PHSX 491: HW08

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Question 1

a) *Show that* $\Gamma^{\lambda}_{\mu\nu} = 0$.

Let us begin with the equation of the Christoffel symbol

$$\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2} g^{\beta\lambda} \left[\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\nu\mu}\right].$$

Taking the fact that the metric we are looking at is diagonal, then the equation simplifies as

$$\begin{aligned}
\Gamma^{\lambda}_{\nu\mu} &= \frac{1}{2} g^{\beta\lambda} \left[\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\nu\mu}\right] \\
&= \frac{1}{2} g^{\lambda\lambda} \left[\cancelto{0}{\partial_\mu g_{\nu\lambda}} + \cancelto{0}{\partial_\nu g_{\mu\lambda}} - \cancelto{0}{\partial_\lambda g_{\nu\mu}}\right] \\
&= \boxed{0}. \qquad \checkmark
\end{aligned}$$

b) *Show that* $\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2} (g_{\lambda\lambda})^{-1} \partial_\lambda \, g_{\mu\mu}$.

This starts at the same point, but not everything cancels out now:

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\mu} &= \frac{1}{2} g^{\beta\lambda} \left[\partial_\mu g_{\mu\beta} + \partial_\mu g_{\mu\beta} - \partial_\beta g_{\mu\mu}\right] \\
&= \frac{1}{2} g^{\lambda\lambda} \left[\cancelto{0}{\partial_\mu g_{\mu\lambda}} + \cancelto{0}{\partial_\mu g_{\mu\lambda}} - \partial_\lambda g_{\mu\mu}\right] \\
&= \frac{1}{2} g^{\lambda\lambda} \left[-\partial_\lambda g_{\mu\mu}\right] \\
&= \boxed{-\frac{1}{2} (g_{\lambda\lambda})^{-1} \partial_\lambda \, g_{\mu\mu}} \qquad \checkmark
\end{aligned}$$

where the last line is justified by realizing that the metric is diagonal and, thus, the inverse is the reciprocal of the diagonal elements.

c) *Show that* $\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu \left(\ln\left(\sqrt{|g_{\lambda\lambda}|}\right)\right)$.

This one is quite clever, I like it! Let's start where we have started the last two parts (but skipping a couple cancellations in the beginning):

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\lambda} &= \frac{1}{2} g^{\lambda\lambda} \left[\cancelto{0}{\partial_\lambda g_{\mu\lambda}} + \partial_\mu g_{\lambda\lambda} \right] \\
&= \frac{1}{2} \left(g_{\lambda\lambda}\right)^{-1} \partial_\mu \, g_{\lambda\lambda} \\
&= \frac{1}{2} \left(g_{\lambda\lambda}\right)^{-1/2} \left(g_{\lambda\lambda}\right)^{-1/2} \partial_\mu g_{\lambda\lambda} \\
&= \boxed{\partial_\mu \left(\ln\left(\sqrt{|g_{\lambda\lambda}|} \right) \right)}. \qquad \checkmark
\end{aligned}$$

d) *Show that* $\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda \left(\ln\left(\sqrt{|g_{\lambda\lambda}|} \right) \right)$.

If we take the previous part and let $\mu \to \lambda$ then we get to

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda \left(\ln\left(\sqrt{|g_{\lambda\lambda}|} \right) \right).$$

This, however, does overlook some of the cancellations of $g_{\mu\lambda}$. So, let's recalculate the beginning of the derivation.

$$\begin{aligned}
\Gamma^{\lambda}_{\lambda\lambda} &= \frac{1}{2} g^{\lambda\beta} \left[\partial_\lambda g_{\lambda\beta} + \partial_\lambda g_{\lambda\beta} - \partial_\beta g_{\lambda\lambda} \right] \\
&= \frac{1}{2} g^{\lambda\lambda} \left[\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda} \right] \\
&= \frac{1}{2} g^{\lambda\lambda} \left[\partial_\lambda g_{\lambda\lambda} \right]
\end{aligned}$$

and, at this point, we have gotten back to the calculation done in the previous part. So, we will just jump to the conclusion of

$$\Gamma^{\lambda}_{\lambda\lambda} = \boxed{\partial_\lambda \left(\ln\left(\sqrt{|g_{\lambda\lambda}|} \right) \right)}.$$

e) *Find the Christoffel symbols for the Schwarzschild spacetime.*

There are a total of nine that turn out to be non-negative (13 if you count identical ones), those are:

