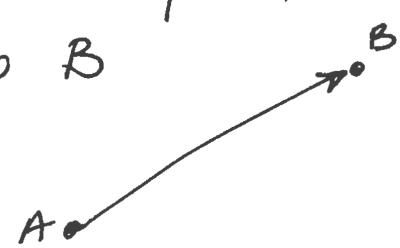
Flash review :

In lieu of theoretical math, I will rely on your intuition for things like:

- what is a point
- 2 line segments $\overline{AB} = \overline{CD}$ having equal lengths
- 2 line segments $\overline{AB} \parallel \overline{CD}$ are parallel

here we go ... !

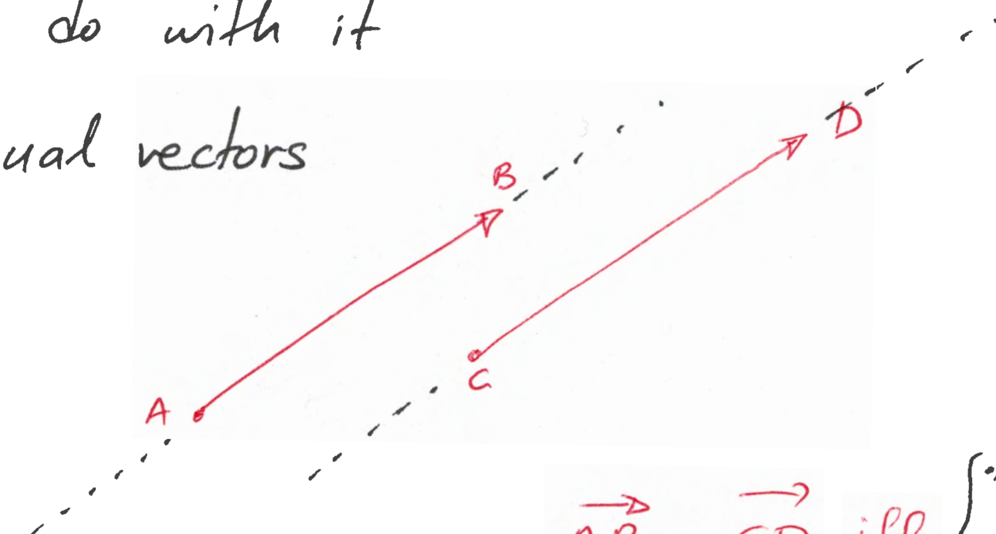
- The point is the mark that your pencil leaves behind. A list of numbers like $(3,-7.5)$ is not the point, it is its name (and we could have many names for the same point depending on what "language", i.e. coordinate system, we are using)
- A vector is represented by an oriented line segment $\overrightarrow{AB}$ (A: initial point, B: terminal point). We draw them as arrows from A to B



But what *is* a vector?

- Intuitively: it's the "motion" that would take A (initial point) to B (terminal point).
- Mathematically: a vector is best defined by what it does (operations) and what we can do with it

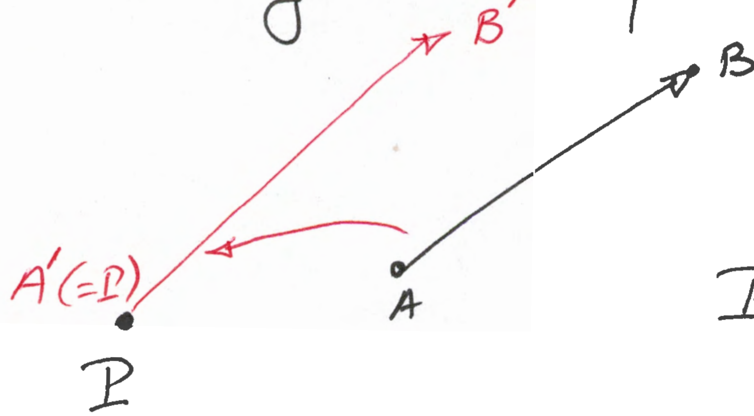
* Equal vectors



$$\vec{AB} = \vec{CD} \text{ iff } \begin{cases} \cdot \text{ lines } AB, CD \text{ parallel} \\ \cdot \text{ length}(AB) = \text{length}(CD) \\ \cdot\ \vec{AB}, \vec{CD} \text{ point in same direction} \end{cases}$$

(we use notation $\vec{u}$ $(= \vec{AB} = \vec{CD})$ if we want to de-empasize where vector is centered)

