Derivation of EWMA-based variance:

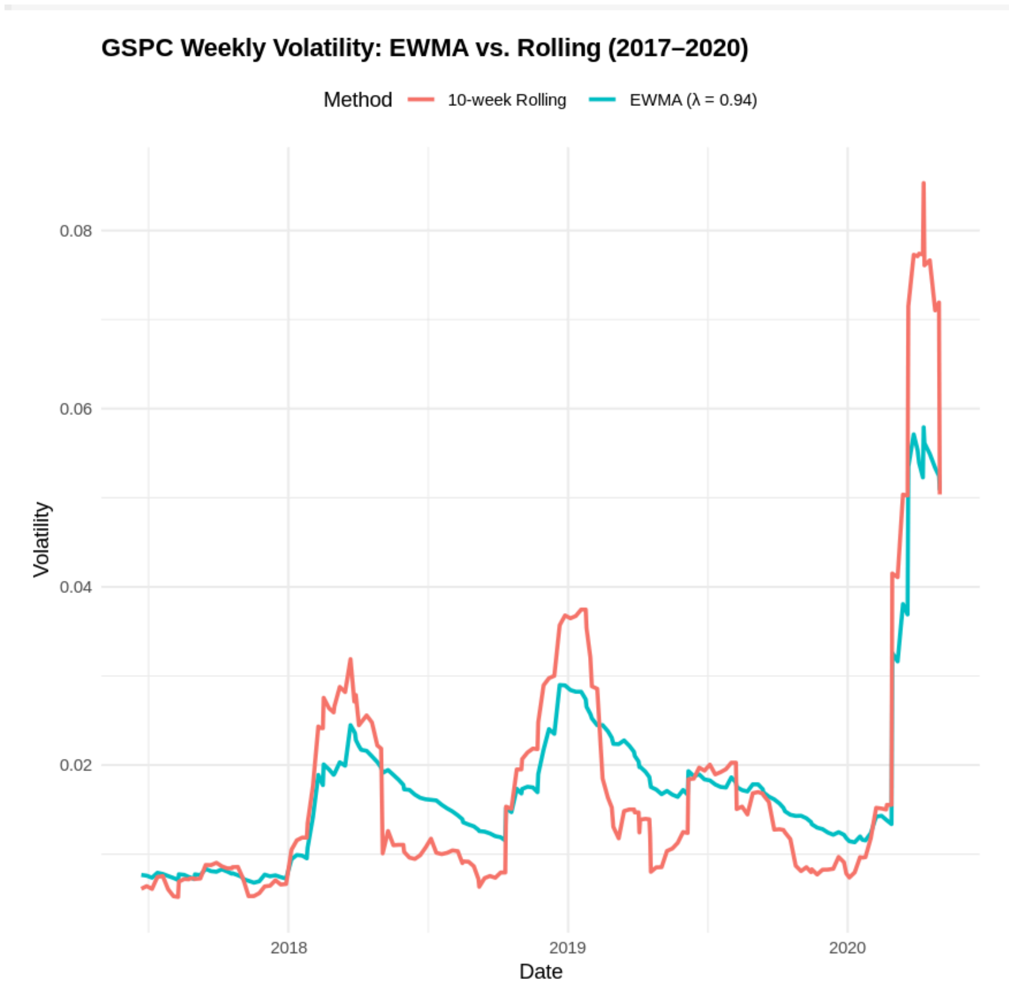
# 1

$$
\begin{aligned}
\sigma^2_{\mathrm{ewma}}(t) &= \lambda \sigma^2_{\mathrm{ewma}}(t-1) + (1-\lambda) r^2_{t-1} \\
&= \lambda \left[ \lambda \sigma^2_{\mathrm{ewma}}(t-2) + (1-\lambda) r^2_{t-2} \right] + (1-\lambda) r^2_{t-1} \\
&= \lambda^2 \sigma^2_{\mathrm{ewma}}(t-2) + (1-\lambda) \left( r^2_{t-1} + \lambda r^2_{t-2} \right) \\
&= \cdots \\
&= \lambda^N \sigma^2_{\mathrm{ewma}}(t-N) + (1-\lambda) \sum_{k=1}^{N} \lambda^{k-1} r^2_{t-k}
\end{aligned}
$$

Since $0 < \lambda < 1$, $\lambda^N$ is close to zero for large enough $N$. Therefore,

$$
\sigma^2_{\mathrm{ewma}}(t) = (1-\lambda) \sum_{k=1}^{N} \lambda^{k-1} r^2_{t-k}
$$

We compared the estimated volatility of GSPC weekly returns from 2017 to 2020 using two methods: EWMA with λ = 0.94 and a 10-week rolling window (standard averaging recent samples of square returns). Both methods generally follow similar trends, but the EWMA-based estimate reacts more smoothly and promptly to sudden changes in market conditions. For example, during the sharp market decline in early 2020, EWMA volatility rises rapidly, while the rolling estimate lags and exhibits more jagged fluctuations. This demonstrates the EWMA method's advantage in capturing recent volatility shocks with greater sensitivity, making it more suitable for real-time risk management.



GSPC Weekly Volatility: EWMA vs. Rolling (2017–2020)
Method
10-week Rolling
EWMA (λ = 0.94)
Volatility
0.08
0.06
0.04
0.02
2018
2019
2020
Date

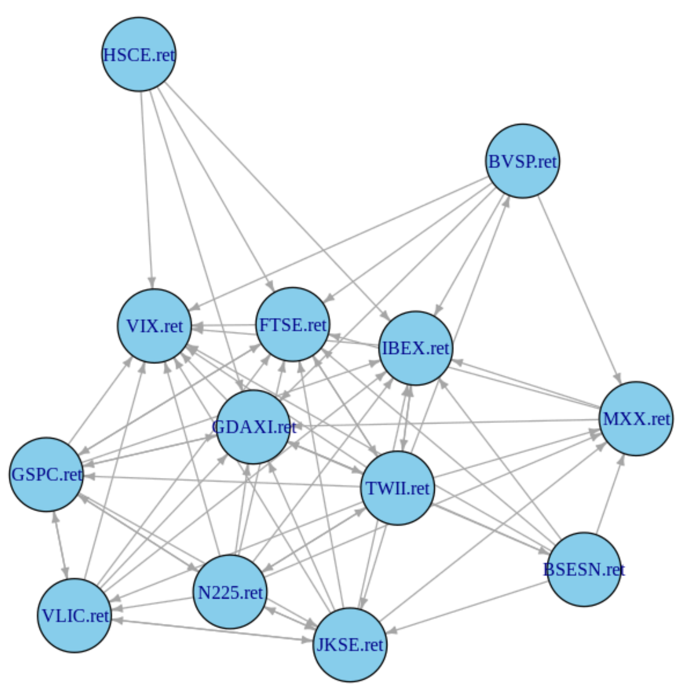
Results of causality analysis:

**<u>Weekly returns:</u>**

A data.frame: 13 × 13

| | BSESN.ret | BVSP.ret | FTSE.ret | GDAXI.ret | GSPC.ret | HSCE.ret | IBEX.ret | JKSE.ret | MXX.ret | N225.ret | TWII.ret | VIX.ret | VLIC.ret |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> |
| **BSESN.ret** | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) |
| **BVSP.ret** | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,1,1) | (0,0,0) | (0,0,0) |
| **FTSE.ret** | (0,1,1) | (0,1,1) | - | (0,0,0) | (1,1,1) | (0,1,1) | (0,0,0) | (0,1,1) | (0,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (0,1,1) |
| **GDAXI.ret** | (0,1,1) | (0,1,1) | (0,0,0) | - | (1,1,1) | (0,1,1) | (0,0,0) | (0,1,1) | (0,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (0,1,1) |
| **GSPC.ret** | (0,0,0) | (0,0,0) | (1,0,0) | (1,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,1,1) | (0,1,1) | (0,0,0) | (1,1,0) |
| **HSCE.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **IBEX.ret** | (0,1,1) | (0,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,1,1) | - | (1,1,1) | (0,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (0,1,1) |
| **JKSE.ret** | (0,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,1,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (1,1,1) | (0,1,1) | (0,0,0) | (1,1,1) |
| **MXX.ret** | (0,1,0) | (0,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | - | (0,1,1) | (0,1,1) | (0,0,0) | (0,0,0) |
| **N225.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | - | (1,1,1) | (0,0,0) | (0,0,0) |
| **TWII.ret** | (0,0,0) | (0,0,0) | (0,1,0) | (0,1,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | - | (0,0,0) | (0,0,0) |
| **VIX.ret** | (0,1,1) | (0,1,1) | (0,1,1) | (0,1,1) | (0,1,1) | (1,1,1) | (0,1,1) | (0,1,0) | (0,0,0) | (0,1,1) | (1,1,1) | - | (0,1,1) |
| **VLIC.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (0,1,0) | (0,0,0) | (0,1,1) | (1,1,1) | (0,0,0) | - |

