Two Approaches to the String Wrapping Problem

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1 Problem

A cylinder of length 12 units and circumference 4 units has a piece of string wrapped perfectly around it, such that it completes exactly 4 revolutions of the body. The task is to find the length of the string. This is visualised in fig. 1.

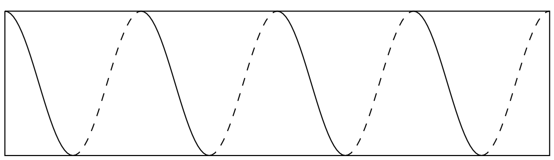

Figure 1: Side on view of the problem

2 Sensible approach

The sensible approach is to realise that unrolling the cylinder simplifies the computation greatly; once this is done, a simple application of Pythagoras' theorem gives the length of the string as $4 \times \sqrt{3^2 + 4^2} = 4 \times 5 = 20$. This is because each revolution corresponds to the hypotenuse of a right-angled triangle with sides of length 4 (from the circumference) and 3 (horizontal distance along cylinder; 4 revolutions with 12 units travelled $\Rightarrow 12/4 = 3$ units per revolution). Figure 2 depicts the situation.

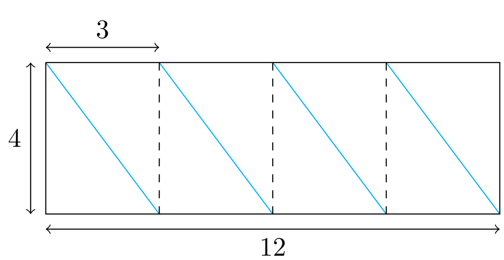

Figure 2: The easy way to solve the string wrapping problem on the unrolled cylinder

3 More fun approach

Alternatively, we can use the techniques developed in `MATH329` to solve the problem more directly; that is, without unrolling the cylinder. Ensuring that the centre of the cylinder lies along the z axis, we let the string be represented by a curve in $\mathbb{R}^3$ that will take the form

$$\boldsymbol{\gamma}(t) : (0, 12) \to \mathbb{R}^3;\ t \mapsto (r \sin at, r \cos at, t),$$

where r is the radius of the cylinder and a is a constant to be determined.

Since the circumference of the cylinder is known to be 4, it follows that $r = 4/2\pi = 2/\pi$. Considering a, we know that multiplication by a in the trigonometric functions divides the period by a. Since we need the period to equal 3,

$$\begin{aligned} 3 &= 2\pi/a \\ \Rightarrow a &= 2\pi/3 \end{aligned}$$

Thus

$$\boldsymbol{\gamma}(t) : (0, 12) \to \mathbb{R}^3;\ t \mapsto (\frac{2}{\pi} \sin \frac{2\pi}{3} t, \frac{2}{\pi} \cos \frac{2\pi}{3} t, t)$$

The difficult work is done. Now all that's left is to apply the arc-length formula:

$$\ell(\vec{ab}) = \int_a^b |\boldsymbol{\gamma}'| \, \mathrm{d}t \tag{1}$$

First, we differentiate $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma}'(t) : (0, 12) \to \mathbb{R}^3;\ t \mapsto (\frac{4}{3} \cos \frac{2\pi}{3} t, -\frac{4}{3} \sin \frac{2\pi}{3} t, 1)$$

Now apply eq. (1):

$$\begin{aligned} \ell &= \int_0^{12} \sqrt{\frac{4^2}{3^2} \left(\cos^2 \frac{2\pi}{3} t + \sin^2 \frac{2\pi}{3} t \right) + 1^2} \, \mathrm{d}t \\ &= \int_0^{12} \sqrt{\frac{4^2}{3^2} + \frac{3^2}{3^2}} \, \mathrm{d}t \\ &= 12 \times \frac{5}{3} = 20 \end{aligned}$$

So we arrive at the same answer as before, namely that the string is 20 units long.