602_hw2

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3.1

a)

$$P(Y_1 = y_1, \ldots, Y_{100} = y_{100}|\theta) = \text{by independence} = \prod_{i=1}^{n} P(Y_i|\theta) =$$

$$\prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i}\theta^{\Sigma_{i=1}^{n} y_i}(1-\theta)^{100-\Sigma_{i=1}^{n} y_i}; y = 0, 1$$

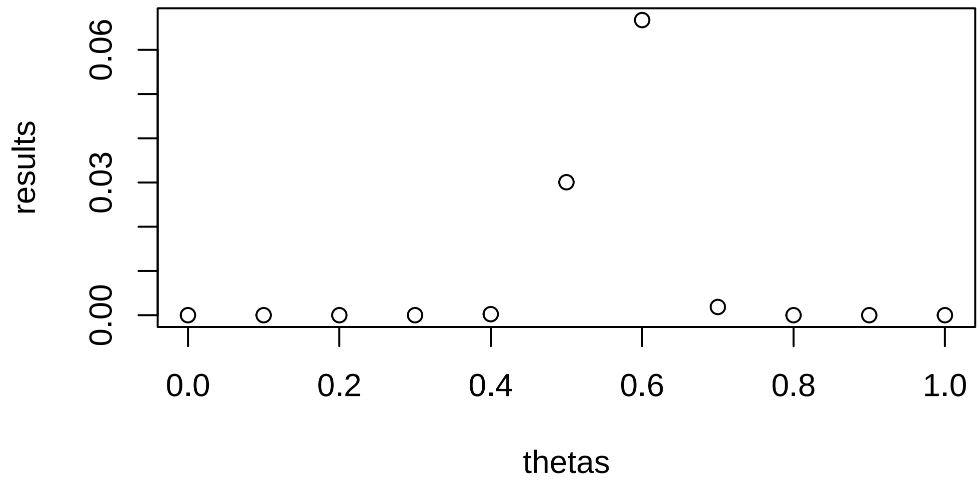
Finding the distribution of $P(\sum_{i=1}^{n} Y_i = y|\theta)$

$$M_{\sum Y_i = y|\theta}(t) = \text{by independence} = \prod_{i=i}^{n} M_{Y_i|\theta}(t) = \prod_{i=i}^{n}(1-p+pe^t) = (1-p+pe^t)^n = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$= \binom{100}{57}\theta^{57}(1-\theta)^{43}; \theta \in [0,1] \text{assuming a uniform prior?}$$

b)

```
thetas = seq(0.0,1.0,by=0.1)
results = dbinom(57,100,thetas)

plot(thetas,results)
```



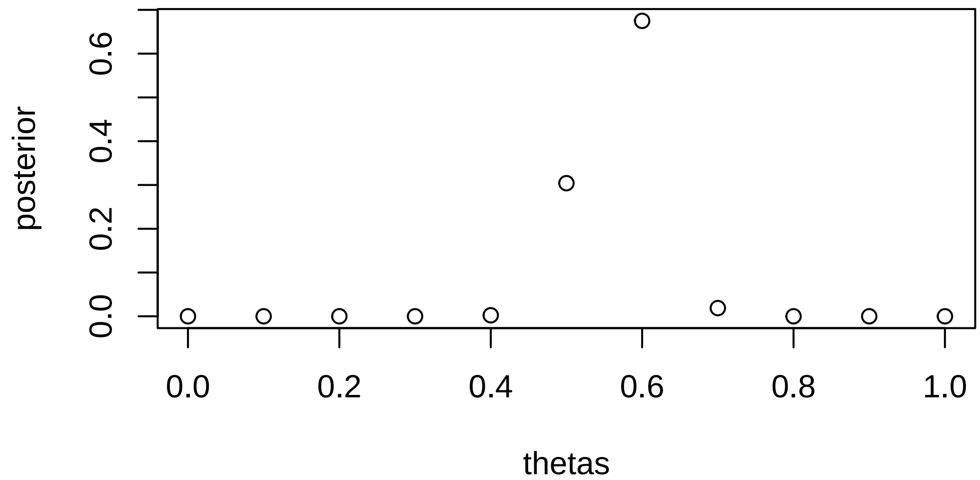
c)

