PHSX 462: HW09

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Griffiths 11.2

Let's start here by simply copying the wavefunctions that we will need:

$$\begin{aligned}
\psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a}\,, \\
\psi_{200} &= \frac{1}{\sqrt{8\pi a^3}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}\,, \\
\psi_{210} &= \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} e^{-r/2a} \cos(\theta)\,, \\
\psi_{21\pm1} &= \mp \frac{1}{\sqrt{64\pi a^3}} \frac{r}{a} e^{-r/2a} \sin(\theta) e^{\pm i\phi}\,.
\end{aligned}$$

Since r has the same degree as z ($r^2 = x^2 + y^2 + z^2$), then we can just do some analysis of each as even or odd, and eliminate each on that. For each of the H'_{ii}, they are even in z; thus, each is zero. The same holds for each of the combinations of $|200\rangle, |211\rangle, |21-1\rangle$ with $|100\rangle$. However, the combination of $|210\rangle$ with $|100\rangle$ is not, so let us look at that:

$$\begin{aligned}
H'_{100,210} &= eE \frac{1}{\sqrt{\pi a^3}} \frac{1}{\sqrt{32\pi a^3}} \frac{1}{a} \int r e^{-r/a} e^{-r/2a} \cos(\theta) z r^2 \sin(\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \\
&= eE \frac{1}{a^4 \sqrt{8}\pi} \int_0^\infty r^4 e^{-3r/2a}\, \mathrm{d}r \int_0^\pi \cos^2(\theta)\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \\
&= \frac{eE}{2\sqrt{2} a^4} 4! \left(\frac{2a}{3}\right)^5 \frac{2}{3} \\
&= \left(\frac{2^{17/2}}{3^5}\right) eEa\,.
\end{aligned}$$

So, every matrix element is zero except

$$\boxed{H'_{100,210} = \left(\frac{2^{17/2}}{3^5}\right) eEa}\,.$$

Griffiths 11.8

a) According to first order perturbation theory, we are looking at

$$\begin{aligned}
c_a^{(1)}(t) &= 1 \\
c_b^{(1)}(t) &= -\frac{i}{\hbar}\int_{t_0}^{t} H'_{ba}(t')e^{i\omega_0 t'}\,\mathrm{d}t' \\
&= -\frac{i}{\hbar}\int_{-\infty}^{t} \frac{\alpha}{\sqrt{\pi}\tau}e^{-(t'/\tau)^2}e^{i\omega t'}\,\mathrm{d}t' \\
&= -\frac{i}{\hbar}\frac{\alpha}{\sqrt{\pi}\tau}\int_{-\infty}^{t} e^{-\frac{1}{\tau^2}\left(t'-\frac{i\omega\tau^2}{2}\right)^2-\frac{\omega^2\tau^2}{4}}
\end{aligned}$$

taking into consideration that we will be taking $t \to \infty$

$$\begin{aligned}
c_b^{(1)}(t) &= -\frac{i\alpha}{\hbar\sqrt{\pi}}e^{-(\omega_0\tau/2)^2}\sqrt{\pi} \\
&= -\frac{i\alpha}{\hbar}e^{-(\omega_0\tau/2)^2}\,.
\end{aligned}$$

The probability of the transition will then be

$$\boxed{P_{a\to b} = \left(\frac{\alpha}{\hbar}\right)^2 e^{-(\omega_0\tau)^2/2}}\,.$$

b) Applying a slight modification on the previous part:

$$c_b^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{\infty}\alpha\delta(t)e^{i\omega t}\,\mathrm{d}t = -\frac{i\alpha}{\hbar}\,.$$

This gives a probability of $(\alpha/\hbar)^2$, which is the first order term of the solution of 11.4 $(\sin^2(\alpha/\hbar))$.

c) The probability gets killed off, as $e^{-(\omega\tau)^2/2} \to 0$. Consequently, we see no transition of states and the hinting at an adiabatic-type system.

Griffiths 11.9

This is post homework Will typing this up; this question was a pain in the butt while doing it, but actually super fun on the other side. Since I am straight exhausted, I am going to be skipping some steps to save my sanity...

a) Plugging in what we are given

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b = -\frac{i}{2\hbar} V_{ab} e^{i(\omega-\omega_0)t} c_b$$
$$\dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a = -\frac{i}{2\hbar} V_{ab} e^{i(\omega_0-\omega)t} c_a \,.$$

We must go down before we can go up, so let us differentiate $\dot{c_a}$:

$$\begin{aligned}
\ddot{c}_a &= -\frac{i}{2\hbar} V_{ab} \left[i(\omega-\omega_0) e^{i(\omega-\omega_0)t} c_b + e^{i(\omega-\omega_0)t} \dot{c}_b \right] \\
&= -\frac{i}{2\hbar} V_{ab} \left[i(\omega-\omega_0) \frac{i2\hbar}{V_{ab}} \dot{c}_a + e^{i(\omega-\omega_0)t} \left(-\frac{i}{2\hbar} \right) V_{ba} e^{i(\omega_0-\omega)t} c_a \right] \\
&= i(\omega-\omega_0)\dot{c}_a - \frac{|V_{ab}|^2}{(2\hbar)^2} c_a \,.
\end{aligned}$$

