

Part 3

a) Case 1: BTC cheaper in the pool

$$x_{t+1} = x_t + \Delta x, \quad y_{t+1} = y_t + (1+\gamma) \Delta y$$

$$\frac{P_{t+1}}{1-\gamma} = S_{t+1} \rightarrow \frac{\left(\frac{y_{t+1}}{x_{t+1}}\right)}{1-\gamma} = S_{t+1} \rightarrow y_{t+1} = S_{t+1} (1-\gamma) x_{t+1}$$

$$x_{t+1} \cdot y_{t+1} = K \rightarrow x_{t+1} (S_{t+1} (1-\gamma) x_{t+1}) = K \rightarrow S_{t+1} (1-\gamma) x_{t+1}^2 = K$$

~~$x_{t+1} = \sqrt{\frac{K}{S_{t+1} (1-\gamma)}}$~~

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$$y_{t+1} = S_{t+1} (1-\gamma) x_{t+1} = S_{t+1} (1-\gamma) \sqrt{\frac{K}{S_{t+1} (1-\gamma)}}$$

$$y_{t+1} = \sqrt{K \cdot S_{t+1} (1-\gamma)}, \quad \Delta x = x_t - x_{t+1} = x_t - \sqrt{\frac{K}{S_{t+1} (1-\gamma)}}$$

$$(1-\gamma) \Delta y = y_{t+1} - y_t \quad \text{so,} \quad \Delta y = \frac{y_{t+1} - y_t}{1-\gamma} = \frac{\sqrt{K S_{t+1} (1-\gamma)} - y_t}{1-\gamma}$$

So Case 1 final:

$$x_{t+1} = \sqrt{\frac{K}{S_{t+1} (1-\gamma)}}, \quad y_{t+1} = \sqrt{K S_{t+1} (1-\gamma)}$$

$$\Delta x = x_t - \sqrt{\frac{K}{S_{t+1} (1-\gamma)}}, \quad \Delta y = \frac{\sqrt{K S_{t+1} (1-\gamma)} - y_t}{1-\gamma}$$

$$\text{Fee: } R_1(S_{t+1}) = \frac{\gamma}{1-\gamma} \left(\sqrt{K S_{t+1} (1-\gamma)} - y_t \right)$$

Case 2: BTC cheaper Outside

$$x_{t+1} = x_t + (1-\gamma)\Delta x, \quad y_{t+1} = y_t - \Delta y$$

$$P_{t+1}(1-\gamma) = S_{t+1} \rightarrow \left(\frac{y_{t+1}}{x_{t+1}}\right)(1-\gamma) = S_{t+1} \rightarrow y_{t+1} = \frac{S_{t+1}}{1-\gamma} \cdot x_{t+1}$$

$$x_{t+1} \left(\frac{S_{t+1}}{1-\gamma} \cdot x_{t+1} \right) = k \rightarrow x_{t+1}^2 = k \cdot \frac{1-\gamma}{S_{t+1}}, \quad x_{t+1} = \sqrt{\frac{k(1-\gamma)}{S_{t+1}}}$$

$$y_{t+1} = \frac{S_{t+1}}{1-\gamma} \cdot \sqrt{\frac{k(1-\gamma)}{S_{t+1}}} = \sqrt{k} \cdot \frac{S_{t+1}}{1-\gamma} \cdot \frac{\sqrt{1-\gamma}}{\sqrt{S_{t+1}}} = \sqrt{\frac{k S_{t+1}}{1-\gamma}}$$

$$\Delta y = y_t - y_{t+1} = y_t - \sqrt{\frac{k S_{t+1}}{1-\gamma}}, \quad \Delta x = \frac{x_{t+1} - x_t}{1-\gamma} = \frac{\sqrt{\frac{k(1-\gamma)}{S_{t+1}}} - x_t}{1-\gamma}$$

so Case 2 final:

$$x_{t+1} = \sqrt{\frac{k(1-\gamma)}{S_{t+1}}}, \quad y_{t+1} = \sqrt{\frac{k S_{t+1}}{1-\gamma}}$$

$$\Delta y = y_t - \sqrt{\frac{k S_{t+1}}{1-\gamma}}, \quad \Delta x = \frac{\sqrt{\frac{k(1-\gamma)}{S_{t+1}}} - x_t}{1-\gamma}$$

$$\text{fee: } R_2(S_{t+1}) = \frac{\gamma}{1-\gamma} \left(\sqrt{k S_{t+1}(1-\gamma)} - x_t S_{t+1} \right)$$