$$p(\theta|\Sigma_{i=1}^{n}y_i = 57) = \frac{P(\Sigma_{i=1}^{n}y_i = 57|\theta)P(\theta)}{P(\Sigma_{i=1}^{n}y_i = 57)}\text{each } P(\Theta = \theta) = \frac{1}{11}$$

sum function above
may need to add binomial coefficient and sum out the discrete values of theta?

The posterior distribution and marginal distribution of Y are just scaling constants since the denominator does not depend on theta and we have equal belief for each of $P(\theta)$.

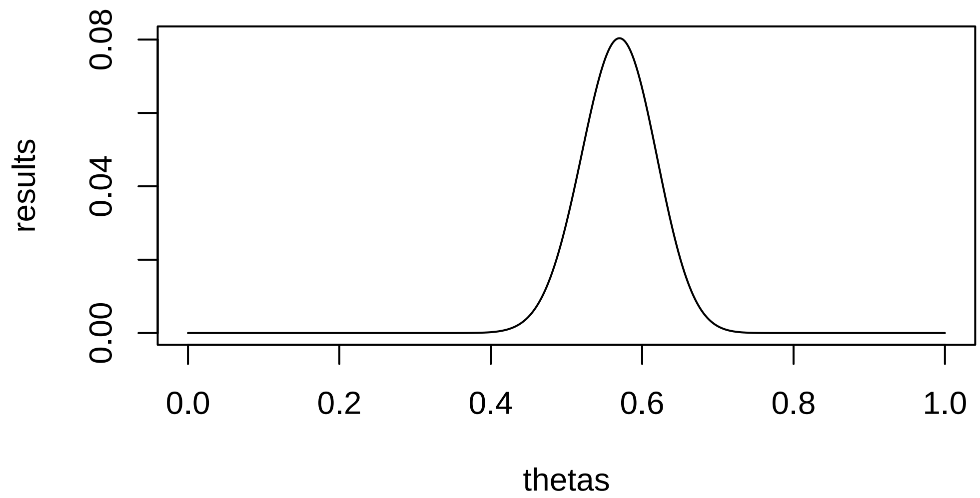
```
marginal_y = sum((1/11) * dbinom(57,100,thetas))
posterior = (results * (1/11))/marginal_y
plot(thetas,posterior)
```



d)

Not sure here on letting theta be any value in the interval. Approximating that with discrete values below but not sure if that's correct?

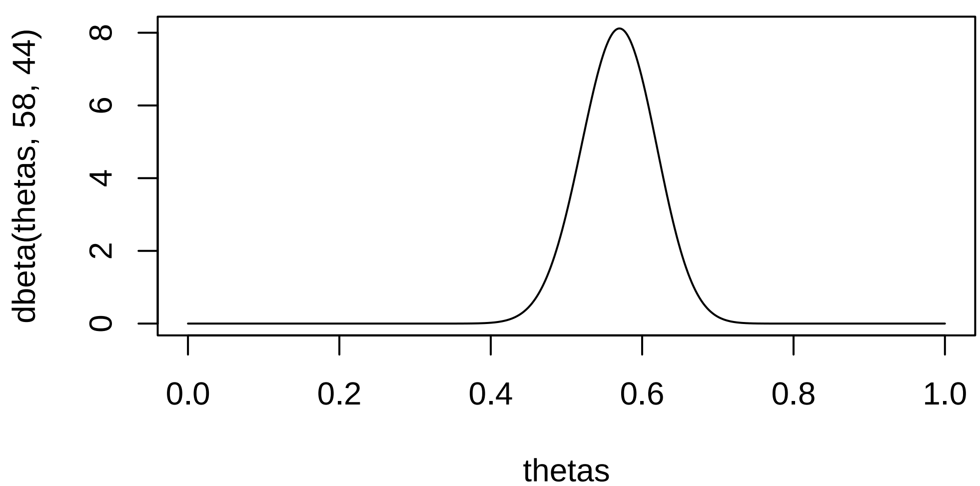
```
thetas = seq(0,1,by=0.001) #U(0,1)
results = dbinom(57,100,thetas)
plot(thetas,results,type="l")
```



e)

