FE621 Homework 1

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2026-02-15

I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/ examination. I further pledge that I have not copied any material from a book, article, the Internet or any other source except where I have expressly cited the source. -Tasha Khosla

NOTE: There is a Python file attached in Canvas that contains the code used for this assignment.

PART 1

1.

The data source used for this assignment is Yahoo Finance. Further details of the data download are in the attached Python code.

2.

Using the function created in problem 1, data was downloaded on options and equity for TSLA, SPY, and VIX for the days 2/9/2026 and 2/10/2026 during the trading day. Although traditional options are maturing the third Friday of each month, there are a lot more options available to download. These additional maturities allow traders to target specific time horizons for hedging and other strategies. For instance, a one-week option would be beneficial for a trader looking to trade around short-term news (e.g. the release of an earnings report).

3.

The symbols being downloaded are TSLA, SPY, and ^VIX. TSLA represents Tesla Inc., which is a publicly traded U.S. equiy and its option contracts are American (calls and puts). SPY represents SPDR S&P 500, which is an exchange traded fund (ETF) that tracks the performance of the S&P 500. The S&P 500 index tracks the stock performance of 500 leading companies in the U.S. An ETF is a basket of securities (stocks, bonds, etc.) that are traded on an exchange. So, the purpose of the SPY ETF is to provide investors with diversified exposure to U.S. equities across market sectors since it is tracking the S&P500. Options on SPY are very liquid and can be used for hedging portfolio risk. ^VIX represents Chicago Board Option Exchange's (CBOE) Volatility Index, which measures the U.S. stock market's expected volatility over the next 30 days based on S&P500 index options. It is often called the "fear index" because higher volatility corresponds to higher investor uncertainty. The VIX can be used for different purposes such as assessing market sentiment, hedging, managing risk, etc. As for when each option expires, this assignment focuses on TSLA and SPY options that are expiring on the third Friday for the next three months. Options on VIX typically expire on Wednesdays, according to CBOE.

4.

The underlying price at the exact moment when the rest of the data is downloaded is recorded for both days. The short-term interest rate was taken from the Federal Reserve site linked in the assignment instructions. The federal funds (effective) rate was used consistently in this assignment and the percentage is converted to a decimal. Time to maturity was calculated by taking the difference between expiration date and download date and then dividing by 365 to convert the time in days to years.

PART 2

5.

The Black-Scholes formulas are implemented in Python as a function of the current stock price S_0, volatility σ, time to expiration T_t, strike price K, and short term interest rate r.

6.

The bisection method is implemented to find the root of arbitrary functions to calculate the implied volatility on the first day the data was downloaded. The formula from lecture used for the function is $f(\sigma) = C(K, T, S, r, \sigma) = \frac{C_B + C_A}{2}$.

The tolerance level used was 10^{-6}.

For TSLA, there were two values for implied volatility at the money, as shown in the output below. The two values are very close (0.424434 and 0.439182.)

```
At the money for TSLA:
     moneyness  implied_volatility  implied_volatility_newtons
58        1.04            0.424434                         0.2
213       1.04            0.439182                         0.2
```

The average implied volatility for all options between in the money and out of the money was 0.4391.

For SPY, there are several values for implied volatility at the money, as displayed in the below output. These values range from 0.13 to 0.158 and are relatively close to one another.

```
      moneyness  implied_volatility  implied_volatility_newtons
1061       0.36            0.137321                    0.137322
1274       0.36            0.139782                    0.139781
1446       0.36            0.143720                    0.143720
1672       0.36            0.153702                    0.153702
1814       0.36            0.142608                    0.142608
2009       0.36            0.157617                    0.157616
```

The average implied volatility for all options between in the money and out of the money was 0.1706.

7.

Applying Newton's method to the above options, the time it takes to get the root is 0.95 seconds whereas the Bisection method takes 16.77 seconds. This is consistent with the course textbook, which states that Newton's method converges faster than the bisection method since its convergence is quadratic instead of linear.

8.

