From Rainbows to Resurgence: Asymptotics of the Airy Function

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Supernumerary, or Spurious, Rainbow

Historical Preface

- In 1838, G. B. Airy found a theoretical model for the illumination, involving (the square of) the integral:

$$W(m) = \int_0^\infty \cos\left(\frac{\pi}{2}(w^3 - mw)\right) dw$$

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- In 1841, W. H. Miller observed/measured 30 dark bands for the primary bow
- In 1850, Stokes employed another method, managing to calculate 50 zeroes!

Stokes' Method I

After some rescaling/changing variables, Stokes worked with:

$$u = e^{-i\frac{\pi}{6}} \int_0^\infty e^{-(z^3 - pz)} dx = U - iU'$$

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As noted by Stokes, though, this is "not convenient" when n becomes large.

Stokes' Method II

Stokes had the idea to write:[1]

$$U = e^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left(An^{\alpha} + Bn^{\beta} + Cn^{\gamma} + ...\right)$$

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From the equation, he deduced that:

$$U = An^{-\frac{1}{4}}e^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left(1 - \frac{1\cdot 5\cdot i}{16\sqrt{3n^2}} + \frac{1\cdot 5\cdot 7\cdot 11}{1\cdot 2}\left(\frac{i}{16\sqrt{3n^2}}\right)^2 + ...\right)$$

and similarly for negative values $n' = -n$.

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Just one small problem... the series diverges.

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Stokes' Method III

Here's what Stokes' original series looks like:

$$U = An^{-\frac{1}{4}}\,\epsilon^{\frac{2}{3}\sqrt{-\frac{n^3}{3}}}\left\{1 - \frac{1\,.\,5}{1}\,\frac{\sqrt{-1}}{16\sqrt{(3n^3)}} + \frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^2 - \frac{1\,.\,5\,.\,7\,.\,11\,.\,13\,.\,17}{1\,.\,2\,.\,3}\left(\frac{\sqrt{-1}}{16\sqrt{(3n^3)}}\right)^3 + \ldots\right\}. \quad . \quad . \qquad (14)$$

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Secondly, suppose n negative, and equal to $-n'$. Then, writing $-n'$ for n in (14), and changing the arbitrary constant, and the sign of the radical, we get

$$U = Cn'^{-\frac{1}{4}}\,\epsilon^{-\frac{2}{3}\sqrt{\frac{n'^3}{3}}}\left\{1 - \frac{1\,.\,5}{1\,.\,16\,(3n^3)^{\frac{1}{2}}} + \frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2\,.\,16\,.\,3n^3} - \ldots\ldots\right\}. \quad . \quad . \quad . \quad (17)$$

Stokes' Method IV

G. G. Stokes

When n or n' is at all large, the series [for U] are at first rapidly convergent, but they are ultimately in all cases hypergeometrically divergent. Notwithstanding this divergence, we may employ the series in numerical calculation, provided we do not take in the divergent terms.

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This is exactly what in modern days we would deem optimal (or least) truncation of an asymptotic expansion.

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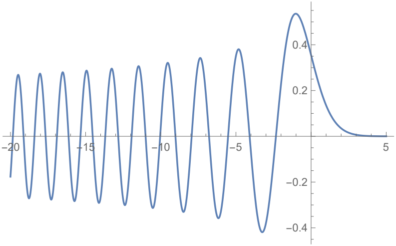
Today, we do know how to deduce the behavior for all n from the integral itself!

The Airy Function (Real Inputs)

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(xt + \frac{t^3}{3}\right)} dt$$



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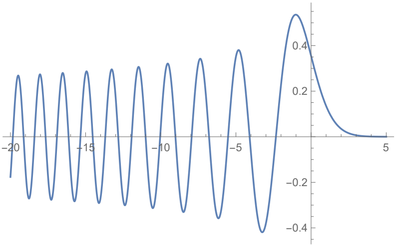

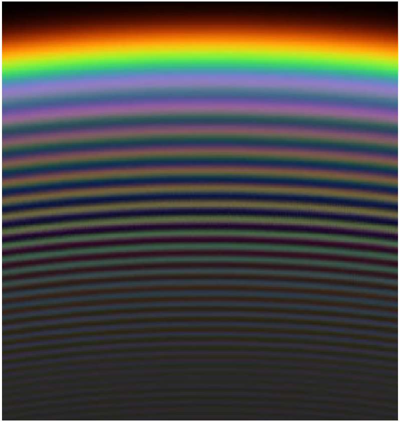

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The method of steepest descent for an integral of the form:

$$\int_C f(t) e^{xg(t)} dt, \quad x \gg 1$$

involves deforming the integration contour to pass along the direction of steepest descent (i.e. parallel to $-\nabla u$, where $g = u + iv$) to pass by saddle points (viz. near where the integral is maximal, or at least less rapid oscillations cause less cancellation.)