Same thing as d)

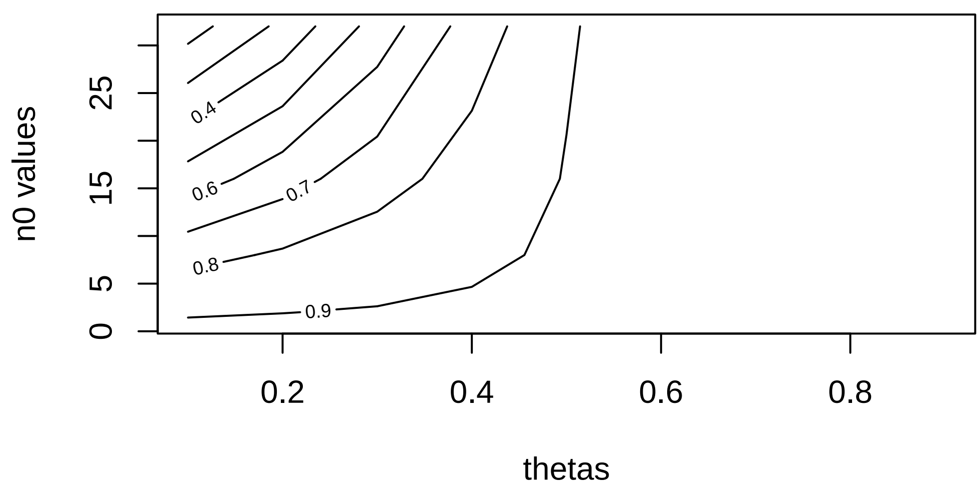
```
plot(thetas,dbeta(thetas,58,44),type='l')
```



3.2

Looks almost correct here but some kind of calculation is off?

```
theta0 = seq(0.1,0.9,by=0.1)
n0 = c(1,2,8,16,32)

data=c()

for (i in theta0) {
for (j in n0) {

    a = i * j
    b = (1-i)*j

    p = pbeta(.5,a+57,b+43,lower.tail=FALSE) #posterior (theta > .5 | sum = 57)
    data = append(data,p)

}
}

probability_data = matrix(data,nrow=9,ncol=5,byrow=TRUE)
contour(theta0,n0,probability_data, xlab="thetas",ylab='n0 values')
```



3.4

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

a)

calculations for posterior with prior beta(2,8) and beta(8,2) are here, with plots for part a) and part b) following this chunk.

```
beta_mean = function(a,b){
  print("mean:")
  a / (a+b)
}

beta_mode = function(a,b){
  print('mode:')
  (a-1) / (a+b-2)
}

beta_sd = function(a,b){
  print('standard deviation:')
  var = (a*b) / ((a+b)^2 * (a+b+1))
  sd = sqrt(var)
  return(sd)
}

CI_28 = c(qbeta(.025,17,36),qbeta(.975,17,36))
CI_82 = c(qbeta(.025,23,30),qbeta(.975,23,30))

#data for the posterior w/ 2,8 prior and posterior a = 17, posterior b = 36
print("using alpha = 17 and beta = 36 with beta(2,8) prior")
```

```
[1] "using alpha = 17 and beta = 36 with beta(2,8) prior"
```

```
beta_mean(17,36)
```

```
[1] "mean:"
```

```
[1] 0.3207547
```

```r
beta_mode(17,36)
```

```
[1] "mode:"
```

```
[1] 0.3137255
```

```r
beta_sd(17,36)
```

```
[1] "standard deviation:"
```

```
[1] 0.0635189
```

```r
print(c("95% CI",CI_28))
```

```
[1] "95% CI"            "0.203297787819103" "0.451023982216632"
```

```r
#with 8,2 prior
print(" ")
```

```
[1] " "
```

```r
print("================================================")
```

```
[1] "================================================"
```

```r
print("using alpha = 23, beta = 30 with beta(8,2) prior")
```

```
[1] "using alpha = 23, beta = 30 with beta(8,2) prior"
```

```r
beta_mean(23,30)
```

```
[1] "mean:"
```

```
[1] 0.4339623
```

```r
beta_mode(23,30)
```

```
[1] "mode:"
```

```
[1] 0.4313725
```

```r
beta_sd(23,30)
```

```
[1] "standard deviation:"
```

