Stochastic averaging of dynamical systems with multiple time scales forced with α-stable noise

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Introduction

α-**stable Lévy processes** are **heavy-tailed stochastic processes** observed in several different aspects of **climate variability**.

- Observed α-stable noise in continental aridity proxy measured from Greenland ice core [2].
- Stratospheric particle dynamics are superdiffusive (i.e. α-stable) due to eddy transport [4].
- Measuring dispersion rates of Lagrangian particles indicates superdiffusive processes in the extratropical troposphere [3].

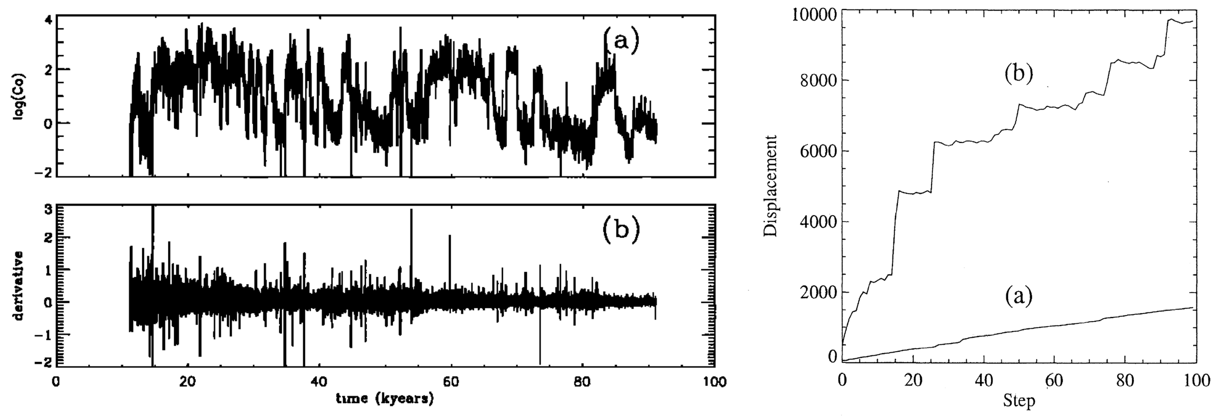

FIGURE 1: Left: GRIP Calcium signal. [2], Right: (b) shows displacement of a superdiffusive particle. [4]

- α-stable dynamics are generally associated with **non-equilibrium statistical physics**, **turbulent fluid dynamics**.
- Many climate systems have **interacting dynamical time scales** (weather influences climate and vice-versa).
- In this poster, we display a **stochastic averaging procedure for multiple time scale systems with α-stable forcing**. Full results are presented in [6].

α-stable noise processes

Continuous-time stochastic processes with increments that are distributed according to an **α-stable distribution**.

- **Heavy-tailed** $\Rightarrow$ infinite variance, large jumps.
- Three parameters: **stability parameter** $\alpha \in (0,2]$ (tail decay), **skewness parameter** $\beta \in [-1,1]$, and **scale parameter** $\gamma > 0$ (width).

Stochastic averaging and α-stable noise

Stochastic averaging is a method of approximating the distribution of the slow variables of slow-fast stochastic dynamical systems. We assume $1 < \alpha \le 2$ and consider the following stochastic slow-fast system

$$dx_t = f(x_t, y_t)\,dt, \quad \epsilon\, dy_t = g(x_t, y_t)\,dt + \epsilon^{\gamma} b\, dL_t^{(\alpha,\beta)},$$

where $\gamma = 1 - 1/\alpha$, $t \ge 0$, $0 < \epsilon \ll 1$ and

$$f(x,y) = f_1(x) + \epsilon^{-\gamma} f_2(x) y, \quad g(x,y) = \epsilon^{\gamma} g_1(x) + g_2(x) y.$$

- An asymptotic analysis is performed on **the characteristic function** for this system.
- The resulting (L) approximation for this system is $x \sim \overline{x} + \xi$ where

$$\begin{cases} d\overline{x}_t = \overline{f}(\overline{x}_t)\,dt, \quad \overline{x}_0 = x_0, \quad \overline{f}(x) = \mathbb{E}^{(y|x)}\left[f(x,y)\right] \\ d\xi_t = \overline{f}'(\overline{x}_t)\xi_t\,dt + b\left|\frac{f_2(\overline{x}_t)}{g_2(\overline{x}_t)}\right| dL_t^{(\alpha,\beta^*)}, \quad \xi_0 = 0 \end{cases}$$

