Rainbows, Quantum Billiards, and the Birth of Reflections

Stokes Phenomenon Exemplified

Will Hoffer

University of California, Riverside

math@willhoffer.com

October 22, 2020

Abstract

In this talk...

- What is the Stokes Phenomenon?

Abstract

In this talk...

- What is the Stokes Phenomenon?

- Example One: Rainbows/The Airy function

Abstract

In this talk...

- What is the Stokes Phenomenon?
- Example One: Rainbows/The Airy function
- Example Two: Birth of Reflections/Helmholtz Equation

Abstract

In this talk...

- What is the Stokes Phenomenon?
- Example One: Rainbows/The Airy function
- Example Two: Birth of Reflections/Helmholtz Equation
- Example Three: Quantum Billiards & Weyl Expansions

Preliminaries

Asymptotic Expansions, in the sense of Poincaré

The following are equivalent. As $z \to \infty$:

$$f(z) \sim \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$$

Asymptotic Expansions, in the sense of Poincaré

The following are equivalent. As $z \to \infty$:

$$f(z) \sim \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$$

$$f(z) = \sum_{n=1}^{N} a_n \frac{1}{z^n} + O\left(\frac{1}{z^{N+1}}\right)$$

Preliminaries

Asymptotic Expansions, in the sense of Poincaré

The following are equivalent. As $z \to \infty$:

$$f(z) \sim \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$$

$$f(z) = \sum_{n=1}^{N} a_n \frac{1}{z^n} + O\left(\frac{1}{z^{N+1}}\right)$$

$$f(z) = \sum_{n=1}^{N} a_n \frac{1}{z^n} + o\left(\frac{1}{z^N}\right)$$

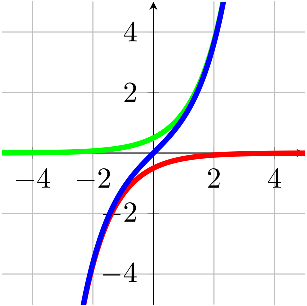
Preliminary Example: The Hyperbolic Sine Function

Let $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$ where $z \in \mathbb{C}$.

Preliminary Example: The Hyperbolic Sine Function

Let $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$ where $z \in \mathbb{C}$.

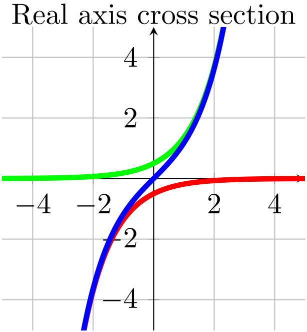
Real axis cross section



Preliminary Example: The Hyperbolic Sine Function

Let $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$ where $z \in \mathbb{C}$. Observe, as $z \to \infty$:

$$\sinh(z) \sim \begin{cases} \frac{1}{2}e^z & \Re(z) > 0 \\ -\frac{1}{2}e^{-z} & \Re(z) < 0 \end{cases}$$

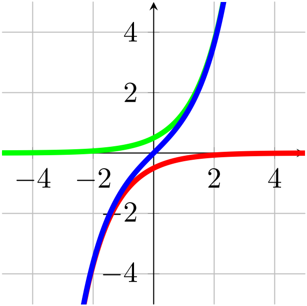


Preliminary Example: The Hyperbolic Sine Function

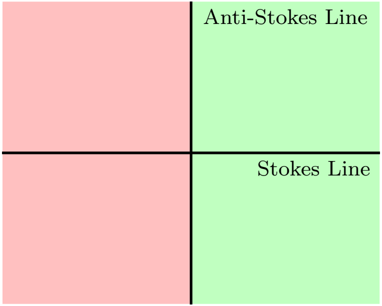
Let $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$ where $z \in \mathbb{C}$. Observe, as $z \to \infty$:

$$\sinh(z) \sim \begin{cases} \frac{1}{2}e^z & \Re(z) > 0 \\ -\frac{1}{2}e^{-z} & \Re(z) < 0 \end{cases}$$

Real axis cross section



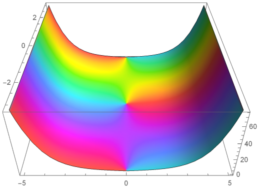
Stokes regions in $\mathbb{C}$



Sinh in the Complex Plane

Observe the change in behavior across the imaginary axis.

