CS 6351 DATA COMPRESSION

THIS LECTURE:
TRANSFORMS PART II

Instructor: Abdou Youssef

OBJECTIVES OF THIS LECTURE

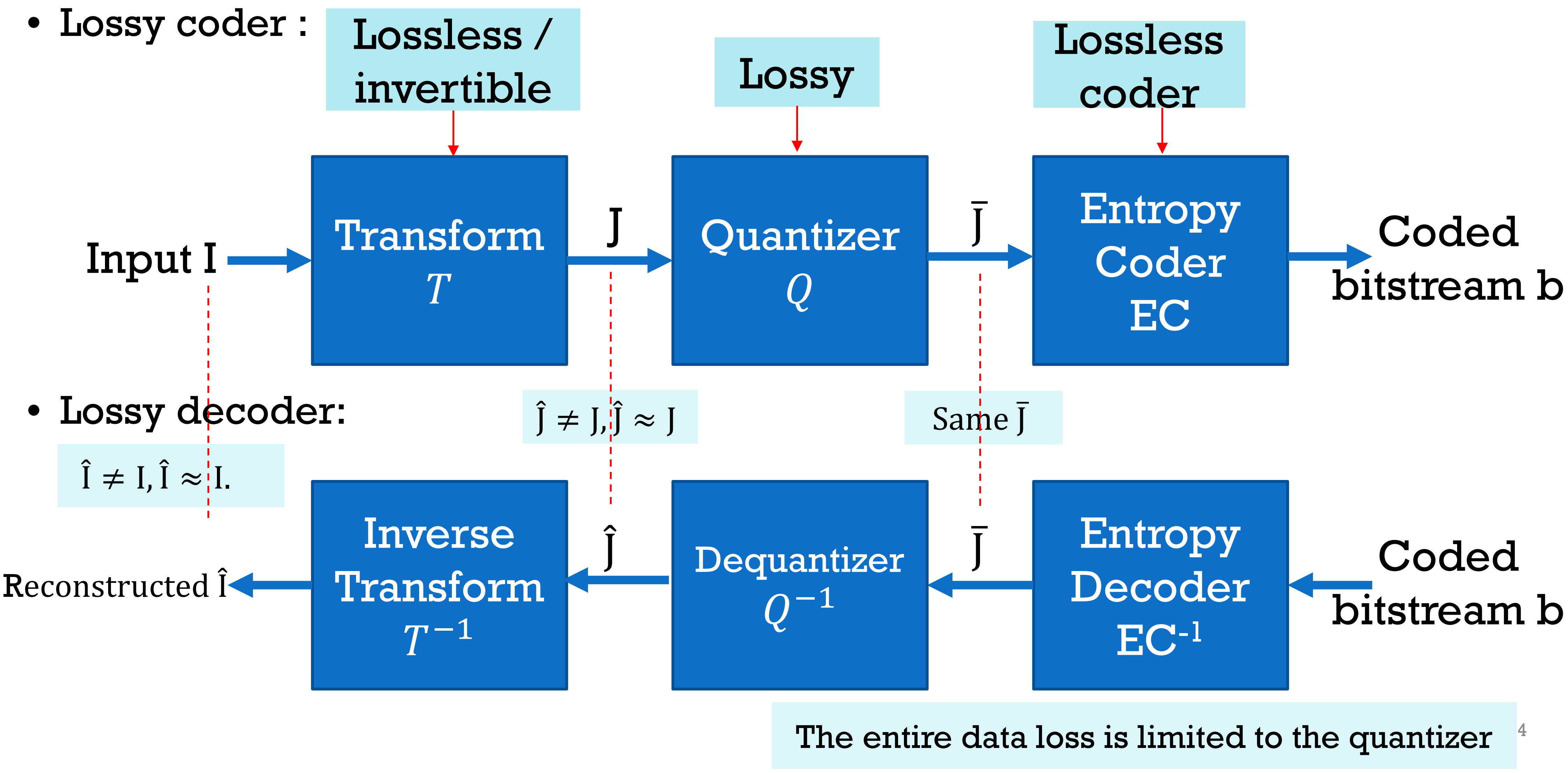
By the end of this lecture, you will be able to:

- Describe vector spaces, and related concepts such as linear independence, linear combinations, bases (axes), and dimensionality of vector spaces
- Draw connections between vectors and digital audio/visual signals, and also between vectors and analog signals
- Explain and appreciate the connections b/w vector space concepts and relevant audio/visual concepts
- Tie matrix-column multiplication with expressing vectors as linear combinations of basis vectors
- Prove that transforms are but a change of bases, a change of coordinate systems
- Explicate the connection between transforms, bases, human senses, and the kind of basis conducive to effective lossy compression

OUTLINE

- Vector spaces
- Vector space perspective of transforms
- Connection between
  - transforms,
  - Vector space bases,
  - human senses, and
  - the kind of basis conducive to effective lossy compression

RECALL: GENERAL SCHEME OF LOSSY COMPRESSION



VECTOR SPACES OVER $\mathbb{R}$
-- PRELIMINARIES (1/2) --

- A vector space V is a set of objects (called *vectors*) with two operations:
  - Vector addition (+): for all $u, v \in V$, the sum $u + v$ is a new vector in V
  - Scalar multiplication (.) of a number by a vector: for all $a \in \mathbb{R}$ and $u \in V$, the product $a.u$ (typically written au) is a new vector in V

  such that certain properties (called *axioms*) are satisfied (to be seen)
- It is a generalization of 2D /3D vectors you studied in Physics, and how to add/subtract them, and extend/shrink them by multiplying them by a real number

VECTOR SPACES OVER $\mathbb{R}$
-- PRELIMINARIES (2/2) --

- One big take-away for data compression is the notion of ***basis*** of vector space
  - A basis is a generalization of axes (like x-y axes and x-y-z axes)
  - There is a special connection between transforms and bases
  - Interest: a basis that "aligns" with the human eyes/ears

DEFINITION OF A VECTOR SPACE OVER $\mathbb{R}$

-- FORMAL DEFINITION--

A vector space $(V, +, ., O)$ is a non-empty set V with two operation (+) and (.) in V that satisfy the following properties (called axioms):

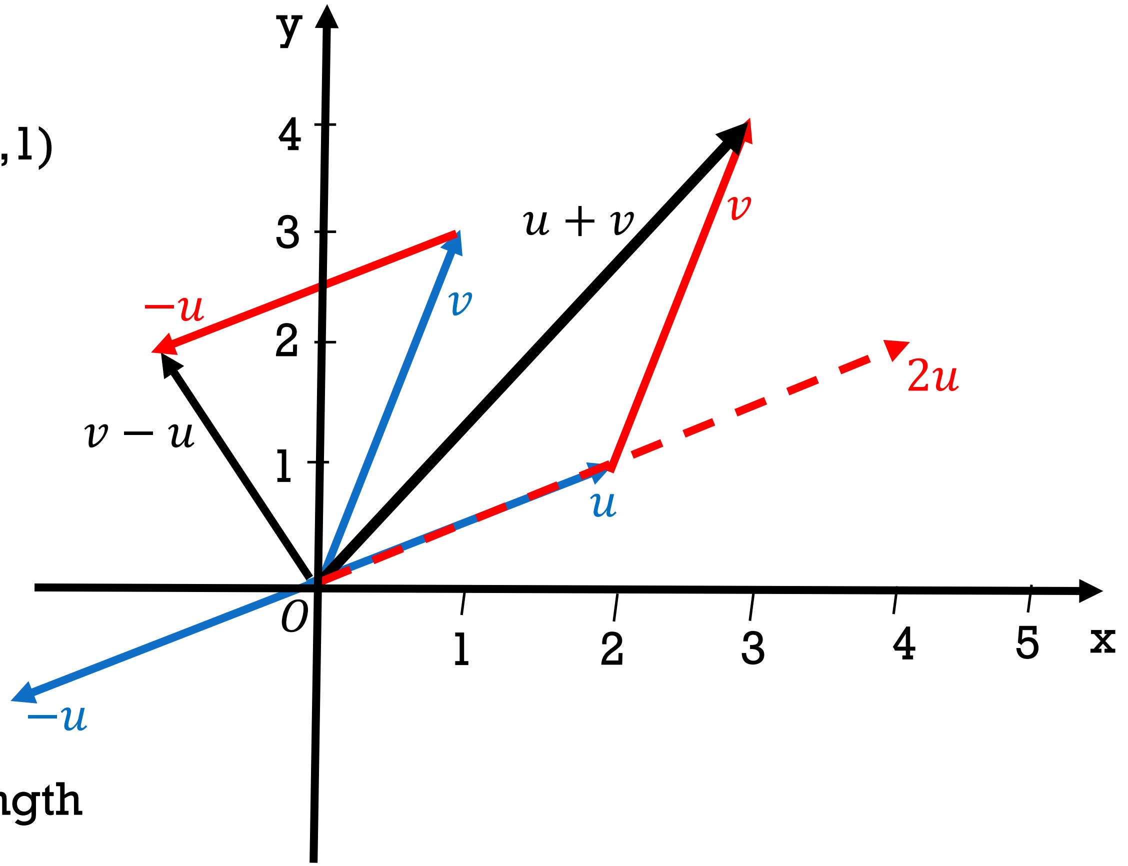
1. $\forall\, u, v \in V, u + v = v + u$ (Commutativity, i.e., + is commutative)
2. $\forall\, u, v, w \in V, (u + v) + w = u + (v + w)$ (Associativity, i.e., + is associative)
3. $\exists O \in V$ such that $\forall u \in V, u + O = O + u = u$ (O is called the *zero vector*)
4. $\forall u \in V, \exists v \in V$ such that $u + v = v + u = O;\ v$ is called the opposite vector of u and is denoted $-u$ (so $u + (-u) = (-u) + u = O$)
5. $\forall a \in \mathbb{R}$ and $\forall u, v \in V, a(u + v) = au + av$
6. $\forall a, b \in \mathbb{R}$ and $\forall u \in V, (a + b)u = au + bu$
7. $\forall a, b \in \mathbb{R}$ and $\forall u \in V, a(bu) = (ab)u$
8. $\forall u \in V, 0u = O, and\ 1u = u$