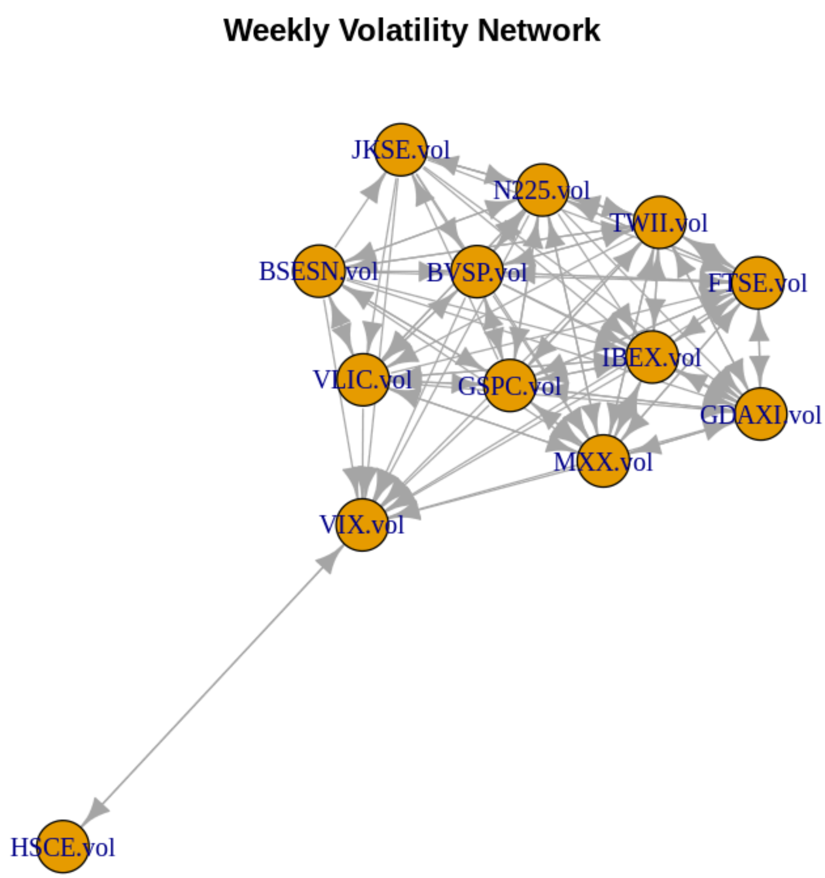
**Weekly Return Causality Network**



## Weekly volatility:

A data.frame: 13 × 13

| | BSESN.vol | BVSP.vol | FTSE.vol | GDAXI.vol | GSPC.vol | HSCE.vol | IBEX.vol | JKSE.vol | MXX.vol | N225.vol | TWII.vol | VIX.vol | VLIC.vol |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> |
| **BSESN.vol** | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (1,0,1) | (0,0,0) | (0,1,1) |
| **BVSP.vol** | (1,1,1) | - | (0,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (0,0,1) | (0,0,0) | (1,1,1) | (1,0,1) | (0,0,0) | (0,1,1) |
| **FTSE.vol** | (1,1,1) | (1,1,1) | - | (0,1,1) | (1,1,1) | (0,0,0) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) |
| **GDAXI.vol** | (1,1,1) | (1,1,1) | (0,1,1) | - | (1,1,1) | (0,0,0) | (0,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) |
| **GSPC.vol** | (0,1,1) | (0,0,1) | (0,0,1) | (0,0,1) | - | (0,0,0) | (0,1,1) | (0,0,0) | (0,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (0,1,1) |
| **HSCE.vol** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | (0,0,0) |
| **IBEX.vol** | (1,1,1) | (1,1,1) | (1,1,1) | (0,1,1) | (1,1,1) | (0,0,0) | - | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) |
| **JKSE.vol** | (0,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (1,0,1) | (0,0,0) | (0,0,0) | - | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (0,0,0) |
| **MXX.vol** | (1,1,1) | (1,1,1) | (0,0,1) | (0,0,1) | (1,1,1) | (0,0,0) | (0,0,1) | (1,1,1) | - | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) |
| **N225.vol** | (1,1,1) | (1,1,1) | (0,1,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (1,1,1) | - | (1,1,1) | (0,0,0) | (1,1,1) |
| **TWII.vol** | (1,1,1) | (0,1,1) | (0,1,1) | (0,1,1) | (1,1,1) | (0,0,0) | (1,1,1) | (1,1,1) | (0,0,1) | (1,1,1) | - | (0,0,0) | (0,0,0) |
| **VIX.vol** | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | - | (1,1,1) |
| **VLIC.vol** | (1,1,1) | (0,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,1) | (0,1,1) | (0,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | - |

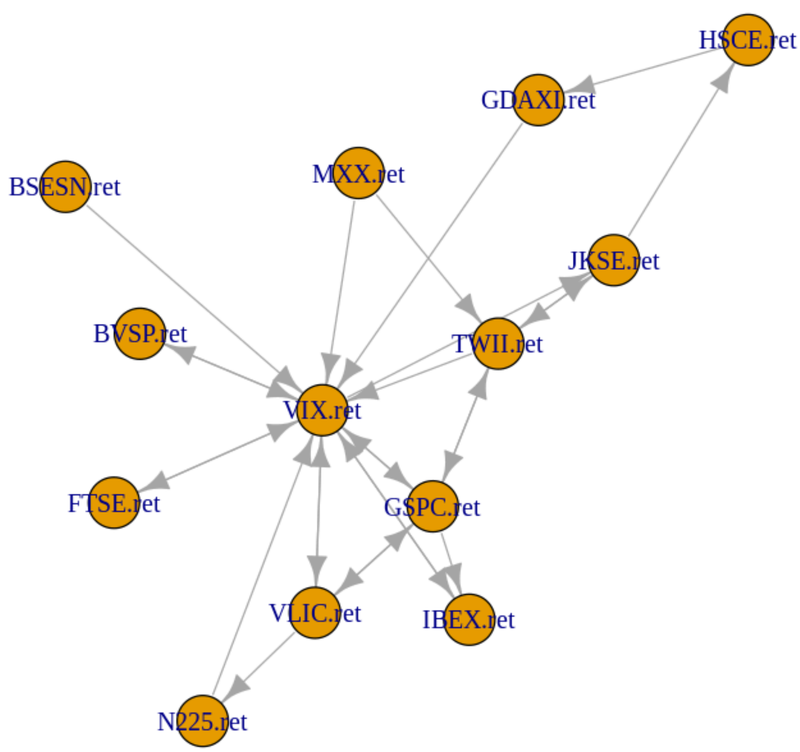


**Monthly returns:**

A data.frame: 13 × 13

| | BSESN.ret | BVSP.ret | FTSE.ret | GDAXI.ret | GSPC.ret | HSCE.ret | IBEX.ret | JKSE.ret | MXX.ret | N225.ret | TWII.ret | VIX.ret | VLIC.ret |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> |
| **BSESN.ret** | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **BVSP.ret** | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | (0,0,0) |
| **FTSE.ret** | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,0,1) | (0,0,0) |
| **GDAXI.ret** | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **GSPC.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | (1,1,1) | (1,0,1) |
| **HSCE.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (1,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **IBEX.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,0,0) | (0,0,0) |
| **JKSE.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,1,1) | (1,0,0) | (0,0,0) |
| **MXX.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **N225.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (1,1,0) |
| **TWII.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,1) | (0,0,0) | (0,0,0) | (1,1,0) | (0,1,1) | (0,0,0) | - | (0,0,0) | (0,0,0) |
| **VIX.ret** | (1,1,1) | (1,0,0) | (1,1,1) | (1,0,1) | (1,0,1) | (0,0,0) | (1,0,1) | (0,0,0) | (1,0,1) | (0,1,0) | (1,1,1) | - | (1,1,1) |
| **VLIC.ret** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,0,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | - |

**Monthly Return Network**

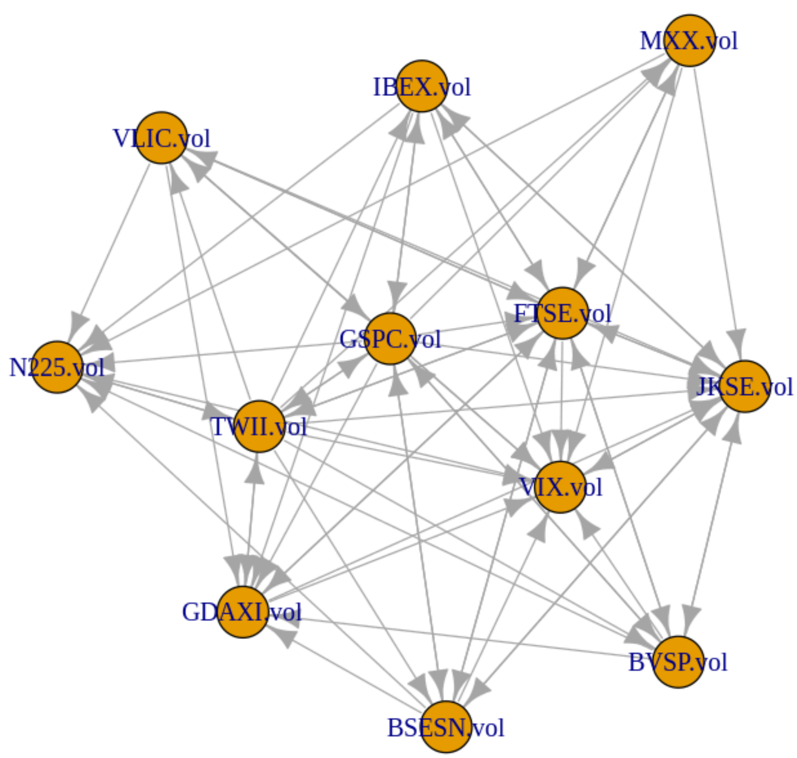


## Monthly volatility:

A data.frame: 13 × 13

| | BSESN.vol | BVSP.vol | FTSE.vol | GDAXI.vol | GSPC.vol | HSCE.vol | IBEX.vol | JKSE.vol | MXX.vol | N225.vol | TWII.vol | VIX.vol | VLIC.vol |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> | <chr> |
| **BSESN.vol** | - | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,0) | (0,0,0) | (0,0,0) |
| **BVSP.vol** | (0,0,0) | - | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,0) | (0,0,0) | (0,0,0) |
| **FTSE.vol** | (1,1,1) | (1,1,1) | - | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) |
| **GDAXI.vol** | (1,1,1) | (1,1,1) | (1,0,0) | - | (1,1,1) | (0,0,0) | (0,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) |
| **GSPC.vol** | (0,1,1) | (0,1,1) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,1,1) |
| **HSCE.vol** | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) |
| **IBEX.vol** | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | - | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,0) | (0,0,0) | (0,0,0) |
| **JKSE.vol** | (1,1,1) | (1,1,1) | (0,1,1) | (1,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | - | (0,0,1) | (0,0,0) | (1,1,1) | (1,0,0) | (1,0,1) |
| **MXX.vol** | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,1) | (0,0,0) | (0,0,0) | (0,0,0) | - | (0,0,0) | (1,1,0) | (0,0,0) | (0,0,0) |
| **N225.vol** | (1,1,1) | (1,1,1) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | - | (1,1,1) | (0,0,0) | (1,1,1) |
| **TWII.vol** | (0,0,0) | (0,0,0) | (1,1,1) | (1,1,0) | (0,0,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,1,1) | - | (0,0,0) | (0,0,0) |
| **VIX.vol** | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (0,0,0) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | (1,1,1) | - | (1,1,1) |
| **VLIC.vol** | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | (1,1,1) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (0,0,0) | (1,1,1) | (0,0,0) | - |