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(In particular, this enables one to use Laplace's method to estimate the integral.)

Airy Function, Deformed Contour

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Thus, we write:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_\gamma e^{\frac{t^3}{3}-zt}dt, \qquad \mathrm{Im}(\gamma) = (\infty e^{-\frac{\pi i}{3}}, 0] \cup [0, \infty e^{\frac{\pi i}{3}})$$

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$$= \frac{1}{2\pi i}\int_{\infty e^{-\frac{i\pi}{3}}}^{\infty e^{\frac{i\pi}{3}}} e^{\frac{t^3}{3}-zt}dt$$

Airy Function Asymptotics

Asymptotic Expansions:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\zeta^n} \qquad |\arg(z)| < \pi$$

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$$\begin{aligned}\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \Bigg(& \sin(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\zeta^{2n}} \\ & - \cos(\zeta + \frac{\pi}{4}) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\zeta^{2n+1}} \Bigg) \qquad |\arg(z)| < \frac{2}{3}\pi\end{aligned}$$

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Notation:

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_n = \frac{\Gamma(3n + \frac{1}{2})}{54^n \Gamma(n+1) \Gamma(n + \frac{1}{2})}$$

Asymptotics on the Real Line

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} & x > 0 \\ \frac{1}{\sqrt{\pi}} (-x)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right) & x < 0 \end{cases}$$

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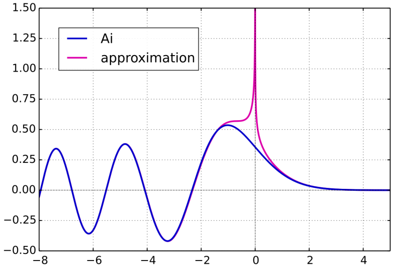
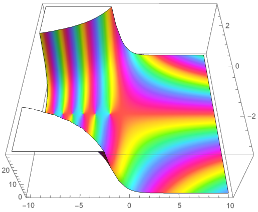


Asymptotics in the Complex Plane

Complex plots of the approximations and where they agree.

$\mathrm{Ai}(z)$

(Entire)



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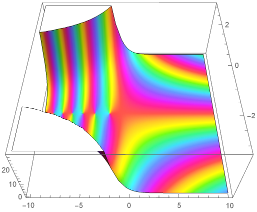
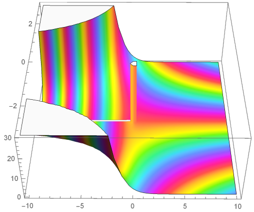


$$\frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{\frac{3}{2}}}$$

$$|\arg(z)| < \pi$$



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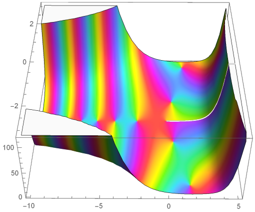
$$\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right)$$

$$|\arg(-z)| < \frac{2\pi}{3}$$



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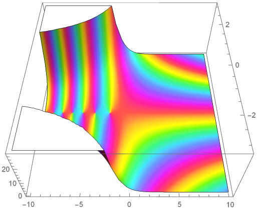

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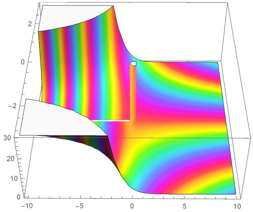

Resumming the Asymptotic Series

The Airy function is governed by the asymptotic expansion:

$$\varphi_{\mathrm{Ai}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} \frac{1}{z^n}$$

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(This is the same formula as before, but with a change of variables to make growth akin to $\Gamma(n)$ more manifest.)

$$a_n = \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\Gamma(n+1)} = \left(-\frac{2}{3}\right)^{-n} \frac{\Gamma(3n+\frac{1}{2})}{54^n\Gamma(n+1)\Gamma(n+\frac{1}{2})}$$

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More remarks:

- φ_{Ai} is factorially divergent (of Gevrey-class one.)

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More remarks:

- φ_{Ai} is factorially divergent (of Gevrey-class one.)
- $z = k^{\frac{3}{2}}$ is a natural change of variables for ensuing resummation.