Below is a table reporting the implied volatility values obtained for each maturity, option type and stock. Since this table has many lines, an abridged version of the table is provided. To view the full dataset, refer to the attached Python code.

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
939	SPY	2026-02-20	10	call	335.0	2.07355223880597	4.999999403953671
1126	SPY	2026-02-20	10	put	335.0	2.07355223880597	1.2473858947566747
1127	SPY	2026-02-20	10	put	340.0	2.0430588235294116	1.2231196566289662
940	SPY	2026-02-20	10	call	345.0	2.0134492753623188	4.999999403953671
1128	SPY	2026-02-20	10	put	345.0	2.0134492753623188	1.1992015095571282
941	SPY	2026-02-20	10	call	350.0	1.9846857142857142	4.999999403953671
1129	SPY	2026-02-20	10	put	350.0	1.9846857142857142	1.1756195326145886
1130	SPY	2026-02-20	10	put	355.0	1.9567323943661972	1.1523653811527492
1131	SPY	2026-02-20	10	put	360.0	1.9295555555555555	1.1294295184303522
942	SPY	2026-02-20	10	call	365.0	1.9031232876712327	4.999999403953671

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
1514	SPY	2026-03-20	38	put	245.0	2.835265306122449	0.9757497014490366
1318	SPY	2026-03-20	38	call	250.0	2.77856	4.999999403953671
1515	SPY	2026-03-20	38	put	250.0	2.77856	0.9575321414633988
1516	SPY	2026-03-20	38	put	255.0	2.724078431372549	0.9396781697381735
1320	SPY	2026-03-20	38	call	260.0	2.671692307692308	4.999999403953671
1517	SPY	2026-03-20	38	put	260.0	2.671692307692308	0.9221746732541325
1321	SPY	2026-03-20	38	call	265.0	2.6212830188679246	4.999999403953671
1518	SPY	2026-03-20	38	put	265.0	2.6212830188679246	0.9050061548067332
1322	SPY	2026-03-20	38	call	270.0	2.5727407407407408	4.999999403953671
1519	SPY	2026-03-20	38	put	270.0	2.5727407407407408	0.8881606934694055

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
1895	SPY	2026-04-17	66	put	305.0	2.2775081967213113	0.6917670044919253
1896	SPY	2026-04-17	66	put	310.0	2.240774193548387	0.6822827153121234
1897	SPY	2026-04-17	66	put	315.0	2.205206349206349	0.6758191889253855
1898	SPY	2026-04-17	66	put	320.0	2.17075	0.6661334360865356
1899	SPY	2026-04-17	66	put	325.0	2.137353846153846	0.6536891808391809
1900	SPY	2026-04-17	66	put	330.0	2.1049696969696967	0.6467774276133778
1901	SPY	2026-04-17	66	put	335.0	2.07355223880597	0.6395604986673593
1902	SPY	2026-04-17	66	put	340.0	2.0430588235294116	0.6320932302633524
1903	SPY	2026-04-17	66	put	345.0	2.0134492753623188	0.6201150832449198
1722	SPY	2026-04-17	66	call	350.0	1.9846857142857142	4.999999403953671

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
155	TSLA	2026-02-20	10	put	100.0	4.1854000000000005	2.4795400176800486
1	TSLA	2026-02-20	10	call	110.0	3.8049090909090912	4.999999403953671
156	TSLA	2026-02-20	10	put	110.0	3.8049090909090912	2.3175143600064523
2	TSLA	2026-02-20	10	call	120.0	3.4878333333333336	4.999999403953671

157	TSLA	2026-02-20	10	put	120.0	3.4878333333333336	2.1699094472056633
3	TSLA	2026-02-20	10	call	130.0	3.2195384615384617	4.999999403953671
158	TSLA	2026-02-20	10	put	130.0	3.2195384615384617	2.0343506307047603
4	TSLA	2026-02-20	10	call	140.0	2.9895714285714288	4.999999403953671
159	TSLA	2026-02-20	10	put	140.0	2.9895714285714288	1.9089985115338564
5	TSLA	2026-02-20	10	call	150.0	2.7902666666666667	4.999999403953671

