Homework-1 Report

- Data Collection Method:

  The data was obtained using R to access the Bloomberg Terminal (Have attached the script)
  DATA1 and DATA2 were collected on two consecutive trading days during market hours: Feb 6 and Feb 9

Observations

- Implied volatility varies across strikes and is not constant. The standard deviation of IV across strikes is significant for each maturity.
- Bisection and Newton methods produce virtually identical implied volatility values. The maximum difference between the two methods is on the order of 10^-7, confirming numerical consistency.
- Put–call parity holds almost exactly. The median absolute deviation is approximately 1.25 × 10^-7, indicating negligible pricing discrepancies.
- Analytical Greeks closely match finite-difference approximations. Mean absolute errors are very small for Delta, Gamma, and Vega.
- Gamma is not the most sensitive Greek numerically in this dataset; Delta shows the largest mean absolute difference among the three.
- Using DATA1 implied volatilities to price DATA2 options results in noticeable pricing differences. The average absolute pricing error is approximately 1.92, with errors reaching over 5 in some cases.
- Overall, the results confirm correct implementation of the Black–Scholes model and numerical methods, while highlighting the time-varying nature of implied volatility.

(a) (10 pts) Derive the swap amounts

Under both Case 1 and Case 2, use the corresponding boundary condition, and the reserve updates above, to derive the swap size:

$$\Delta x(S_{t+1}; x_t, y_t, \gamma, k), \quad \Delta y(S_{t+1}; x_t, y_t, \gamma, k)$$

such that the one-step fee revenue is:

$$R(S_{t+1}) = \mathbf{1}_{\{S_{t+1} > \frac{P_t}{1-\gamma}\}} \gamma \, \Delta y + \mathbf{1}_{\{S_{t+1} < P_t(1-\gamma)\}} \gamma \, \Delta x \, S_{t+1}.$$

where we used the notation $1_{\{.\}}$ for the indicator function.

Case 1:-

Reserve Updates

$x_{t+1} = x_t - \Delta x$, $y_{t+1} = y_t + (1-\gamma)\Delta y$, $\Delta x, \Delta y > 0$

$x_{t+1} y_{t+1} = k.$ → constant product constraint

Boundary condition

$\frac{P_{t+1}}{1-\gamma} = s$, where $P_{t+1} = \frac{y_{t+1}}{x_{t+1}}$.

Substituting P_{t+1}

$$\frac{1}{1-\gamma} \cdot \frac{y_{t+1}}{x_{t+1}} = s$$

$$\frac{y_{t+1}}{x_{t+1}} = (1-\gamma)s$$

$$y_{t+1} = (1-\gamma) x_{t+1} s$$

using constant product constraint

$$x_{t+1} y_{t+1} = K$$

$$x_{t+1}^2 (1-\gamma) s = k$$

$$x_{t+1}^2 = \frac{K}{s(1-\gamma)}$$

$$x_{t+1} = \sqrt{\frac{K}{s(1-\gamma)}} \rightarrow 1$$

$$y_{t+1} = \frac{k}{x_{t+1}}$$

$$= \frac{k}{\sqrt{\frac{k}{s(1-\gamma)}}}$$

$$y_{t+1} = \sqrt{k(1-\gamma)s} \longrightarrow (2)$$

$$\therefore \quad \Delta x = x_t - x_{t+1}$$

$$= x_t - \sqrt{\frac{k}{(1-\gamma)s}}$$

$$y_{t+1} = y_t + (1-\gamma)\Delta y$$

$$\Delta y = \frac{y_{t+1} - y_t}{1-\gamma}$$

$$= \frac{\sqrt{k(1-\gamma)s} - y_t}{1-\gamma}$$

Fee Revenue

$$R(s) = \gamma \Delta y$$

$$= \gamma \cdot \frac{\sqrt{k(1-\gamma)s} - y_t}{1-\gamma} \quad , \text{ when } \quad s > \frac{P_t}{1-\gamma}$$