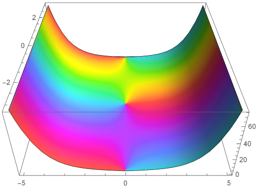
$$f(z) = \sinh(z)$$



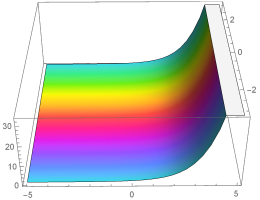
Sinh in the Complex Plane

Observe the change in behavior across the imaginary axis.

$$f(z) = \sinh(z)$$



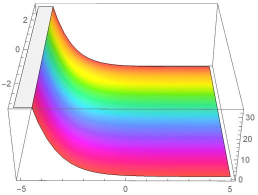
$$f(z) = \frac{1}{2}e^{z}$$



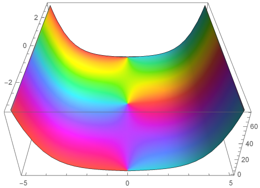
Sinh in the Complex Plane

Observe the change in behavior across the imaginary axis.

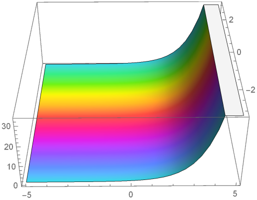
$$f(z) = -\frac{1}{2}e^{-z}$$



$$f(z) = \sinh(z)$$



$$f(z) = \frac{1}{2}e^{z}$$



What is the Stokes Phenomenon?

Broadly speaking, the Stokes phenomenon is that asymptotic expansions may change behavior in the complex plane.

What is the Stokes Phenomenon?

Broadly speaking, the Stokes phenomenon is that asymptotic expansions may change behavior in the complex plane.

More strictly, a Stokes phenomenon is a change arising from the "conception and subsequent birth" of terms that appear and become active with the changing phase.

What is the Stokes Phenomenon?

Broadly speaking, the Stokes phenomenon is that asymptotic expansions may change behavior in the complex plane.

More strictly, a Stokes phenomenon is a change arising from the "conception and subsequent birth" of terms that appear and become active with the changing phase.

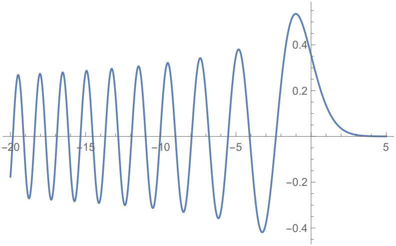
R. B. Dingle's Description

At a certain phase drawn in the complex plane as a "Stokes ray", an "associated function" appears, disappears or changes its numerical multiplier."

Supernumerary Rainbow

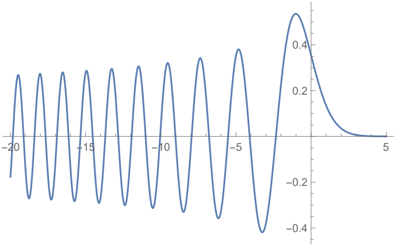
The Airy Model

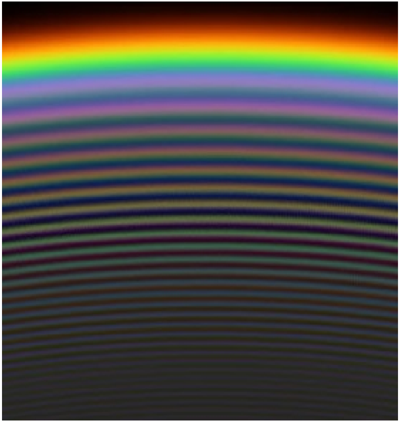
$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt$$



The Airy Model

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt$$





Stokes' Work

Here, U denotes a solution to the Airy equation, $\frac{d^2U}{dn^2} + \frac{n}{3}U = 0$.

$$U = An^{-\frac{1}{4}} \epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1 - \frac{1.5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}} + \frac{1.5.7.11}{1.2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2 - \frac{1.5.7.11.13.17}{1.2.3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3 + \ldots\right\} \quad . \quad . \quad . \quad (14)$$

Stokes' Work

Here, U denotes a solution to the Airy equation, $\frac{d^2U}{dn^2} + \frac{n}{3}U = 0$.

$$U = An^{-\frac{1}{4}}\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1 - \frac{1.5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}} + \frac{1.5.7.11}{1.2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2 - \frac{1.5.7.11.13.17}{1.2.3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3 + \ldots\right\}. \quad . \quad . \quad (14)$$