- A similar analysis yields the analogue to the (N+) approximation: $x \sim X$, where

$$dX_t = \overline{f}(X_t)\,dt + b\left(\frac{f_2(X_t)}{-g_2(X_t)}\right) \diamond dL_t^{(\alpha,\beta)}, \quad X_0 = x_0.$$

and '$\diamond$' denotes **Marcus' canonical stochastic integration**, the generalization of Stratonovich integration to jump processes [1]:

$$\int_0^t \kappa(z_s) \diamond dL_s^{(\alpha,\beta)} = \sum_{s \le t}\left[\theta(1;\Delta L_s, z_{s-}) - z_{s-}\right].$$

where $\theta(r;\Delta L_s, z_{s-})$ satisfies

$$\frac{d\theta(r;\Delta L_s, z_{s-})}{dr} = \Delta L_s\, \kappa(\theta(r;\Delta L_s, z_{s-})), \quad \theta(0;\Delta L_s, z_{s-}) = z_{s-}$$

- The **autocovariance function does not exist** for heavy-tailed processes, so we estimate the analogous **autocodifference function**, A, of the full system and the approximations to compare temporal dependence [5]:

$$A(z,\tau) = \log\left(\mathbb{E}\left[\exp(i(z_{t+\tau} - z_t))\right]\right) - \log\left(\mathbb{E}\left[\exp(iz_{t+\tau})\right]\right) - \log\left(\mathbb{E}\left[\exp(-iz_t)\right]\right),$$

Examples

We apply our method of averaging to one linear and three nonlinear systems. For all subsequent figures, $\epsilon = 10^{-2}$.

- **Linear system**:

$$\begin{cases} dx_t = \left(-x_t + \epsilon^{-\gamma} a y_t\right) dt, \quad x_0 = 0 \\ \epsilon\, dy_t = \left(\epsilon^{\gamma} c x_t - y_t\right) dt + \epsilon^{\gamma} b\, dL_t^{(\alpha,\beta)}, \quad y_0 = 0 \end{cases}$$

(L): $\overline{x}_t = 0, \quad d\xi_t = -(1 - ac)\xi_t\,dt + ab\,dL_t^{(\alpha,\beta)}, \quad \xi_0 = 0$

(N+) is equivalent to (L) in stationary limit.

