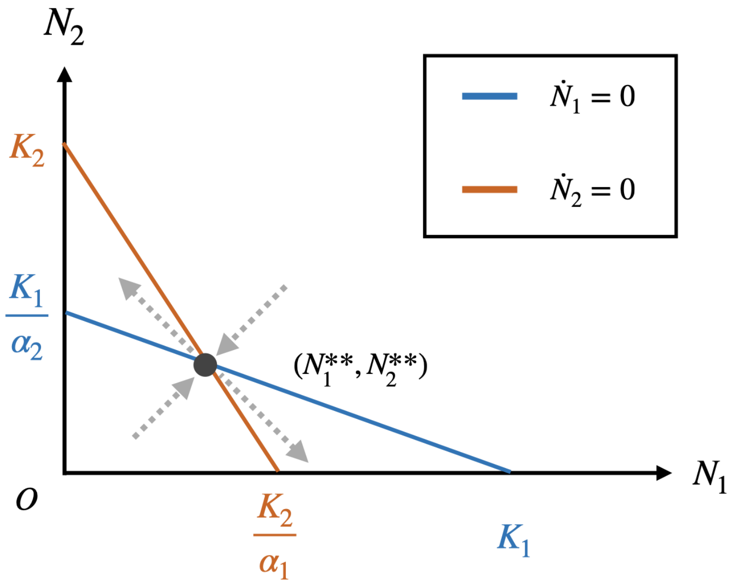

Figure 3: Vector space around the equilibrium in the Lotka-Volterra model when Equation 4 is met, derived from Figure 2 . The dotted gray arrows indicate the direction of the vector after combining the previous 2 arrows.

As we can see from the figure, the equilibrium is a saddle point, where it is stable in one axis and unstable in its orthogonal axis.

1.4 Stability Analysis of the Equilibrium

Question 2 presents an unstable state in which the population of organism 2 (N_2) dominates and reduces the population of organism 1(N_1) to local extinction. The proposed intervention involved reducing the populations of both N_1 and N_2 by a constant factor $\rho \in (0, 1)$ The question is, is this intervention effective and if so, is there a range of value of ρ in which the intervention is effective?

More formally the state of the system is represented by a vector

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

The system is an an region of the system that moves towards an equilibrium where $N_1 = 0$. The question is then: is there some scalar value ρ which results in the system moving towards an equilibirum where $N_1, N_2 > 0$

The result of the reduction reduction by rho is a vector **N'**

$$\mathbf{N'} = \begin{bmatrix} N_1' \\ N_2' \end{bmatrix}$$

Since **N** is multiplied by a scalar, **N'** will have the same direction as **N**, but a smaller magnitude.

From the initial wording of the question we know that $N1$ goes to 0. In order for an intervention to be successful the scaled **N'** must fall within a region such that the evolution will not cause the population of N_1 to go to 0. Fig 4 depicts this process.

This is not possible for the following reason. The derivative $\dot{N_1}$ constantly decreases going from positive to negative as it crossed the N_1 nullcline. An intervention can change the magnitude of **N** but not the direction so the resulting intervention will pull **N** the same distance or closer towards $N_1 = 0$. As a result no intervention and no value of rho can prevent the population of species 1 from collapsing.