Rewriting this gives

$$\ddot{c}_a + i(\omega_0-\omega)\dot{c}_a + \frac{|V_{ab}|^2}{(2\hbar)^2} c_a = 0 \,.$$

This is a differential equation, that if we propose an exponential solution in the form of $e^{\lambda t}$, gives

$$\lambda^2 + i(\omega_0-\omega)\lambda + \frac{|V_{ab}|^2}{(2\hbar)^2} = 0$$

and, via the quadratic formula

$$\lambda = -\frac{1}{2} \left[i(\omega-\omega_0) \pm \sqrt{-(\omega_0-\omega)^2 - \frac{4|V_{ab}|^2}{(2\hbar)^2}} \right] = -i \left[\frac{(\omega-\omega_0)}{2} \pm \omega_r \right] \,.$$

the general equation for c_a is then

$$c_a(t) = A e^{-i((\omega-\omega_0)/2+\omega_r)t} + B e^{-i((\omega-\omega_0)/2-\omega_r)t} \,.$$

Using the initial conditions gives $A + B = 1$.

By a very similar derivation, the general solution for c_b is

$$c_a(t) = C e^{-i((\omega-\omega_0)/2+\omega_r)t} + D e^{-i((\omega-\omega_0)/2-\omega_r)t} \,.$$

Using the initial conditions here gives $C + D = 0$.

To solve for the coefficients in the c_b equation, let us take the derivative:

$$\begin{aligned}
\dot{c}_b &= -i\left[\frac{(\omega-\omega_0)}{2}+\omega_r\right]Ce^{-i\left[\frac{(\omega-\omega_0)}{2}+\omega_r\right]} - i\left[\frac{(\omega-\omega_0)}{2}-\omega_r\right]De^{-i\left[\frac{(\omega-\omega_0)}{2}-\omega_r\right]} \\
-\frac{i}{2\hbar}V_{ba}e^{i(\omega_0-\omega)t}c_a &= \quad \text{'' ''} \\
c_a &= \left[\frac{2\hbar}{V_{ba}}\right]\left[\left(\frac{(\omega-omega_0)}{2}+\omega_r\right)Ce^{-i\omega_r t} + \left(\frac{(\omega-omega_0)}{2}-\omega_r\right)De^{i\omega_r t}\right] \\
1 &= \left[\frac{2\hbar}{V_{ba}}\right]\left[\left(\frac{(\omega-omega_0)}{2}+\omega_r\right)C + \left(\frac{(\omega-omega_0)}{2}-\omega_r\right)D\right] \\
1 &= \left[\frac{2\hbar}{V_{ba}}\right][C-D]\omega_r
\end{aligned}$$

$$C - D = \frac{V_{ba}}{2\hbar\omega_r}\,.$$

Combining this with an initial condition gives

$$2C = \frac{V_{ba}}{2\hbar\omega_r} \to C = \frac{V_{ba}}{4\hbar\omega_r} \text{ and } D = -\frac{V_{ba}}{4\hbar\omega_r}\,.$$

At this point I am realizing my flaw in keeping everything in complex exponentials. On my scratch work I worked out what this is, specifically when I plugged in C and D;

$$c_b(t) = -\frac{i}{2\hbar\omega_r}V_{ba}\left[e^{i((\omega-\omega_0)/2)t}\right]\sin(\omega_r t)\,.$$

A very similar process will need to be done with c_a. I will skip typing it all up, but we get

$$\begin{aligned}
c_b &= \frac{V_{ab}}{2\hbar}\left[\left(\frac{\omega_0-\omega}{2}+\omega_r\right)Ae^{-i\omega_r t} + \left(\frac{\omega_0-\omega}{2}-\omega_r\right)Be^{i\omega_r t}\right] \\
0 &= \frac{V_{ab}}{2\hbar}\left[\left(\frac{\omega_0-\omega}{2}+\omega_r\right)A + \left(\frac{\omega_0-\omega}{2}-\omega_r\right)B\right] \\
0 &= \frac{\omega_0-\omega}{2} + \omega_r[A-B]\,.
\end{aligned}$$

Combining this equation with the initial values from before we get

$$2A = 1 + \frac{\omega-\omega_0}{2\omega_r} \to A = \frac{1}{2} + \frac{\omega-\omega}{4\omega_r} \text{ and } B = \frac{1}{2} - \frac{\omega-\omega}{4\omega_r}\,.$$

Plugging these into c_a, and doing some simplifying,

$$c_a(t) = e^{-i((\omega_0-\omega)/2)t}\left[\cos(\omega_r t) + \frac{i(\omega_0-\omega)}{2\omega_r}\sin(\omega_r t)\right]\,.$$

Finally, the two constants are

$$\boxed{\begin{aligned} c_a(t) &= e^{-i((\omega_0-\omega)/2)t}\left[\cos(\omega_r t) + \tfrac{i(\omega_0-\omega)}{2\omega_r}\sin(\omega_r t)\right] \\ c_b(t) &= -\tfrac{i}{2\hbar\omega_r}V_{ba}\left[e^{i((\omega-\omega_0)/2)t}\right]\sin(\omega_r t)\end{aligned}}$$

Griffiths 11.11

Griffiths 11.13