VECTOR SPACE EXAMPLES (1/4)

- Take $V = \mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$, that is, every vector u is of the form (x, y), where
  - $(x, y) + (x', y') \stackrel{\text{def}}{=} (x + x', y + y')$ (note: $\stackrel{\text{def}}{=}$ means "by definition")
  - $a(x, y) \stackrel{\text{def}}{=} (ax, ay)$
  - $O \stackrel{\text{def}}{=} (0,0)$
- You can easily verify that the 8 axioms are satisfied, for example
  - $(x, y) + (x', y') = (x', y') + (x, y)$ because $(x + x', y + y') = (x' + x, y' + y)$
  - $O + (x, y) = (x, y)$ because $O + (x, y) = (0,0) + (x, y) = (0 + x, 0 + y) = (x, y)$
  - $(x, y) + (-x, -y) = (x - x, y - y) = (0,0) = O$
  - $\forall a, b \in \mathbb{R}$ and $\forall (x, y) \in V$, we have $(a + b)(x, y) = \big((a + b)x, (a + b)y\big) = (ax + bx, ay + by) = (ax, ay) + (bx, by) = a(x, y) + b(x, y)$, therefore $(a + b)(x, y) = a(x, y) + b(x, y)$
  - $0.(x, y) = O$ because $0.(x, y) = (0.x, 0.y) = (0,0) = O$
  - $1.(x, y) = (x, y)$ because $1.(x, y) = (1.x, 1.y) = (x, y)$

VECTOR SPACE EXAMPLES (2/4)
-- VISUAL/GRAPHICAL REPRESENTATIONS --

- Draw x-y axes: each point has two coordinates. Origin=(0,0)= O
- $u = (2,1), \quad v = (1,3)$
  - Vector (2,1): arrow from point (0,0) to point (2,1)
- $u + v = (3,4)$, visually:
  - Make a "copy" of v starting from the tip of u
  - Take arrow from (0,0) to the tip of copy of v
- $-u = (-2,-1)$, visually:
  - Take reflection of u from the origin
- $2u = (4,2)$, visually:
  - Same angle and orientation as u, twice the length
- $v - u = (-1,2)$, same as $v + (-u)$



VECTOR SPACE EXAMPLES (3/4)

- Take $V = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x \in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}\}$, that is, every vector u is of the form (x, y, z), where
  - $(x, y, z) + (x', y', z') \stackrel{\text{def}}{=} (x + x', y + y', z + z')$
  - $a(x, y, z) \stackrel{\text{def}}{=} (ax, ay, az)$
  - $O \stackrel{\text{def}}{=} (0,0,0)$
- You can easily verify that the 8 axioms are satisfied
- Note: We typically denote
  - $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ // set of couples or pairs of real numbers
  - $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ // set of triples
  - $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \mathbb{R}^n$ // set of n-tuples of the form $(x_1, x_2, \dots, x_n)$ where every $x_i \in \mathbb{R}$

VECTOR SPACE EXAMPLES (4/4)

- Take $V = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{R} \text{ for all } i = 1,2, \dots, n\}$, that is, every vector u is of the form $(x_1, x_2, \dots, x_n)$, where
  - $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
  - $a(x_1, x_2, \dots, x_n) \stackrel{\text{def}}{=} (ax_1, ax_2, \dots, ax_n)$
  - $O \stackrel{\text{def}}{=} (0,0, \dots, 0)$
- You can easily verify that the 8 axioms are satisfied

SOUND SIGNALS AS “VECTORS”

- A digital audio signal, e.g., a sound recording, is a sequence of sound samples, i.e., a sequence of real numbers $x_1, x_2, \dots, x_n$
- Each such signal can be viewed as a vector $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$
- **Adding two sound signals** $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ into $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ is like **playing the two sound recordings <u>at the same time</u>**
- **Multiplying a sound signal** $(x_1, x_2, \dots, x_n)$ by some number a (e.g., a=3) to get $a(x_1, x_2, \dots, x_n) \stackrel{\text{def}}{=} (ax_1, ax_2, \dots, ax_n)$ is like playing the sound 3 times louder, i.e., **raising the volume**
  - **If $0 < a < 1$**, as for example $a = \frac{1}{2}$, is like **reducing the volume** (e.g., making the sound less loud)

IMAGES AS "VECTORS"

- A digital image I of n rows and m columns can be viewed as a vector in $\mathbb{R}^{n\times m}$
  - Like stacking the rows one after another, or
  - Stacking the columns one on top of the other
- **Adding two vectors** is like **superposing the two images** on top of each other
- **Multiplying a vector** by some number a (e.g., a=3 or $a = \frac{1}{2}$) is like **making the colors (or gray shades)** proportionaly **darker or lighter**
- The **opposite** $-I$ is what we typically call the **negative of an image/photo**

CAN ANALOG SIGNALS BE VIEWED AS VECTORS?

- We just saw that digital signals (whether audio or visual) can be viewed as vectors
- How about analog signals (like those captured by old recorders or old cameras)?
  - Can you view them as vectors in a vector space?
  - If so, what would an analog signal correspond to mathematically?
- Why should we care, since analog technology is old, obsolete, and deprecated?
  - Also, even if some preserved old recordings/photos are still of interest, they can be digitized, so why bother with analog signals as vectors?
- Answer: Analysis of analog signals is like analysis of continuous functions, which is easier and more insightful, in some respects, than discrete functions
  - Using Calculus and Mathematical Analysis

ANALOG AUDIO SIGNALS AS VECTORS

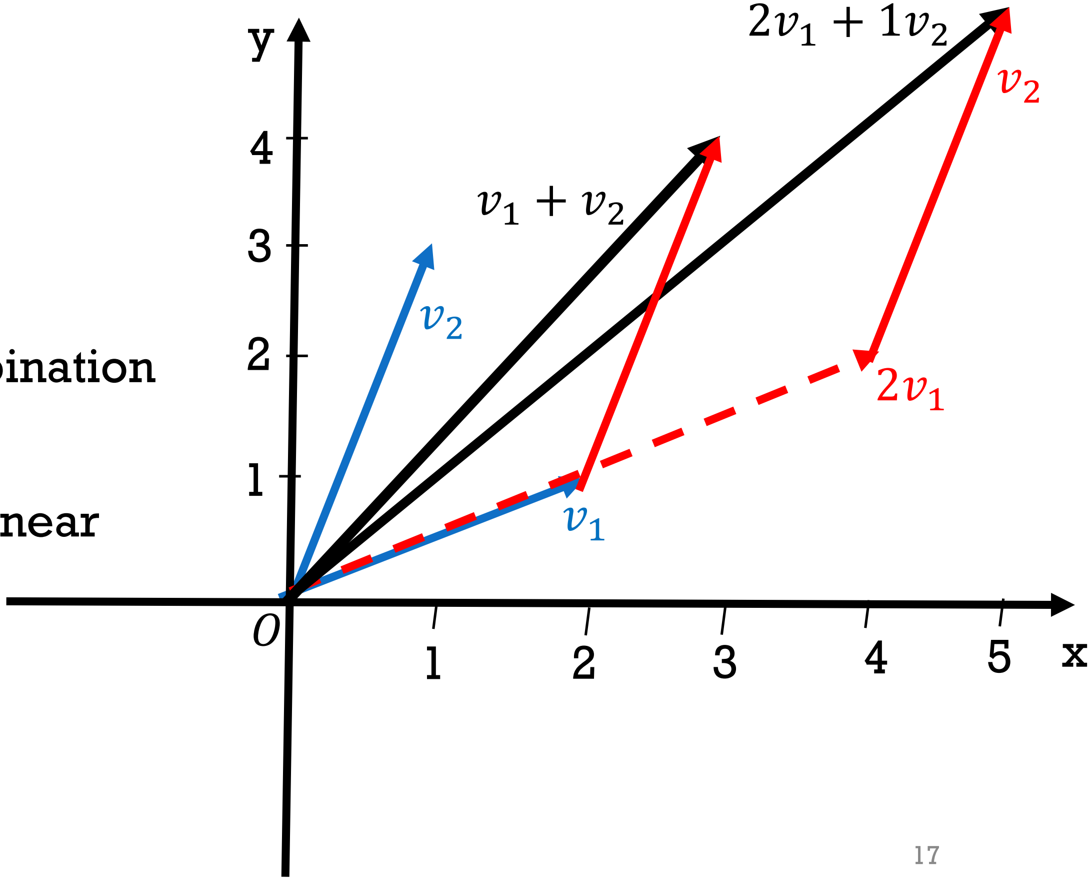
- Take V as the set of all functions $f(t)$ of one real variable t (t is like time)
- View each such function f as an (abstract) vector
- Addition of two vectors is addition of two functions: $f+g$ is such that
  - $(f+g)(t) \stackrel{\text{def}}{=} f(t)+g(t)$
  - Example: like playing two analog sounds simultaneously
- Multiplying a vector (i.e., a function f) by a number a gives us a new vector (or function) af where $(af)(t) \stackrel{\text{def}}{=} a.f(t)$
  - Example: like making the volume of an analog (old) radio louder or quieter
- The zero vector (or function) O is the zero function f where $f(t)=0\ \forall t$
  - The zero function is the equivalent of "dead silence"
- The opposite vector (or function) of f is $-f$ where $(-f)(t) \stackrel{\text{def}}{=} -f(t)$
- One can show that this $(V,+,.,O)$ is a vector space, i.e., satisfies all the 8 axioms

