

PHSX 462: HW09

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Griffiths 11.2

Let's start here by simply copying the wavefunctions that we will need:

$$\begin{aligned}\psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \\ \psi_{200} &= \frac{1}{\sqrt{8\pi a^3}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}, \\ \psi_{210} &= \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} e^{-r/2a} \cos(\theta), \\ \psi_{21\pm 1} &= \mp \frac{1}{\sqrt{64\pi a^3}} \frac{r}{a} e^{-r/2a} \sin(\theta) e^{\pm i\phi}.\end{aligned}$$

Since r has the same degree as z ($r^2 = x^2 + y^2 + z^2$), then we can just do some analysis of each as even or odd, and eliminate each on that. For each of the H'_{ii} , they are even in z ; thus, each is zero. The same holds for each of the combinations of $|200\rangle, |211\rangle, |21-1\rangle$ with $|100\rangle$. However, the combination of $|210\rangle$ with $|100\rangle$ is not, so let us look at that:

$$\begin{aligned}H'_{100,210} &= eE \frac{1}{\sqrt{\pi a^3}} \frac{1}{\sqrt{32\pi a^3}} \frac{1}{a} \int r e^{-r/a} e^{-r/2a} \cos(\theta) z r^2 \sin(\theta) dr d\theta d\phi \\ &= eE \frac{1}{a^4 \sqrt{8\pi}} \int_0^\infty r^4 e^{-3r/2a} dr \int_0^\pi \cos^2(\theta) d\theta \int_0^{2\pi} d\phi \\ &= \frac{eE}{2\sqrt{2}a^4} 4! \left(\frac{2a}{3}\right)^5 \frac{2}{3} \\ &= \left(\frac{2^{17/2}}{3^5}\right) eEa.\end{aligned}$$

So, every matrix element is zero except

$$\boxed{H'_{100,210} = \left(\frac{2^{17/2}}{3^5}\right) eEa}.$$

Griffiths 11.8

a) According to first order perturbation theory, we are looking at

$$\begin{aligned}
 c_a^{(1)}(t) &= 1 \\
 c_b^{(1)}(t) &= -\frac{i}{\hbar} \int_{t_0}^t H'_{ba}(t') e^{i\omega_0 t'} dt' \\
 &= -\frac{i}{\hbar} \int_{-\infty}^t \frac{\alpha}{\sqrt{\pi}\tau} e^{-(t'/\tau)^2} e^{i\omega t'} dt' \\
 &= -\frac{i}{\hbar} \frac{\alpha}{\sqrt{\pi}\tau} \int_{-\infty}^t e^{-\frac{1}{\tau^2} \left(t' - \frac{i\omega\tau^2}{2}\right)^2 - \frac{\omega^2\tau^2}{4}} dt'
 \end{aligned}$$

taking into consideration that we will be taking $t \rightarrow \infty$

$$\begin{aligned}
 c_b^{(1)}(t) &= -\frac{i\alpha}{\hbar\sqrt{\pi}} e^{-(\omega_0\tau/2)^2} \sqrt{\pi} \\
 &= -\frac{i\alpha}{\hbar} e^{-(\omega_0\tau/2)^2}.
 \end{aligned}$$

The probability of the transition will then be

$$P_{a \rightarrow b} = \left(\frac{\alpha}{\hbar} \right)^2 e^{-(\omega_0\tau)^2/2}.$$

b) Applying a slight modification on the previous part:

$$c_b^{(1)} = -\frac{i}{\hbar} \int_{-\infty}^{\infty} \alpha \delta(t) e^{i\omega t} dt = -\frac{i\alpha}{\hbar}.$$

This gives a probability of $(\alpha/\hbar)^2$, which is the first order term of the solution of 11.4 ($\sin^2(\alpha/\hbar)$).

c) The probability gets killed off, as $e^{-(\omega\tau)^2/2} \rightarrow 0$. Consequently, we see no transition of states and the hinting at an adiabatic-type system.