Secondly, suppose n negative, and equal to $-n'$. Then, writing $-n'$ for n in (14), and changing the arbitrary constant, and the sign of the radical, we get

$$U = Cn'^{-\frac{1}{4}}\epsilon^{-\frac{2}{3}\sqrt{\frac{n'^3}{3}}}\left\{1 - \frac{1.5}{1.16\,(3n^3)^{\frac{1}{2}}} + \frac{1.5.7.11}{1.2.16.3n^3} - \ldots\ldots\right\}. \quad . \quad . \quad . \quad (17)$$

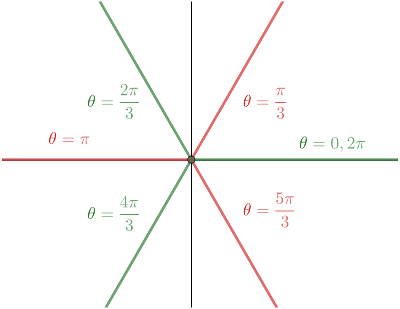
Stokes Behavior of the Airy Function

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_\gamma e^{i(zt-\frac{1}{3}t^3)}dt, \quad \mathrm{Im}(\gamma) = (-\infty e^{-\frac{2}{3}i\pi}, 0] \cup [0, \infty)$$

Stokes Behavior of the Airy Function

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_\gamma e^{i(zt-\frac{1}{3}t^3)}dt, \quad \mathrm{Im}(\gamma) = (-\infty e^{-\frac{2}{3}i\pi}, 0] \cup [0, \infty)$$

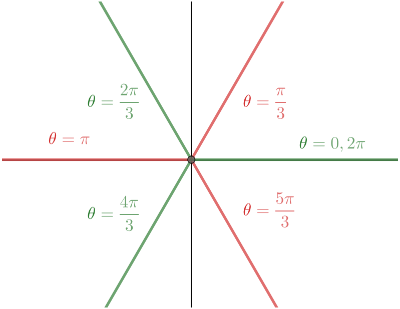
The Stokes behavior of the Airy function:

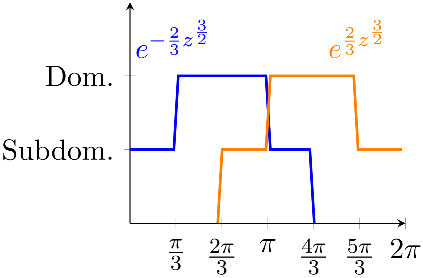


Stokes Behavior of the Airy Function

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_\gamma e^{i(zt-\frac{1}{3}t^3)}dt, \quad \mathrm{Im}(\gamma) = (-\infty e^{-\frac{2}{3}i\pi}, 0] \cup [0, \infty)$$

The Stokes behavior of the Airy function:





Airy Function Asymptotics

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

Airy Function Asymptotics

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

$$\begin{aligned}\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \Bigg(& \sin(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\zeta^{2n}} \\ & - \cos(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\zeta^{2n+1}} \Bigg) \qquad |\arg(z)| < \frac{2}{3}\pi\end{aligned}$$

Airy Function Asymptotics

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

$$\begin{aligned} \mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \Bigg(& \sin(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\zeta^{2n}} \\ & - \cos(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\zeta^{2n+1}} \Bigg) \qquad |\arg(z)| < \frac{2}{3}\pi \end{aligned}$$

Notation:

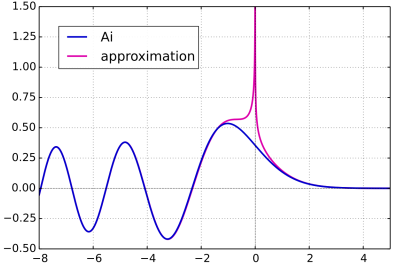
$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_n = \frac{\Gamma(3n + \frac{1}{2})}{54^n \Gamma(n + \frac{1}{2})} = \frac{(2n+1)(2n+3)\ldots(6n-1)}{216^n n!}$$

Asymptotics on the Real Line

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}} & x > 0 \\ \frac{1}{\sqrt{\pi}} (-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right) & x < 0 \end{cases}$$

Asymptotics on the Real Line

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} & x > 0 \\ \frac{1}{\sqrt{\pi}} (-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right) & x < 0 \end{cases}$$

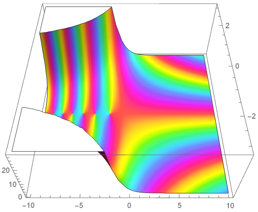


Airy Function in the Complex Plane

Complex plots of the approximations and where they agree.