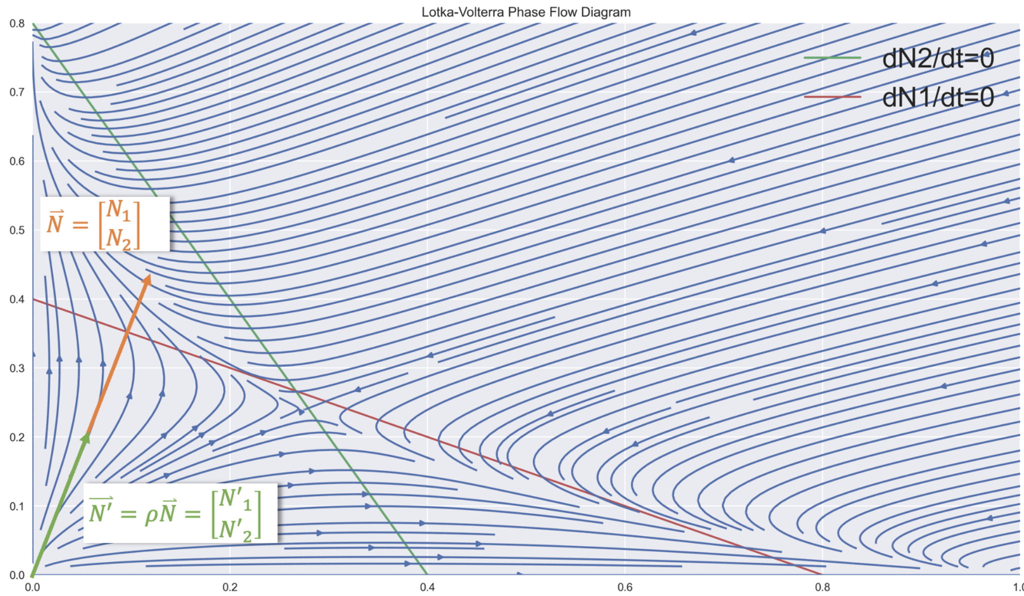

Figure 4: A phase flow diagram of the Lotka Volterra Model, the state of the system at any time is specified by a vector $\vec{N}$, the reduction by rho then corresponds to multiplying this vector by a scalar quantity ρ to a resulting vector $\vec{N}'$

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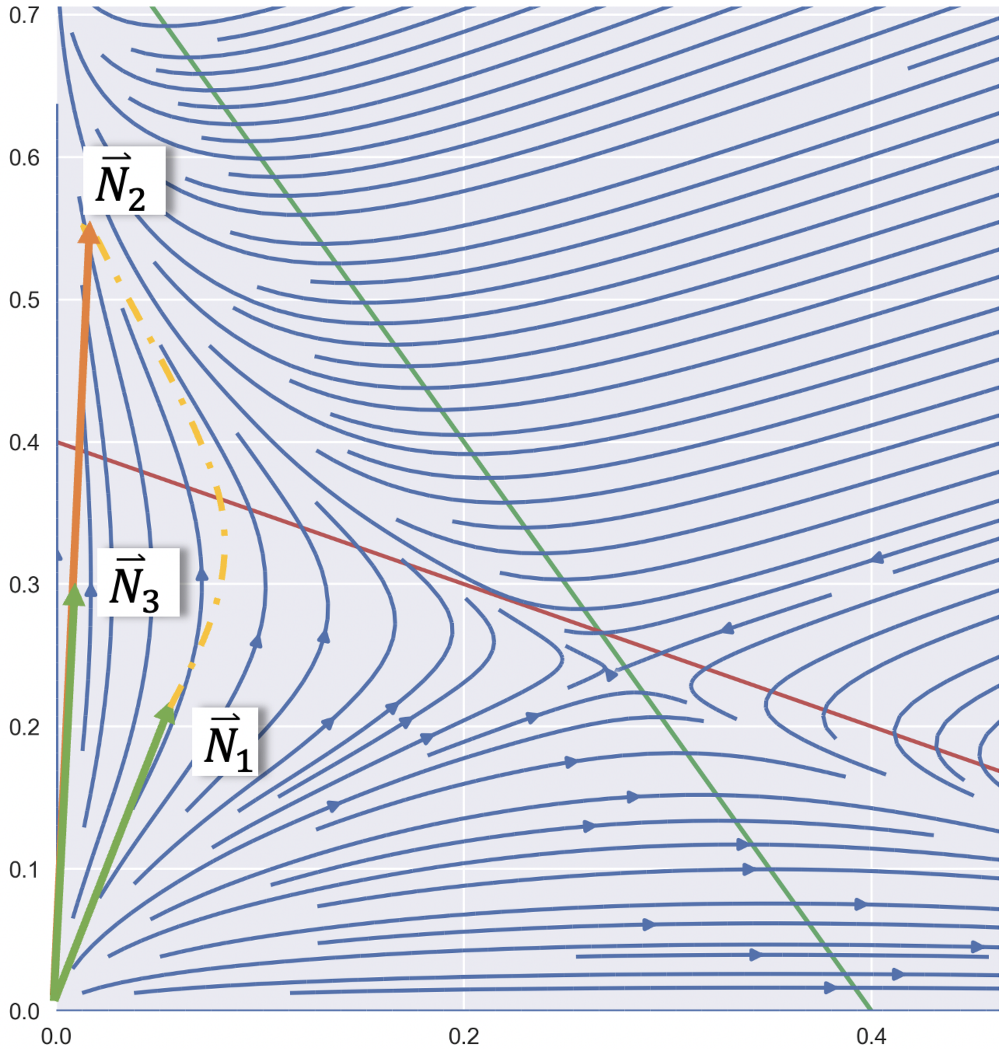

