Announcements

Make sure to sign in on the google form (I send a list of which section questions are useful for which pset questions afterwards)

Pset 7 due Friday 11/10

Introductions

- One question or thought related to lecture last week (LASSO, Ridge, cross validation)

Weighted least squares regression

This question is based on a conversation with Skyler Wu.

Consider a least squares model where, rather than weighting all residuals equally, we are going to assign different weights to different residuals. That is, we want to minimize

$$\sum_{i=1}^{n}[w_i(Y_i - \hat{Y}_i)]^2$$

Equivalently, letting $\mathbf{W}$ be a diagonal matrix of weights, letting $\vec{Y} = \mathbf{X}\vec{\beta} + \vec{\epsilon}$ with $\vec{\epsilon} \sim \text{MVN}_n(0, \sigma^2 I_n)$, and using the fact that $\hat{\vec{Y}} = \mathbf{X}\hat{\vec{\beta}}$, we want to minimize

$$||\mathbf{W}(\vec{Y} - \mathbf{X}\hat{\vec{\beta}})||^2$$

Expanding and taking the gradient gives the following:

$$\begin{aligned}
0 &= \frac{\partial}{\partial \hat{\vec{\beta}}}((\vec{Y} - \mathbf{X}\hat{\vec{\beta}})^T \mathbf{W}^T \mathbf{W}(\vec{Y} - \mathbf{X}\hat{\vec{\beta}})) \\
&= -2\mathbf{X}^T\mathbf{W}^T\mathbf{W}(\vec{Y} - \mathbf{X}\hat{\vec{\beta}}) \\
\implies \mathbf{X}^T\mathbf{W}^T\mathbf{W}\mathbf{X}\hat{\vec{\beta}} &= \mathbf{X}^T\mathbf{W}^T\mathbf{W}\vec{Y} \\
\implies \hat{\vec{\beta}} &= (\mathbf{X}^T\mathbf{W}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}^T\mathbf{W}\vec{Y}
\end{aligned}$$

This is our new weighted least-squares regression $\hat{\vec{\beta}}$, which we will be studying in this problem.

Here are a few facts that will be useful here and on the homework:

- For matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, of allowable dimensions, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- If $\mathbf{A}$ is of full column rank, $\mathbf{A}^T\mathbf{A}$ is symmetric and invertible
- If $\mathbf{A}$ is symmetric, and $\mathbf{B}$ is of allowable dimensions, $\mathbf{B}^T\mathbf{AB}$ is symmetric
- For an invertible and symmetric matrix $\mathbf{A}$, $\mathbf{A}^{-1} = (\mathbf{A}^{-1})^T$
- For $\vec{Y} = \vec{c} + \mathbf{B}\vec{X}$ with $\vec{c}$ and $\mathbf{B}$ constant and $\vec{X}$ random, $E(\vec{Y}) = \vec{c} + \mathbf{B}E(\vec{X})$
- For a vector $\vec{X}$ of length n, $\text{Cov}(\vec{X})$ is an $n \times n$ matrix whose i, j entry is $\text{Cov}(X_i, X_j)$
- For $\vec{Y} = \vec{c} + \mathbf{B}\vec{X}$ with $\vec{c}$ and $\mathbf{B}$ constant and $\vec{X}$ random, $\text{Cov}(\vec{Y}) = \mathbf{B}\text{Cov}(\vec{X})\mathbf{B}^T$

1. Verify that using the usual weights for least squares regression, this formula reduces to the usual estimator for $\hat{\vec{\beta}}$.

2. If $\mathbf{W} = c\mathbf{I}$ for some non-zero constant c, what is $\hat{\vec{\beta}}$? What does this say about when $\mathbf{W}$ is useful?

3. Find the bias of $\hat{\vec{\beta}}$ for $\vec{\beta}$.

4. Find the variance-covariance matrix of $\hat{\vec{\beta}}$ in matrix form. What does this reduce to when $\mathbf{W} = \mathbf{I}$? You may find it useful to let $\mathbf{A} = (\mathbf{X}^T\mathbf{W}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}^T\mathbf{W}$ throughout.