### Monthly Volatility Network



We conducted the Granger causality analysis at the 5% significance level. While dealing with the data we decided to completely omit NA values instead of using interpolation or forward-fill as simply filling in the missing values could artificially create autocorrelation or a kind of bias, which would significantly alter the results for causality testing.

Overall, analysis revealed substantial interdependence between global equity markets, especially in weekly data. Weekly return networks indicate that developed markets like the United States (GSPC), United Kingdom (FTSE), Germany (GDAXI), and Japan (N225) frequently cause movements in emerging markets such as India (BSESN), Mexico (MXX), and Indonesia (JKSE). Volatility causality is even more pervasive than return causality, with the United States Volatility Index (VIX) acting as a global driver of volatility across nearly all markets — consistent with its role as a proxy for global risk aversion.

The monthly return network shows far sparser causality, suggesting that return predictability weakens at lower frequencies, though volatility spillovers persist strongly. Notably, contemporaneous correlation appears in many bidirectional relationships (e.g., United Kingdom ↔ Germany, United States ↔ VLIC), reflecting high levels of financial integration across developed markets.

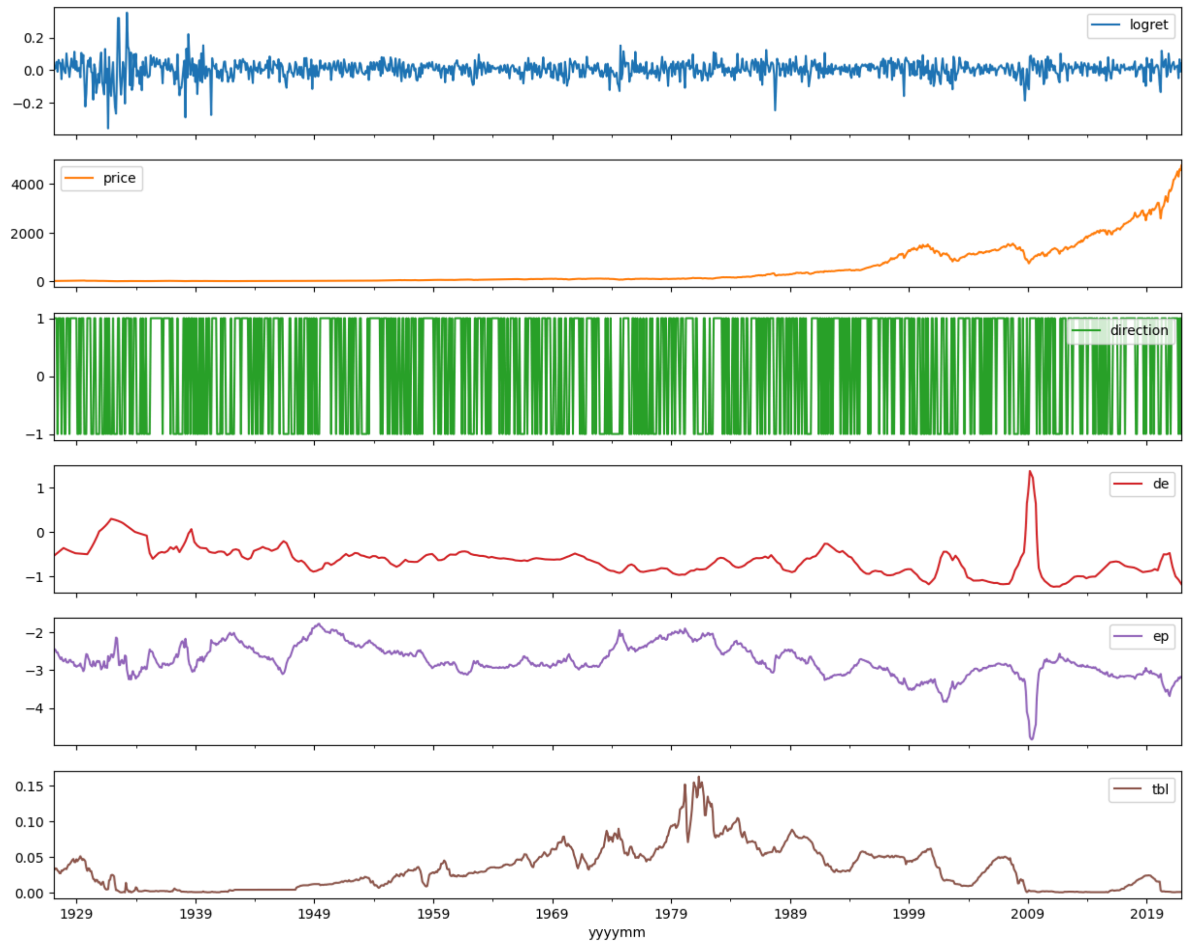
These causalities align closely with major economic events during 2017–2020, such as the COVID-19 pandemic, which triggered synchronized spikes in global volatility, as well as US-China trade tensions and Brexit uncertainty, which heightened interconnectedness between major markets. The four directed network graphs clearly illustrate these findings, with central nodes (United States, VIX) exerting widespread influence in both return and volatility networks.

### General Overview:

Financial indicators assigned to our team were **ep, de, tbl** which denote Earnings to Price Ratio, Dividend Payout Ratio and Treasury Bill Rate respectively.

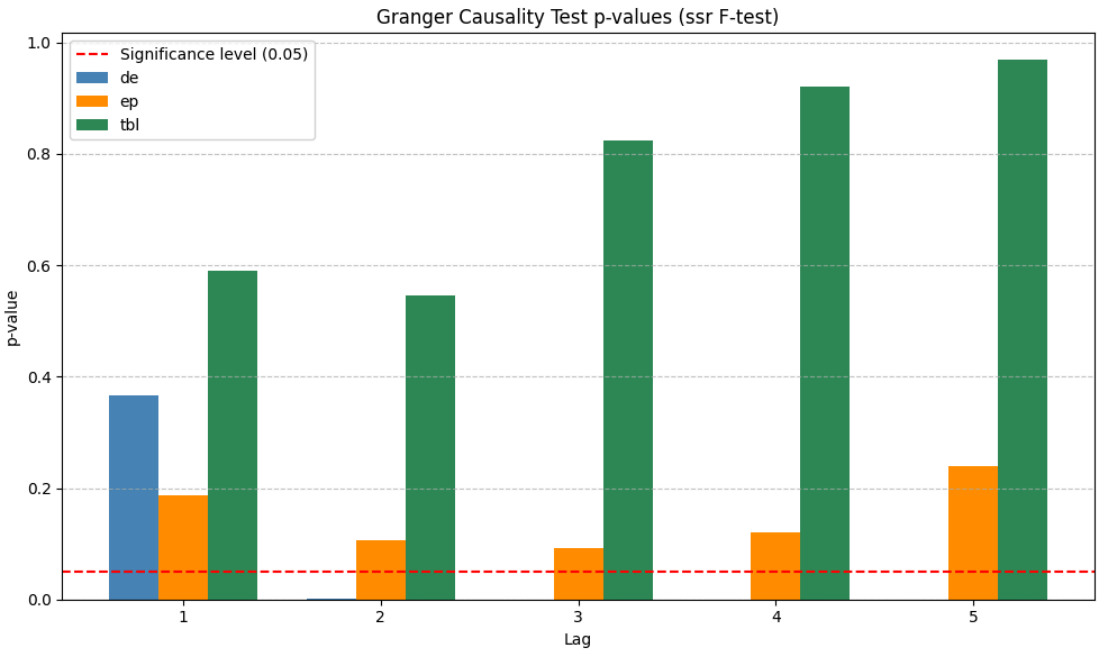
We made predictions for SP500 price, returns and direction using Neural Network, Long Short Term Memory and Gaussian Process.

Overview of targets and main explanatory variables:




To select appropriate lags and features we conducted Granger Causality tests.