Figure 5: The result of an intervention by reducing the population by ρ. The state begins at $\vec{N}_1$ and evolves forward in time. The system evolves in time forward crossing the nullcline to into the region on the top right.

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2 Numerical Methods

2.1 Question 3

The numerical integrator with and functions for both Euler's and Heun's methods of integration is in the file `numerical_integration.jl`. The SIS model and the solutions are in a notebook `sis_model.ipynb`. For all practical purposes discrete time models are identical to numerical solutions to continuous equations. The only difference is the inclusion of the term ΔT in the latter which does not necessarily need to appear in the former.

2.2 Question 4

The SIS model has the following form

$$\begin{cases} \dot{S} = -\beta SI + \gamma I \\ \dot{I} = \beta SI - \gamma I \end{cases} \tag{7}$$

Where S is the number of susceptible individuals in the population, I is the number of infected individuals, β is the infection rate and γ is the recovery rate.

The equations are numerically solved with two algorithms. First, is Euler's method:

$$x_e(t + x) = x(t) + f(x(t))h \tag{8}$$

Secondly, the results are calculated using Heun's method. Heun's method is more accurate and has a global error $\mathcal{O}(h^2)$

$$x_h(t + h) = x(t) + \frac{h}{2}[\ f(x_e(t + x)) + f(x(t)] \tag{9}$$

The solutions to the SIS model using Heun's method are displayed in Fig. 6. Each run is run until a max time of $T = 10$ regardless of the number of steps the integrator takes. This allows us to compare the behavior of the algorithm at the same scale, regardless of the

For the smallest step size of 0.01, the two methods diverge only slightly across all three values of β. Increasing values of β increase the rate at which S drops. The difference in the algorithms can be most clearly seen in $step\_size : 0.5|\beta = 0.03$. Here, Heun's method remains smooth while visible artifacts from the integration are seen with Euler's method. The strangest result is $step\_size : 0.5|\beta = 0.06$. Here, Heun's method oscillates between two values while Euler's method appears to be chaotic. The behaviors here are reminiscent of the behavior of the logistic map, with Heun's method analogous to the logistic map oscillating between two values and Euler's method being analogous to chaotic behavior of the logistic map. Since both numerical integration of the SIS model and the logistic map are discrete time models, it is plausible there is a formal connection.

