Problem 1

find the coefficient x^{105} in Maclaurin Series $f(x) = (1+x^8)\sin x$, then determine $f^{(105)}(0)$

Maclaurin for $\sin x$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}(1+x^8)$

multiply numerator by $(1+x^8$ $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}+x^{2n+9}}{(2n+1)!}\right)$

Coefficient will equal x^{2n+1} so $x^{2n+1} = x^{105}$
or $2n+1 = 105$, $n = 52$
same for $2n+9$
$2n+9 = 105$, $n = 48$

So in total if we set the numerator with the n's found above

$$(-1)^{52}\frac{x^{105}}{105!} + (-1)^{48}\frac{x^{105}}{97!} \quad \text{or} \quad \frac{x^{105}}{105!} + \frac{x^{105}}{97!}$$

$$x^{105}\left(\frac{1}{105!}+\frac{1}{97!}\right) \quad \text{and} \quad f^{(105)}(0) = \left(\frac{1}{105!}+\frac{1}{97!}\right)$$

Problem 2

for $f(x) = x^{-1/2}$ find the taylor polynomial centered at $a=4$ of degree 2 then approximate $\frac{1}{\sqrt{4.5}}$, then bound the error

first find taylor polynomial of degree 2

$$T_2 = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2}$$

$f'(x) = -\frac{1}{2x^{3/2}}$ $\quad f''(x) = \frac{3}{4x^{5/2}}$

$f(4) = 1/2$ $\quad f'(4) = -\frac{1}{2(4)^{3/2}} = -1/16$

$f''(x) = \frac{3}{4(4)^{5/2}} = 3/128$

$$T_2 = 1/2 + -1/16(x-4) + 3/256\,(x-4)^2$$

$(x^2-8x+16)$

$$T_2 = 1/2 - 1/16x + 1/4 + 3/256x^2 - 3/32x + 3/16$$

$$T_2 = 3/256x^2 - 5/32x + 15/16$$

to approximate $\frac{1}{\sqrt{4.5}}$ for $\frac{1}{\sqrt{x}}$ substitute 4.5 for x in T_2

$$T_2 = 1/2 + -1/16\,(4.5-4) + 3/256\,(4.5-4)^2$$

$T_2 = 1/2 - 1/32 + 3/256 \cdot 1/4$ $\quad T_2 = \frac{512}{1024} - \frac{32}{1024} + \frac{3}{1024}$

$\boxed{T_2 = 483/1024}$

Taylor Remainder Theorem $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

in our case $x = 4.5$ and $a = 4$

Problem 2 cont'd

$R_2(4.5) \leq \frac{f'''(c)}{3!}(4.5-4)^3$

Find the upper bound with maximum value of f''' over interval of $(4, 4.5)$ is $f(4.5) = -.009$

$f'''(c) = -\frac{15}{8x^{7/2}}$

$R_2(4.5) \leq \frac{-.009}{6}(.5)^3$

$R_2(4.5) \leq -.0002$

Problem 3

find a polynomial $p(x)$ so that $p(x)$ approximates $f(x) = \sin x$ for $x \in [-1, 1]$ with error $< .01$

Sin in radians

Use taylor's theorem with remainder

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Approximate using 0

$$P(x) = \sin(0) + x\cos(0) - \frac{x^2\sin(0)}{2!} - \frac{x^3\cos(0)}{3!}$$

$$P(x) = x - \tfrac{1}{6} \cdot x^3$$

$$R_n(x) = |f(x) - P(x)|$$

bound using 1 as the largest c

$\sin(1) - (1) - \tfrac{1}{6}(1)^3 = .008138$ which is $\leq .01$