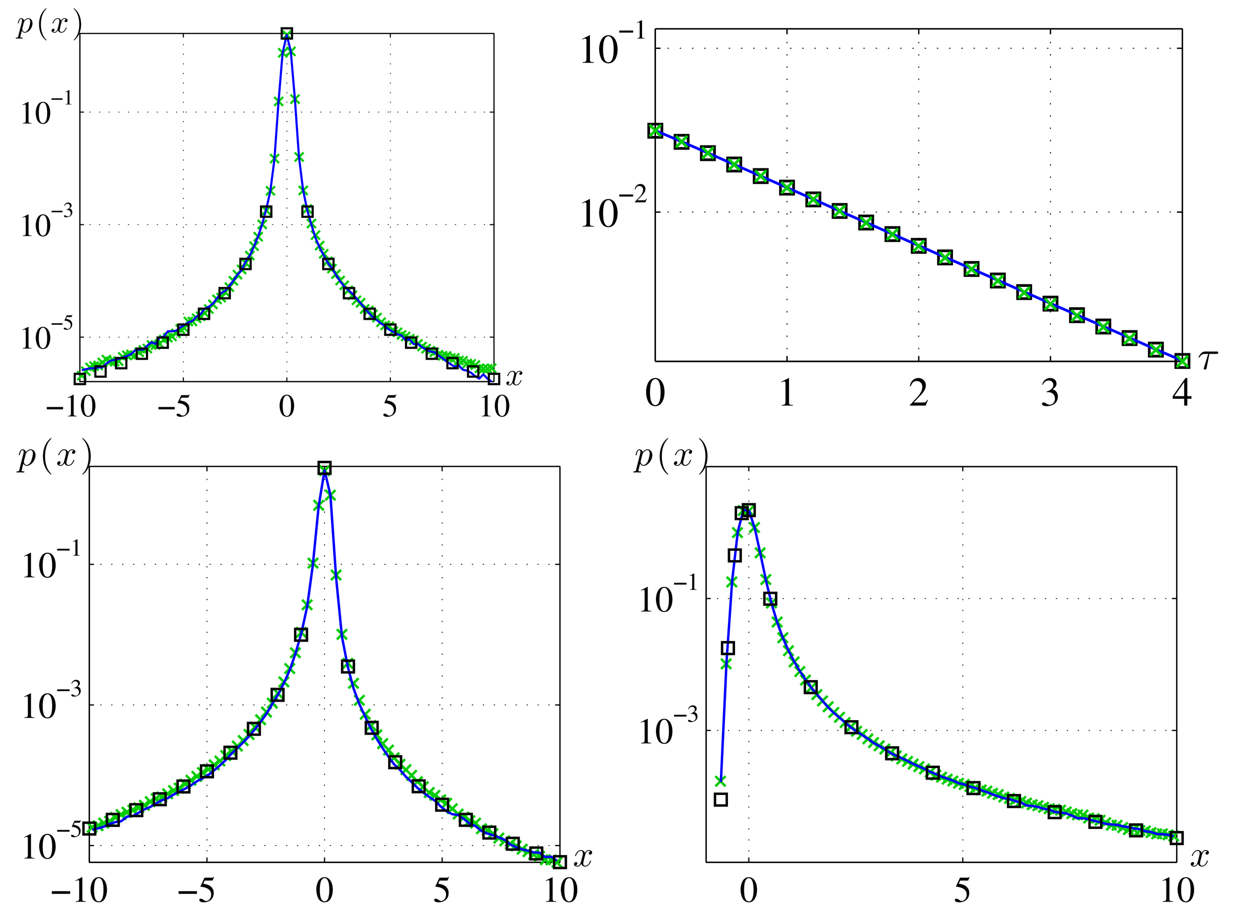

FIGURE 2: *Top*: Estimated densities (left) and autocodiff. function (right) for the linear system where $(\alpha,\beta) = (1.9, 0)$. *Bottom*: Estimated densities where $\alpha = 1.7$ and $\beta = -0.5$ (left) and $\beta = 1$ (right). The "exact" values of each function for ξ are plotted in black squares.

- **Nonlinear system 1**:

$$\begin{cases} dx_t = \left(c - x_t + \epsilon^{-\gamma} x_t y_t\right) dt, \quad x_0 = c, \\ dy_t = -\frac{y_t}{\epsilon a}\,dt + \frac{b}{\epsilon^{1/\alpha}}\,dL_t^{(\alpha,0)}, \quad y_0 = 0 \end{cases}$$

(L): $\overline{x}_t = c, \quad d\xi_t = -\xi_t\,dt + abc\,dL_t^{(\alpha,0)}, \quad \xi_0 = 0.$

(N+): $dX_t = (c - X_t)\,dt + abX_t \diamond dL_t^{(\alpha,0)}, \quad X_0 = c.$

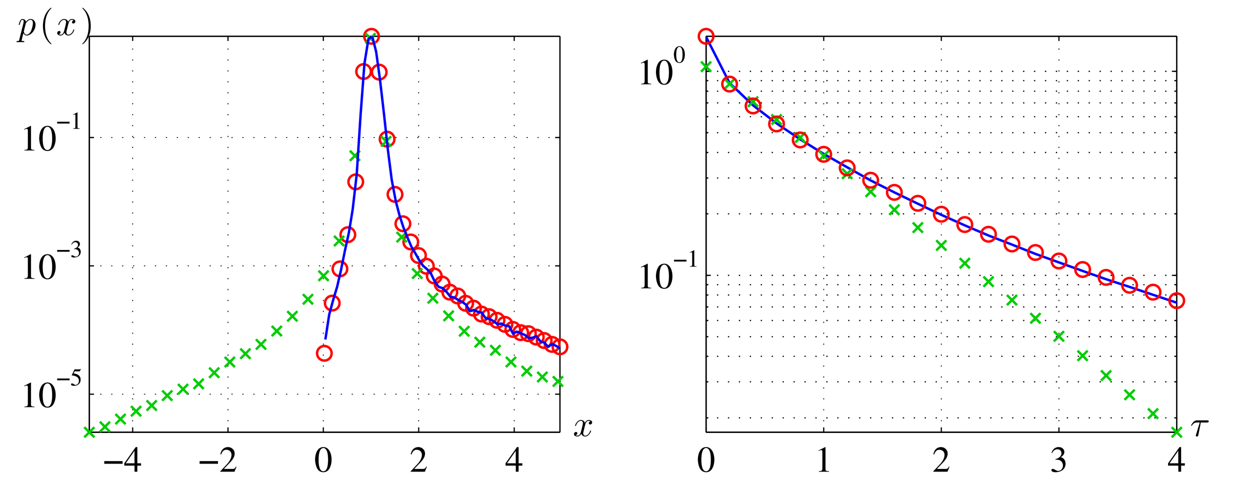

FIGURE 3: Estimated densities (left) and autocodiff. functions (right) for nonlinear system 1 with $(\alpha, a, b, c) = (1.9, 1, 1, 1)$.