ANALOG IMAGES AS VECTORS

- Take V as the set of all functions $f(x, y)$ of two real variables (x, y), where
  - (x, y) are the coordinates of a point in the x-y plane, i.e., the location of a "pixel"
  - $f(x, y)$ represents the value of that "pixel"
  - "Pixels" are "points with intensity/color" rather than squares with average intensity/color
- View each such function f as an (abstract) vector
- Addition of two vectors is addition of two functions: $f + g$
  - $(f + g)(x, y) \stackrel{\text{def}}{=} f(x, y) + g(x, y)$
- Multiplying a vector (i.e., a function f) by a number a: $(af)(x, y) \stackrel{\text{def}}{=} a.f(x, y)$
- The zero vector (or function) O is the zero function f where $f(x, y) = 0\ \forall(x, y)$
- The opposite vector of f is $-f$ where $(-f)(x, y) \stackrel{\text{def}}{=} -f(x, y)$ // negative image
- One can show that this $(V, +, ., O)$ is a vector space, i.e., satisfies all the 8 axioms

VECTOR SPACE CONCEPTS
-- LINEAR COMBINATIONS --

- Let $(V, +, ., O)$ be a vector space, and let n be a positive integer or infinity ∞
- A vector v is called a *linear combination* of vectors $v_1, v_2, \dots, v_n$ if we can find n numbers $x_1, x_2, \dots, x_n$ such that
  - $v = x_1 v_1 + x_2 v_2 + \cdots, + x_n v_n$
- Examples:
  - Take $v_1 = (2,1)$, and $v_2 = (1,3)$
  - The vector $v = 2v_1 + 1v_2 = (5,5)$ is a linear combination of v_1 and v_2
  - Also: $v = 1v_1 + 1v_2 = v_1 + v_2 = (3,4)$ is another linear combination of v_1 and v_2

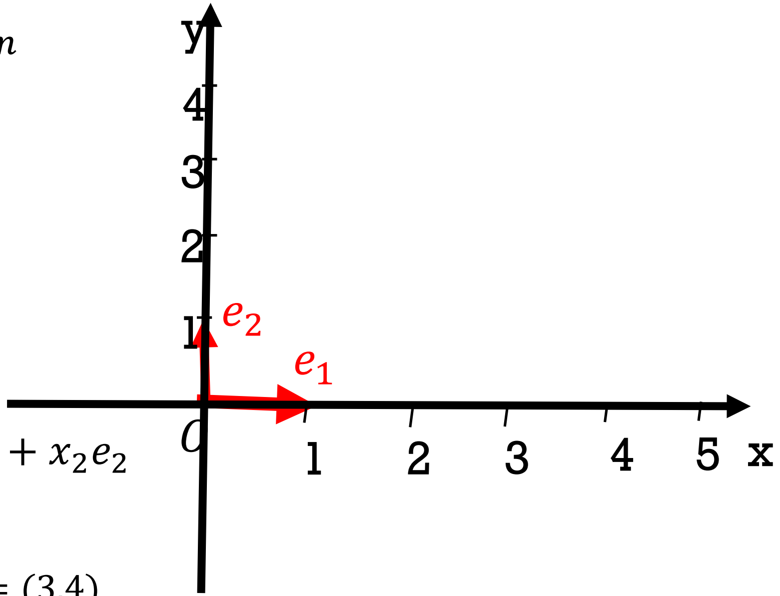


VECTOR SPACE CONCEPTS
-- LINEAR INDEPENDENT VECTORS--

- Let $(V, +, ., O)$ be a vector space, and let n be a positive integer or infinity ∞
- A subset of non-zero vectors $v_1, v_2, \dots, v_n$ are called ***linearly independent*** if
  - No vector in that subset can be expressed as a linear combination of other vectors in that subset
  - That is, if $x_1 v_1 + x_2 v_2 + \cdots, +x_n v_n = 0$, then we must have $x_1 = x_2 = \cdots = x_n = 0$
- Examples:
  - Take $v_1 = (2,1), v_2 = (1,3)$
    - v_1, v_2 are linearly independent (simply prove that if $x_1 v_1 + x_2 v_2$=0, then we must have $x_1 = x_2 = 0$)
  - Take $v_1 = (2,1), v_2 = (1,3), v_3 = (5,5)$
    - v_1, v_2, v_3 are NOT linearly independent because v_3 can indeed be expressed as a linear combination of v_1, v_2: $v_3 = 2v_1 + 1v_2$
    - Equivalently: $2v_1 + 1v_2 + (-1)v_3 = 0$ even though $2 \neq 0, 1 \neq 0, -1 \neq 0$
    - In that case, we say that v_1, v_2, v_3 are ***linearly dependent***

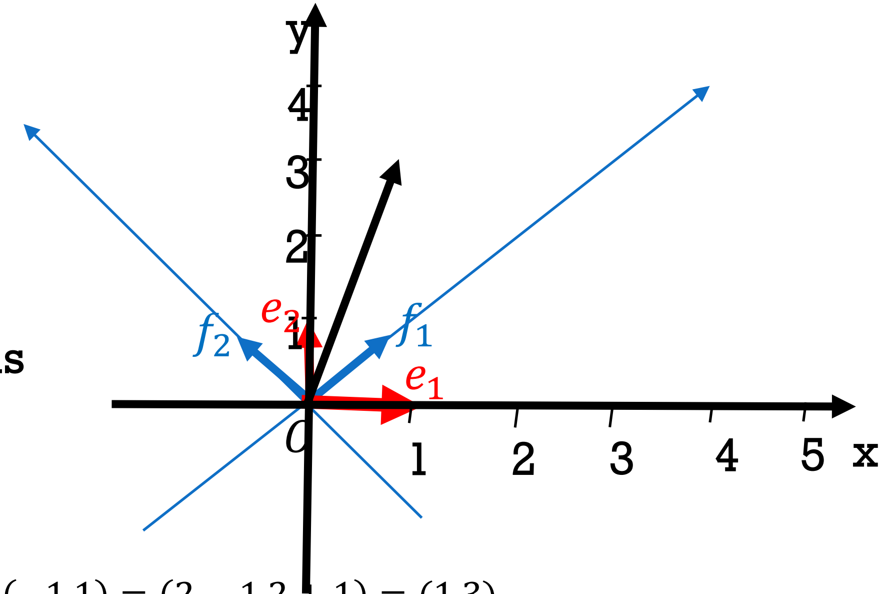
VECTOR SPACE CONCEPTS
-- BASIS --

- Let $(V, +, ., O)$ be a vector space, and let n be a positive integer or infinity ∞
- A subset of non-zero vectors $v_1, v_2, \dots, v_n$ is called ***a basis*** of V if
  - $v_1, v_2, \dots, v_n$ are linearly independent, and
  - For every vector $v \in V$, v can be expressed as a linear combination of $v_1, v_2, \dots, v_n$, that is, we can find n numbers $x_1, x_2, \dots, x_n$ such that $v = x_1 v_1 + x_2 v_2 + \cdots, +x_n v_n$
- Example:
  - Take $V = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
  - The vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ form a basis of $\mathbb{R}^2$
    - You can show that e_1, e_2 are linearly independent (prove it)
    - Also, for every vector $v = (x, y) = (x_1, x_2)$, we can express $v = x_1 e_1 + x_2 e_2$
      - Example: (3,4)=$3e_1 + 4e_2$
        - Proof: $3e_1 + 4e_2 = 3(1,0) + 4(0,1) = (3,0) + (0,4) = (3 + 0, 0 + 4) = (3,4)$



VECTOR SPACE CONCEPTS
-- BASIS IS NOT UNIQUE NOR GUARANTEED --

- Let $(V, +, ., O)$ be a vector space, and let n be a positive integer or infinity ∞
- V is not guaranteed to have a basis
- On the other hand, if V has a basis, it can have many bases
- Example: Take $V = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
  - The vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ form a basis of $\mathbb{R}^2$
  - Also, the vectors $f_1 = (1,1)$ and $f_2 = (-1,1)$ form another basis
    - For every vector $v = (x, y)$, we can express v as
      - $v = x_1 f_1 + x_2 f_2$ where $x_1 = \frac{x+y}{2}$ and $x_2 = \frac{y-x}{2}$ (prove it)
      - Ex: $(1,3) = 2f_1 + 1f_2$ because $2f_1 + 1f_2 = 2(1,1) + 1(-1,1) = (2,2) + (-1,1) = (2-1, 2+1) = (1,3)$



