PHSX 491: HW09

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Question 1

a) *Why won't two dimensional flat space be an interesting calculation for the elements of the Riemann Tensor and curvature scalar?*

Without actually calculating the value, the first guess is that it become trivial, probably saying that the curvator is just the θ component. There also isn't a three dimensional analysis we could do, like in part **e)**.

b) *Final all non-zero Christoffel Symbols for the metric* $\mathrm{d}s^2 = R^2\,\mathrm{d}\theta^2 + R^2\sin^2(\theta)\,\mathrm{d}\phi^2$

We know from the last homework

$$\Gamma^\lambda_{\nu\mu} = 0\,, \qquad \Gamma^\lambda_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu}\,,$$
$$\Gamma^\lambda_{\mu\lambda} = \partial_\mu\left(\ln\!\left(\sqrt{|g_{\lambda\lambda}|}\right)\right)\,,\ \text{and} \qquad \Gamma^\lambda_{\lambda\lambda} = \partial_\lambda\left(\ln\!\left(\sqrt{|g_{\lambda\lambda}|}\right)\right)\,,$$

where the metric is diagonal. Using this for this metric:

$$\begin{aligned}
\Gamma^\phi_{\theta\theta} &= -\frac{1}{2}\left(\frac{1}{R^2\sin^2(\theta)}\right)\partial_\phi(R^2) = 0\,,\\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \partial\left(\ln\!\left(\sqrt{R^2\sin^2(\theta)}\right)\right) = \frac{1}{\sqrt{R^2\sin^2(\theta)}}\frac{1}{2}\frac{1}{\sqrt{R^2\sin^2(\theta)}}2\sin(\theta)\cos(\theta)\\
&= \cot(\theta)\,,\\
\Gamma^\phi_{\phi\phi} &= \partial_\phi\left(\ln\!\left(\sqrt{R^2\sin^2(\theta)}\right)\right) = 0\,,\\
\Gamma^\theta_{\theta\theta} &= \partial\left(\ln(R^2)\right) = 0\,,\\
\Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} &= \partial_\phi\left(\ln\!\left(\sqrt{R^2}\right)\right) = 0\,,\\
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}\frac{1}{R^2}\partial_\phi(R^2) = 0\,,\ \text{and}\\
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}\left(\frac{1}{R^2}\right)\partial_\theta\left(R^2\sin^2(\theta)\right)\\
&= -\sin(\theta)\cos(\theta)\,.
\end{aligned}$$

From this, there are two Christoffel Symbols (three if you don't count symmetry) that are non-zero. They are:

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot(\theta)\,,\ \text{and} \qquad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta)\,.$$

c) *For a two dimensional space how many components of the Riemann Tensor will there be? Find the non-zero components fo the Riemann Tensor.* There are a total of $2^4 = 16$ elements that we "need" to consider. But, if we look at symmetries there are really only six, that is:

$$\begin{gathered}\phi\phi\phi\phi\,,\\ \phi\theta\theta\theta = -\theta\phi\theta\theta = \theta\theta\phi\theta = -\theta\theta\theta\phi\,,\\ \phi\theta\theta\phi = -\phi\theta\phi\theta = -\theta\phi\theta\phi = \theta\phi\phi\theta\,,\\ \phi\phi\theta\theta = \theta\theta\phi\phi\,,\\ \theta\phi\phi\phi = -\phi\theta\phi\phi = \phi\phi\theta\phi = -\phi\phi\phi\theta\,,\ \text{and}\\ \theta\theta\theta\theta\,.\end{gathered}$$

Saving both of us the laborious task of sifting through the non-zero terms, there are only two interesting lines. They are the $\phi\phi\theta\theta$ element:

$$\begin{aligned}R^{\phi}_{\phi\theta\theta} &= \partial_\theta\Gamma^{\phi}_{\phi\theta} - \partial_\theta\Gamma^{\phi}_{\phi\theta} + \Gamma^{\phi}_{\theta\gamma}\Gamma^{\gamma}_{\phi\theta} - \Gamma^{\phi}_{\theta\sigma}\Gamma^{\sigma}_{\phi\theta}\\ &= \partial_\theta(\cot(\theta)) - \partial_\theta(\cot(\theta))\Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\phi\theta} - \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\phi\theta}\\ &= 0\,,\end{aligned}$$

and the $\phi\theta\theta\phi$ term:

$$\begin{aligned}R^{\phi}_{\theta\theta\phi} &= \partial_\theta\Gamma^{\phi}_{\theta\phi} - \partial\Gamma^{\phi}_{\theta\theta} + \Gamma^{\phi}_{\theta\gamma}\Gamma^{\gamma}_{\theta\phi} - \Gamma^{\phi}_{\phi\sigma}\Gamma^{\sigma}_{\theta\theta}\\ &= \partial_\theta\Gamma^{\phi}_{\theta\phi} + \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\theta\phi}\\ &= \partial_\theta\left(\cot(\theta)\right) + \left(\cot(\theta)\right)^2\\ &= -\csc^2(\theta) + \cot^2(\theta)\\ &= -1\,.\end{aligned}$$

So, the only non-zero element out of the six unique elements of the Riemann Tensor is

$$R^{\phi}_{\theta\theta\phi} = -R^{\phi}_{\theta\phi\theta} = -R^{\theta}_{\phi\theta\phi} = R^{\theta}_{\phi\phi\theta} = -1\,.$$

d) *Find the components of the Ricci tensor for this metric. From this, what is the curvature scalar?*

The elements of the Ricci tensor are $R^{\alpha}_{\beta\alpha\nu}$. Using this, it is easy to get that the Ricci tensor is

$$R_{\beta\nu} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

This gives us a Ricci scalar, or curvature scalar, of

$$\begin{aligned}R = g^{\beta\nu}R_{\beta\nu} &= \frac{1}{R^2} + \frac{1}{R^2\sin^2(\theta)}\\ &= \frac{1}{R^2}\left(1 + \frac{1}{\sin^2(\theta)}\right)\\ &= \frac{1}{R^2}\left(1 + \csc^2(\theta)\right)\,.\end{aligned}$$

e) *Consider the two limits $R \to 0$ and $R \to \infty$. Comment on what happens to the curvature scalar in these two limits. Does this make sense physically?*

As $R \to 0$, the curvature scalar goes to infinity, which makes sense. As we collapse to a point all of the curving happens at once.

As $R \to \infty$, the curvature scalar goes to zero. This also makes sense, as the surface of the sphere becomes more and more flat as the sphere becomes larger (meaning less curving).