Interlude: Borel Summation Schematic

Formal Power Series

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n z^{-(n+1)}$$

Analytic Function, $\mathrm{Re}(z) > 0$

$$f(z) = \int_0^\infty e^{-z\zeta} \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n d\zeta$$

Asymptotic Expansion as $z \to \infty$

Borel-Laplace Summation $\mathcal{LB}$

Laplace Transform

$$\mathcal{L}[\mathcal{B}[\tilde{\phi}]](z) = \int_0^\infty e^{-z\zeta} \mathcal{B}[\tilde{\phi}](\zeta) d\zeta$$

Borel Transform

$$z^{n+1} \mapsto \frac{\zeta^n}{n!}$$

Germ of Analytic Function

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Borel Summation Example

Formal Power Series

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} (-1)^n n! z^{-(n+1)}$$

Asymptotic Expansion as $z \to \infty$

Borel-Laplace Summation $\mathcal{LB}$

Analytic Function, $\mathrm{Re}(z) > 0$

$$\int_0^{\infty} \frac{e^{-z\zeta}}{1+\zeta} d\zeta = e^z E_1(z)$$

Laplace Transform

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Borel Transform

$$z^{n+1} \mapsto \frac{\zeta^n}{n!}$$

Germ of Analytic Function

$$\mathcal{B}[\tilde{\phi}](\zeta) = \sum_{n=0}^{\infty} (-\zeta)^n = \frac{1}{1+\zeta}$$

Airy Series: Borel Summation

- The minor of φ_{Ai} is its (formal) Borel transform, forgetting the constant term:

$$\tilde{\varphi}_{\mathrm{Ai}} := \mathcal{B}[\varphi_{\mathrm{Ai}}] = \sum_{n=1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$$

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- $\tilde{\varphi}_{\mathrm{Ai}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \left\{0, -\frac{4}{3}\right\}$
- For any direction θ not along the negative real axis, the following converges for $\mathrm{Re}(ze^{i\theta}) > 0$:

$$S_\theta \varphi_{\mathrm{Ai}}(z) := a_0 + \mathcal{L}_\theta \mathcal{B}[\varphi_{\mathrm{Ai}}](z) = a_0 + \int_0^{\infty e^{i\theta}} \tilde{\varphi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} d\zeta$$

A Borel Resummed Expansion

Where before:

$$\mathrm{Ai}(k) \sim \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

We now have:

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This resummation is valid for $|\arg(k)| < \frac{\pi}{3}$, $|k| > 0$.

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$$\mathrm{Ai}(k) \sim \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

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$$\mathrm{Ai}(k) = \frac{1}{2\sqrt{\pi}} k^{-\frac{1}{4}} e^{-\frac{2}{3}k^{\frac{3}{2}}} S_0 \varphi_{\mathrm{Ai}}(k^{\frac{3}{2}})$$

This resummation is valid for $|\arg(k)| < \frac{\pi}{3}$, $|k| > 0$.

One can rotate the direction of summation for new regions of validity.

Contours near the Singularity at $-\frac{4}{3}$

Rotation of summation is fine up until one encounters the singularity on the negative real line.

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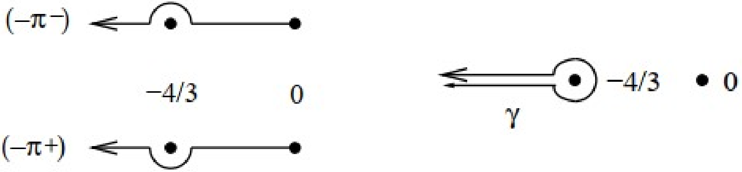

FIGURE 2. Right and left Borel-resummation.

One can compares right and left-resummations, since

$$S_{-\pi^-}\varphi_{Ai}(z) = S_{-\pi^+}\varphi_{Ai}(z) + \int_\gamma \widetilde{\varphi_{Ai}}(\zeta) e^{-z\zeta}\, d\zeta \tag{4}$$

“Alien” Calculus & Behavior across the Singularity

The Hankel contour γ can be expressed using the so-called alien derivative:

$$\int_\gamma \tilde{\varphi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} d\zeta = e^{+\frac{4}{3}z} S_{-\pi} \left(\Delta^{z}_{-\frac{4}{3}} \varphi_{\mathrm{Ai}} \right)(z)$$

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φ_{Bi} is also Gevrey-1 and its minor $\tilde{\varphi}_{\mathrm{Bi}}$ extends analytically to the universal cover of $\mathbb{C} \setminus \{0, +\frac{4}{3}\}$.

Airy Function on the Negative Real Line

Deducing the behavior Ai for negative real inputs.

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Once can rewrite the LHS as the resummed version of the second expansion we saw previously.

Zeroes “Resurge” from the Original Expansion

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- This procedure follows the original calculations of Stokes, but now by manipulating the full series itself. The mathematical justification for the resummation comes from resurgent analysis.
- In particular, the behavior on the negative real line is manifestly contained in the expansion on the positive real line— an example of resurgence.

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