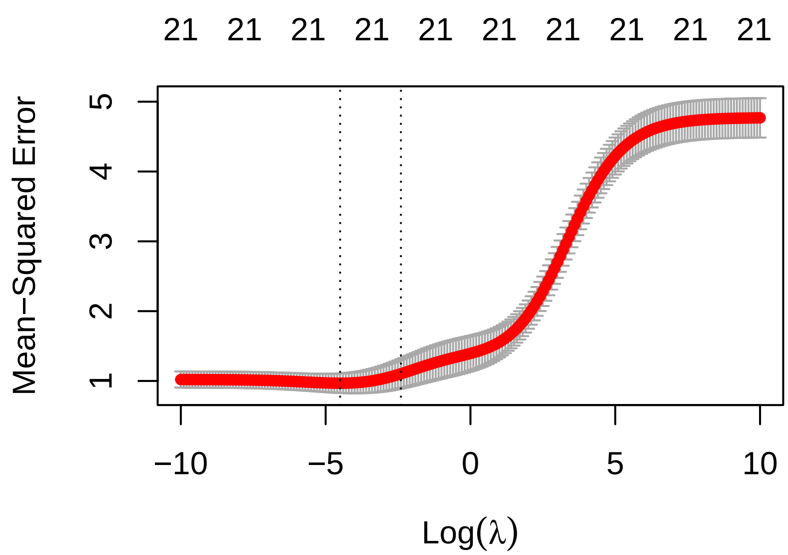
Ridge, LASSO, optimizing λ, and β trajectories

These problems will deal with a dataset of country-level statistics from UNdata, Varieties of Democracy, and the World Bank.

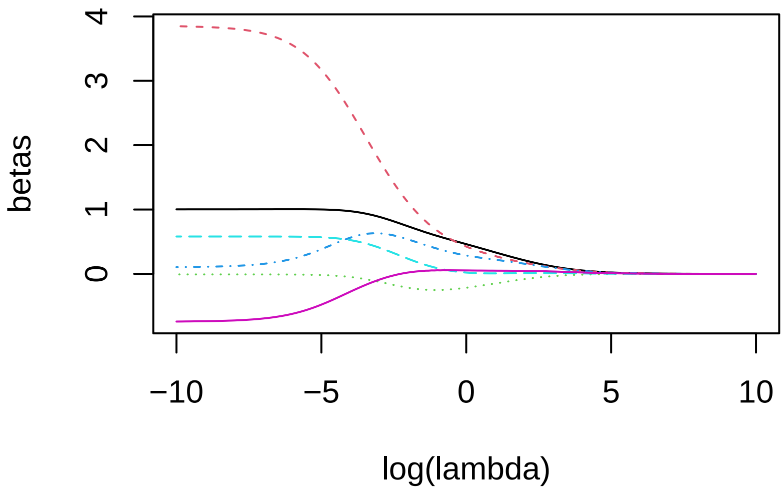
1. First, we'll fit a well-tuned Ridge regression model via `cv.glmnet` for predicting log2 GDP per capita from a country's urban population, its proportion of people 60+, its arable land, its energy supply, its unemployment rate, and its number of tourists and visitors. We'll perform 10-fold cross validation to choose the optimal λ. What is the optimal model?

```
##
## Call:  cv.glmnet(x = X, y = y, lambda = exp(seq(-10, 10, 0.1)), nfolds = 10,      alpha = 0)
##
## Measure: Mean-Squared Error
##
##      Lambda Index Measure     SE Nonzero
## min 0.01111   146   0.968 0.1390      21
## 1se 0.09072   125   1.103 0.2071      21
```

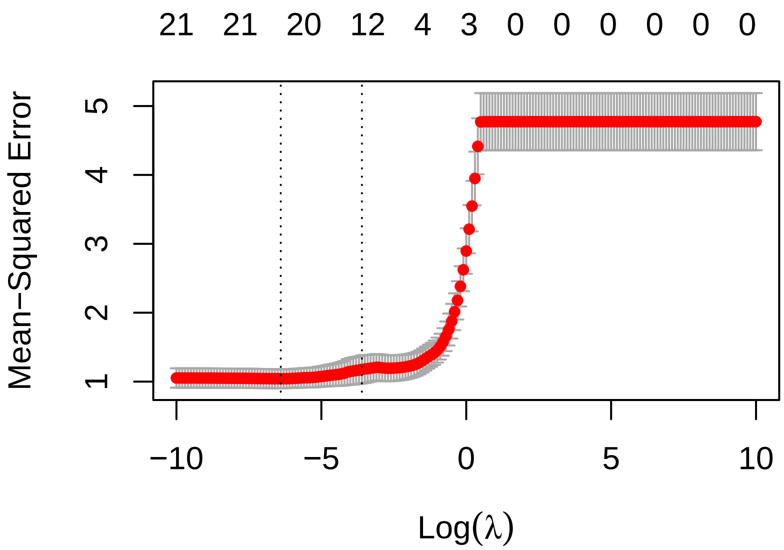
2. The following is a plot of MSE on the validation sets against the λ's from the previous part. Justify the previous λ with this plot.



