STA 602 HW 4

William Tirone

3.9

a) To find the form of the prior, we can express the distribution in its exponential family form and pick out the components needed to construct the prior (using the notation from the book):

$$Need : p(y) = h(y)c(\phi)e^{\phi t(y)}$$

$$\begin{aligned}
h(y) &= \frac{2}{\Gamma(a)} y^{2a-1} \\
c(\phi) &= \phi^a \\
t(y) &= -y^2 \\
\phi &= \theta^2
\end{aligned}$$

Now we know the prior will have the form:

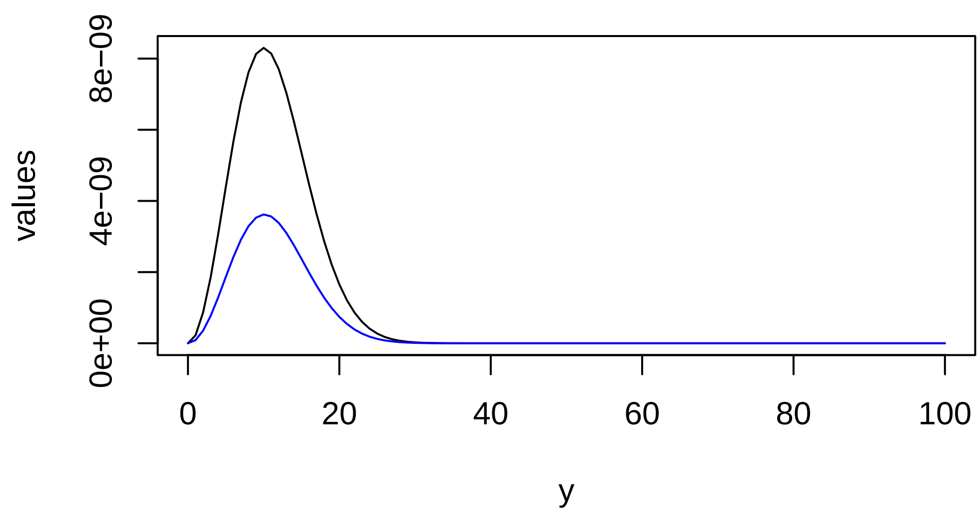
$$p(\phi) = k(n_0, t_0)c(\phi)^{n_0} e^{n_0 t_0 \phi} = k(n_0, t_0)\phi^{a n_0} e^{n_0 t_0 \theta^2}$$

Now using the change of variables formula to get the prior on $\theta = \sqrt{\phi}$

$$\begin{aligned}
p(\theta) &= p(\phi) * |\frac{d\phi}{d\theta}| \\
&= k(n_0, t_0)\theta^{2a*n_0} e^{n_0 t_0 \theta^2} * 2\theta \\
&\propto \theta^{2a*n_0+1} e^{n_0 t_0 \theta^2} \\
&\sim galenshore(a n_0 + 1, \sqrt{-n_0 t_0})
\end{aligned}$$

Now attempting to plot these priors:

```r
y = seq(0,100,1)

n0 = 0.1
t0 = -0.1
a = 5
theta= sqrt(-n0 * t0)


values = (2/gamma(a*n0 + 1)) * theta ^(2*a) * y^(2*a*n0+1) * exp(1)^(-theta^2 * y^2)
plot(y,values,type='l')

n0 = 0.1
t0 = -0.1
a = 5.2
theta= sqrt(-n0 * t0)

values2 = (2/gamma(a*n0 + 1)) * theta ^(2*a) * y^(2*a*n0+1) * exp(1)^(-theta^2 * y^2)

lines(y,values2,type='l',col='blue')
```



b)

