

Part 3

a) Case 1: BTC cheaper in the pool

$$x_{t+1} = x_t - \Delta x, \quad y_{t+1} = y_t + (1-\gamma) \Delta y$$

$$\frac{p_{t+1}}{1-\gamma} = S_{t+1} \rightarrow \frac{\left(\frac{y_{t+1}}{x_{t+1}}\right)}{1-\gamma} = S_{t+1} \rightarrow y_{t+1} = S_{t+1} (1-\gamma) x_{t+1}$$

$$x_{t+1} \cdot y_{t+1} = k \rightarrow x_{t+1} (S_{t+1} (1-\gamma) x_{t+1}) = k \rightarrow S_{t+1} (1-\gamma) x_{t+1}^2 = k$$

~~$$x_{t+1} = \sqrt{\frac{k}{S_{t+1} (1-\gamma)}}$$~~

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$$y_{t+1} = S_{t+1} (1-\gamma) x_{t+1} = S_{t+1} (1-\gamma) \sqrt{\frac{k}{S_{t+1} (1-\gamma)}}$$

$$y_{t+1} = \sqrt{k S_{t+1} (1-\gamma)}, \quad \Delta x = x_t - x_{t+1} = x_t - \sqrt{\frac{k}{S_{t+1} (1-\gamma)}}$$

$$(1-\gamma) \Delta y = y_{t+1} - y_t \text{ so, } \Delta y = \frac{y_{t+1} - y_t}{1-\gamma} = \frac{\sqrt{k S_{t+1} (1-\gamma)} - y_t}{1-\gamma}$$

So Case 1 final:

$$x_{t+1} = \sqrt{\frac{k}{S_{t+1} (1-\gamma)}}, \quad y_{t+1} = \sqrt{k S_{t+1} (1-\gamma)}$$

$$\Delta x = x_t - \sqrt{\frac{k}{S_{t+1} (1-\gamma)}}, \quad \Delta y = \frac{\sqrt{k S_{t+1} (1-\gamma)} - y_t}{1-\gamma}$$

$$\text{Fee: } R_1(S_{t+1}) = \frac{\gamma}{1-\gamma} (\sqrt{k S_{t+1} (1-\gamma)} - y_t)$$

Case 2: BTC cheaper Outside

$$X_{t+1} = X_t + (1-\gamma)\Delta x, \quad Y_{t+1} = Y_t - \Delta y$$

$$P_{t+1}(1-\gamma) = S_{t+1} \rightarrow \left(\frac{Y_{t+1}}{X_{t+1}}\right)(1-\gamma) = S_{t+1} \rightarrow Y_{t+1} = \frac{S_{t+1}}{1-\gamma} \cdot X_{t+1}$$

$$X_{t+1} \left( \frac{S_{t+1}}{1-\gamma} \cdot X_{t+1} \right) = k \rightarrow X_{t+1}^2 = k \cdot \frac{1-\gamma}{S_{t+1}}, \quad X_{t+1} = \sqrt{\frac{k(1-\gamma)}{S_{t+1}}}$$

$$Y_{t+1} = \frac{S_{t+1}}{1-\gamma} \cdot \sqrt{\frac{k(1-\gamma)}{S_{t+1}}} = \sqrt{k} \cdot \frac{S_{t+1}}{1-\gamma} \cdot \frac{\sqrt{1-\gamma}}{\sqrt{S_{t+1}}} = \sqrt{\frac{k S_{t+1}}{1-\gamma}}$$

$$\Delta y = Y_t - Y_{t+1} = Y_t - \sqrt{\frac{k S_{t+1}}{1-\gamma}}, \quad \Delta x = \frac{X_{t+1} - X_t}{1-\gamma} = \frac{\sqrt{\frac{k(1-\gamma)}{S_{t+1}}} - X_t}{1-\gamma}$$

So Case 2 Final:

$$X_{t+1} = \sqrt{\frac{k(1-\gamma)}{S_{t+1}}}, \quad Y_{t+1} = \sqrt{\frac{k S_{t+1}}{1-\gamma}}$$

$$\Delta y = Y_t - \sqrt{\frac{k S_{t+1}}{1-\gamma}}, \quad \Delta x = \frac{\sqrt{\frac{k(1-\gamma)}{S_{t+1}}} - X_t}{1-\gamma}$$

$$\text{fee: } R_2(S_{t+1}) = \frac{\gamma}{1-\gamma} \left( \sqrt{k S_{t+1} (1-\gamma)} - X_t S_{t+1} \right)$$