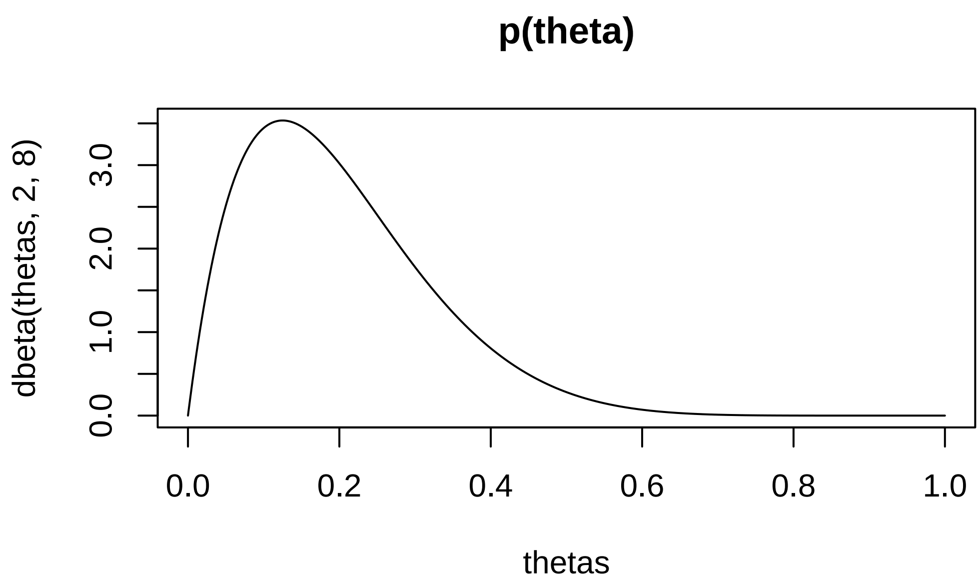
```
[1] 0.06744532
```

```r
print(c("95% CI",CI_82))
```

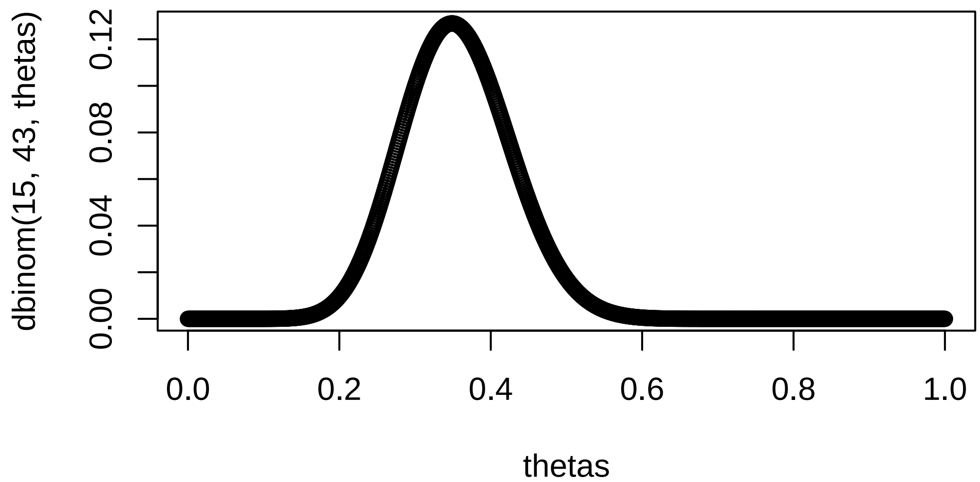
```
[1] "95% CI"            "0.304695624711747" "0.567952795996458"
```