12:21 PM Thu Sep 29 91%

Hw4
Sep 28, 2022 at 9:58 PM

$$p(\theta \mid y_1 \dots y_n) \propto p(y_1 \dots y_n \mid \theta)\, p(\theta)$$

$$\propto c(\theta, a) \prod_{i=1}^{n} y_i^{2a-1} e^{-\theta^2 y_i^2} \cdot y^{2(a n_0 + 1) - 1} \cdot e^{n_0 t_0 y^2}$$

$$\propto \left(\Sigma y_i^{2a-1}\right) + y^{2(a n_0 + 1) - 1} \cdot e^{-\theta^2 \Sigma y_i^2 + n_0 t_0 y^2}$$

$$(2a-1) + 2(a n_0 + 1) - 1 = 2a - 1$$

$$2a \cancel{-1} + 2a n_0 \cancel{+2} \cancel{-1} = 2a - 1$$

$$2a + 2a n_0 = 2a - 1$$

$$2a(1 + n_0) = 2a - 1$$

$$\frac{2a(1 + n_0) + 1}{2} = a$$

$$a(1 + n_0) + 1/2 = a$$

1 / 5

$$-\theta^2 \Sigma y_i^2 + n_0 t_0 y^2 = -\theta^2 y^2$$

$$= n_0 t_0 + n \bar{t}(y)$$

- attempting to reconcile this with the formula on p. 51 in Hoff.

$$\sim \text{Galenshore}\left(a(1+n_0)+1/2,\ n_0 t_0 + n \bar{t}_y\right)$$

c) and d)

c)

$$\frac{P(\theta_a|Y_1\ldots Y_n)}{P(\theta_b|Y_1\ldots Y_n)} = \frac{\theta_a^{2a}}{\theta_b^{2a}}\,\frac{e^{-\theta_a^2}}{e^{-\theta_b^2}}$$

· We've known the sufficient statistic from the beginning, it is just $t(y) = -y^2$.

d) Using the formula given,

$$E(\theta|y_1\ldots y_n)$$

$$= \frac{\Gamma(a(1+n_0)+1)}{(n_0t_0+n\bar{t}y)\,\Gamma(a(1+n_0)+1)}$$

e)

e)

e)

$$p(\tilde{y} \mid y_1 \ldots y_n) = \int p(\tilde{y} \mid \theta)\, p(\theta \mid y_1 \ldots y_n)\, d\theta$$

$$\frac{2}{\Gamma(a)} \frac{2}{\Gamma(a(1+n_0)+1/2)} \int \theta^{2a+2(a(1+n_0)+1/2} \cdot y^{2a-1+2(a(1+n_0)+1/2)} \cdot e^{-(n_0 t_0 + n\bar{E}y)^2 y^2} d\theta$$

$$= \frac{2}{\Gamma(a)\Gamma(a(1+n_0)+1/2} \cdot \frac{\Gamma(a+a(1+n_0)+1/2)}{2}$$

4 / 5

2

a)

Let $\sum x_i = X$ We have that $\delta(x) = \frac{X+10}{n+1}$, so the bias, is:

$$E(\delta(X)) - \theta = \frac{10 + E(X)}{1+n} - \theta = \frac{10-\theta}{11}$$

Above using that we have n=10, and that n iid poisson result in $X \sim POIS(n\theta)$ so the expected value is $n\theta$

With variance:

$$Var(\frac{X+10}{n+1}) = \frac{Var(X)}{(11)^2} = \frac{10\theta}{121}$$

So the MSE is then:

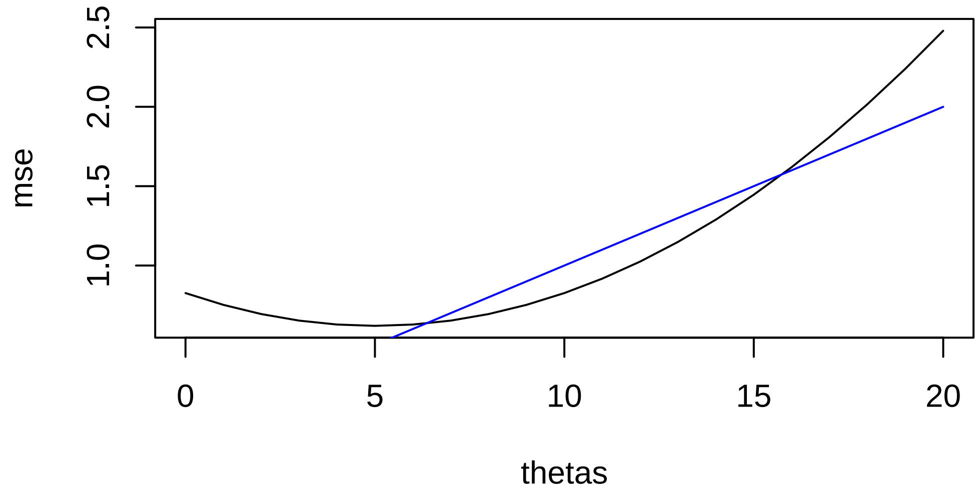
$$\frac{(10-\theta)^2 + 10\theta}{121}$$

b)

The MSE for the Bayes estimator under squared error loss, the posterior mean, is lower than the MSE for the MLE from about theta = 6 to theta = 16, so this is "outperforming" the MLE in that narrow range of thetas.

I believe this is the case where the bias does not make as much of a difference, so we're doing a little better by accepting some bias for a lower variance. However, as theta grows, the tradeoff is no longer worth it, and the MSE is lower for the MLE because of its unbiasedness.

```
thetas = seq(0,20,1)
values = ((10-thetas)^2 + 10 * thetas)/121
x_bar_vals = thetas / 10

plot(thetas,values,type='l',ylab='mse')
lines(thetas,x_bar_vals,type='l',col='blue')
```



3

a)

Bias is calculated below, and is clearly biased though slightly difficult to simplify:

$$E(\delta_5(X)) = \frac{E(X) + \sqrt{n}/2}{n + \sqrt{n}} - \theta$$
$$= \frac{n\theta + \sqrt{n}/2}{n + \sqrt{n}} - \theta$$

variance:

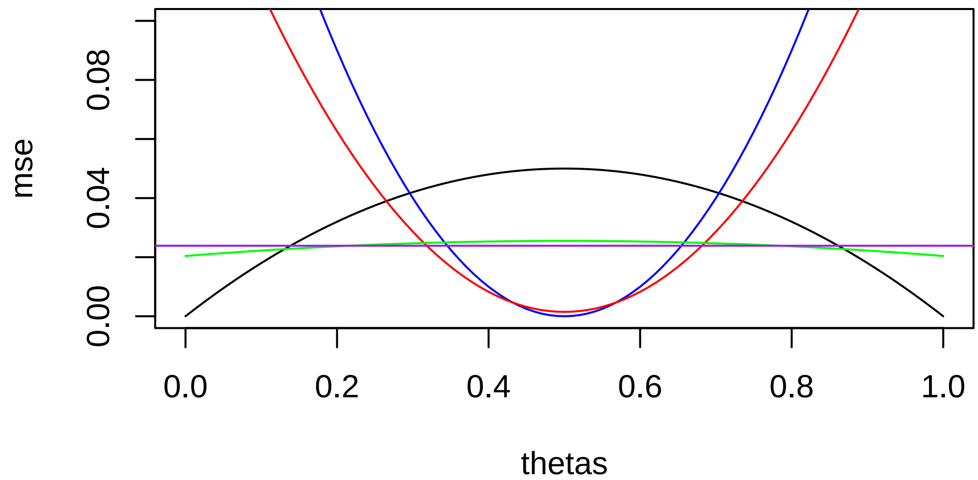
$$Var(\delta_5(X)) = \frac{Var(X)}{(n + \sqrt{n})^2}$$
$$= \frac{n\theta(1-\theta)}{(n + \sqrt{n})^2}$$

and MSE:

$$= (\frac{n\theta + \sqrt{n}/2}{n + \sqrt{n}} - \theta)^2 + \frac{n\theta(1-\theta)}{(n + \sqrt{n})^2}$$
$$= \frac{1}{4(1 + \sqrt{n})^2}$$