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
466	TSLA	2026-03-20	38	put	5.0	83.708	4.085594002499461
304	TSLA	2026-03-20	38	call	10.0	41.854	4.999999403953671
467	TSLA	2026-03-20	38	put	10.0	41.854	3.389514410727858
305	TSLA	2026-03-20	38	call	15.0	27.90266666666667	2.9118643407000304
468	TSLA	2026-03-20	38	put	15.0	27.90266666666667	2.9975388478723755
306	TSLA	2026-03-20	38	call	20.0	20.927	4.999999403953671
469	TSLA	2026-03-20	38	put	20.0	20.927	2.7253948230928176
307	TSLA	2026-03-20	38	call	25.0	16.741600000000002	4.999999403953671
470	TSLA	2026-03-20	38	put	25.0	16.741600000000002	2.517422338137031
308	TSLA	2026-03-20	38	call	30.0	13.951333333333334	4.999999403953671

	underlying_symbol	expiration	days_to_expiry	option_type	strike	moneyness_ratio	implied_volatility
779	TSLA	2026-04-17	66	put	5.0	83.708	3.1022784929095506
780	TSLA	2026-04-17	66	put	10.0	41.854	2.5739954793645143
781	TSLA	2026-04-17	66	put	20.0	20.927	2.439345037468315
625	TSLA	2026-04-17	66	call	30.0	13.951333333333334	4.999999403953671
626	TSLA	2026-04-17	66	call	35.0	11.958285714285715	4.999999403953671
783	TSLA	2026-04-17	66	put	35.0	11.958285714285715	1.9918274935337308
784	TSLA	2026-04-17	66	put	40.0	10.4635	1.7169654974960091
785	TSLA	2026-04-17	66	put	45.0	9.300888888888889	1.7851734630566838
628	TSLA	2026-04-17	66	call	50.0	8.370800000000001	4.804716745332598
786	TSLA	2026-04-17	66	put	50.0	8.370800000000001	1.5515387994717358

Examining the table, TSLA's implied volatility values are much higher than SPY's, which makes sense since SPY is diversified and TSLA is subject to higher idiosyncratic risk and larger price changes.

The current value of the VIX is 17.24%. As shown in a previous problem, the average implied volatility for all TSLA options between in the money and out of the money was 43.91% and for SPY it was 17.06%. As these results show, SPY's implied volatility average is much closer to the VIX value than TSLA's since SPY is diversified and TSLA is not.

The trend is not easily visible in the small snippets of the data above, but as shown in the plots in Part 10 below, as maturity increases, implied volatility appears to generally decrease.

9.

For eac option in the table, the price of the different type of option was calculated using put-call parity
$P + S_0 = C + Ke^{-rT}$

$C - P = S_0 - Ke^{(-rT}$

The table below shows a sample of the data.

[1976 rows x 7 columns]

	option_type	mid	put_call_parity	bid	ask
0	call	318.150	200.290324	317.60	318.70
1	call	327.000	201.430356	325.70	328.30
2	call	298.500	239.920388	297.35	299.65
3	call	288.025	260.385421	286.05	290.00
4	call	278.550	279.850453	277.40	279.70
...	...	...	...	...	...
2042	put	75.685	5.376434	73.93	77.44
2043	put	80.625	5.349236	78.87	82.38
2044	put	85.800	5.557037	84.05	87.55
2045	put	105.955	5.843244	104.40	107.51
2046	put	110.955	5.876045	109.40	112.51

While some of the values are close, many are not which makes sense because the put-call parity formula is meant for European options but the securities used for this assignment are American. It is known that an American call behaves like a European call (due to early exercise not being optimal), which explains why the put call parity values for the calls are not as far off from the bid and ask prices as the put values are.

10.

Below is the two dimensional plot of implied volatilities vs. strike K for the closest to maturity options.