plots for part a)

```r
#plotting prior p(\theta)
thetas = seq(0,1,by=0.001) #U(0,1)

plot(thetas, dbeta(thetas, 2,8), type='l',main="p(theta)")
```



```r
#plotting p(y=15|\theta)
#plot a binomial here
plot(thetas, dbinom(15,43,thetas))
```



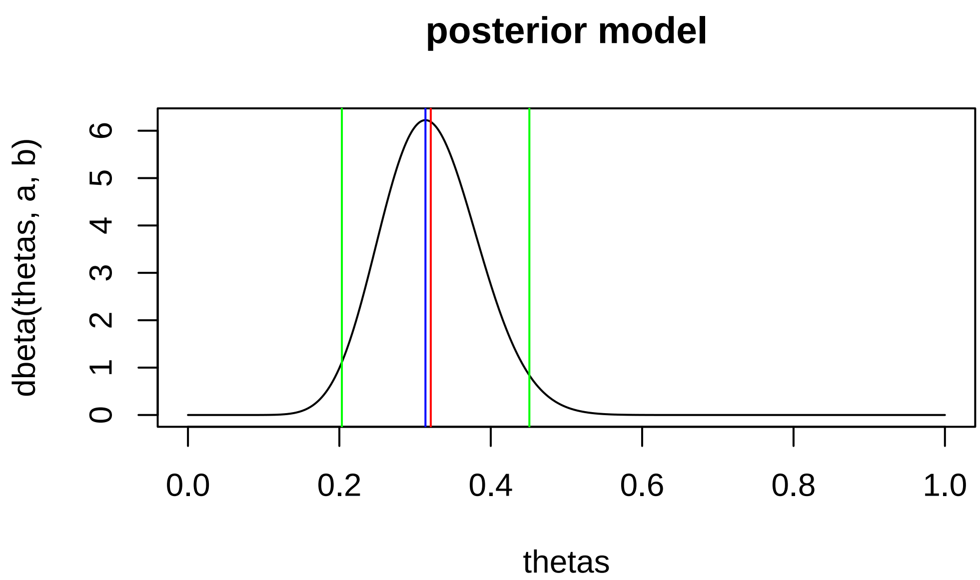

```r
#posterior which is beta(2 + success, 8 + failure) = beta()
a=2+15
b=8+28
plot(thetas, dbeta(thetas,a,b),type='l',main="posterior model")
abline(v=beta_mean(a,b), col='red') #mean
```

```
[1] "mean:"
```

```r
abline(v=beta_mode(a,b), col='blue') #mode
```

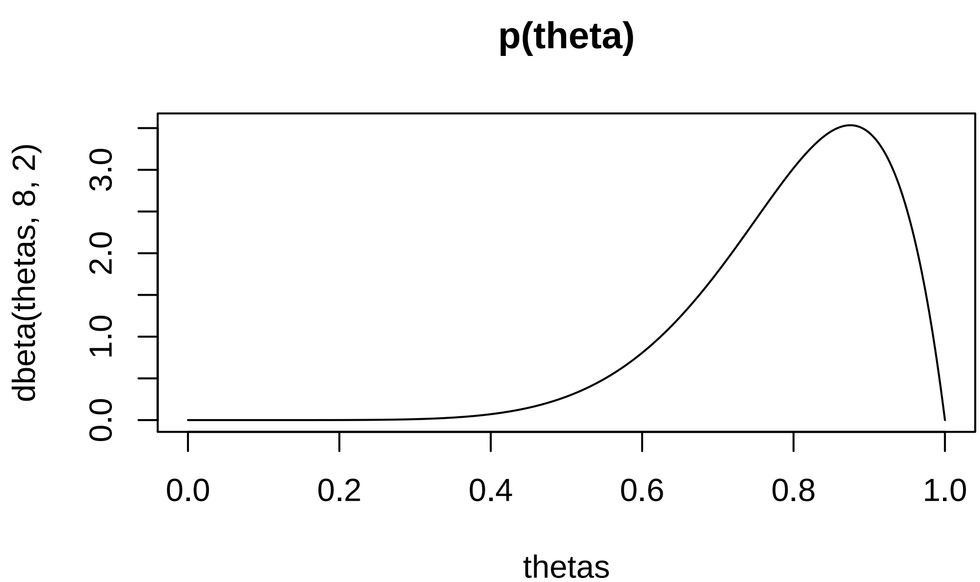
```
[1] "mode:"
```

```r
# CI
abline(v=qbeta(.975,a,b),col='green') #lower bound
abline(v=qbeta(.025,a,b),col='green') #upper bound
```