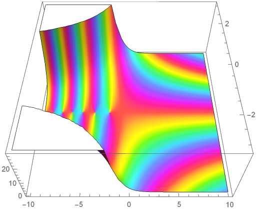
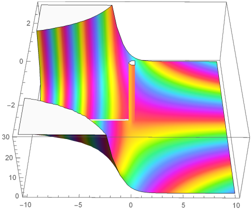
$$\mathrm{Ai}(z)$$

(Entire)



Airy Function in the Complex Plane

Complex plots of the approximations and where they agree.

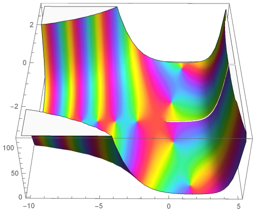
$\mathrm{Ai}(z)$	$\dfrac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$
(Entire)	$\lvert \arg(z) \rvert < \pi$
	

Airy Function in the Complex Plane

Complex plots of the approximations and where they agree.

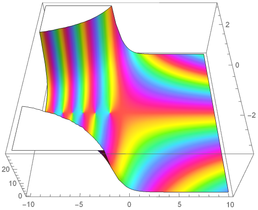
$$\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right)$$

$$|\arg(-z)| < \frac{2\pi}{3}$$



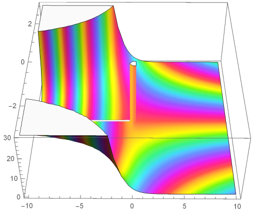
$$\mathrm{Ai}(z)$$

(Entire)



$$\frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$$

$$|\arg(z)| < \pi$$



Example Two: WKB Solution to a Helmholtz Equation

Consider this one-dimensional Helmholtz Equation:

$$\frac{d^2u}{dz^2}(z) = k^2 R^2(z) u(z)$$

Example Two: WKB Solution to a Helmholtz Equation

Consider this one-dimensional Helmholtz Equation:

$$\frac{d^2u}{dz^2}(z) = k^2 R^2(z) u(z)$$

Medium of Varying Refractive Index μ:

$$R(z) = i\mu(z), \quad \mu(x) > 0, \quad \mu(x) \to 1 \text{ as } x \to \infty$$

Example Two: WKB Solution to a Helmholtz Equation

Consider this one-dimensional Helmholtz Equation:

$$\frac{d^2u}{dz^2}(z) = k^2 R^2(z)u(z)$$

Medium of Varying Refractive Index μ:

$$R(z) = i\mu(z), \quad \mu(x) > 0, \quad \mu(x) \to 1 \text{ as } x \to \infty$$

Exponentially weak reflections arise:

M. Berry

We are able to answer this question [i.e. where and how does the reflected wave arise on the x-axis], because the birth of a reflection is simply the switching-on of a subdominant multiplier.

Stokes Lines for the Equation

Stokes lines arise from zeroes of R, say z_j, wherein:

$$\operatorname{Im} w(z) = 0, \quad w(z) := \int_{z_j}^{z} R(t)dt$$

Stokes Lines for the Equation

Stokes lines arise from zeroes of R, say z_j, wherein:

$$\operatorname{Im} w(z) = 0, \quad w(z) := \int_{z_j}^{z} R(t)dt$$

M. Berry

Stokes lines lie at the heart of the asymptotics of [this equation.] They are the locus of greatest disparity between the dominant and subdominant fundamental phase-integral approximate solutions attached to z_j:

$$u_{\pm} \approx \exp(\pm k w(z))/R^{\frac{1}{2}}(z).$$

Behavior of the Waves

Dominant u_+ corresponds to the incident wave,
Subdominant u_- to the reflected wave.

$$u_\pm \approx \exp(\pm kw(z))/R^{\frac{1}{2}}(z).$$

Behavior of the Waves

Dominant u_+ corresponds to the incident wave,
Subdominant u_- to the reflected wave.

$$u_\pm \approx \exp(\pm k w(z))/R^{\frac{1}{2}}(z).$$

WKB Approximate Solution:

$$u(z) \approx a_+(z)u_+(z) + a_-(z)u_-(z)$$

Across a Stokes line, the multiplier a_- jumps by ia_+.

Example Three: Quantum Billiards

Suppose a quantum particle moves freely in planar region $\mathscr{B}$ with reflection at the boundary $\partial\mathscr{B}$.

Example Three: Quantum Billiards

Suppose a quantum particle moves freely in planar region $\mathscr{B}$ with reflection at the boundary $\partial\mathscr{B}$.