- A given basis $v_1, v_2, \ldots, v_n$ is like an axis system:
  - The coefficients of $x_1, x_2, \ldots, x_n$ of $v = x_1 v_1 + x_2 v_2 + \cdots, + x_n v_n$ are viewed as the **coordinates** of v wrt to that basis (or that axis system)
    - Ex: the basis $e_1 = (1,0)$ and $e_2 = (0,1)$ corresponds to the x-y axes
    - Vector $(1,3) = 1e_1 + 3e_2$ has coordinates 1 and 3 wrt basis e_1 and e_2
- Another basis is like another axis system

Vector $(1,3) = 2f_1 + 1f_2$ has coordinates 2 and 1 wrt basis f_1, f_2

MORE ON VECTOR SPACE BASES
-- CANONICAL BASIS OF $\mathbb{R}^n$ --

- $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ **has as a basis** e_1, e_2, e_3 where
  - $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$
- $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{ times}}$ **has as a basis** $e_1, e_2, \dots, e_n$ where
  - $e_1 = (1,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_i = (0,\dots 0,1,0,\dots,0), and\ e_n = (0,\dots,0,1)$
  - We call this basis the ***canonical basis*** of $\mathbb{R}^n$ (i.e., the "natural" basis)
  - That is because for any vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the coordinates of that vector wrt the canonical basis $e_1, e_2, \dots, e_n$ are simply the components of the vector:
    - $(x_1, x_2, \dots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$
  - Whereas for any other basis $f_1, f_2, \dots, f_n$, the coordinates of $(x_1, x_2, \dots, x_n)$ wrt $f_1, f_2, \dots, f_n$ are not $x_1, x_2, \dots, x_n$

DIMENSION OF A VECTOR SPACE

- If a vector space $(V, +, ., O)$ has a basis of some n vectors, then
  - **Theorem**: Every basis of V will have n vectors, for the same value of n
  - We say n is the ***dimension*** of V, and V an n-dimensional (nD) space
  - n can be finite or infinite
- Examples:
  - $\mathbb{R}^n$ is of dimension n (because the canonical basis $e_1, e_2, \dots, e_n$ of $\mathbb{R}^n$ has n vectors)
  - $\mathbb{R}^{n\times m}$ is a vector space of dimension $n \times m$
- Notes about terminology:
  - Before, we would refer to audio signals as 1D signals, and to images as 2D signals
  - Now, in vectors spaces, 1D audio signals of n samples form an n-dimensional (nD) space, and 2D images form an $n \times m$- dimensional space
  - This can be confusing, but the context makes things clear:
    - 1D *signals* vs. nD *vectors*
    - 2D *images* vs. $(n \times m)D$ *vectors*

CANONICAL BASIS VS. OTHER BASES OF $\mathbb{R}^n$ -- COMPRESSION-DESIRABLE BASES OF $\mathbb{R}^n$ --

- The canonical basis of $\mathbb{R}^n$ is great in geometry and form a convenient representation of vectors, but it is terribly poor for compression
- We seek alternative bases that will be more suitable for lossy compression
- It would be great to have a basis $f_1, f_2, \dots, f_n$ for $\mathbb{R}^n$ such that for any "natural" signal $x = (x_1, x_2, \dots, x_n)$, we can express x as a linear combination of $f_1, f_2, \dots, f_n$ such that
  - $x = y_1 f_1 + y_2 f_2 + \cdots y_n \, f_n$
  - The human eyes/ears are more sensitive to certain $f_i' s$ and less sensitive to other $f_i' s$, *e.g.,*
    - More sensitive to f_1, f_2, f_3 and less sensitive to $f_4, f_5, \dots$
  - I.e., some basis vectors are more important, and others less important, to human senses
  - Then, we can drop the less important $f_i' s$ (i.e., replace the corresponding $y_i' s$ with 0's)
  - And approximate x by the remaining (important vectors), e.g., $x \approx y_1 f_1 + y_2 f_2 + y_3 f_3$
  - Or quantize the less important $y_i' s$ aggressively (to get more compression), and the more important $y_i' s$ lightly (to retain reconstruction quality)

EXISTANCE OF COMPRESSION-DESIRABLE BASES OF $\mathbb{R}^n$

- We will see that such compression-desirable bases do exist
- We will see also how the transforms we studied are connected to desirable bases
- But, first, let's see how a linear transform relates to bases in vector spaces
- That is the subject of the next few slides

1D SIGNALS AND COLUMN/MATRIX NOTATION

- Recall that a 1D signal x (like an audio signal) of n components is viewed as a vector $x = (x_1, x_2, \dots, x_n)$ in an n-dimensional vector space
- We will represent each such signal as a column vector:
  - $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
  - Note: In some situations, it is more convenient to start the indexing from 0 instead of 1
  - To save on vertical space, we sometimes write $x = [x_1, x_2, \dots x_n]^T$
- Also, each vector in $\mathbb{R}^n$, including each basis vector (for whatever basis), will be represented as a column vector

MATRICES AS A SEQUENCE OF COLUMN VECTORS

- A matrix $A = \left(a_{ij}\right)_{\substack{1\le i\le n \\ 1\le j\le m}}$, or simply $A = (a_{ij})$ when the number of rows and columns of A are known, is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

// again, sometimes the indexing starts from 0

- The matrix A can be viewed as a sequence of columns:

$$A = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \\ a_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{matrix}} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ C_1 & C_2 & \qquad & C_m \end{matrix} \end{array} = [C_1\ C_2\ \ \ldots C_m] \text{ where } C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, C_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \ldots, C_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}$$

MATRIX MULTIPLICATION

- Let $A = (a_{ij})$ be an $n \times m$ matrix, and $B = (b_{ij})$ be a $m \times r$ matrix
- The product AB is a $n \times r$ matrix $C = (c_{ij})$ where
  - $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{im}b_{mj}$ (inner product of row i of A and column j of B)
- $C = \begin{array}{c} \\ \\ i \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{array}{c} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{bmatrix} \\ \begin{array}{cccc} & & j & \end{array} \end{array} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \ddots & \boxed{c_{ij}} & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nr} \end{bmatrix}$

MATRIX-COLUMN MULTIPLICATION

- Let $A = (a_{ij})$ be an $n \times n$ matrix, $x = [x_1, x_2, \ldots x_n]^T$ be a column
- The product Ax is a column $y = [y_1, y_2, \ldots y_n]^T = Ax$ where
  - $y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$$

- $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \ddots \\ a_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{matrix}} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ C_1 & C_2 & & C_n \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [C_1\ C_2\ \ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + \cdots x_nC_n$

- Similarly, $x = A^{-1}y = \begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{matrix}} & \boxed{\begin{matrix} d_{12} \\ d_{22} \\ \ddots \\ d_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} d_{1n} \\ d_{2n} \\ \vdots \\ d_{nn} \end{matrix}} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ f_1 & f_2 & & f_n \end{matrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [f_1\ f_2\ \ldots f_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y_1f_1 + y_2f_2 + \cdots y_nf_n$

CONNECTION WITH TRANSFORMS

- Recall that a transform is characterized by a square matrix $A = (a_{ij})$
- The transform of a signal $x = [x_1, x_2, \dots x_n]^T$ is $y = Ax$
- The inverse transform of $y = [y_1, y_2, \dots y_n]^T$ is $x = A^{-1}y$
- Based on the previous slide, we have:

$$x = A^{-1}y = \left[\begin{array}{cccc} \boxed{\begin{matrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{matrix}} & \boxed{\begin{matrix} d_{12} \\ d_{22} \\ \ddots \\ d_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} d_{1n} \\ d_{2n} \\ \vdots \\ d_{nn} \end{matrix}} \\ f_1 & f_2 & & f_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [f_1\ f_2\ \dots f_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y_1 f_1 + y_2 f_2 + \cdots y_n f_n$$

- $\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_n$ are a new basis of $\mathbb{R}^n$
- $\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_n$ are the columns of the inverse matrix A^{-1} of the transform

- That is, $x = y_1 f_1 + y_2 f_2 + \cdots y_n f_n$

- Therefore: **$y_1, y_2, \dots y_n$ are the coordinates of vector x wrt to the basis $\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_n$**

TRANSFORMS AND BASES

- We saw earlier for vector/signal $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$:
  - $x_1, x_2, \ldots x_n$ are the coordinates of vector x wrt to the canonical basis $e_1, e_2, \ldots, e_n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

  - $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 \ldots + x_n e_n = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

- From the previous slide,

$$\text{if } y = Ax, A^{-1} = \begin{bmatrix} \underbrace{\boxed{\begin{matrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{matrix}}}_{f_1} & \underbrace{\boxed{\begin{matrix} d_{12} \\ d_{22} \\ \ddots \\ d_{n2} \end{matrix}}}_{f_2} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \underbrace{\boxed{\begin{matrix} d_{1n} \\ d_{2n} \\ \vdots \\ d_{nn} \end{matrix}}}_{f_n} \end{bmatrix}, \text{ then:}$$