Case 2

Boundary condition

$$P_{t+1}(1-\gamma) = s \quad , \quad P_{t+1} = \frac{y_{t+1}}{x_{t+1}}$$

$$(1-\gamma)\frac{y_{t+1}}{x_{t+1}} = s$$

$$y_{t+1} = \frac{s}{1-\gamma} \cdot x_{t+1}$$

$$x_{t+1} y_{t+1} = k$$

$$x_{t+1}^2 \frac{s}{1-r} = k$$

$$x_{t+1} = \sqrt{\frac{k(1-r)}{s}} \longrightarrow ③$$

$$y_{t+1} = \frac{k}{x_{t+1}}$$

$$= \frac{k}{\sqrt{\frac{k(1-r)}{s}}}$$

$$y_{t+1} = \sqrt{\frac{ks}{1-r}} \longrightarrow ④$$

$$x_{t+1} = x_t + (1-r)\Delta x$$

$$\Delta x = \frac{x_{t+1} - x_t}{1-r}$$

$$\Delta x = \frac{\sqrt{\frac{k(1-r)}{s}} - x_t}{1-r}$$

$$y_{t+1} = y_t - \Delta y$$

$$\Delta y = y_t - y_{t+1}$$

$$= y_t - \sqrt{\frac{ks}{1-r}}$$

Fee Revenue

$$R(s) = r\Delta x \cdot s$$

$$R(s) = r\left(\frac{\sqrt{\frac{k(1-r)}{s}} - x_t}{1-r}\right)s$$

Part 4. (**Bonus** 10 points) Consider the following functions:

$$f_1(x, y) = xy$$
$$f_2(x, y) = e^{x+y}$$

1. Analytically solve the following integral for both f_1 and f_2

$$\int_0^1 \int_0^3 f_i(x, y)\mathrm{d}y\mathrm{d}x$$

2. Calculate the numerical integral of the f_1 and f_2 by applying the trapezoidal rule for double integral as discussed in [3], Page 118-119. Please choose four different pairs of values for $(\Delta x, \Delta y)$. Use these values to approximate the double integral for both f_1 and f_2. Calculate the error of the approximation for each choice of values. Comment on the results.

   Hint 1 First, discretize the x domain into n+1 points Δx apart, where $x_0 = 0$ and $x_{n+1} = 1$, and the y domain into $m + 1$ points Δy apart, where $y_0 = 0$ and $y_{m+1} = 3$. The composite trapezoidal rule approximates the integral as

$$\int_0^1 \int_0^3 f(x, y)\mathrm{d}y\mathrm{d}x \approx \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{\Delta x \Delta y}{16}[f(x_i, y_j)+f(x_i, y_{j+1})+f(x_{i+1}, y_j)$$
$$+f(x_{i+1}, y_{j+1})+2\left(f(\frac{x_i + x_{i+1}}{2}, y_j) + f(\frac{x_i + x_{i+1}}{2}, y_{j+1}) + f(x_i, \frac{y_j + y_{j+1}}{2})\right.$$
$$\left.+f(x_{j+1}, \frac{y_j + y_{j+1}}{2})\right) + 4f(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2})]$$

   Hint 2. Please note the numbers chosen for $(\Delta x, \Delta y)$ should be specific to each student in the class.

(1) $f_1(x, y) = xy$

$$\int_0^1 \int_0^3 xy \, dy \, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^3 dx$$

$$= \int_0^1 x \cdot 9/2 \, dx$$

$$= \frac{9}{2}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{9}{4} = 2.25$$

(ii) $f_2(x,y) = e^{x+y}$

$$\int_0^1 \int_0^3 e^{x+y}\, dy\, dx$$

$$= \int_0^1 e^x \left[e^y\right]_0^3 dx$$

$$= \int_0^1 e^x (e^3 - 1)\, dx$$

$$= (e^3-1)\left[e^x\right]_0^1$$

$$= (e^3-1)(e-1)$$

$$\approx 32.794.$$