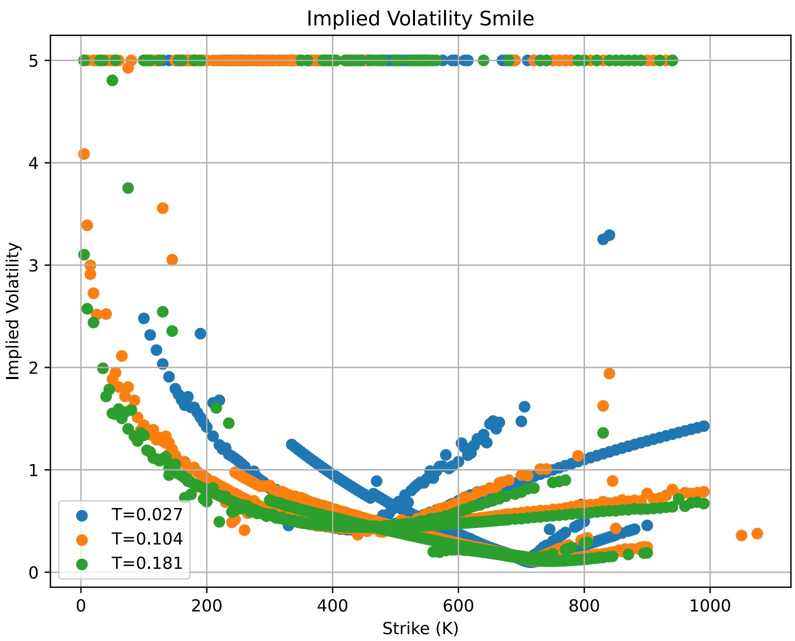


Abb. 1: 2D plot of volatility vs strike at different maturities

The plot shows that the implied volatility is lowest at the money and it increases as the strike moves in the money or out the money, thus creating the volatility smile. However, there are many values at the imposed upper bound (5), which could correspond to options that were very out of the money or in the money. Additionally, the shortest maturity (blue) has the steepest curve whereas the longer maturities (orange and green) are less steep, showing that the volatility smile becomes flatter as volatility increases.

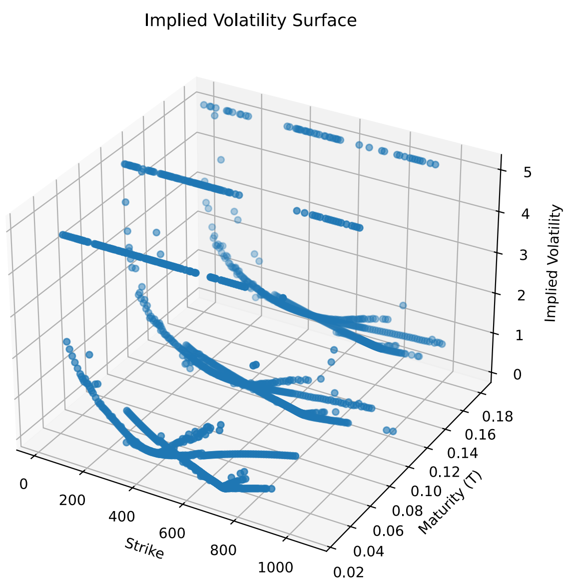


Abb. 2: Bonus 3D plot of the same

This plot contains the same patterns observed in the two dimensional plot.

11.