$y_1, y_2, \ldots y_n$ are the coordinates of vector x wrt to the basis $f_1, f_2, \ldots, f_n$

$$x = y_1 f_1 + y_2 f_2 + \cdots y_n f_n$$

- Therefore, a **transform** characterized by a matrix A is:
  - **A change of basis, from the canonical basis $e_1, e_2, \ldots, e_n$ to a new basis $f_1, f_2, \ldots, f_n$ determined by A^{-1}**
  - **That is, a change of coordinate systems**

VISUALIZATION OF THE BASIS VECTORS OF A TRANSFORM

- Take the transform matrix A
- Get its transform matrix A^{-1}
- Extract the columns of A^{-1}: $f_1\ f_2\ \dots f_n$
  - Or, indexing from 0, rename them $f_0\ f_1\ \dots f_{n-1}$
- Plot each f_i where the horizontal axis is $0, 1, 2, \dots, n-1$, and the vertical axis corresponds to the values of the components of f_i
- There should be n plots, one per basis vector

THE BASIS VECTORS OF THE FOURIER TRANSFORM FOR N=8

- $a_{kl} = \sqrt{\frac{1}{N}} e^{-\frac{2\pi i}{N}kl} = \sqrt{\frac{1}{N}} \left(\cos\frac{2\pi}{N}kl - i\sin\frac{2\pi}{N}kl\right), \forall k, l = 0, 1, 2, \dots, N-1$
- $a = \frac{\sqrt{2}}{2}(1+i)$ and $\bar{a} = \frac{\sqrt{2}}{2}(1-i)$

$$A_8 = \sqrt{\frac{1}{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \bar{a} & -i & -a & -1 & -\bar{a} & i & a \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -a & i & \bar{a} & -1 & a & -i & -\bar{a} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\bar{a} & -i & a & -1 & \bar{a} & i & -a \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & a & i & -\bar{a} & -1 & -a & -i & \bar{a} \end{bmatrix}$$

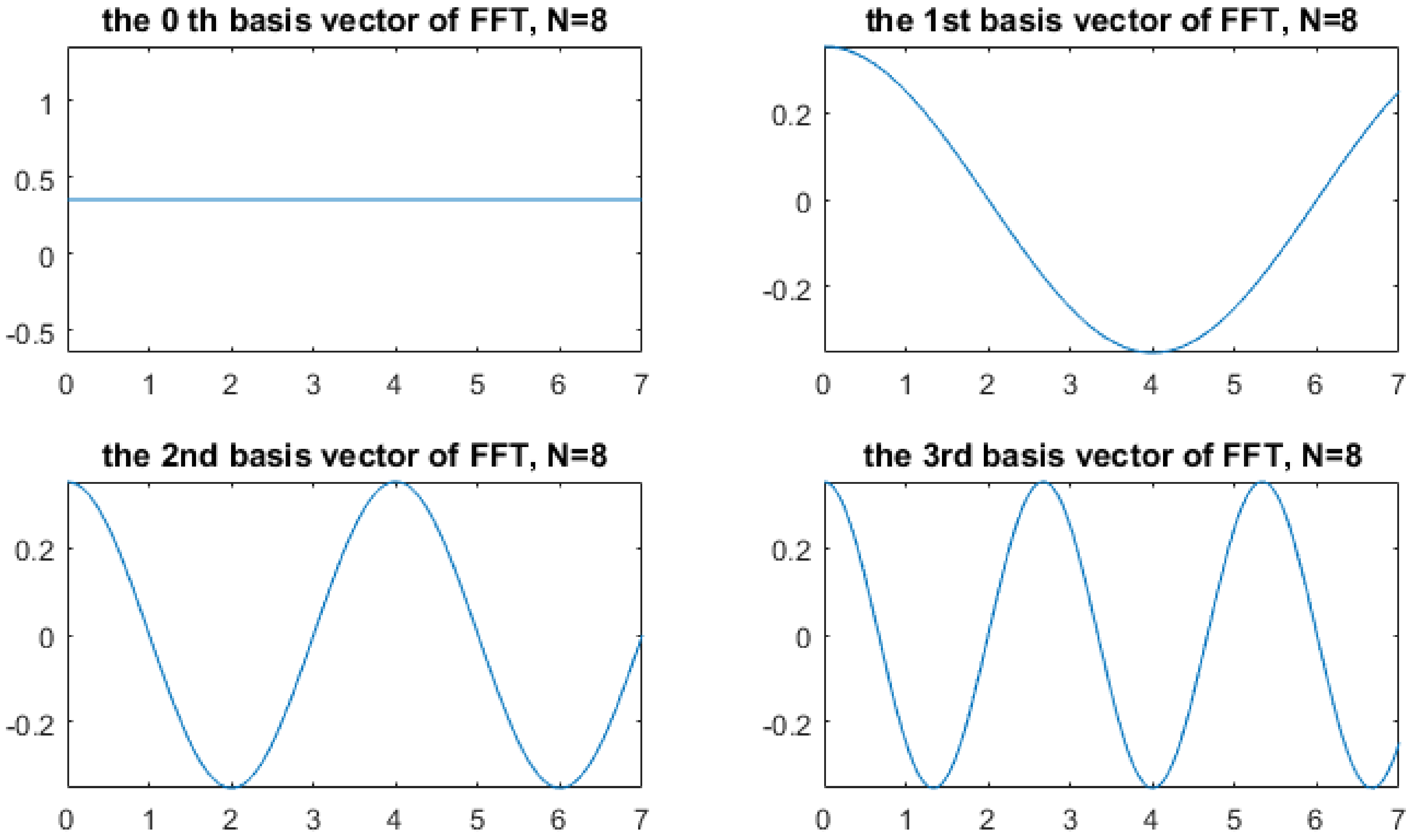
$$A^{-1} = \sqrt{\frac{1}{8}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & i & -\bar{a} & -1 & -a & -i & \bar{a} \\ 1 & i & -1 & -i & 1 & i & 1 & -i \\ 1 & -\bar{a} & -i & a & -1 & \bar{a} & i & -a \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -a & i & \bar{a} & -1 & a & -i & -\bar{a} \\ 1 & -i & 1 & i & 1 & -i & -1 & i \\ 1 & \bar{a} & -i & -a & -1 & -\bar{a} & i & a \end{vmatrix}$$

THE BASIS VECTORS OF FOURIER TRANSFORM

- $A_N^{-1} = conjugate(A_N)$=$(\sqrt{\frac{1}{N}}e^{\frac{2\pi i}{N}kl}) = \sqrt{\frac{1}{N}}(\cos\frac{2\pi}{N}kl + i\sin\frac{2\pi}{N}kl), \forall k,l = 0,1,2,\dots,N-1$
- Because the entries involve complex numbers, we can't plot them
- But we can plot the real-part: $(\sqrt{\frac{1}{N}}\cos\frac{2\pi}{N}kl)\ \forall k,l = 0,1,2,\dots,N-1$
- $\forall l = 0,1,2,\dots,N-1$, take real-part of the l^{th} basis vector: $f_l(k) = \sqrt{\frac{1}{N}}\cos\frac{2\pi}{N}kl$
- It is more insightful to plot each basis vector $f_l(k) = \sqrt{\frac{1}{N}}\cos\frac{2\pi}{N}kl$ as a continuous function of k (or of x): $f_l(x) = \sqrt{\frac{1}{N}}\cos\frac{2\pi}{N}lx$, for $N = 8, l = 0,1,\dots,7$

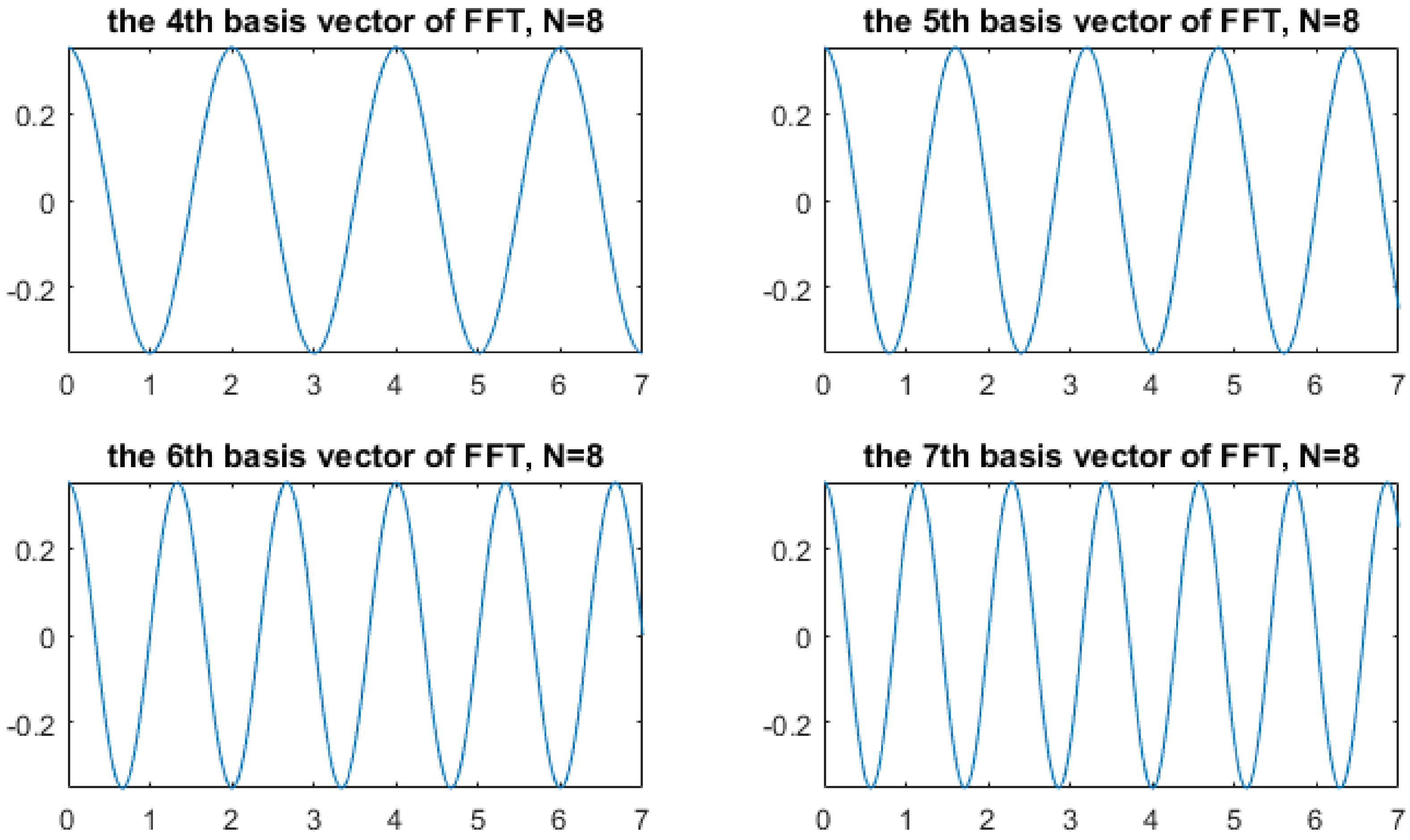
THE REAL-PART OF THE BASIS VECTORS OF FFT, N=8 (1/2)