Consider the Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta\phi_n(r) = E_n\phi_n(r), & r = (x, y) \in \mathscr{B} \\ \phi_n = 0 & r \in \partial\mathscr{B} \end{cases}$$

Example Three: Quantum Billiards

Suppose a quantum particle moves freely in planar region $\mathscr{B}$ with reflection at the boundary $\partial\mathscr{B}$.

Consider the Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta\phi_n(r) = E_n\phi_n(r), & r = (x, y) \in \mathscr{B} \\ \phi_n = 0 & r \in \partial\mathscr{B} \end{cases}$$

Regularized Resolvent:

$$g(s) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{E_n + s^2} - \frac{\mathscr{A}}{4\pi}\log\left(\frac{E_N}{s^2}\right)\right)$$

Example Three: Quantum Billiards

Weyl Expansion:

$$g(s) = \sum_{r=1}^{\infty} \frac{c_r}{s^r}$$

Example Three: Quantum Billiards

Weyl Expansion:

$$g(s) = \sum_{r=1}^{\infty} \frac{c_r}{s^r}$$

One may truncate to the least term, obtaining an exponentially small remainder.

$$g(s) = \sum_{r=1}^{r^*} \frac{c_r}{s^r} + R(s)$$

Example Three: Quantum Billiards

Weyl Expansion:

$$g(s) = \sum_{r=1}^{\infty} \frac{c_r}{s^r}$$

One may truncate to the least term, obtaining an exponentially small remainder.

$$g(s) = \sum_{r=1}^{r^*} \frac{c_r}{s^r} + R(s)$$

Across Stokes lines, the remainder changes behavior becoming oscillatory, then large.

$$e^{-s} \quad \longrightarrow \quad e^{-is} \quad \longrightarrow \quad e^{s}, \quad s \in \mathbb{R}^+$$

Next Time: Segue into Resurgence

Next Time: Segue into Resurgence

- Quantum Billiards Examples

Next Time: Segue into Resurgence

- Quantum Billiards Examples
- Spectral Resurgence

Next Time: Segue into Resurgence

- Quantum Billiards Examples
- Spectral Resurgence
- Deducing Stokes behavior from expansions

Next Time: Segue into Resurgence

- Quantum Billiards Examples
- Spectral Resurgence
- Deducing Stokes behavior from expansions
- And more!

Historical References

- G. B. Airy, “On the Intensity of Light in the neighbourhood of a Caustic,” Trans. Cambridge Phil. Soc. Vol. 6, Pt. 3, 397-402 (1838)
- G. G. Stokes, “On the numerical Calculation of a Class of Definite Integrals and Infinite Series,” Trans. Cambridge Phil. Soc. Vol. 9 Pt. 2, 166-187 (1850)
- G. G. Stokes, “On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series (A letter to the Editor),” Acta Math. Stockholm 26, 393-397 (1902)

References

- A. B. Olde Daalhuis, S. J. Chapman, J. R. King, J. R. Ockendon, and R. H. Tew, “Stokes phenomenon and matched asymptotic expansions,” SIAM J. Appl. Math, Vol. 55, No. 6, 1469-1483 (1995)
- M. V. Berry, “Uniform asymptotic smoothing of Stokes’ discontinuities,” Proc. R. Soc. Lond. A 422, 7-21 (1989)
- M. V. Berry, “Waves near Stokes lines,” Proc. R. Soc. Lond. A 427, 265-280 (1990)
- M. V. Berry and C. J. Howls, “High orders of the Weyl Expansion for quantum billiards: resurgence of periodic orbits, and the Stokes phenomenon,” Proc. R. Soc. Lond. A 447, 527-555 (1994)
- O. Costin, *Asymptotics and Borel Summability*, Chapman & Hall/CRC Press, 2009.
- Abramowitz, M. and Stegun, I. A., eds. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* ”Chapter 10,” Applied Mathematics Series, 55, (Ninth reprint, 1983). Washington D.C.; New York: United States Department of Commerce, National Bureau of Standards; Dover Publications. p. 446-447

Images

- Mika-Pekka Markkanen, Supernumerary Rainbows [Photograph], 23 May 2010, Wikimedia Commons. License: CC BY-SA 4.0.
- Les Cowley, Bow from 0.75mm diameter drops illuminated by a distant point source [Simulated Image, Cropped], n.d., Atmospheric Optics, atopics.co.uk. Copyright of Les Cowley, reproduced under Fair Use clause.
- Geek3, Airy Ai Asymptotic [SVG, converted to PNG], 7 Feb. 2015, Wikimedia Commons. License: CC BY 3.0.