Below is a table calculating the derivatices (delta, vega, and gamma) using Black-Scholes and finite differences. The full table is able for viewing in the Python code.

```
[1976 rows x 5 columns]
    underlying_symbol expiration  strike         T     sigma  delta_bs  delta_fd   vega_bs   vega_fd  gamma_bs  gamma_fd
0                TSLA 2026-02-20   100.0  0.027397  4.999999  0.984016  0.984013  2.770567  2.770567  0.000115  0.000115
1                TSLA 2026-02-20   110.0  0.027397  4.999999  0.978804  0.978801  3.523399  3.523399  0.000147  0.000147
2                TSLA 2026-02-20   120.0  0.027397  4.999999  0.972855  0.972851  4.337464  4.337464  0.000181  0.000181
3                TSLA 2026-02-20   130.0  0.027397  4.999999  0.966210  0.966206  5.200447  5.200447  0.000217  0.000217
4                TSLA 2026-02-20   140.0  0.027397  4.999999  0.958917  0.958912  6.100716  6.100716  0.000254  0.000254
..                ...        ...     ...       ...       ...       ...       ...       ...       ...       ...       ...
990               SPY 2026-04-17   840.0  0.180822  0.152872  0.002642  0.002730  2.409874  2.409881  0.000181  0.000183
991               SPY 2026-04-17   845.0  0.180822  0.155049  0.002262  0.002338  2.093936  2.093943  0.000155  0.000157
992               SPY 2026-04-17   870.0  0.180822  0.173519  0.001724  0.001773  1.636616  1.636621  0.000108  0.000109
993               SPY 2026-04-17   895.0  0.180822  0.185621  0.001008  0.001035  1.001816  1.001819  0.000062  0.000063
994               SPY 2026-04-17   900.0  0.180822  0.189428  0.000989  0.001015  0.984894  0.984897  0.000060  0.000060
```

As these results show, the values are very close, indicating that either method works to calculate the derivatives.

12.

Using the second dataset and the implied volatility from the first dataset, the option prices are calculated and displayed in the table below. To view the full table, refer to the attached Python code.

```
[995 rows x 11 columns]
     data_tag download_time_local underlying_symbol  underlying_spot_at_download expiration option_type  ... currency      mid  days_to_expiry         T  implied_volatility  price_using_day1
0       DATA2 2026-02-10 10:33:45              TSLA                       424.21 2026-02-20        call  ...      USD  321.200               9  0.024658            4.999999        326.256791
1       DATA2 2026-02-10 10:33:45              TSLA                       424.21 2026-02-20        call  ...      USD  327.000               9  0.024658            4.999999        317.092862
2       DATA2 2026-02-10 10:33:45              TSLA                       424.21 2026-02-20        call  ...      USD  300.450               9  0.024658            4.999999        308.119852
3       DATA2 2026-02-10 10:33:45              TSLA                       424.21 2026-02-20        call  ...      USD  290.875               9  0.024658            4.999999        299.348388
4       DATA2 2026-02-10 10:33:45              TSLA                       424.21 2026-02-20        call  ...      USD  281.025               9  0.024658            4.999999        290.786480
...       ...                 ...               ...                          ...        ...         ...  ...      ...      ...             ...       ...                 ...               ...
1972    DATA2 2026-02-10 10:33:45               SPY                       696.13 2026-04-17         put  ...      USD   74.325              65  0.178082            0.220305         74.382956
1973    DATA2 2026-02-10 10:33:45               SPY                       696.13 2026-04-17         put  ...      USD   79.330              65  0.178082            0.229454         79.312968
1974    DATA2 2026-02-10 10:33:45               SPY                       696.13 2026-04-17         put  ...      USD   84.335              65  0.178082            0.241763         84.480341
1975    DATA2 2026-02-10 10:33:45               SPY                       696.13 2026-04-17         put  ...      USD  104.345              65  0.178082            0.282096        104.606257
1976    DATA2 2026-02-10 10:33:45               SPY                       696.13 2026-04-17         put  ...      USD  109.345              65  0.178082            0.291310        109.599826
```

PART 3

(a)

Case 1: $S_{t+1} > P_t \frac{1}{1-\gamma}$ (BTC cheaper in the pool)

Using the boundary condition and rearranging:

$P_{t+1}\frac{1}{1-\gamma} = \left(\frac{y_{t+1}}{x_{t+1}}\right)\frac{1}{1-\gamma} = S_{t+1}$