- The first 4 basis vectors (shown as continuous plots of the real part)



THE REAL-PART OF THE BASIS VECTORS OF FFT, N=8 (2/2)

- The second 4 basis vectors (shown as continuous plots of the real part)



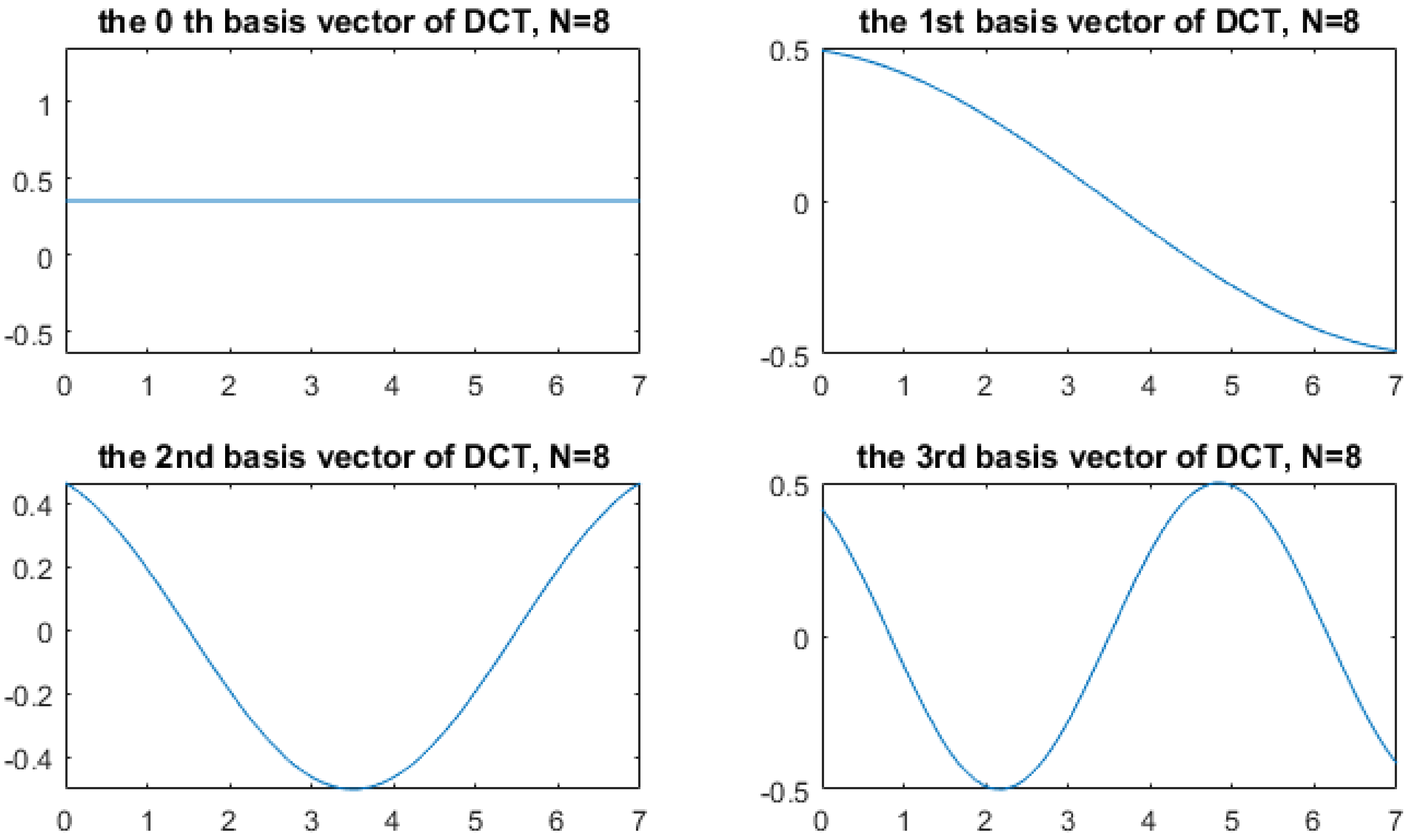
THE BASIS VECTORS OF DCT OF SIZE N=8

- $a_{0l} = \sqrt{\frac{1}{N}}$ for all $l = 0, 1, \ldots, N-1;$
- $a_{kl} = \sqrt{\frac{2}{N}} \cos \frac{\left(l+\frac{1}{2}\right)k\pi}{N}$ $\forall\, k = 1,2, \ldots, N-1,$ and $l = 0,1, \ldots, N-1$
- $A_N^{-1} = A_N^T$
- f_k = Col$_k(A_N^{-1})$=Row$_k(A_N)$ =(a_{kl}) for $l = 0,1, \ldots, N-1$
- $f_0 = \sqrt{\frac{1}{N}}\,[1\ 1\ \ldots 1]^T,$ and for $k \geq 1$:
- $f_k = \sqrt{\frac{2}{N}}\left[\cos\frac{\left(\frac{1}{2}\right)k\pi}{N},\ \cos\frac{\left(1+\frac{1}{2}\right)k\pi}{N},\ \cos\frac{\left(2+\frac{1}{2}\right)k\pi}{N},\ \ldots\ \cos\frac{\left(N-1+\frac{1}{2}\right)k\pi}{N}\right]^T$

- Let $\alpha_0 = \sqrt{\frac{1}{N}},$ *and* $\alpha_k = \sqrt{\frac{2}{N}}$ for $k \geq 1$
- $f_k(x) = \alpha_k \cos\frac{\left(x+\frac{1}{2}\right)k\pi}{N}$
- We will plot $f_k(x)$ as a continuous function of x, for N=8, $k = 1,2, \ldots, 7$