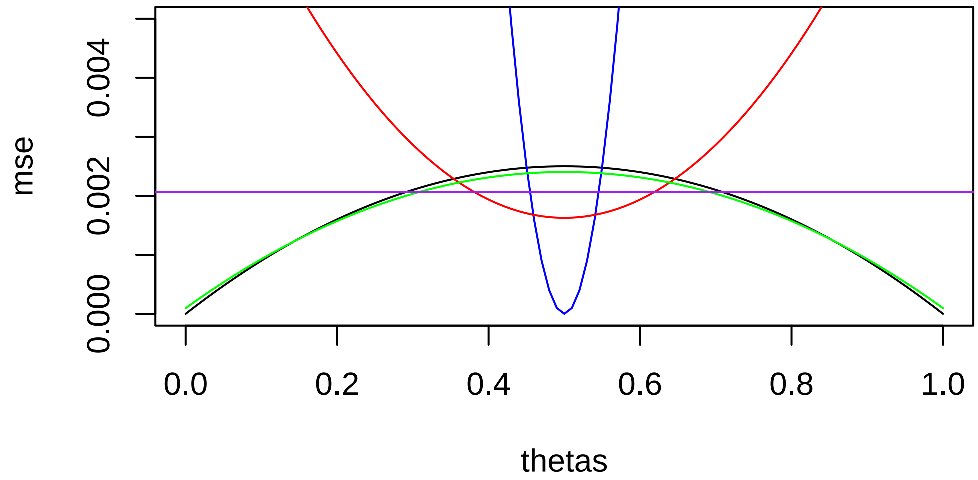
b)

```r
thetas = seq(0,1,0.01)
n=5

d1 = (thetas * (1-thetas))/n
d2 = (.5 - thetas)^2
d3 = (n * thetas * (1-thetas) + 144 * (1-2*thetas)^2) / (n+24)^2
d4 = (n * thetas * (1-thetas)  + (1-2*thetas)^2) / (n+2)^2
d5 = 1 / (4 * (sqrt(n) + 1)^2)

plot(thetas,d1,type='l',col='black',ylab='mse',ylim=c(0,0.10))
lines(thetas,d2,col='blue')
lines(thetas,d3,col='red')
lines(thetas,d4,col='green')
abline(h=d5,col='purple')
```




```r
n=100

d1 = (thetas * (1-thetas))/n
d2 = (.5 - thetas)^2
d3 = (n * thetas * (1-thetas) + 144 * (1-2*thetas)^2) / (n+24)^2
d4 = (n * thetas * (1-thetas)  + (1-2*thetas)^2) / (n+2)^2
d5 = 1 / (4 * (sqrt(n) + 1)^2)

plot(thetas,d1,type='l',col='black',ylab='mse',ylim=c(0,0.005))
lines(thetas,d2,col='blue')
lines(thetas,d3,col='red')
```

```r
lines(thetas,d4,col='green')
abline(h=d5,col='purple')
```



c)

$$\frac{n}{n+\sqrt{n}}\frac{X}{n}+\frac{\sqrt{n}}{n+\sqrt{n}}\frac{1}{2}$$

Just to convince myself, I calculated the weights under n=5 and n=100 below. Even in the low sample size case, we place a low weight of about .3 on our "dumb" prior of 1/2, since this is effectively a made up number. Still, though, we give it some weight if we have almost no observations. With a larger n, we almost completely ignore the static prior.

Since unbiasedness is not useful at all when we have a very small sample size, we don't really care that 1/2 is biased since it has no variance.

```r
n = 5

n / (sqrt(n) + n)
```

```
[1] 0.690983
```

```r
sqrt(n) / (sqrt(n) + n)
```

```
[1] 0.309017
```

```r
n = 100

n / (sqrt(n) + n)
```

```
[1] 0.9090909
```

```r
sqrt(n) / (sqrt(n) + n)
```

```
[1] 0.09090909
```