## Results of the Granger Test For Price:




```
                            AutoReg Model Results
==============================================================================
Dep. Variable:                  price   No. Observations:                 1139
Model:                    AutoReg(12)   Log Likelihood               -5744.149
Method:               Conditional MLE   S.D. of innovations             39.564
Date:                Sat, 03 May 2025   AIC                          11516.298
Time:                        10:10:18   BIC                          11586.680
Sample:                    02-01-1928   HQIC                         11542.892
                         - 12-01-2021
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -1.0162      1.417     -0.717      0.473      -3.793       1.760
price.L1       0.9390      0.030     31.304      0.000       0.880       0.998
price.L2      -0.0202      0.041     -0.488      0.626      -0.102       0.061
price.L3       0.0594      0.042      1.405      0.160      -0.023       0.142
price.L4       0.0608      0.044      1.382      0.167      -0.025       0.147
price.L5      -0.0267      0.045     -0.594      0.553      -0.115       0.061
price.L6      -0.0144      0.044     -0.325      0.745      -0.101       0.073
price.L7       0.1929      0.044      4.346      0.000       0.106       0.280
price.L8      -0.0718      0.045     -1.601      0.109      -0.160       0.016
price.L9      -0.1499      0.045     -3.311      0.001      -0.239      -0.061
price.L10      0.1321      0.045      2.917      0.004       0.043       0.221
price.L11     -0.0874      0.045     -1.934      0.053      -0.176       0.001
price.L12     -0.0050      0.032     -0.156      0.876      -0.068       0.058
```

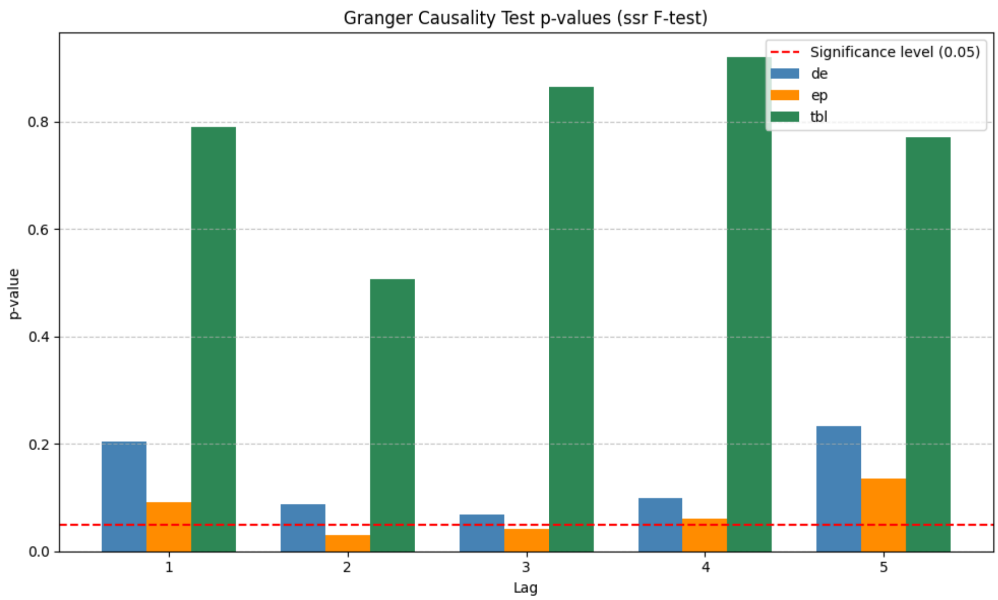
As we can see from the graph, only the lags of DE (except for the 1st one) showed causality for price. Also when checking for causality of lags of the target itself only 1st, 7th, 9th and 10th lags were significant.

Additionally, We did an OLS using selected variables to see the significance once again. As we can see from the results below, all explanatory variables are significant at 5% level except for the 10th lag of the target. (we decided to omit 4th and 5th lags of de as they had extremely high p-values and showed no statistical significance due to their correlation with 2nd and 3rd lags).

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                  price   R-squared:                       0.998
Model:                            OLS   Adj. R-squared:                  0.998
Method:                 Least Squares   F-statistic:                 8.119e+04
Date:                Sat, 03 May 2025   Prob (F-statistic):               0.00
Time:                        10:10:20   Log-Likelihood:                -5751.2
No. Observations:                1127   AIC:                         1.152e+04
Df Residuals:                    1120   BIC:                         1.155e+04
Df Model:                           6
Covariance Type:            nonrobust
================================================================================
                  coef    std err          t      P>|t|      [0.025      0.975]
--------------------------------------------------------------------------------
const         532.6787      1.190    447.757      0.000     530.344     535.013
price_lag1    796.5817     11.921     66.823      0.000     773.192     819.971
price_lag7    133.3478     19.793      6.737      0.000      94.512     172.184
price_lag9   -143.2028     27.532     -5.201      0.000    -197.222     -89.183
price_lag10    44.4862     23.769      1.872      0.062      -2.150      91.122
de_lag2       -20.2379     10.003     -2.023      0.043     -39.864      -0.611
de_lag3        22.8862      9.945      2.301      0.022       3.372      42.400
==============================================================================
Omnibus:                      501.756   Durbin-Watson:                   2.056
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            31701.475
Skew:                          -1.211   Prob(JB):                         0.00
Kurtosis:                      28.869   Cond. No.                         61.2
==============================================================================
```

It is important to note that the price target was not normalized (standardized) while the lags as features were. This is why the regression coefficients in the table above are so big.

## Results of the Granger Test For Returns:




```
                            AutoReg Model Results
==============================================================================
Dep. Variable:                 logret   No. Observations:                 1139
Model:                    AutoReg(12)   Log Likelihood                1703.872
Method:               Conditional MLE   S.D. of innovations              0.053
Date:                Wed, 30 Apr 2025   AIC                          -3379.745
Time:                        14:48:35   BIC                          -3309.362
Sample:                    02-01-1928   HQIC                         -3353.150
                         - 12-01-2021
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0040      0.002      2.429      0.015       0.001       0.007
logret.L1      0.0761      0.030      2.556      0.011       0.018       0.134
logret.L2     -0.0194      0.030     -0.650      0.516      -0.078       0.039
logret.L3     -0.0890      0.030     -2.980      0.003      -0.148      -0.030
logret.L4      0.0365      0.030      1.217      0.223      -0.022       0.095
logret.L5      0.0755      0.030      2.521      0.012       0.017       0.134
logret.L6     -0.0449      0.030     -1.497      0.134      -0.104       0.014
logret.L7      0.0398      0.030      1.326      0.185      -0.019       0.099
logret.L8      0.0519      0.030      1.734      0.083      -0.007       0.111
logret.L9      0.0356      0.030      1.188      0.235      -0.023       0.094
logret.L10     0.0061      0.030      0.204      0.838      -0.052       0.065
logret.L11     0.0042      0.030      0.142      0.887      -0.054       0.063
logret.L12     0.0176      0.030      0.592      0.554      -0.041       0.076
```

As we can see from the graph, only two of the variables showed causality for logret which are 2nd and 3rd lags of EP. Also when checking for causality of lags of the target itself only 1st, 3rd and 5th lags were significant.

Additionally, We did an OLS using selected variables to see the significance once again. As we can see from the results below, only the lags of the target are significant at 5% level.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                 logret   R-squared:                       0.021
Model:                            OLS   Adj. R-squared:                  0.017
Method:                 Least Squares   F-statistic:                     4.781
Date:                Wed, 30 Apr 2025   Prob (F-statistic):           0.000250
Time:                        14:49:05   Log-Likelihood:                 1699.8
No. Observations:                1127   AIC:                            -3388.
Df Residuals:                    1121   BIC:                            -3357.
Df Model:                           5
Covariance Type:            nonrobust
================================================================================
                  coef    std err          t      P>|t|      [0.025      0.975]
--------------------------------------------------------------------------------
const           0.0050      0.002      3.109      0.002       0.002       0.008
ep_lag2         0.0165      0.010      1.607      0.108      -0.004       0.037
ep_lag3        -0.0146      0.010     -1.415      0.157      -0.035       0.006
logret_lag1     0.0038      0.002      2.371      0.018       0.001       0.007
logret_lag3    -0.0042      0.002     -2.593      0.010      -0.007      -0.001
logret_lag5     0.0037      0.002      2.290      0.022       0.001       0.007
==============================================================================
Omnibus:                      209.773   Durbin-Watson:                   1.989
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             2182.420
Skew:                          -0.526   Prob(JB):                         0.00
Kurtosis:                       9.736   Cond. No.                         12.8
==============================================================================
```

Anyway, we decided to keep the lags of ep as features, as they passed the Granger Causality test.

## Results of the Feature Selection For Direction:

As direction is a binary variable, we cannot perform Granger Causality tests or run LASSO regressions on it, so we decided to fit a Random Forest model to select features and these are the top 6 features by importance:

```
Top 6 features:
ep_lag1     0.066091
ep_lag5     0.065608
ep_lag4     0.062833
de_lag4     0.061309
ep_lag3     0.060794
de_lag1     0.060587
dtype: float64
```

**Before running the models we splitted the sample into train (75%) and test (25%).*

## Results of Neural Network:

Normalized RMSE table by different number of layers and activation functions:

*For Price:*

| | 1 layer(s) | 2 layer(s) | 3 layer(s) | 4 layer(s) |
|---|---|---|---|---|
| identity | 0.052310 | 0.047692 | 0.051467 | 0.049700 |
| logistic | 0.047410 | 0.340115 | 0.420011 | 0.430247 |
| tanh | 0.052827 | 0.060945 | 0.109541 | 0.128435 |
| relu | 0.079015 | 0.069603 | 0.093733 | 0.144652 |