Griffiths 11.9

This is post homework Will typing this up; this question was a pain in the butt while doing it, but actually super fun on the other side. Since I am straight exhausted, I am going to be skipping some steps to save my sanity...

a) Plugging in what we are given

$$\begin{aligned}\dot{c}_a &= -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b = -\frac{i}{2\hbar} V_{ab} e^{i(\omega - \omega_0)t} c_b \\ \dot{c}_b &= -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a = -\frac{i}{2\hbar} V_{ab} e^{i(\omega_0 - \omega)t} c_a.\end{aligned}$$

We must go down before we can go up, so let us differentiate \dot{c}_a :

$$\begin{aligned}\ddot{c}_a &= -\frac{i}{2\hbar} V_{ab} \left[i(\omega - \omega_0) e^{i(\omega - \omega_0)t} c_b + e^{i(\omega - \omega_0)t} \dot{c}_b \right] \\ &= -\frac{i}{2\hbar} V_{ab} \left[i(\omega - \omega_0) \frac{i2\hbar}{V_{ab}} \dot{c}_a + e^{i(\omega - \omega_0)t} \left(-\frac{i}{2\hbar} \right) V_{ba} e^{i(\omega_0 - \omega)t} c_a \right] \\ &= i(\omega - \omega_0) \dot{c}_a - \frac{|V_{ab}|^2}{(2\hbar)^2} c_a.\end{aligned}$$

Rewriting this gives

$$\ddot{c}_a + i(\omega_0 - \omega) \dot{c}_a + \frac{|V_{ab}|^2}{(2\hbar)^2} c_a = 0.$$

This is a differential equation, that if we propose an exponential solution in the form of $e^{\lambda t}$, gives

$$\lambda^2 + i(\omega_0 - \omega) \lambda + \frac{|V_{ab}|^2}{(2\hbar)^2} = 0$$

and, via the quadratic formula

$$\lambda = -\frac{1}{2} \left[i(\omega - \omega_0) \pm \sqrt{-(\omega_0 - \omega)^2 - \frac{4|V_{ab}|^2}{(2\hbar)^2}} \right] = -i \left[\frac{(\omega - \omega_0)}{2} \pm \omega_r \right].$$

the general equation for c_a is then

$$c_a(t) = A e^{-i((\omega - \omega_0)/2 + \omega_r)t} + B e^{-i((\omega - \omega_0)/2 - \omega_r)t}.$$

Using the initial conditions gives $A + B = 1$.

By a very similar derivation, the general solution for c_b is

$$c_b(t) = C e^{-i((\omega - \omega_0)/2 + \omega_r)t} + D e^{-i((\omega - \omega_0)/2 - \omega_r)t}.$$

Using the initial conditions here gives $C + D = 0$.

To solve for the coefficients in the c_b equation, let us take the derivative:

$$\begin{aligned}
\dot{c}_b &= -i \left[\frac{(\omega - \omega_0)}{2} + \omega_r \right] C e^{-i \left[\frac{(\omega - \omega_0)}{2} + \omega_r \right] t} - i \left[\frac{(\omega - \omega_0)}{2} - \omega_r \right] D e^{-i \left[\frac{(\omega - \omega_0)}{2} - \omega_r \right] t} \\
-\frac{i}{2\hbar} V_{ba} e^{i(\omega_0 - \omega)t} c_a &= \quad \quad \quad \text{''} \quad \text{''} \\
c_a &= \left[\frac{2\hbar}{V_{ba}} \right] \left[\left(\frac{(\omega - \omega_0)}{2} + \omega_r \right) C e^{-i\omega_r t} + \left(\frac{(\omega - \omega_0)}{2} - \omega_r \right) D e^{i\omega_r t} \right] \\
1 &= \left[\frac{2\hbar}{V_{ba}} \right] \left[\left(\frac{(\omega - \omega_0)}{2} + \omega_r \right) C + \left(\frac{(\omega - \omega_0)}{2} - \omega_r \right) D \right] \\
1 &= \left[\frac{2\hbar}{V_{ba}} \right] [C - D] \omega_r
\end{aligned}$$

$$C - D = \frac{V_{ba}}{2\hbar\omega_r}.$$

Combining this with an initial condition gives

$$2C = \frac{V_{ba}}{2\hbar\omega_r} \rightarrow C = \frac{V_{ba}}{4\hbar\omega_r} \text{ and } D = -\frac{V_{ba}}{4\hbar\omega_r}.$$

At this point I am realizing my flaw in keeping everything in complex exponentials. On my scratch work I worked out what this is, specifically when I plugged in C and D ;

$$c_b(t) = -\frac{i}{2\hbar\omega_r} V_{ba} \left[e^{i((\omega - \omega_0)/2)t} \right] \sin(\omega_r t).$$