$$\begin{aligned}
\Gamma^{t}_{rt} = \Gamma^{t}_{tr} &= \partial_r \left(\ln \sqrt{|g_{tt}|} \right) \\
&= \left(1 - \frac{2GM}{r} \right)^{-1} \left(\frac{2GM}{r^2} \right) \\
\Gamma^{r}_{rr} &= \left(1 - \frac{2GM}{r} \right)^{-1} \left(\frac{2GM}{r^2} \right) \\
\Gamma^{r}_{\theta\theta} &= -\frac{1}{2} \left(1 - \frac{2GM}{r} \right) 2r \\
&= -(r - 2GM) \\
\Gamma^{r}_{\phi\phi} &= -(r - 2GM) \sin^2(\theta)
\end{aligned}$$

$$\begin{aligned}
\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} &= \partial_r \, \ln\!\left(\sqrt{r^2}\right) = \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}\left(\frac{1}{r^2}\right)\partial_\theta \, r^2 \sin^2(\theta) \\
&= -\sin(\theta)\cos(\theta) \\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \partial_\theta \, \ln\!\left(r^2 \sin^2(\theta)\right) \\
&= \frac{1}{\sin^2(\theta)} \sin(\theta)\cos(\theta) = \cot(\theta) \\
\Gamma^r_{tt} &= \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right) \\
\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} &= \partial_r \, \ln(r\sin(\theta)) = \frac{1}{r} \,.
\end{aligned}$$

Wow, that's a lot of mess. Let's state them as (removing the duplicates) and letting $\theta = \pi/2$:

$$\begin{aligned}
&\Gamma^t_{rt} = \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r^2}\right) && \Gamma^r_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r^2}\right) && \Gamma^r_{\theta\theta} = -(r - 2GM) \\
&\Gamma^r_{tt} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right) && \Gamma^r_{\phi\phi} = -(r - 2GM) && \Gamma^\theta_{r\theta} = \frac{1}{r} \\
&\Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta) = 0 && \Gamma^\phi_{\theta\phi} = \cot(\theta) = 0 && \Gamma^\phi_{r\phi} = \frac{1}{r} \,.
\end{aligned}$$

The rest of the Christoffel symbols are zero.

f) *Verify your answer to part **d** with the geodesic equation.*

So, I found it easier to rederive some of the values instead of manipulating our new equations into the form discussed in class. I start with the previous geodesics equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$

and derive the appropriate part in parallel to the derivation from the equivalent geodesics equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \,.$$

I will be skipping quite a few steps on the former (right) derivations for brevity's sake and both of our sanities'.

For t:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} - \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r^2}\right)\frac{\mathrm{d}x^t}{\mathrm{d}\tau}\frac{\mathrm{d}x^r}{\mathrm{d}\tau} = 0$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{tt}\frac{\mathrm{d}x^t}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} &= 0 \\
\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}g_{tt} + g_{tt}\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} - 0 &= 0 \\
-\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r^2}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} &= 0
\end{aligned}$$

For θ:

$$\frac{\mathrm{d}^2x^\theta}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}x^r}{\mathrm{d}\tau}\frac{\mathrm{d}x^\theta}{\mathrm{d}\tau}=0$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)-\frac{1}{2}\partial_\theta g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}&=0\\ \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}r^2+\frac{\mathrm{d}\theta}{\mathrm{d}\tau}2r\frac{\mathrm{d}r}{\mathrm{d}\tau}&=0\\ \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}&=0\end{aligned}$$

For ϕ:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}=0$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)-\frac{1}{2}\partial_\phi g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}&=0\\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}r^2+\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}2r&=0\\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}&=0\end{aligned}$$

For r:

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\left(\frac{GM}{r^2}\right)\left(1-\frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2&\\ -(r-2GM)\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]&=0\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)-\frac{1}{2}\partial_r g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}&=0\\ \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}g_{rr}+\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}g_{rr}-\frac{1}{2}\left[\partial_r g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2+\partial_r g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\partial_\theta g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\partial_\phi g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]&=0\\ \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\left(\frac{GM}{r^2}\right)\left(1-\frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2&\\ -(r-2GM)\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]&=0\end{aligned}$$

It may be messy, but it works. All four of the free variable, choices gave the same value for both geodesic equations.

Copying down the four geodesics provide:

$$\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2}-\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r^2}\right)\frac{\mathrm{d}x^t}{\mathrm{d}\tau}\frac{\mathrm{d}x^r}{\mathrm{d}\tau}1=0$$

$$\frac{\mathrm{d}^2x^\theta}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}x^r}{\mathrm{d}\tau}\frac{\mathrm{d}x^\theta}{\mathrm{d}\tau}=0$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+\frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}=0$$

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\left(\frac{GM}{r^2}\right)\left(1-\frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2&\\ -(r-2GM)\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]&=0\end{aligned}$$