*For Returns:*

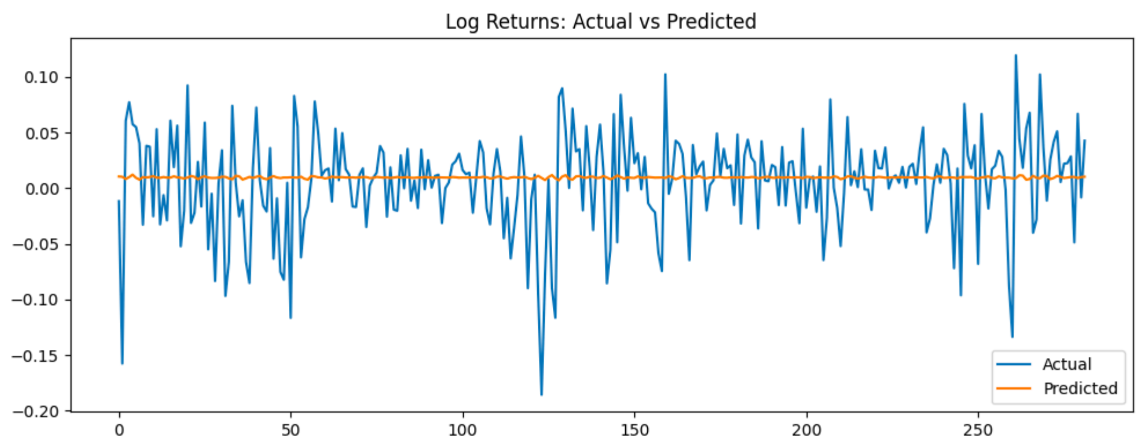
| | 1 layer(s) | 2 layer(s) | 3 layer(s) | 4 layer(s) |
|---|---|---|---|---|
| identity | 0.148306 | 0.260459 | 0.149757 | 0.152067 |
| logistic | 0.150550 | 0.148527 | 0.147157 | 0.148175 |
| tanh | 0.149461 | 0.846593 | 0.148328 | 0.148902 |
| relu | 0.163749 | 0.184807 | 0.166829 | 0.183144 |

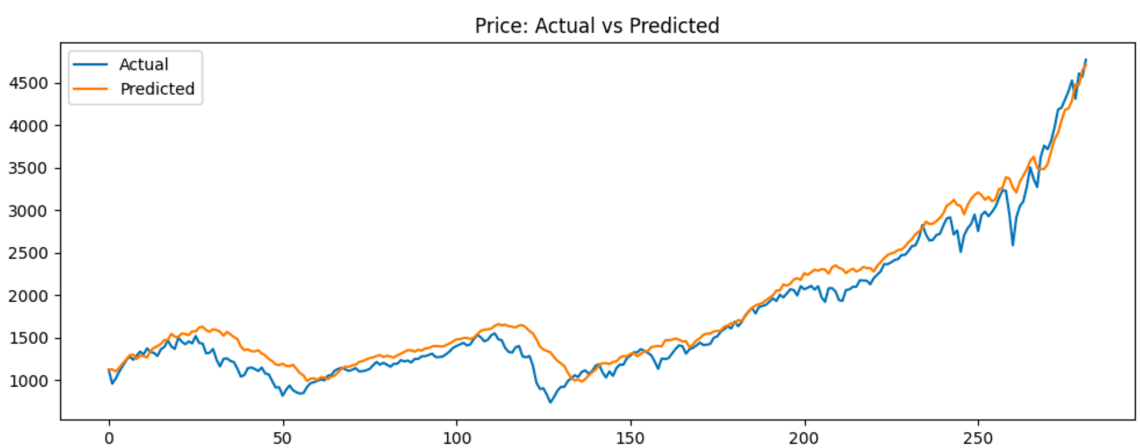
*Accuracy for Direction:*

| | 1 layer(s) | 2 layer(s) | 3 layer(s) | 4 layer(s) |
|---|---|---|---|---|
| identity | 0.606383 | 0.602837 | 0.606383 | 0.606383 |
| logistic | 0.606383 | 0.624113 | 0.624113 | 0.624113 |
| tanh | 0.602837 | 0.609929 | 0.595745 | 0.613475 |
| relu | 0.624113 | 0.631206 | 0.599291 | 0.613475 |

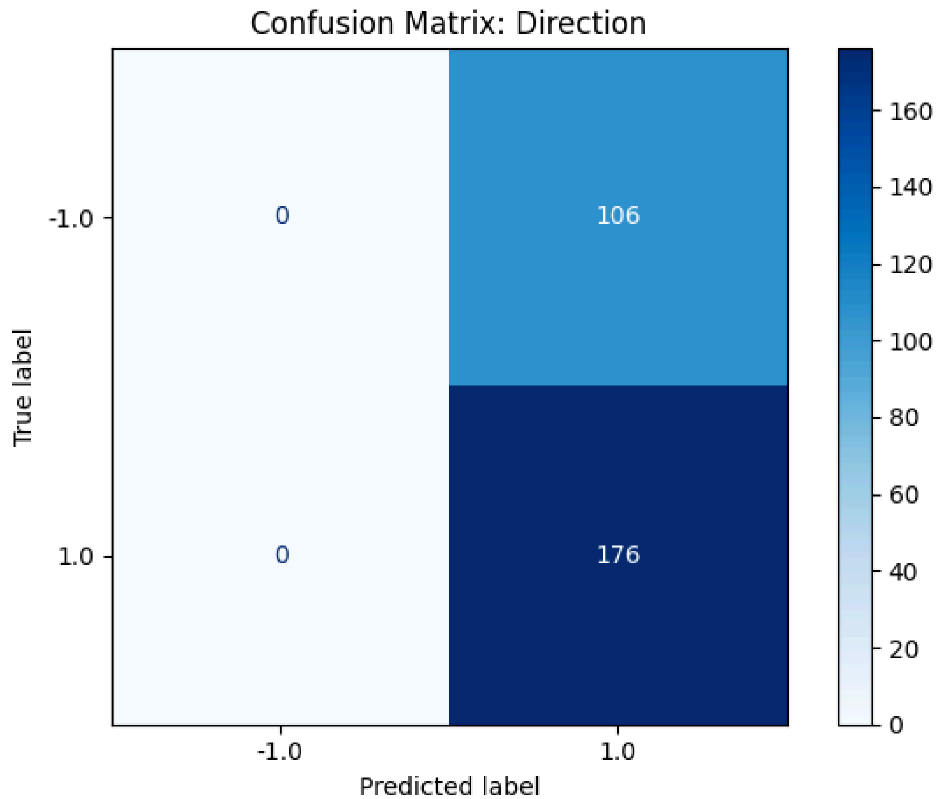
As we can see, Neural Networks with the lowest NRMSE were the ones using Logistic function with 1 layer for Price, Logistic function with 3 layers for Returns. For direction, the model having the highest accuracy is using the Logistic function with 2 layers for Direction. In this last case, we can see that many other models perform at the same accuracy. This is because all that models are predicting direction to be up as many observations in the training dataset were up. As we will see, this is a problem quite recurrent for all models.

## Actual vs Predicted (Neural Network):




As we can see, predicted prices closely follow the actual trend with minor lag, indicating good performance. On the other hand, for log returns predictions are nearly flat around zero, suggesting the neural network struggles to learn from the noisy return data.




The model predicts all cases as upward movement, resulting in high true positives. But it has very poor detection of downward movement, with 106 false positives. As financial data is mostly trending up, the model has bias toward the positive class. Especially in indices like the S&P 500, positive movements outweigh negatives in frequency and magnitude over time.

We also conducted the **sensitivity analysis** of the target with three best models for 1% change in explanatory variables.

*Sensitivity table for Price Target:*

| | price_lag1 | price_lag7 | price_lag9 | price_lag10 | de_lag2 | de_lag3 |
|---|---|---|---|---|---|---|
| +1% | 2.083937 | 1.320659 | 1.083932 | 1.084929 | 0.004724 | 0.006 |
| -1% | -2.083937 | -1.320659 | -1.083932 | -1.084929 | -0.004724 | -0.006 |

*Sensitivity table for Returns Target:*

| | ep_lag2 | ep_lag3 | logret_lag1 | logret_lag3 | logret_lag5 |
|---|---|---|---|---|---|
| +1% | 6.385091e-07 | -4.772658e-08 | 0.000002 | -4.216226e-07 | 0.000006 |
| -1% | -6.385184e-07 | 4.771405e-08 | -0.000002 | 4.216228e-07 | -0.000006 |

*Sensitivity table for Direction Target:*

| | ep_lag1 | ep_lag2 | ep_lag3 | ep_lag4 | ep_lag5 |
|---|---|---|---|---|---|
| +1% | -6.434170e-08 | -6.441038e-08 | -4.804403e-08 | -5.999324e-08 | -5.955772e-08 |
| -1% | 6.435526e-08 | 6.442690e-08 | 4.805377e-08 | 6.000775e-08 | 5.957038e-08 |

| | de_lag5 |
|---|---|
| +1% | 2.464131e-08 |
| -1% | -2.464518e-08 |

For the Price Target, the model is most sensitive to price lags while de_lag2 and de_lag3 have a minimal effect, suggesting that Dividend-Equity Ratio lags are less important for price prediction in this network.

For the Returns Target, sensitivity values are very small implying weak influence of explanatory variables. Main contributors are logret_lag1, logret_lag5, but their effect is still very small.

For the Direction Target, all values are tiny suggesting that no single feature strongly drives predicted class probability.

Overall, the sensitivity analysis reveals that the neural network for price prediction is most responsive to its own recent lags, particularly price_lag1 and price_lag10, suggesting a strong autoregressive structure. In contrast, the models for returns and direction show minimal sensitivity to individual explanatory variables, indicating either model underfitting or the need for more relevant features. These findings reinforce that NN performance varies by target, with clear strengths in price forecasting but weaker explanatory power for returns and direction.

