MAT 275 Laboratory 5
The Mass-Spring System

In this laboratory we will examine *harmonic* oscillation. We will model the motion of a mass-spring system with differential equations.
Our objectives are as follows:

1. Determine the effect of parameters on the solutions of differential equations.
2. Determine the behavior of the mass-spring system from the graph of the solution.
3. Determine the effect of the parameters on the behavior of the mass-spring.

The primary MATLAB command used is the `ode45` function.

Mass-Spring System without Damping

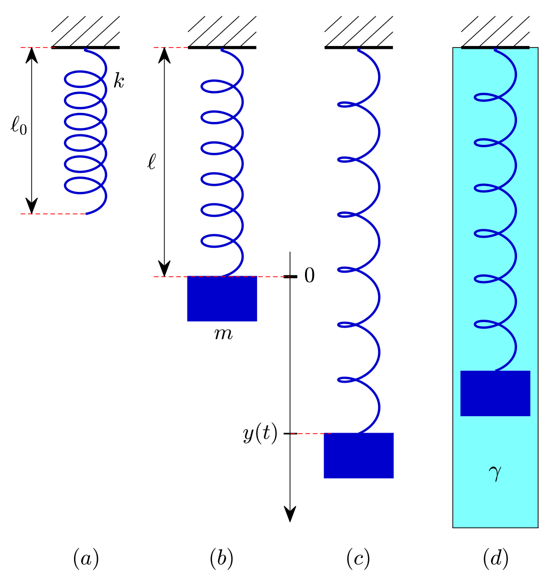
The motion of a mass suspended to a vertical spring can be described as follows. When the spring is not loaded it has length ℓ_0 (situation (a)). When a mass m is attached to its lower end it has length ℓ (situation (b)). From the first principle of mechanics we then obtain

$$\underbrace{mg}_{\text{downward weight force}} + \underbrace{-k(\ell-\ell_0)}_{\text{upward tension force}} = 0. \tag{1}$$

The term g measures the gravitational acceleration ($g \approx 9.8m/s^2 \approx 32ft/s^2$). The quantity k is a spring constant measuring its stiffness. We now pull downwards on the mass by an amount y and let the mass go (situation (c)). We expect the mass to oscillate around the position $y = 0$. The second principle of mechanics yields

$$\underbrace{mg}_{\text{weight}} + \underbrace{-k(\ell+y-\ell_0)}_{\text{upward tension force}} = \underbrace{m\frac{d^2(\ell+y)}{dt^2}}_{\text{acceleration of mass}}, \text{ i.e., } \quad m\frac{d^2y}{dt^2} + ky = 0 \tag{2}$$

using (1). This is a second-order ODE.



Equation (2) is rewritten

$$\frac{d^2y}{dt^2} + \omega_0^2\, y = 0 \tag{3}$$

where $\omega_0^2 = k/m$. Equation (3) models simple harmonic motion. A numerical solution in the interval $0 \le t \le 15$ with initial conditions $y(0) = 0.8$ meter and $y'(0) = -0.3$, (i.e. the mass is initially stretched downward 0.8 meters and released with an initial upward velocity of 30 cm/s; see setting (c) in figure), $m = 9, k = 16$, is obtained by first reducing the ODE to first-order ODEs (see Laboratory 4).

Let $v = y'$. Then $v' = y'' = -\omega_0^2\, y = -\frac{16}{9}y$. Also, $v(0) = y'(0) = -0.3$. The MATLAB program `LAB05ex1` implements the problem.

LAB05ex1.m

```
clear all;      % clear all variables
m = 9;  % mass [kg]
k = 16;  % spring constant [N/m]
omega0 = sqrt(k/m);
y0 =0.8;  v0 = -0.3; % initial conditions
[t,Y] = ode45(@f,[0,15],[y0,v0],[],omega0); % solve for 0<t<15
y = Y(:,1); v = Y(:,2);  % retrieve y, v from Y
figure(1); plot(t,y,'bo-',t,v,'r+-');% time series for y and v
grid on; axis tight;
%---------------------------------------------------
function dYdt = f(t,Y,omega0); % function defining the DE
y = Y(1); v = Y(2);
dYdt=[ v ; - omega0^2*y ];
end
```

Note that the parameter ω_0 was passed as an argument to `ode45` rather than set to its value $\omega_0 = \frac{4}{3}$ directly in the function `f`. The advantage is that its value can easily be changed in the driver part of the program rather than in the function, for example when multiple plots with different values of ω_0 need to be compared in a single MATLAB figure window.

NOTE: for the following exercises, just like for all the other lab reports, unlesss otherwise specified, include in your lab report all M-files, figures, MATLAB input commands, the corresponding output, and the answers to the questions.

1. From the graph in Fig. 1 answer the following questions.

   (a) Which curve represents $y = y(t)$? How do you know?

   (b) What is the period of the motion? Answer this question first graphically (by reading the period from the graph) and then analytically (by finding the period using ω_0).

   (c) We say that the mass comes to rest if, after a certain time, the position of the mass remains within an arbitrary small distance from the equilibrium position. Will the mass ever come to rest? Why?

   (d) What is the amplitude of the oscillations for y?