- **Nonlinear system 2**:

$$\begin{cases} dx_t = \left(-x_t + \epsilon^{-\gamma} y_t\right) dt, \quad x_0 = 0, \\ dy_t = -\frac{(1+|x_t|)}{\epsilon} y_t\,dt + \frac{b}{\epsilon^{1/\alpha}}\,dL_t^{(\alpha,0)}, \quad y_0 = 0. \end{cases}$$

(L): $\overline{x}_t = 0, \quad d\xi_t = -\xi_t\,dt + b\,dL_t^{(\alpha,0)}, \quad \xi_0 = 0$

(N+): $dX_t = -X_t\,dt + b(1 + |X_t|)^{-1} \diamond dL_t^{(\alpha,0)}, \quad X_0 = 0.$

- **Nonlinear system 3**:

$$\begin{cases} dx_t = \left(-x_t - x_t^3 + \epsilon^{-\gamma} y_t\right) dt, \quad x_0 = 0, \\ dy_t = -\frac{y_t}{\epsilon a}\,dt + \frac{b}{\epsilon^{1/\alpha}}\,dL_t^{(\alpha,0)}, \quad y_0 = 0. \end{cases}$$

(L): $\overline{x}_t = 0, \quad d\xi_t = -\xi_t\,dt + ab\,dL_t^{(\alpha,0)}, \quad \xi_0 = 0.$

(N+): $dX_t = (-X_t - X_t^3)\,dt + ab\,dL_t^{(\alpha,0)}, \quad X_0 = 0.$

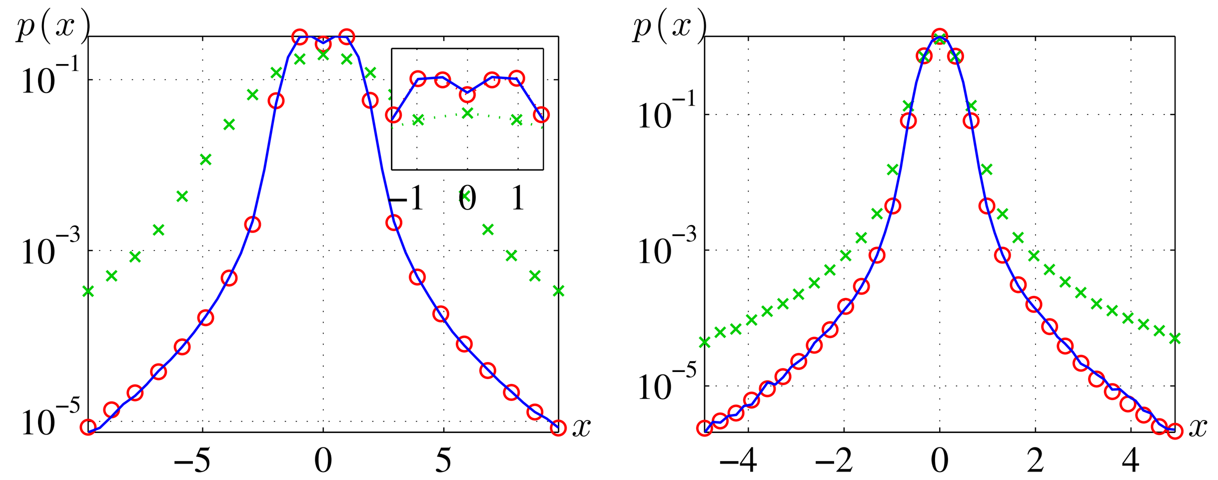

FIGURE 4: Estimated densities with $\alpha = 1.9$ for nonlin. system 2 with $b = 10$ (left, inset on linear scale) and nonlin. system 3 with $(a,b) = (1,1)$ (right).

Conclusions and future goals

- We present a method for approximating the slow component of fast-slow dynamical systems with α-stable noise forcing.
- Current research explores the application of this work to heavy-tailed stochastic dynamical systems resulting from multiple time scale behaviour in atmospheric dynamics, specifically correlated additive and multiplicative Gaussian forcing.

References

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Acknowledgements

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