### **Results of Gaussian Process using Non-stationary Kernel :**

```
NRMSE for logret: 1.0283
NRMSE for price: 0.6037
Direction accuracy: 0.6099
```

*We also conducted sensitivity analysis of the Price Target for 1% change in explanatory variables, as the non-stationary prediction of price performs better than the stationary one:*

| | price_lag1 | price_lag7 | price_lag9 | price_lag10 | de_lag2 | de_lag3 |
|---|---|---|---|---|---|---|
| +1% | 1.266297 | 2.267301 | 1.059538 | 1.037050 | 0.149695 | 1.052427 |
| -1% | -1.283029 | -2.265563 | -1.057772 | -1.038753 | -0.145917 | -1.056356 |

### **Results of Gaussian Process using Stationary Kernel :**

```
NRMSE for logret: 1.0050
NRMSE for price: 0.9728
Direction accuracy: 0.6241
```

*We also conducted sensitivity analysis of the Returns Target for 1% change in explanatory variables, as the stationary prediction of returns performs better than the non-stationary one:*

| | ep_lag2 | ep_lag3 | logret_lag1 | logret_lag3 | logret_lag5 |
|---|---|---|---|---|---|
| +1% | -1.344498e-07 | -3.373655e-07 | 9.759718e-08 | 1.213024e-09 | -3.988386e-07 |
| -1% | 1.285158e-07 | 3.901377e-07 | -9.598690e-08 | -1.259661e-09 | 4.877763e-07 |

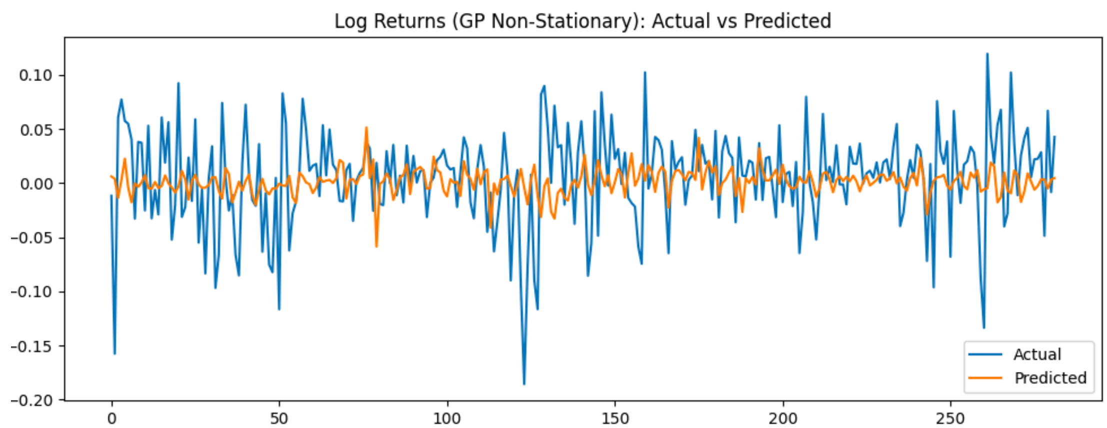
*We also conducted sensitivity analysis of the Direction Target for 1% change in explanatory variables, as the stationary prediction of direction performs better than the non-stationary one:*

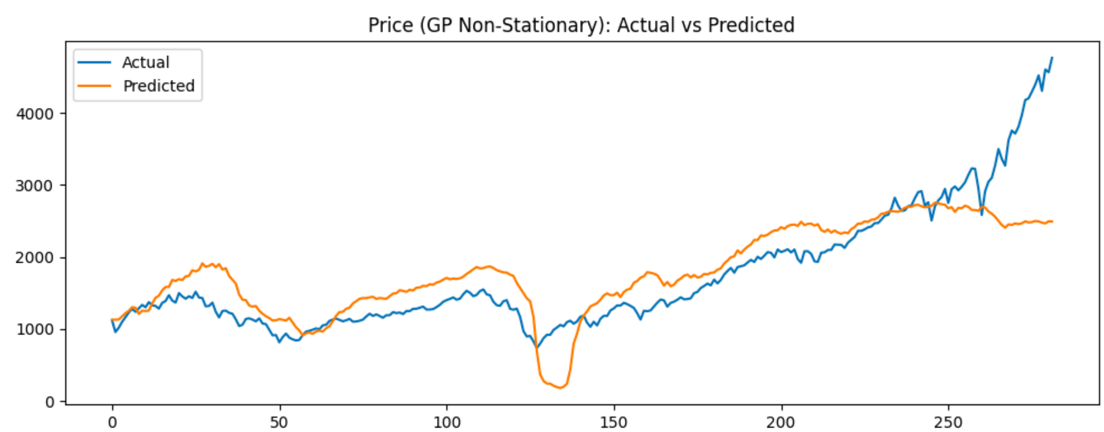
| | ep_lag1 | ep_lag2 | ep_lag3 | ep_lag4 | ep_lag5 |
|---|---|---|---|---|---|
| +1% | 0.000001 | 6.948377e-07 | 7.442336e-07 | 6.555753e-07 | 3.911747e-07 |
| -1% | -0.000001 | -6.735044e-07 | -7.217353e-07 | -6.341151e-07 | -3.733808e-07 |

| | de_lag5 |
|---|---|
| +1% | 0.000002 |
| -1% | -0.000002 |

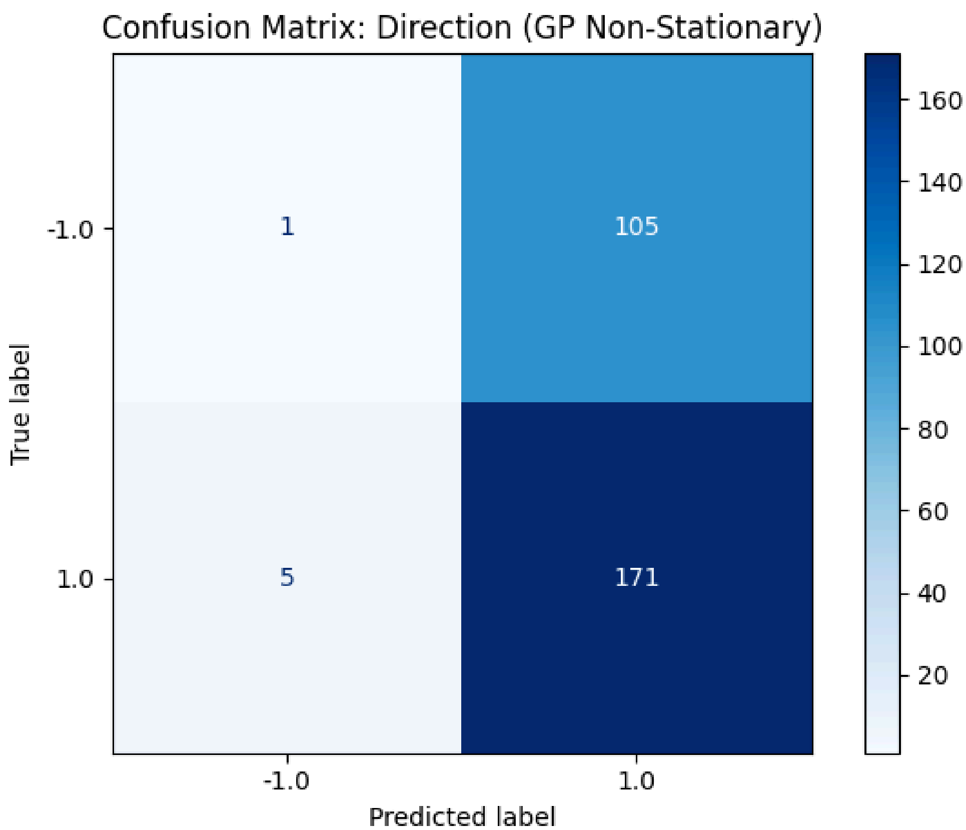
Overall, the sensitivity analysis shows that the non-stationary Gaussian Process model for price reacts most strongly to recent price lags, particularly price_lag7 and price_lag1, indicating that it captures dominant short-term dependencies in past prices. While fundamental variables like de_lag2 and de_lag3 also have some influence, their effect is noticeably smaller. In contrast, the stationary Gaussian Process model for returns shows extremely low sensitivity across all variables, suggesting it is very stable but likely underfits the data. Similarly, for direction prediction, the sensitivity values are extremely small implying that the model does not rely heavily on any individual feature to determine directional movements. This confirms that the non-stationary model is more flexible and responsive to complex patterns, while the stationary model is overly conservative and less adaptive.