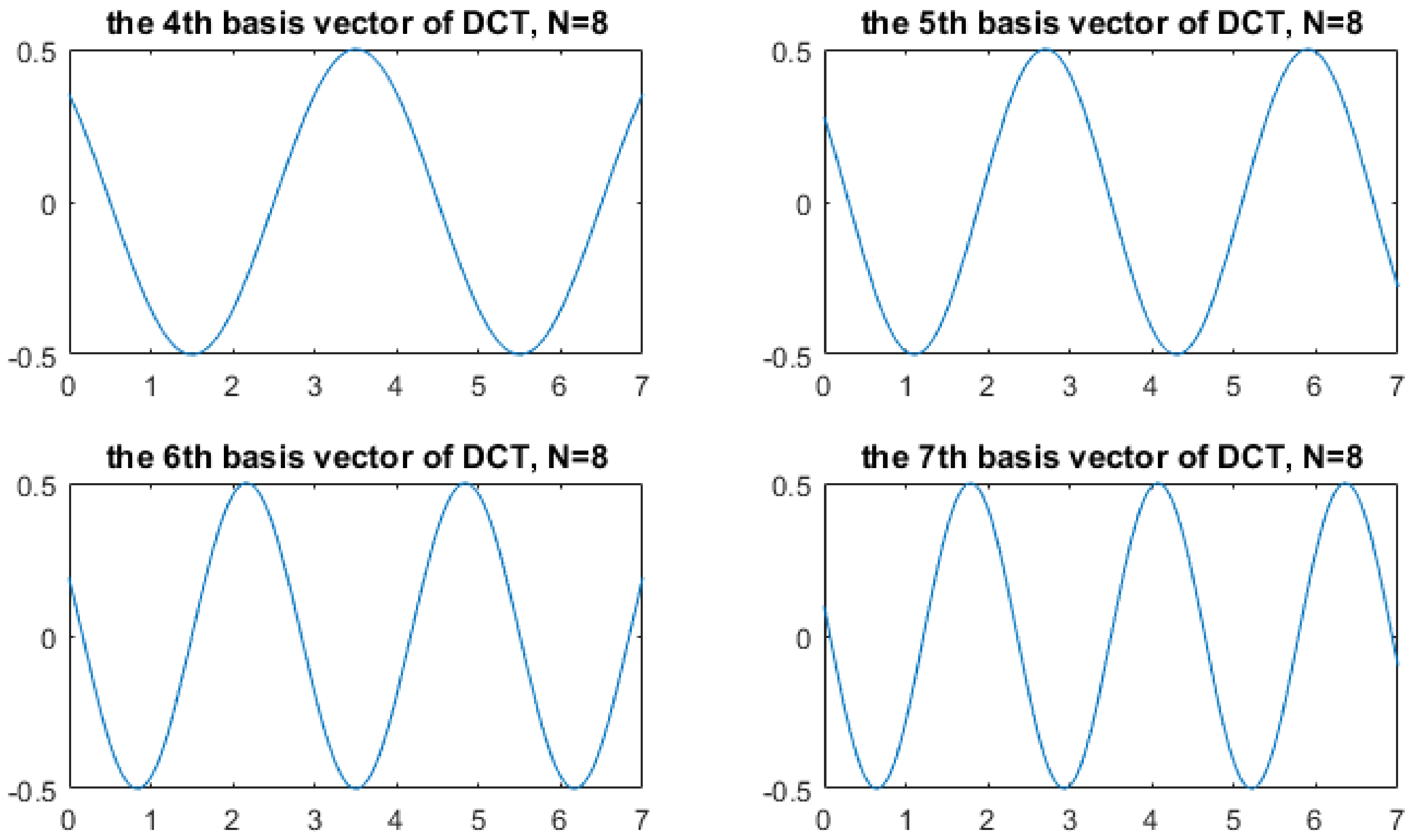
THE BASIS VECTORS OF DCT OF SIZE N=8 (1/2)

- The first 4 basis vectors (shown as continuous plots)



THE BASIS VECTORS OF DCT OF SIZE N=8 (2/2)

- The second 4 basis vectors (shown as continuous plots)



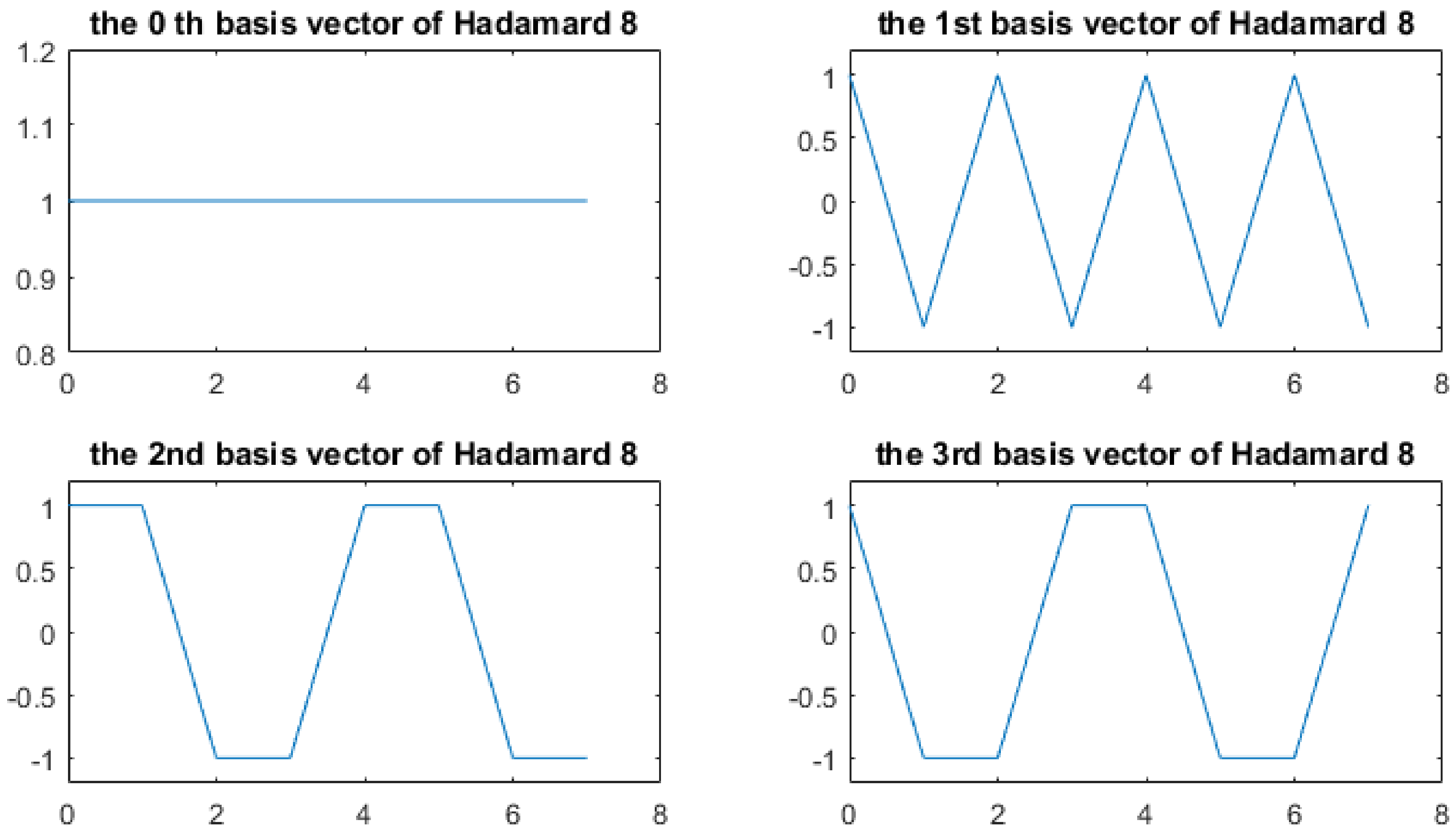
THE MATRIX OF THE HADAMARD TRANSFORM FOR N=8

- $A_N^{-1} = A_N^T = A_N$

$$A_8 = \sqrt{\frac{1}{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

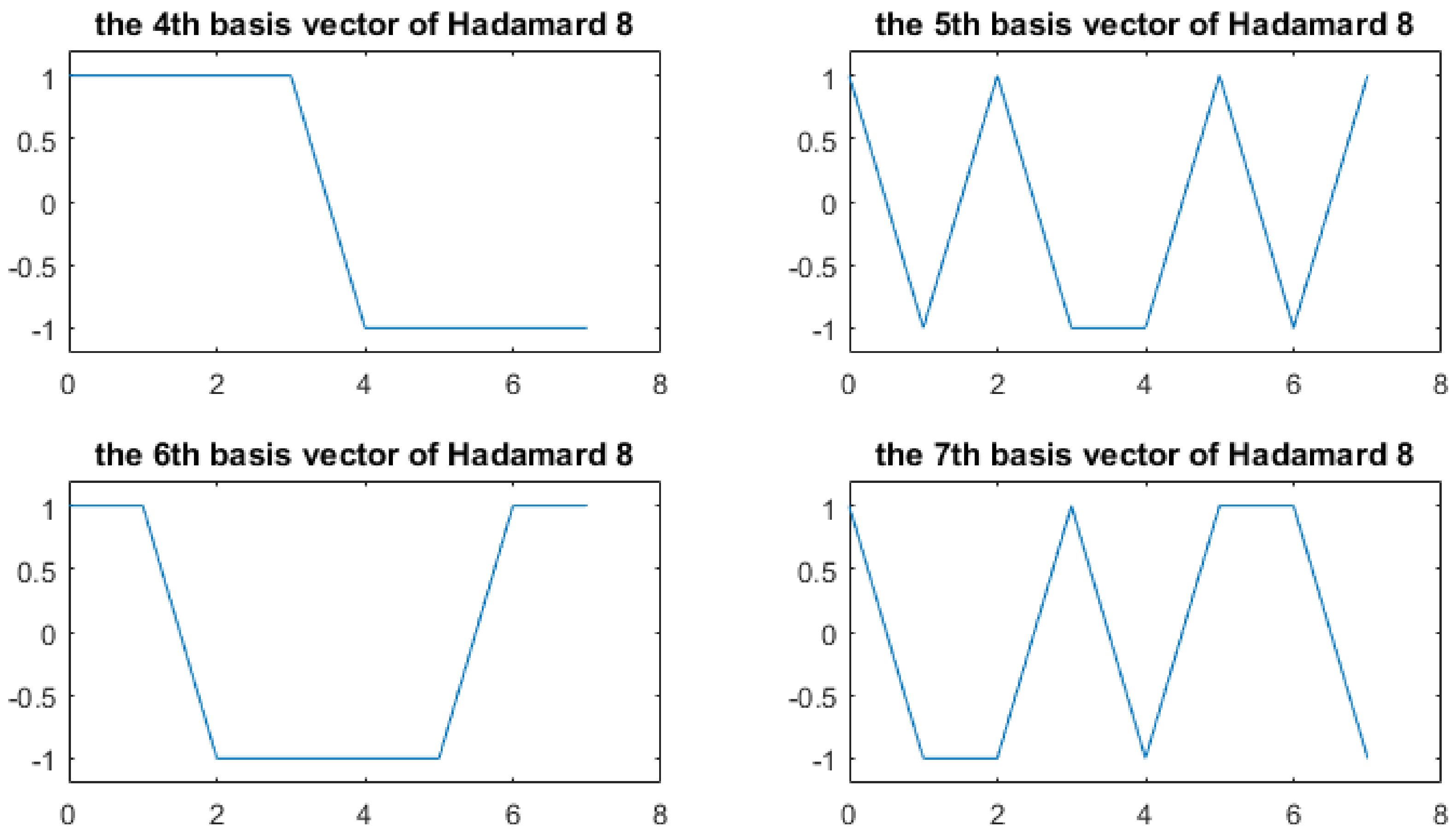
THE BASIS VECTORS OF HADAMARD FOR N=8 (1/2)