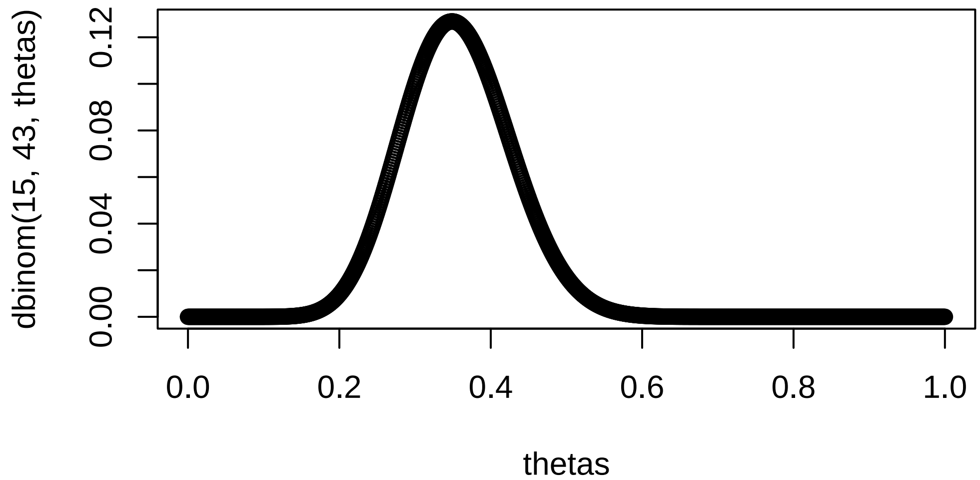


plots for part b)

```
#plotting prior p(\theta)
thetas = seq(0,1,by=0.001) #U(0,1)

plot(thetas, dbeta(thetas, 8,2), type='l',main="p(theta)")
```



```r
#plotting p(y=15|\theta)
#plot a binomial here
plot(thetas, dbinom(15,43,thetas))
```