## Actual vs Predicted (Gaussian Non-stationary):



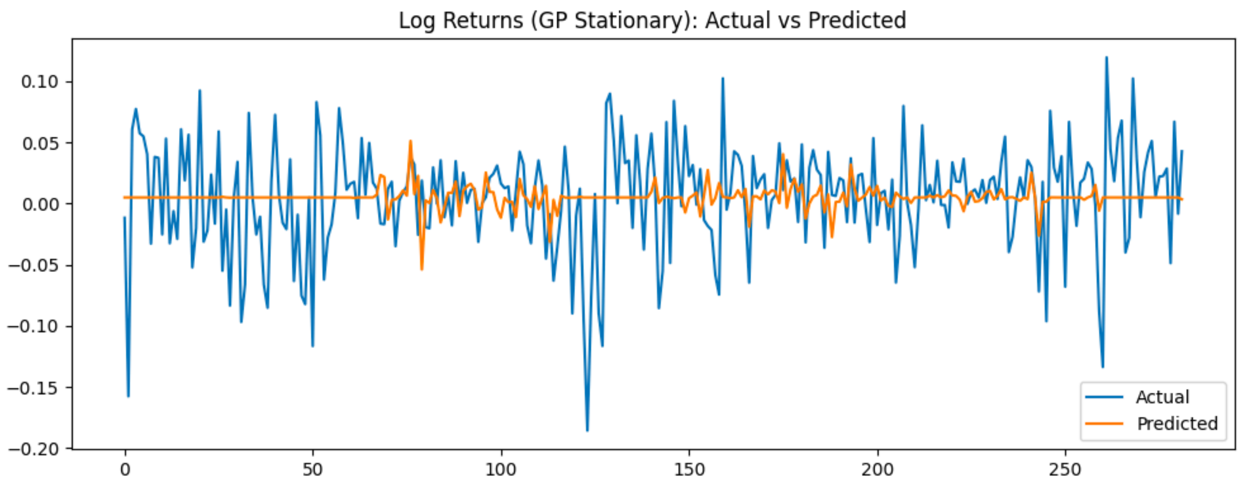



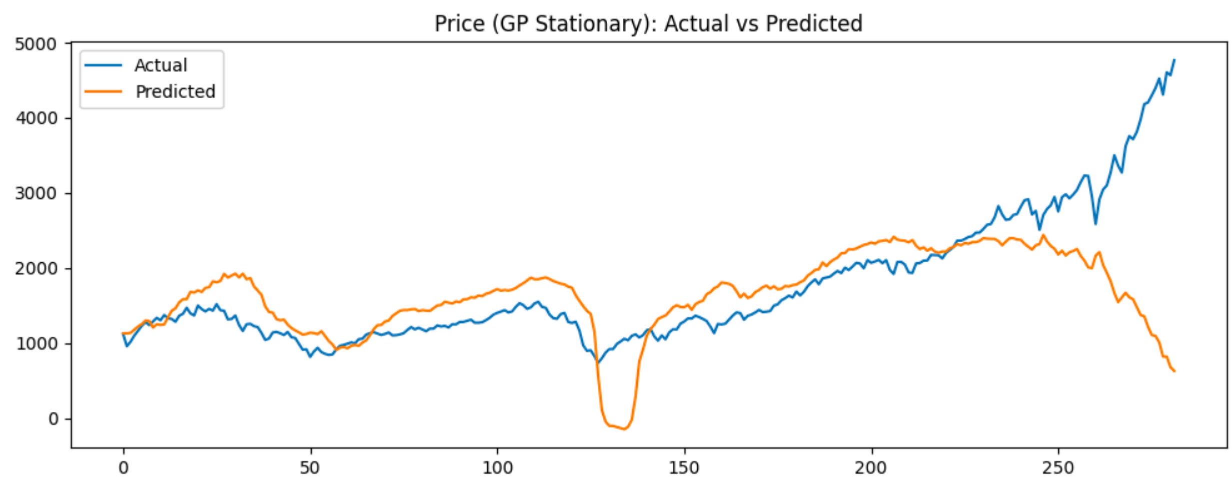

Predicted prices follow the trends reasonably well but fail to capture long-term growth in the second half. For log returns the model predictions are smooth and remain near zero, failing to follow the large fluctuations in actual returns.



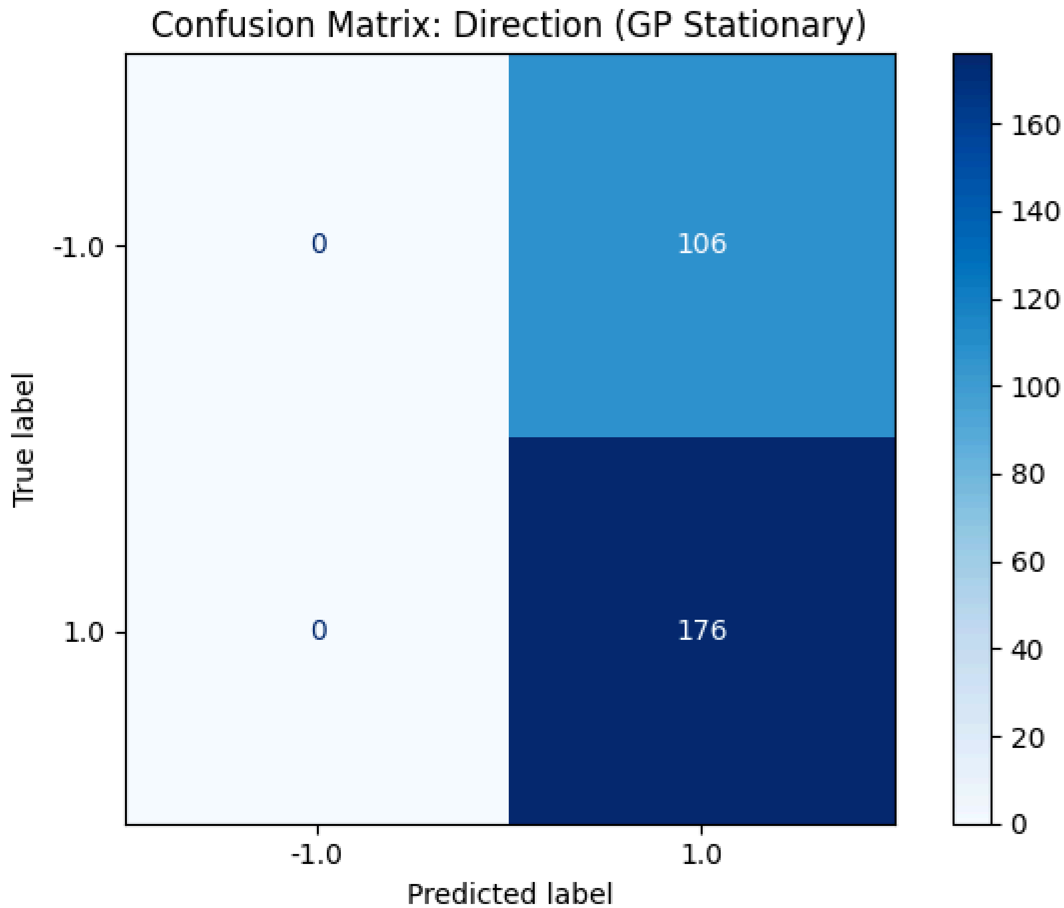

The model almost always predicts +1, missing nearly all down movements, which confirms strong model bias toward the positive class.

## Actual vs Predicted (Gaussian Stationary):




The predictions of the price follow short-term trends but completely miss the strong upward trajectory in the later part of the series. For log returns, the model once again predicts very smooth values, staying close to zero and failing to follow the large spikes and drops in actual returns.




The model always predicts +1, missing all down movements, which confirms strong model bias toward the positive class.

## Results of LSTM:

We conducted LSTM using a different number of past time steps (5,10,20) and layers (1,2,3).

*NRMSE results for Price Target:*

| | 5 | 10 | 20 |
|---|---|---|---|
| 1 | 1.176895 | 1.418805 | 1.535073 |
| 2 | 1.059262 | 1.150496 | 1.27146 |
| 3 | 1.091354 | 1.08978 | 1.221708 |

*NRMSE results for Returns Target:*

| | 5 | 10 | 20 |
|---|---|---|---|
| 1 | 1.709782 | 1.375207 | 1.630545 |
| 2 | 1.604487 | 1.380123 | 1.289464 |
| 3 | 1.329376 | 1.118176 | 1.312448 |