- The first 4 basis vectors



THE BASIS VECTORS OF HADAMARD FOR N=8 (2/2)

- The second 4 basis vectors



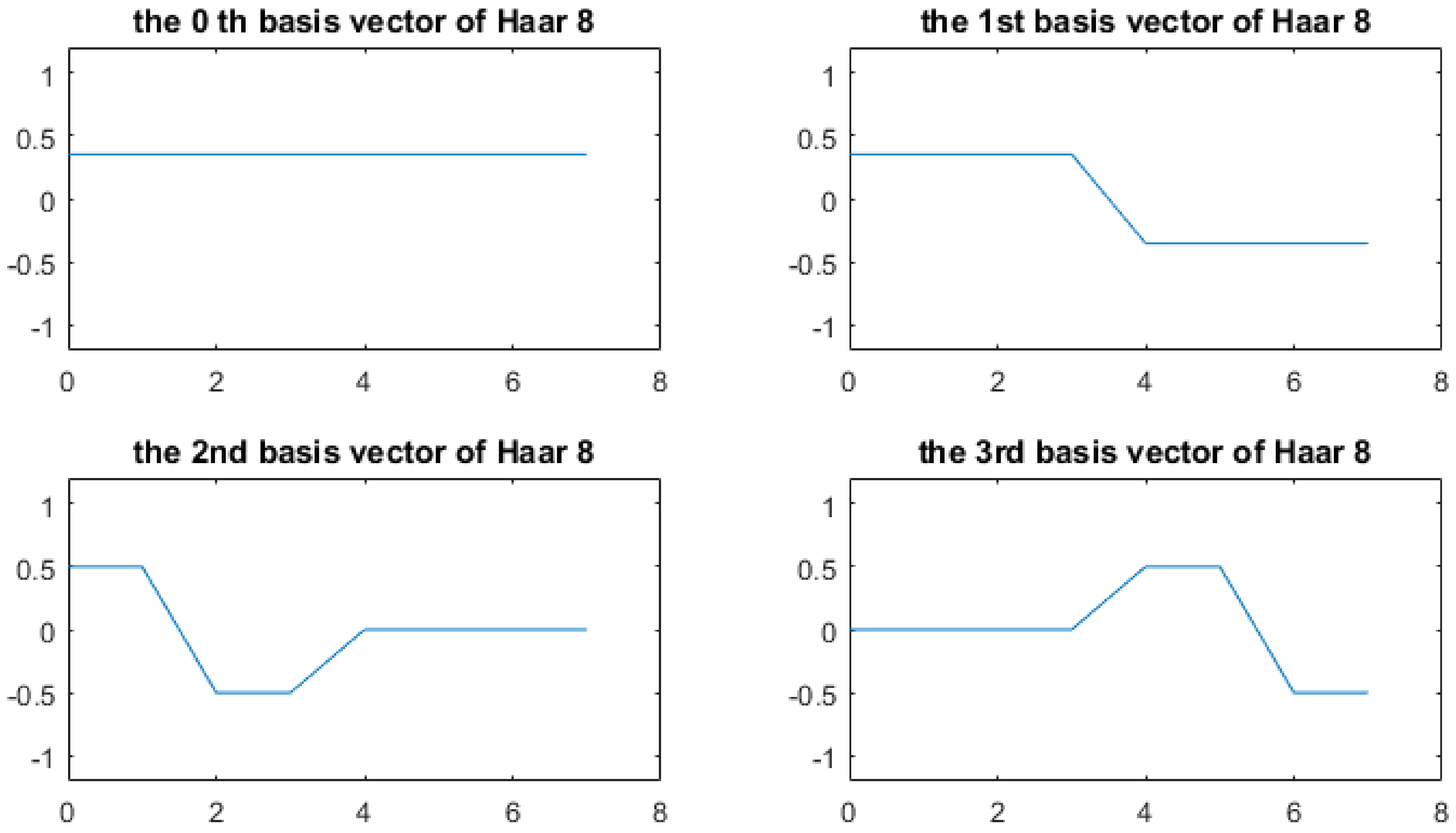
THE MATRIX OF THE HAAR TRANSFORM FOR N=8

- $A_N^{-1} = A_N^T$
- Therefore, $f_k = col_k(A_N^{-1}) = col_k(A_N^T) = [row_k(A_N)]^T$

$$A_8 = \sqrt{\frac{1}{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix}$$

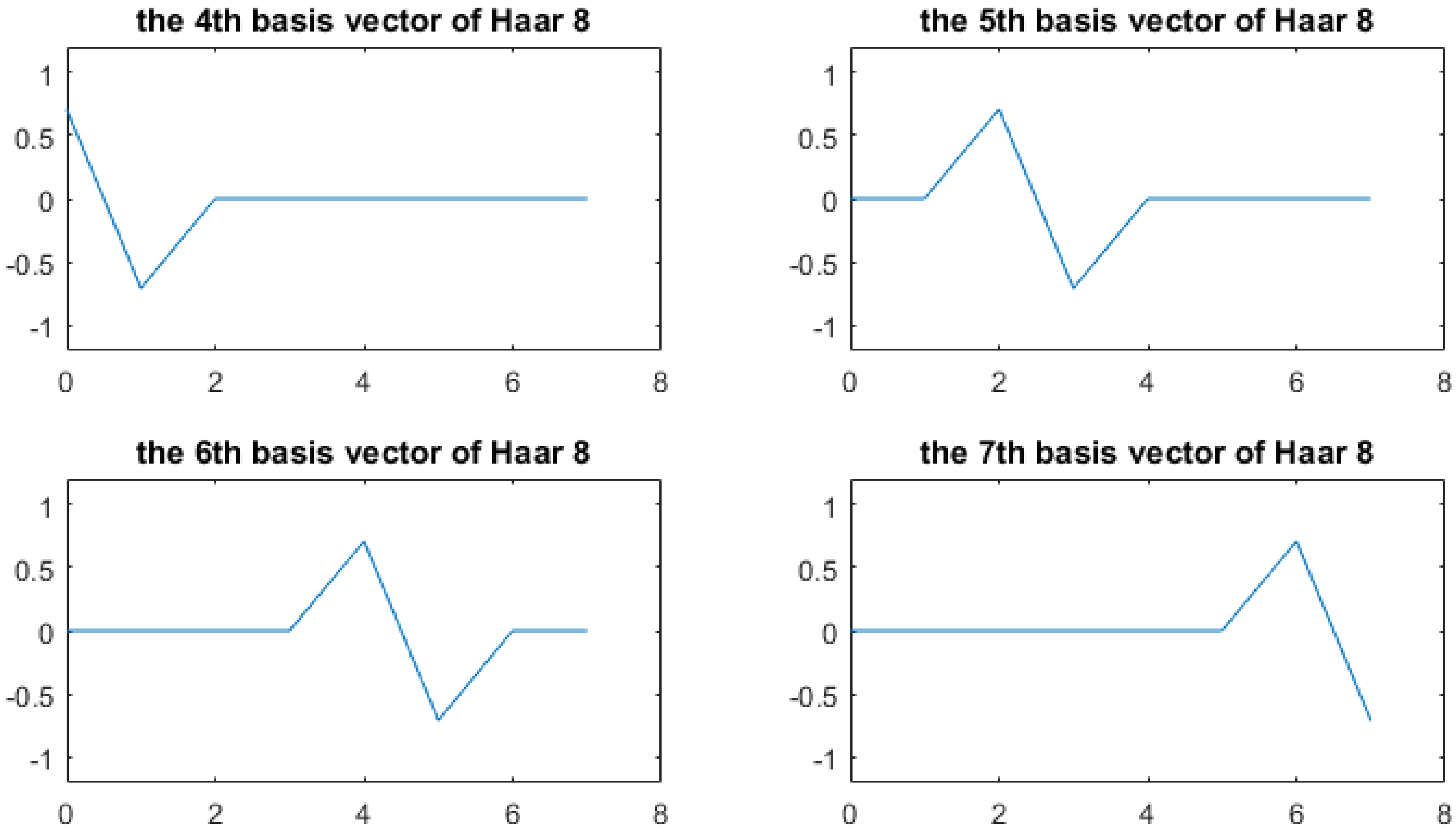
THE BASIS VECTORS OF HAAR FOR N=8 (1/2)

- The first 4 basis vectors



THE BASIS VECTORS OF HAAR FOR N=8 (2/2)

- The second 4 basis vectors



NEXT LECTURE

- Frequency Perspective of Transforms
- Statistical Perspective of Transforms