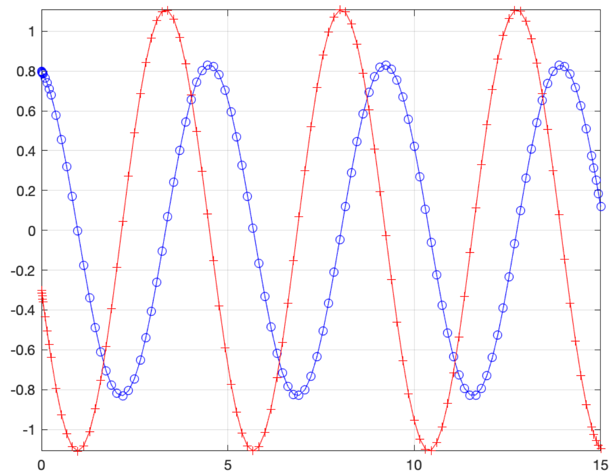


Figure 1: Harmonic motion

(e) What is the maximum velocity (**in magnitude**) attained by the mass, and when is it attained? Make sure you give all the t-values at which the velocity is maximal and the corresponding maximum value. The t-values can be determined by magnifying the MATLAB figure using the magnify button , and by using the periodicity of the velocity function.
How far is the mass from the equilibrium when the maximum velocity is attained?

(f) How does the size of the mass m and the stiffness k of the spring affect the motion? Support your answer first with a theoretical analysis on how ω_0 (and therefore the period of the oscillation) is related to m and k, and then graphically by running `LAB05ex1.m` first with $m = 24$ and $k = 16$, and then with $m = 9$ and $k = 35$. Include the two corresponding graphs.

2. The energy of the mass-spring system is given by the sum of the kinetic energy and the potential energy. In the absence of damping, the energy is conserved.

(a) Add commands to `LAB05ex1` to compute and plot the quantity $E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ as a function of time. What do you observe? (pay close attention to the y-axis scale and, if necessary, use `ylim` to get a better graph). Include at least one plot. Does the graph confirm the fact that the energy is conserved?

(b) Show analytically that $\frac{dE}{dt} = 0$. (Note that this proves that the energy is constant).

(c) Add commands to `LAB05ex1` to plot v vs y (phase plot). Include the plot. Does the curve ever get close to the origin? Why or why not? What does that mean for the mass-spring system?

Mass-Spring System with Damping

When the movement of the mass is damped due to viscous effects (e.g., the mass moves in a cylinder containing oil, situation (d)), an additional term proportional to the velocity must be added. The resulting equation becomes

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} + 2p\frac{dy}{dt} + \omega_0^2 y = 0 \tag{4}$$

by setting $p = \frac{c}{2m}$. The program `LAB05ex1` is updated by modifying the function `f`. Here we set $c = 8$.

LAB05ex2.m

```
clear all;      % clear all variables
m = 9;  % mass [kg]
k = 16;  % spring constant [N/m]
c = 8;  % friction coefficient [Ns/m]
omega0 = sqrt(k/m); p = c/(2*m);
y0 =0.8;  v0 = -0.3; % initial conditions
[t,Y] = ode45(@f,[0,15],[y0,v0],[],omega0, p); % solve for 0<t<15
y = Y(:,1); v = Y(:,2); % retrieve y, v from Y
figure(1); plot(t,y,'bo-',t,v,'r+-');% time series for y and v
grid on; axis tight;
%---------------------------------------------------
function dYdt = f(t,Y,omega0,p); % function defining the DE
y = Y(1); v = Y(2);
dYdt=[ v ; ?? ];   % fill-in dv/dt
end
```

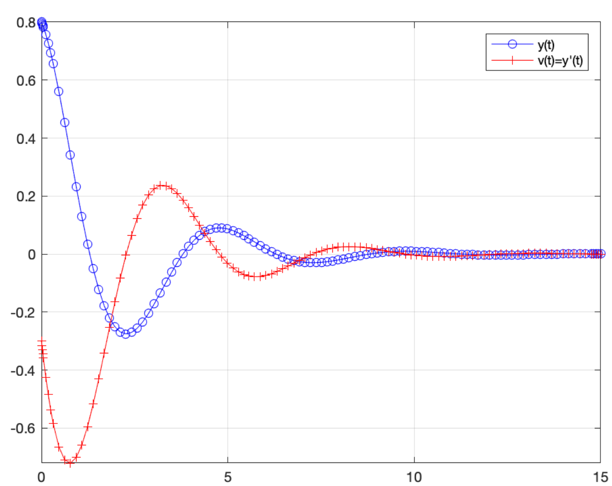


Figure 2: Damped harmonic motion

3. Fill in `LAB05ex2.m` to reproduce Fig. 2 and then answer the following questions.

   (a) For what minimal time t_1 will the mass-spring system satisfy $|y(t)| < 0.06$ for all $t > t_1$? You can answer the question either by magnifying the MATLAB figure using the magnify button, or use the following MATLAB commands (explain):

```
for i=1:length(y)
  M(i)=max(abs(y(i:end)));
end
i = find(M<0.06); i = i(1)
disp(['|y|<0.06 for t > t1 with ' num2str(t(i-1)) '< t1 <' num2str(t(i))])
```

(b) What is the largest (in magnitude) velocity attained by the mass, and when is it attained? Answer by using the magnify button.

(c) How does the size of c affect the motion? To support your answer, run the file `LAB05ex2.m` for $c = 12$, $c = 24$, and $c = 54$. Include the corresponding graphs with a title indicating the value of c used.

(d) Determine analytically the smallest (critical) value of c such that no oscillation appears in the solution.

4. (a) Modify `LAB05ex2.m` to compute and plot the quantity $E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$ as a function of time. What do you observe? Is the energy conserved in this case? Include the plot.

   (b) Show analytically that $\frac{dE}{dt} < 0$ for $c > 0$ while $\frac{dE}{dt} > 0$ for $c < 0$.

   (c) Modify `LAB05ex2.m` to plot v vs y (phase plot). Include the plot. Comment on the behavior of the curve in the context of the motion of the spring. Does the graph ever get close to the origin? Why or why not?