A very similar process will need to be done with c_a . I will skip typing it all up, but we get

$$\begin{aligned}
c_b &= \frac{V_{ab}}{2\hbar} \left[\left(\frac{\omega_0 - \omega}{2} + \omega_r \right) A e^{-i\omega_r t} + \left(\frac{\omega_0 - \omega}{2} - \omega_r \right) B e^{i\omega_r t} \right] \\
0 &= \frac{V_{ab}}{2\hbar} \left[\left(\frac{\omega_0 - \omega}{2} + \omega_r \right) A + \left(\frac{\omega_0 - \omega}{2} - \omega_r \right) B \right] \\
0 &= \frac{\omega_0 - \omega}{2} + \omega_r [A - B].
\end{aligned}$$

Combining this equation with the initial values from before we get

$$2A = 1 + \frac{\omega - \omega_0}{2\omega_r} \rightarrow A = \frac{1}{2} + \frac{\omega - \omega_0}{4\omega_r} \text{ and } B = \frac{1}{2} - \frac{\omega - \omega_0}{4\omega_r}.$$

Plugging these into c_a , and doing some simplifying,

$$c_a(t) = e^{-i((\omega_0 - \omega)/2)t} \left[\cos(\omega_r t) + \frac{i(\omega_0 - \omega)}{2\omega_r} \sin(\omega_r t) \right].$$

Finally, the two constants are

$$\boxed{
\begin{aligned}
c_a(t) &= e^{-i((\omega_0 - \omega)/2)t} \left[\cos(\omega_r t) + \frac{i(\omega_0 - \omega)}{2\omega_r} \sin(\omega_r t) \right] \\
c_b(t) &= -\frac{i}{2\hbar\omega_r} V_{ba} \left[e^{i((\omega - \omega_0)/2)t} \right] \sin(\omega_r t)
\end{aligned}
}$$

- b) To show that the probability of state b is never larger than 1, let us take the square of the component:

$$|c_b(t)|^2 = \frac{1}{4\hbar^2\omega_r^2} |V_{ab}|^2 \sin^2(\omega_r t) \rightarrow \left(\frac{|V_{ab}|}{2\hbar\omega_r} \right)^2.$$

We need to compare the top: $(|V_{ab}|/2\hbar)^2$ to the bottom: $\frac{1}{4}((\omega - \omega_0)^2 + (|V_{ab}|/2\hbar)^2)$. Since the bottom will always be larger than the top, the probability never goes above 1.

To show that the total probability is one, let us take the sum of the square of the components:

$$\begin{aligned} \cos^2(\omega_r t) + \left(\frac{\omega_0 - \omega}{2\omega_r} \right)^2 \sin^2(\omega_r t) + \frac{1}{4\hbar^2\omega_r^2} |V_{ab}|^2 \sin^2(\omega_r t) \\ \cos^2(\omega_r t) + \left(\frac{1}{2\omega_r} \right)^2 ((\omega_0 - \omega)^2 + (|V_{ab}|/2\hbar)^2) \sin^2(\omega_r t) \\ \cos^2(\omega_r t) + \frac{\omega_r^2}{\omega_r^2} \sin^2(\omega_r t) \\ \cos^2(\omega_r t) + \sin^2(\omega_r t) = 1 \quad \checkmark \end{aligned}$$

c)

$$\begin{aligned} |c_b(t)|^2 &= \left(\frac{|V_{ab}|^2}{2\hbar} \right)^2 \frac{4}{(\omega - \omega_0)^2 + (|V_{ab}|/2\hbar)^2} \sin^2 \left(\frac{1}{2} \sqrt{(\omega - \omega_0)^2 + (|V_{ab}|/2\hbar)^2} t \right) \\ &\quad \text{if } (\omega - \omega_0)^2 \gg (|V_{ab}|/2\hbar)^2 \\ &= \left(\frac{|V_{ab}|^2}{2\hbar} \right)^2 \frac{4}{(\omega - \omega_0)^2} \sin^2 \left(\frac{1}{2} (\omega - \omega_0) t \right) \\ &= \frac{|V_{ab}|^2}{\hbar^2 (\omega - \omega_0)^2} \sin^2((\omega - \omega_0)t/2). \end{aligned}$$

This is the solution that we got in the perturbation theory. As already stated, the definition of small here is that $(\omega - \omega_0)^2 \gg (|V_{ab}|/2\hbar)^2$.

- d) The state is cycles on $\omega_r t$. So, the question when what have made one half rotation (since sign doesn't matter here), so $\omega_r t = \pi \rightarrow \boxed{t = \pi/\omega_r}$.

Griffiths 11.11

Griffiths 11.13