$y_{t+1} = S_{t+1} * (1-\gamma) * x_{t+1}$

Also, it is given that

$x_{t+1} * y_{t+1} = k$

So,

$x_{t+1} * [S_{t+1} * (1-\gamma) * x_{t+1}] = k$

$(x_{t+1})^2 * S_{t+1} * (1-\gamma) = k$

$x_{t+1} = \sqrt{\frac{k}{S_{t+1}*(1-\gamma)}}$

$y_{t+1} = S_{t+1} * (1-\gamma) * \sqrt{\frac{k}{S_{t+1}*(1-\gamma)}}$

$= \sqrt{k * S_{t+1} * (1-\gamma)}$

So for Case 1,

$\Delta x(S_{t+1}; x_t, y_t, \gamma, k) = x_t - x_{t+1} = x_t - \sqrt{\frac{k}{S_{t+1}*(1-\gamma)}}$

$\Delta y(S_{t+1}; x_t, y_t, \gamma, k) = \frac{y_{t+1}-y_t}{1-\gamma} = \frac{\sqrt{k*S_{t+1}*(1-\gamma)}-y_t}{1-\gamma}$

Case 2: $S_{t+1} < P_t\frac{1}{1-\gamma}$ (BTC cheaper outside of the pool)

Using the boundary condition and rearranging:

$P_{t+1}(1-\gamma) = \frac{y_{t+1}}{x_{t+1}}(1-\gamma) = S_{t+1}$

$y_{t+1} = \frac{S_{t+1}*x_{t+1}}{1-\gamma}$

Also, it is given that

$x_{t+1} * y_{t+1} = k$

So,

$x_{t+1} * \frac{S_{t+1}*x_{t+1}}{1-\gamma} = k$

$(x_{t+1})^2 = \frac{k*(1-\gamma)}{S_{t+1}}$

$x_{t+1} = \sqrt{\frac{k*(1-\gamma)}{S_{t+1}}}$

$y_{t+1} = \frac{S_{t+1}*\sqrt{\frac{k*(1-\gamma)}{S_{t+1}}}}{1-\gamma}$

$= \sqrt{\frac{k*S_{t+1}}{1-\gamma}}$

So for Case 2,

$\Delta x(S_{t+1}; x_t, y_t, \gamma, k) = \frac{x_{t+1}-x_t}{1-\gamma} = \frac{\sqrt{\frac{k*(1-\gamma)}{S_{t+1}}}-x_t}{1-\gamma}$

$\Delta y(S_{t+1}; x_t, y_t, \gamma, k) = y_t - y_{t+1} = y_t - \sqrt{\frac{k*S_{t+1}}{1-\gamma}}$

For the fee revenue, substituting these values of Δx and Δy for case 1 and case 2 into the given function for $R(S_{t+1})$:

If $S_{t+1} > P_t \frac{1}{1-\gamma}$

$R(S_{t+1}) = \gamma * \Delta y = \gamma * \frac{\sqrt{k*S_{t+1}*(1-\gamma)}-y_t}{1-\gamma}$

If $S_{t+1} < P_t \frac{1}{1-\gamma}$

$R(S_{t+1}) = \gamma * \Delta x * S_{t+1} = \gamma * S_{t+1} * \frac{\sqrt{\frac{k*(1-\gamma)}{S_{t+1}}}-x_t}{1-\gamma}$

Therefore,

$$\boxed{R(S_{t+1}) = \begin{cases} \gamma * \frac{\sqrt{k*S_{t+1}*(1-\gamma)}-y_t}{1-\gamma} \text{ , if } S_{t+1} > P_t \frac{1}{1-\gamma} \\ \gamma * S_{t+1} * \frac{\sqrt{\frac{k*(1-\gamma)}{S_{t+1}}}-x_t}{1-\gamma} \text{ , if } S_{t+1} < P_t \frac{1}{1-\gamma} \\ 0 \text{ , otherwise} \end{cases}}$$

(b)

Using the Trapezoidal Rule from lecture and implementing the problem setup in Python, the numerical approximation of $\mathbb{E}[R(S_{t+1})] =$