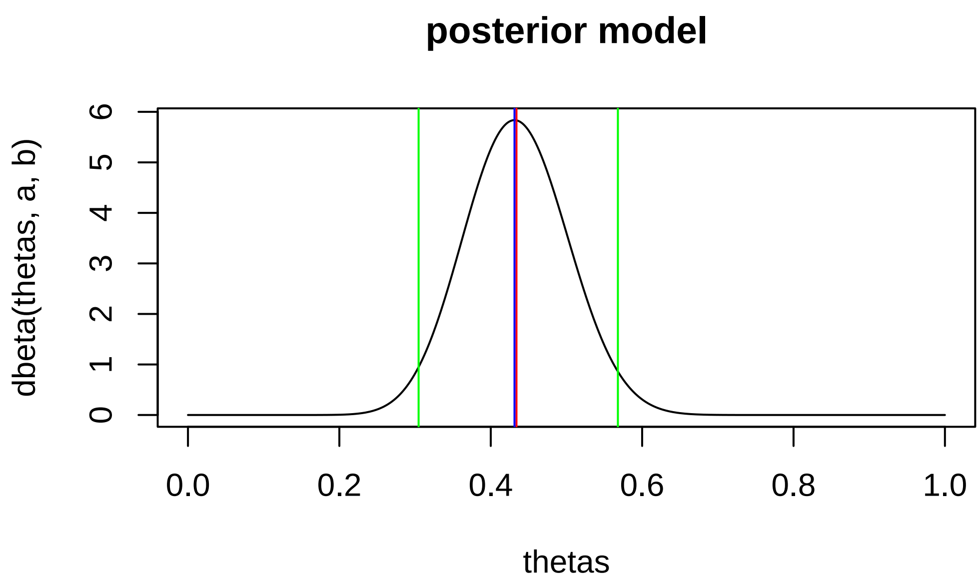

```r
#posterior which is beta(2 + success, 8 + failure) = beta()
a=8+15
b=2+28
plot(thetas, dbeta(thetas,a,b),type='l',main="posterior model")
abline(v=beta_mean(a,b), col='red') #mean
```

```
[1] "mean:"
```

```r
abline(v=beta_mode(a,b), col='blue') #mode
```

```
[1] "mode:"
```

```r
# CI
abline(v=qbeta(.975,a,b),col='green') #lower bound
abline(v=qbeta(.025,a,b),col='green') #upper bound
```

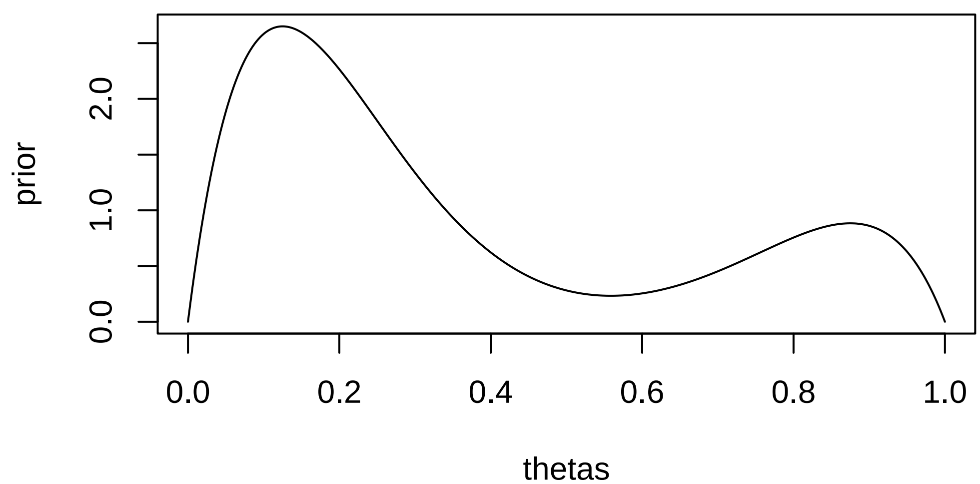


c)

This may represent that you have about 25% confidence that there are going to be 8 cases of recidivism and 2 cases of not, while the beta(2,8) represents you're 75% confident that there will be 2 cases of recidivism and 8 cases of failure respectively. This is if you've only seen 10 prior cases.

Or maybe there were two previous studies with 2 successes and 8 failures or 2 failures and 8 successes respectively.

```
prior = 0.75 * dbeta(thetas,2,8) + 0.25 * dbeta(thetas,8,2)
plot(thetas,prior,type="l")
```



d) i) $$

$$
\begin{aligned}
& p(\theta) * p(y|\theta) && (1)\\
&= \frac{1}{4}\frac{\Gamma(10)}{\Gamma(2)\Gamma(8)}\binom{43}{15}[3\theta^{16}(1-\theta^{35}) + \theta^{22}(1-\theta)^{25}] && (2)
\end{aligned}
$$

$$

ii) This is a mixture of $beta(17, 36)$ and $beta(23, 26)$

iii) plot: