

Homework 1

FE621 Computational Finance

due 23:59, Sunday February 15, 2026

For all the problems in this assignment you need to design and use a computer program, output results and present the results in nicely formatted tables and figures. The computer program may be written in any programming language you want. Please write comments to all the parts of your code. They are a requirement and they will be graded.

You need to submit a PDF containing the report. Please use a word processor such as Microsoft Word, L^AT_EX, or whatever Apple uses to create your report. You will be judged by the quality of the writing and the interpretation of the results.

Part 1. (20 points) Data gathering component

1. Write a function (program) to connect to sources and download data from one of the following sources:
 - (a) GOOGLE finance <http://www.google.com/finance>
 - (b) Yahoo Finance <http://finance.yahoo.com>
 - (c) Bloomberg

Notes. For extra credit you can turn in code to download data from the other two sources. Please note that the program needs to download both option data and equity data. For this problem (and only for this problem) you may use any built in function or toolbox that will facilitate your work. The data will have to be clean (no duplicated values, only one exchange, every column labeled properly, in other words, consolidated).

Note on Bloomberg data. For the Bloomberg source, access to one of Bloomberg terminals in the lab is required. If you use Bloomberg data, you may use the API to download data in Excel automatically and organize the data. However, this should be accomplished with an automatic script. If you use R to interface with the Bloomberg data, a useful package for that is Rblpapi, but there are other packages. For the online students, who do not have access to Bloomberg terminals, you may read about the package quantmod in R to download yahoo and google data automatically.

Bonus (5 points) Create a program that is capable of downloading multiple assets, combine them with the associated time column, and save the data into a csv or excel file.

2. With the function created in problem 1, download data on options and equity for the following symbols:

- TSLA
- SPY
- ^VIX

for two consecutive days (does not matter when, but no later than February 14th) during the trading day (9:30am to 4:00pm). Please record the asset values (both TSLA and SPY) at the time when downloading is done. Please do the same with the ^VIX. For the options please note that the traditional options are maturing third Friday of each month. However note there are a lot more options available to download. Please give your quant student explanation why there are so many maturities available. For the assignment please download option chain data for the next three months (options expiring on the third Friday of the month).

We shall refer to the data sets gathered in the two consecutive days as DATA1 (for the first day) and DATA2 (for the second day) respectively throughout this assignment and the following ones.

3. Write a paragraph describing the symbols you are downloading data for. Explain what is SPY and its purpose. (Hint: look up the definition of an ETF). Explain what is ^VIX and its purpose. Understand the

options' symbols. Understand when each option expires. Write this information and turn it in.

4. The following items will also need to be recorded:

- The underlying equity, ETF, or index price at the exact moment when the rest of the data is downloaded.
- The short-term interest rate which may be obtained here:
<http://www.federalreserve.gov/releases/H15/Current/>. There are a lot of rates posted on the site - they are all yearly, *they are in percents and need to be converted to numbers*. There is no theoretical recommendation on which to use, I used to use 3-months Treasury bills which are not available anymore. Since then I have been using the “Federal funds (effective)” rate but you can go ahead and try others. You should remember to be consistent in your choice. Also, make sure that the interest rate that you use is for the same day when the data you use for the implied volatility was gathered and note that the data is typically quoted in percents (you will need numbers). The same site has a link to past (historical) interest rates.
- Time to Maturity.

Part 2. (50 points) Analysis of the data.

5. Using your choice of computer programming language implement the Black-Scholes formulas as a function of current stock price S_0 , volatility σ , time to expiration $T - t$ (in years), strike price K and short-term interest rate r (annual). Please note that no toolbox function is allowed but you may call the normal CDF function (e.g., `pnorm` in R or `scipy.stats.norm.cdf` in Python).
6. Implement the Bisection method to find the root of arbitrary functions. Apply this method to calculate the implied volatility on the first day you downloaded (DATA1). For this purpose use as the option value the average of bid and ask price if they both exist (and if their corresponding volume is nonzero). Also use a tolerance level of 10^{-6} . Report the implied volatility at the money (for the option with strike price closest to the traded stock price). You need to do it for both the stock and the

ETF data you have (you do not need to do this for $^{\wedge}\text{VIX}$). Then average all the implied volatilities for the options between in-the-money and out-of-the-money.

Note. There is no clearly defined boundary between options at-the-money and out-of-the-money or in-the-money options. If we define moneyness as the ratio between S_0 the stock price today and K the strike price of the option some people use values of moneyness between 0.95 and 1.05 to define the options at the money. Yet, other authors use between 0.9 and 1.1. Use these guidelines if you wish to determine which options' implied volatilities should be averaged.

7. Implement the Newton method/Secant method or Muller method to find the root of arbitrary functions. You will need to discover the formula for the option's derivative with respect to the volatility σ . Apply these methods to the same options as in the previous problem. Compare the time it takes to get the root with the same level of accuracy.
8. Present a table reporting the implied volatility values obtained for every maturity, option type and stock. Also compile the average volatilities as described in the previous point. Comment on the observed difference in values obtained for TSLA and SPY. Compare with the current value of the $^{\wedge}\text{VIX}$. Comment on what happens when the maturity increases. Comment on what happen when the options become in the money respectively out of the money.
9. For each option in your table calculate the price of the different type (Call/Put) using the Put-Call parity (please see Section 4 from [2]). Compare the resulting values with the BID/ASK values for the corresponding option if they exist.
10. Consider the implied volatility values obtained in the previous parts. Create a 2 dimensional plot of implied volatilities versus strike K for the closest to maturity options. What do you observe? Plot all implied volatilities for the three different maturities on the same plot, where you use a different color for each maturity. In total there should be 3 sets of points plotted with different color. (**Bonus** 5 Points) Create a 3D plot of the same implied vols as a function of both maturity and strike, i.e.: $\sigma(\tau_i, K_j)$ where $i = 1, 2, 3$, and $j = 1, 2, \dots, 20$.

11. (Greeks) Calculate the derivatives of the call option price with respect to S (Delta), and σ (Vega) and the second derivative with respect to S (Gamma). First use the Black Scholes formula then approximate these derivatives using an approximation of the partial derivatives. Compare the numbers obtained by the two methods. Output a table containing all derivatives thus calculated.
12. Next we will use the second dataset DATA2. For each strike price in the data use the Stock price for the same day, the implied volatility you calculated from DATA1 and the current short-term interest rate (corresponding to the day on which DATA2 was gathered). Use the Black-Scholes formula, to calculate the option price.

Part 3. (30 points) Numerical Integration of real-valued functions. AMM Arbitrage Fee Revenue

AMMs are decentralized exchanges that quote prices using pool reserves rather than an order book. Compared to Traditional Finance order books, a key advantage is **continuous liquidity from the pool without needing a matching counterparty**. Liquidity Providers (LPs) earn revenue mainly from **swap fees**, so selecting a good fee rate γ under different volatility levels matters [5].

For simplicity we consider the following Constant Product Market Maker (CPMM) for a **BTC/USDC** pool. These are the pool elements:

- x_t : BTC reserves at time t
- y_t : USDC reserves at time t
- pool mid price is calculated as $P_t = \frac{y_t}{x_t}$ (USDC per BTC)
- external market price S_t (USDC per BTC)
- fee rate $\gamma \in (0, 1)$

The pool satisfies the constant-product rule, that is at every moment in time the product of quantities must stay constant.

$$x_{t+1}y_{t+1} = x_ty_t = k.$$

Next we describe the trade mechanics. Suppose someone wants to trade Δx with the pool. That results in a Δy obtained from the pool. This is done so that the updated reserves continue to satisfy the constant product rule. Specifically, using the γ rate that is assesed we must have:

$$(x_t + (1 - \gamma)\Delta x)(y_t - \Delta y) = k$$

This will satisfy the constraint $x_{t+1}y_{t+1} = x_t y_t = k$. Note that the price ratio of the assets is changing after this trade: $P_{t+1} = \frac{y_t - \Delta y}{x_t + (1 - \gamma)\Delta x}$.

Outside this mechanics as long as the quantity of assets in the pool stay constant the price does not change. We are considering a situation where the outside price S_t moves. If S_{t+1} leaves the no-arbitrage band, $\left[P_t(1 - \gamma), \frac{P_t}{1 - \gamma} \right]$ then an arbitrage opportunity arises. We assume that arbitragers act optimally and instantly. As a result the following happens.

Case 1: $S_{t+1} > P_t \frac{1}{1 - \gamma}$ (**BTC cheaper in the pool**)

Arbitragers will swap USDC \rightarrow BTC until the resulting band after the trades moves so that it contains the outside price S_{t+1} . Updates:

$$x_{t+1} = x_t - \Delta x, \quad y_{t+1} = y_t + (1 - \gamma)\Delta y, \quad \Delta x, \Delta y > 0,$$

with boundary condition

$$P_{t+1} \frac{1}{1 - \gamma} = \frac{y_{t+1}}{x_{t+1}} \frac{1}{1 - \gamma} = S_{t+1},$$

The corresponding fee revenue (USDC) is coming from the input asset y . It is thus equal to $\gamma \Delta y$.

Case 2: $S_{t+1} < P_t(1 - \gamma)$ (**BTC cheaper outside**)

Arbitragers will swap BTC \rightarrow USDC and the updated reserves are:

$$x_{t+1} = x_t + (1 - \gamma)\Delta x, \quad y_{t+1} = y_t - \Delta y, \quad \Delta x, \Delta y > 0,$$

with boundary condition

$$P_{t+1}(1 - \gamma) = \frac{y_{t+1}}{x_{t+1}}(1 - \gamma) = S_{t+1},$$

and the fee revenue is $\gamma \Delta x$. After converting BTC fee to USDC using S_{t+1} is calculated as $\gamma \Delta x S_{t+1}$.

Questions:

(a) (10 pts) Derive the swap amounts

Under both Case 1 and Case 2, use the corresponding boundary condition, and the reserve updates above, to derive the swap size:

$$\Delta x(S_{t+1}; x_t, y_t, \gamma, k), \quad \Delta y(S_{t+1}; x_t, y_t, \gamma, k)$$

such that the one-step fee revenue is:

$$R(S_{t+1}) = \mathbf{1}_{\{S_{t+1} > \frac{P_t}{1-\gamma}\}} \gamma \Delta y + \mathbf{1}_{\{S_{t+1} < P_t(1-\gamma)\}} \gamma \Delta x S_{t+1}.$$

where we used the notation $\mathbf{1}_{\{\cdot\}}$ for the indicator function.

(b) (10 pts) Expected fee revenue

Assume the initial BTC/USDC pool reserves are $x_t = y_t = 1000$, so $P_t = \frac{y_t}{x_t} = 1$. Let $S_t = 1$ and $\Delta t = \frac{1}{365}$. Assume the external price follows one-step GBM:

$$S_{t+1} = S_t \exp\left(-\frac{1}{2}\sigma^2 \Delta t + \sigma \sqrt{\Delta t} Z\right), \quad Z \sim N(0, 1),$$

so S_{t+1} is lognormal with density $f_{S_{t+1}}(s)$.

Using the setup above and the solution from (a), the expected one-step fee revenue is:

$$\begin{aligned} \mathbb{E}[R(S_{t+1})] &= \int_{P_t/(1-\gamma)}^{\infty} \gamma \Delta y(S_{t+1}; x_t, y_t, \gamma, k) f_{S_{t+1}}(s) ds \\ &\quad + \int_0^{P_t(1-\gamma)} \gamma \Delta x(S_{t+1}; x_t, y_t, \gamma, k) S_{t+1} f_{S_{t+1}}(s) ds. \end{aligned}$$

Task: Numerically approximate $\mathbb{E}[R(S_{t+1})]$ using trapezoidal rule learned in class.

(c) (10 pts) Optimal Fee Rate under different volatilities

Use $\sigma \in \{0.2, 0.6, 1.0\}$ and $\gamma \in \{0.001, 0.003, 0.01\}$. For each σ , compute $\mathbb{E}[R]$ for the three fee rates and select:

$$\gamma^*(\sigma) = \arg \max_{\gamma} \mathbb{E}[R(S_{t+1})].$$

Construct a table reporting the $\mathbb{E}[R]$ values and the best $\gamma^*(\sigma)$ among the 3 options.

Then for each $\sigma \in [0.1, 1.0]$ on a grid (e.g. 0.01 step), compute the optimal $\gamma^*(\sigma)$. Finally, produce a scatter plot or line plot for σ vs. $\gamma^*(\sigma)$, and comment briefly on the pattern you observe.

Part 4. (Bonus 10 points) Consider the following functions:

$$\begin{aligned}f_1(x, y) &= xy \\f_2(x, y) &= e^{x+y}\end{aligned}$$

1. Analytically solve the following integral for both f_1 and f_2

$$\int_0^1 \int_0^3 f_i(x, y) dy dx$$

2. Calculate the numerical integral of the f_1 and f_2 by applying the trapezoidal rule for double integral as discussed in [3], Page 118-119. Please choose four different pairs of values for $(\Delta x, \Delta y)$. Use these values to approximate the double integral for both f_1 and f_2 . Calculate the error of the approximation for each choice of values. Comment on the results.

Hint 1 First, discretize the x domain into $n+1$ points Δx apart, where $x_0 = 0$ and $x_{n+1} = 1$, and the y domain into $m + 1$ points Δy apart, where $y_0 = 0$ and $y_{m+1} = 3$. The composite trapezoidal rule approximates the integral as

$$\begin{aligned}\int_0^1 \int_0^3 f(x, y) dy dx &\approx \sum_{i=0}^n \sum_{j=0}^m \frac{\Delta x \Delta y}{16} [f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) \\&+ f(x_{i+1}, y_{j+1}) + 2 \left(f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) + f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right) + f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) \right. \\&\quad \left. + f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right) \right) + 4f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right)]\end{aligned}$$

Hint 2. Please note the numbers chosen for $(\Delta x, \Delta y)$ should be specific to each student in the class.

References

- [1] Clewlow, Les and Strickland, Chris. *Implementing Derivative Models (Wiley Series in Financial Engineering)*, John Wiley & Sons 1996.
- [2] Mariani, Maria C. and Florescu, Ionut *Quantitative Finance*, 2020, John Wiley & Sons.
- [3] Rouah, F. D. *The Heston Model and Its Extensions in Matlab and C*, 2013, John Wiley & Sons.
- [4] Mikhailov, Sergei and Nögel, Ulrich. “Heston’s stochastic volatility model: Implementation, calibration and some extensions” *Wilmott Journal*, 2004.
- [5] Angeris, Guillermo and Kao, Hsien-Tang and Chiang, Rei and Noyes, Charlie and Chitra, Tarun. “An analysis of Uniswap markets” *arXiv preprint arXiv:1911.03380*, 2019.