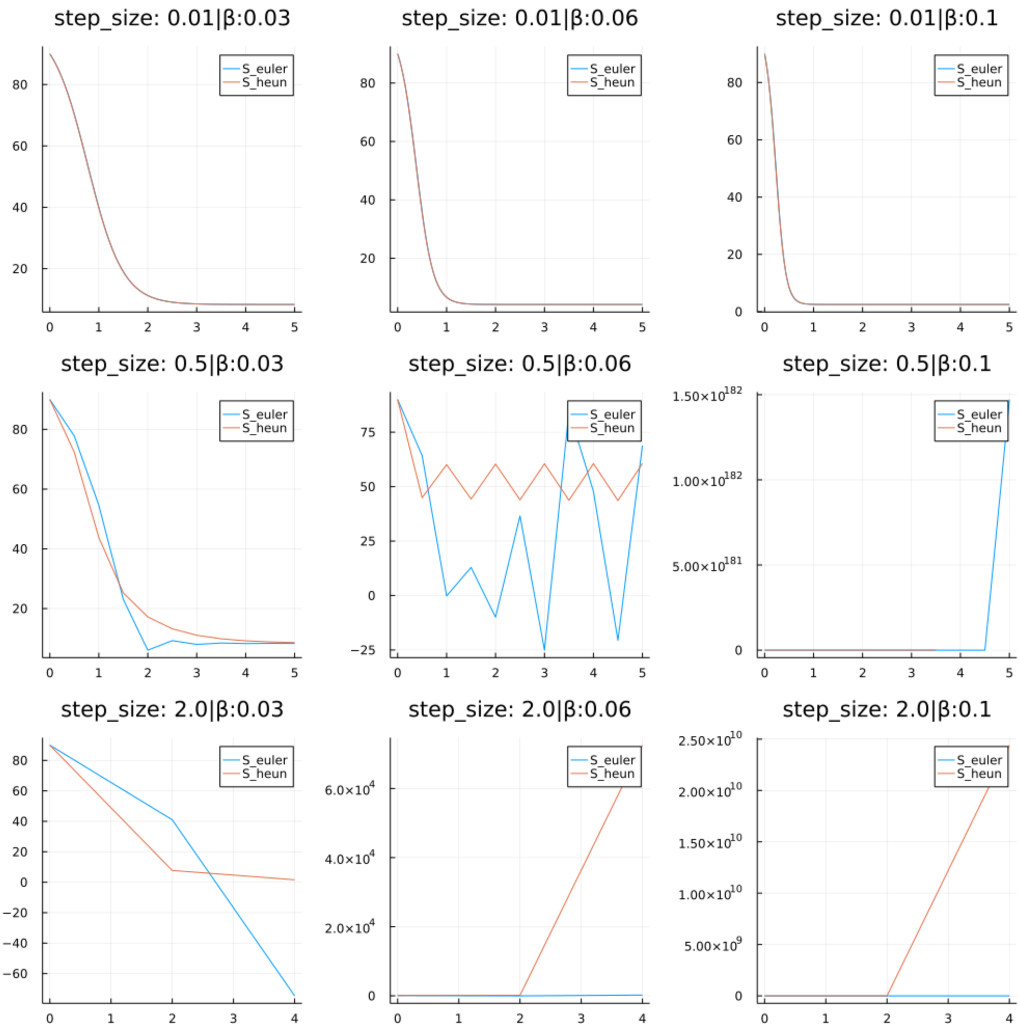

Figure 6: The results of numerical integration of the SIS model with Euler's method and Heun's method. Results for S are plotted for step sizes 0.01,0.5 and 2.0 and for values of β 0.03,0.06,0.1

2.3 Question 5

Show that the global error of Heun's method is of order h^2

We will begin with the definition of Heun's Method

$$x_h(t+h) = x(t) + \frac{h}{2}[\ f(x_e(t+x)) + f(x(t)] \tag{10}$$

Where $x_e(t+h)$ is Euler's method at point $t+h$

$$x_e(t+x) = x(t) + f(x(t))h \tag{11}$$

Now for small values of h($h << 1$) we can make the approximation that

$$f(x_e(t+h)) \approx (x(t+h)) \tag{12}$$

This is equivalent to assuming that for small step sizes, Euler's method predicts the correct next point in the function. We also assume the derivative is linear and can be distributed.

$$f((x(t+h))) = \frac{d}{dt} f(x(t+h)) = f(x(t)) + hf'(x(t)) \tag{13}$$

We substitute eq. 13 into eq. 10 to find the following:

$$x_h(t+h) = x(t) + \frac{h}{2}[\ f(x(t)) + hf'(x(t)) + f(x(t))] \tag{14}$$

Collecting terms and simplifying

$$x_h(t+h) = x(t) + hf(x(t)) + \frac{h^2}{2} f'(x(t)) \tag{15}$$

Which is exactly equivalent to the first three terms of the taylor expansion of $x(t)$

$$x(t+x) = x(t) + hf(x(t)) + \frac{h^2}{2} f'(x(t)) + \mathcal{O}(h^3) \tag{16}$$

So Heun's method is equivalent to the correct time evolution up to terms of order h^3. This means that heun's mathod has a local error $\mathcal{O}(h^3)$.

$GlobalError = LocalError \times TimeSteps$ But $TimeSteps = \frac{T}{h}$ Where T is the total time and h is the step size. As a result

$$GlobalError \propto \frac{\mathcal{O}(h^3)}{h} = \mathcal{O}(h^2) \tag{17}$$

QED