$\boxed{\text{0.00852 USDC for one step}}$

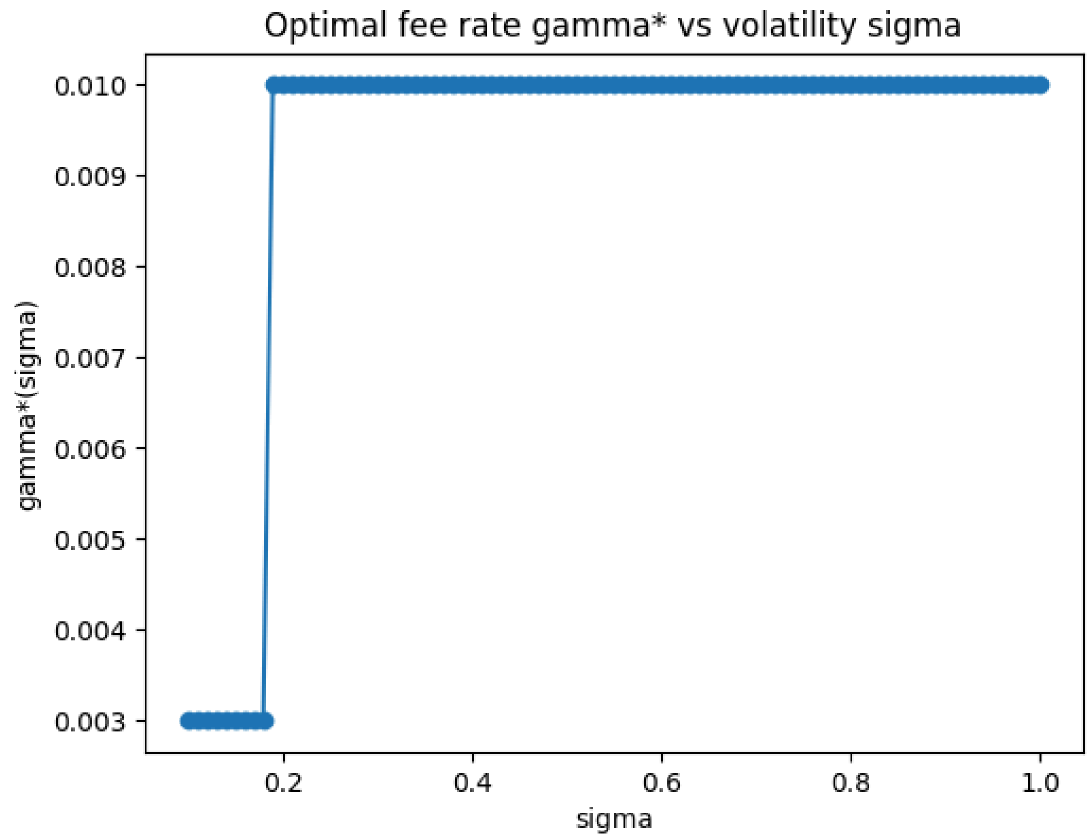
(c)

Below is a table reporting the $\mathbb{E}[R]$ values and the best $\gamma^*(\sigma)$ among the three options.

σ/γ	0.001	0.003	0.01	$\gamma^*(\sigma)$	$\mathbb{E}[R]$
0.2	0.003685	0.008522	0.009430	0.01	0.009430
0.6	0.011923	0.032983	0.081082	0.01	0.081082
1.0	0.011923	0.032983	0.081082	0.01	0.160690

As the table shows, for all three choices of σ, the best choice of $\gamma^*(\sigma)$ is 0.01. Also, as volatility σ increases, the expected fee revenue also increases.

Below is a scatter plot for σ vs. $\gamma^*(\sigma)$.



As the plot shows, for low volatility values (below 0.2), the middle fee rate value of 0.003 maximizes expected revenue. However, as volatility increases, the optimal fee rate is 0.01. Large volatility increases the likelihood and magnitude of large price movements outside of the no-arbitrage band. Charging a higher fee for each arbitrage trade increases revenue, explaining why the table and plot show that as volatility increases, the optimal fee remains high.