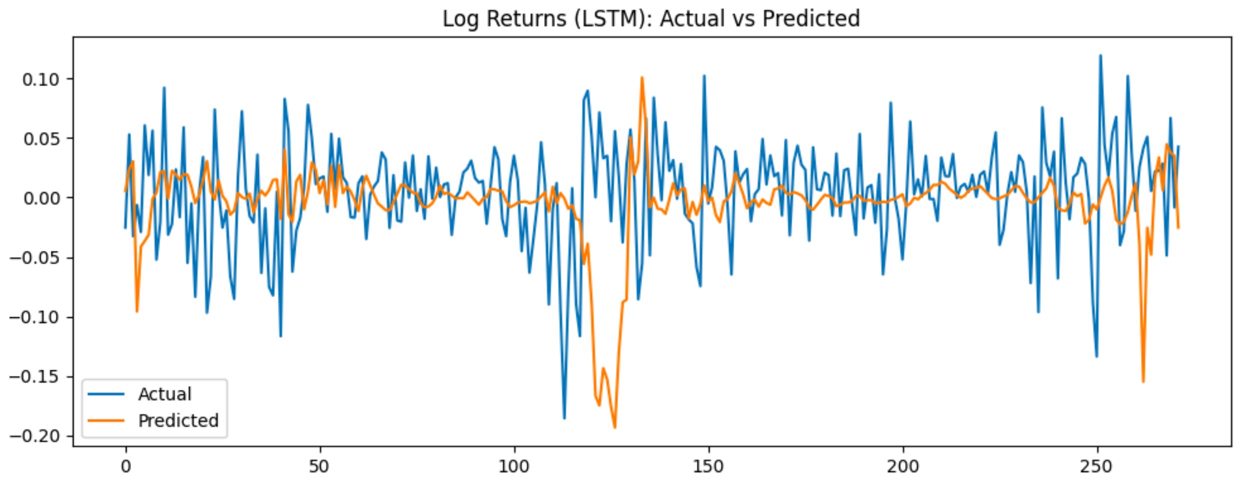
*Accuracy for Direction Target:*

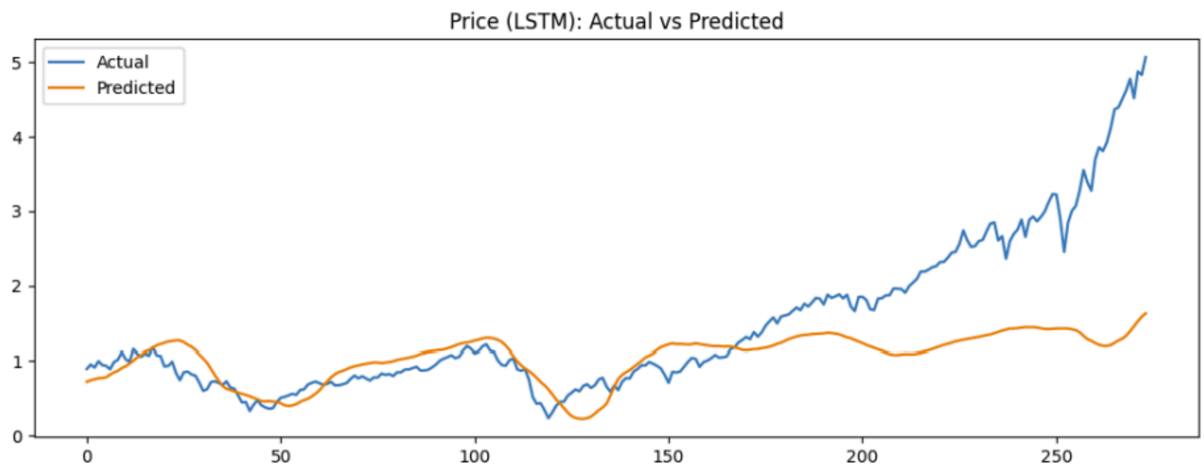
| | 5 | 10 | 20 |
|---|---|---|---|
| 1 | 0.595668 | 0.584559 | 0.564885 |
| 2 | 0.624549 | 0.621324 | 0.629771 |
| 3 | 0.617329 | 0.610294 | 0.572519 |

As we can see, LSTM with the lowest MSE were the ones using 5 past time steps with 2 layers for Price and 10 past time steps with 3 layers for Returns. As for Direction, 20 past time steps with 2 layers had the highest accuracy.

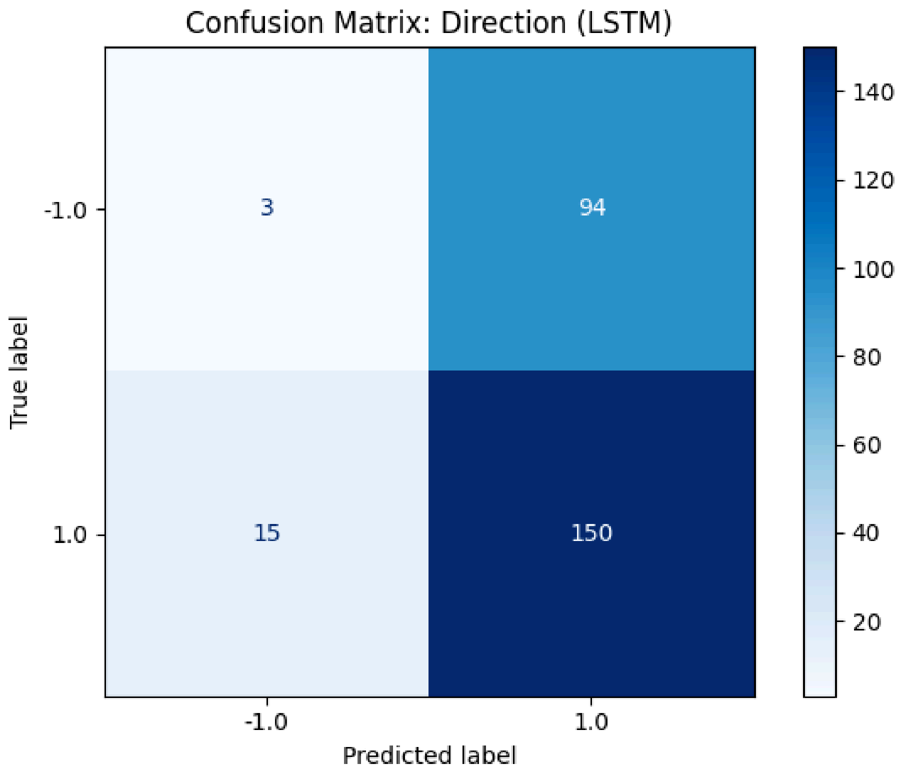
*Note: the results obtained for LSTM prediction on price are not the obtained the last time the code was runned. This is because we undid the scaling of the target variable and it broke the model.*

## Actual vs Predicted (LSTM):




For log returns LSTM captures some local patterns and short-term movements in returns but still misses the sharp spikes of the actual values. The predicted line is smoother and lags slightly behind. For price predictions the model tracks general short- and mid-term price trends well, particularly in flatter segments. However, it fails to capture the strong upward trend in the final section of the test period.




The direction prediction with LSTM slightly outperforms the bold "all-up" guessing of previous models, but still has a great amount of false positives.

*We also conducted the sensitivity analysis of the target with these three chosen models for 1% change in explanatory variables.*

*Sensitivity table for Price Target:*

| | price_lag1 | price_lag7 | price_lag9 | price_lag10 |
|---|---|---|---|---|
| +1% | [0.0006225109] | [7.927418e-05] | [0.00037527084] | [-0.00020575523] |
| -1% | [-0.0006235242] | [-8.016825e-05] | [-0.0003759265] | [0.0002040267] |

| | de_lag2 | de_lag3 |
|---|---|---|
| +1% | [-0.00014042854] | [0.00055128336] |
| -1% | [0.00014013052] | [-0.000551641] |

*Sensitivity table for Returns Target:*

| | ep_lag2 | ep_lag3 | logret_lag1 | logret_lag3 |
|---|---|---|---|---|
| +1% | [0.00028546527] | [0.00024393573] | [-0.0011215806] | [0.00033464655] |
| -1% | [-0.00028535724] | [-0.00024440512] | [0.0011203736] | [-0.00033425912] |

| | logret_lag5 |
|---|---|
| +1% | [0.0016864985] |
| -1% | [-0.0017099865] |

For Price target, LSTM relies primarily on recent price movements, behaving like an advanced autoregressive model with limited influence from fundamental features. For Return prediction, LSTM appears to integrate both fundamental (ep) and lagged return features, though overall sensitivity is still small, reflecting the noisy nature of return prediction.

## ***Summary:***

| ***Model*** | ***Price NRMSE*** | ***Return NRMSE*** | ***Direction Accuracy*** |
|---|---|---|---|
| **Neural Network** | **0.0474** | **0.1472** | 0.6241 |
| **GP (non-stat.)** | 0.6037 | 1.0283 | 0.6099 |
| **GP (stat.)** | 0.9728 | 1.0050 | 0.6241 |
| **LSTM** | 2.2554 | 1.1182 | **0.6298** |

| ***Target*** | ***Best Model*** | ***NRMSE / Accuracy*** |
|---|---|---|
| Price | Neural Network | 0.0474 |
| Log Return | Neural Network | 0.1472 |
| Direction | LSTM | 0.6298 |

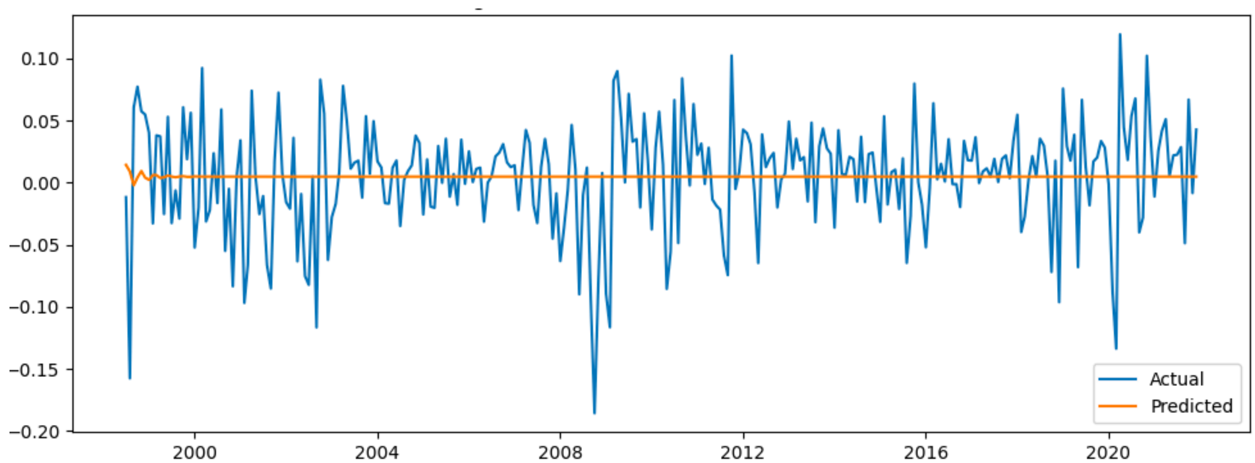
Neural Networks clearly outperform all models for both price and return predictions, achieving the lowest NRMSEs by far. However, we can only say that this performing exercise is outstanding for prices, as for log returns the low NRMSE is due to the flatness of the prediction. GP models perform moderately well for returns as well as for prices. LSTM offers decent returns predictions but struggles with price. For classification (direction), LSTM has the best accuracy, though the differences between models are small.

### *Results of ARMA+GARCH:*

```
Logret ARMA+GARCH NRMSE: 1.0022
Price ARMA+GARCH NRMSE: 1.5060
```

Overall, for returns ARMA+GARCH prediction is a flat line and does not capture any volatility of the actual data, however it still has nrmse close to other models, but still not as low as neural networks. As for the price, it also does not predict well since the function does not take into account the lags of the test data that we have. Overall, Neural Networks are better suited for price forecasting due to their ability to learn non-linear and long-range patterns, even though LSTM’s performance was quite bad.

### *Log Returns (ARMA + GARCH): Actual vs Predicted*




### *Price (ARMA + GARCH): Actual vs Predicted*