3. The following are the $\hat{\beta}$ trajectories of the main (non-interaction) effects from this model. Interpret what you see in 2-3 sentences.

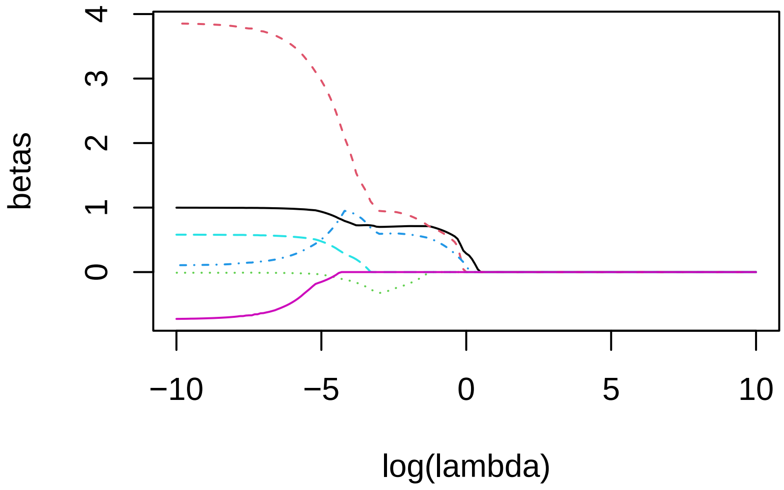


4. Next, we'll fit a well-tuned LASSO regression model for the same question. We'll perform 10-fold cross validation to choose the optimal λ. What is the optimal model? Is this consistent with the plot?




```
##
## Call:  cv.glmnet(x = X, y = y, lambda = exp(seq(-10, 10, 0.1)), nfolds = 10,      alpha = 1)
##
## Measure: Mean-Squared Error
##
##       Lambda Index Measure     SE Nonzero
## min 0.001662   165   1.042 0.1374      21
## 1se 0.027324   137   1.177 0.2037      10
```

5. The following are the $\hat{\beta}$ trajectories of the main (non-interaction) effects from this model. Compare these to the ridge trajectories.



6. What is the best regularized/penalized regression model?

```
##     Ridge, lambda=0.011 LASSO, lambda=0.0017
## MSE           0.9679976             1.04198
```

Penalization functions

Recall that for both Ridge and LASSO, we are trying to minimize something of the form:

$$\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + p(\vec{\hat{\beta}})$$

State whether the following functions could or couldn't be used as penalization functions for $\vec{\hat{\beta}}$. If so, provide a context in which this might be a useful penalization function; if not, explain why it would give undesired behavior.

1. $p(\vec{\hat{\beta}}) = \sum_{i=1}^{k} \hat{\beta}_i$
2. $p(\vec{\hat{\beta}}) = \sum_{i=1}^{k} \hat{\beta}_i^{\,4}$
3. $p(\vec{\hat{\beta}}) = \sum_{i=1}^{k} \log(\hat{\beta}_i)$
4. $p(\vec{\hat{\beta}}) = \sum_{i=1}^{k} \log(|\hat{\beta}_i|)$
5. $p(\vec{\hat{\beta}}) = \sum_{i=1}^{k} 1/|\hat{\beta}_i|$
6. $p(\vec{\hat{\beta}}) = -\sum_{i=1}^{k} 1/|\hat{\beta}_i|$
7. What general requirements do we need for a penalization function?
8. Write a valid penalization function that we haven't studied before.

Miscellaneous

1. For what λs would LASSO and Ridge give the same model?

2. Below are four $\hat{\beta}$ trajectory plots for ridge regressions. Each comes from a data set with 50 data points. One trajectory comes from data with no built-in correlation between the predictors; one comes from data with moderate and equal correlation among all the predictors; one comes from data with moderate random (but fixed) correlation among the predictors; and one is fake (and impossible). Determine which is which.