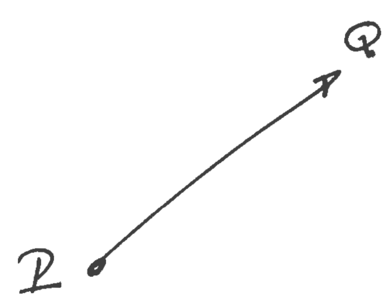
* Adding vector to point



$$\vec{A'B'} = \vec{AB}$$

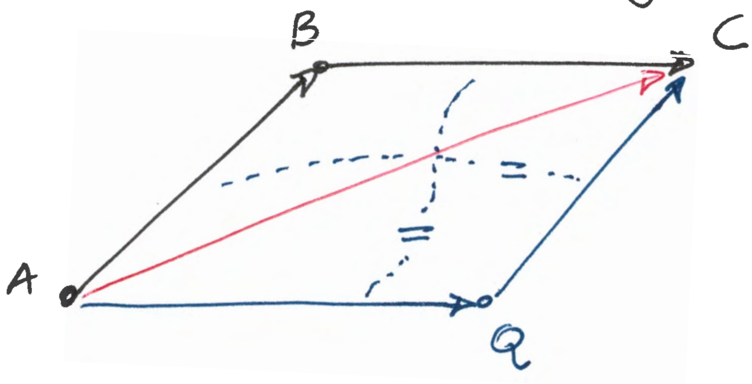
$$P + \vec{AB} = B'$$

(if already aligned



$$P + \vec{PQ} = Q$$

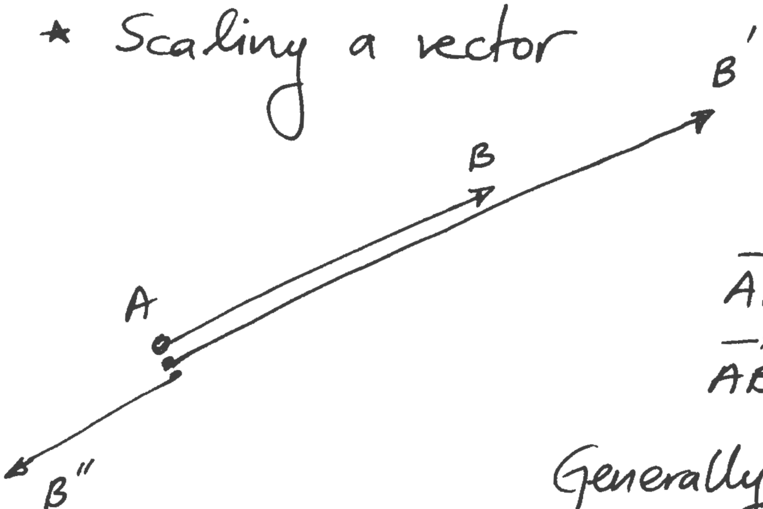
* Adding 2 vectors together:



$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \overrightarrow{AQ} + \overrightarrow{QC}$$
$$= \overrightarrow{BC} + \overrightarrow{AB}$$

must shift if not initially aligned!

* Scaling a vector



$$\overrightarrow{AB'} = 1.5\,\overrightarrow{AB}$$
$$\overrightarrow{AB''} = -.5\,\overrightarrow{AB}$$

Generally: $\overrightarrow{AC} = \gamma\,\overrightarrow{AB}$

iff $\begin{cases} \overline{AC} \parallel \overline{AB} \\ \text{length}(AC) = \gamma\,\text{length}(AB) \\ \text{same direction if } \gamma > 0 \\ \text{opposite if } \gamma < 0 \end{cases}$

* Odds and ends

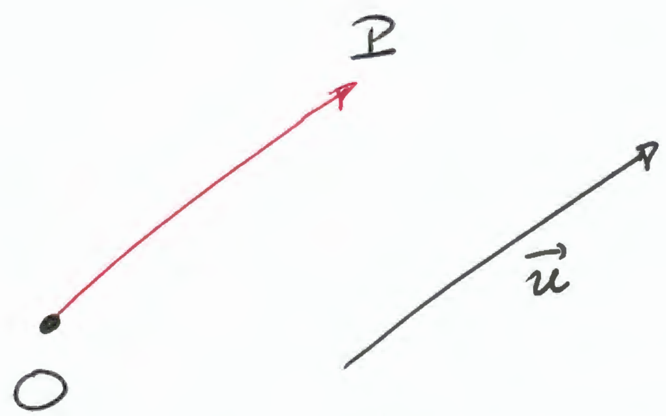
→ $-\overrightarrow{AB} = (-1)\cdot\overrightarrow{AB} = \overrightarrow{BA}$

→ $\overrightarrow{AB} - \overrightarrow{CD} = \overrightarrow{AB} + (-\overrightarrow{CD}) = \overrightarrow{AB} + \overrightarrow{DC}$ ....

2 (ordered points) A, B $\longrightarrow$ prescribe vector $\overrightarrow{AB}$

Vector $\vec{u}$ (e.g. $\vec{u} = \overrightarrow{AB}$)
+ agreed-upon reference point O (origin)

$\Big\} \Rightarrow$ prescribe point $P = O + \vec{u}$



where $\overrightarrow{OP} = \vec{u}$

Summary: 2 points $\Rightarrow$ recipe for vector
1 vector + origin $\Rightarrow$ recipe for point.

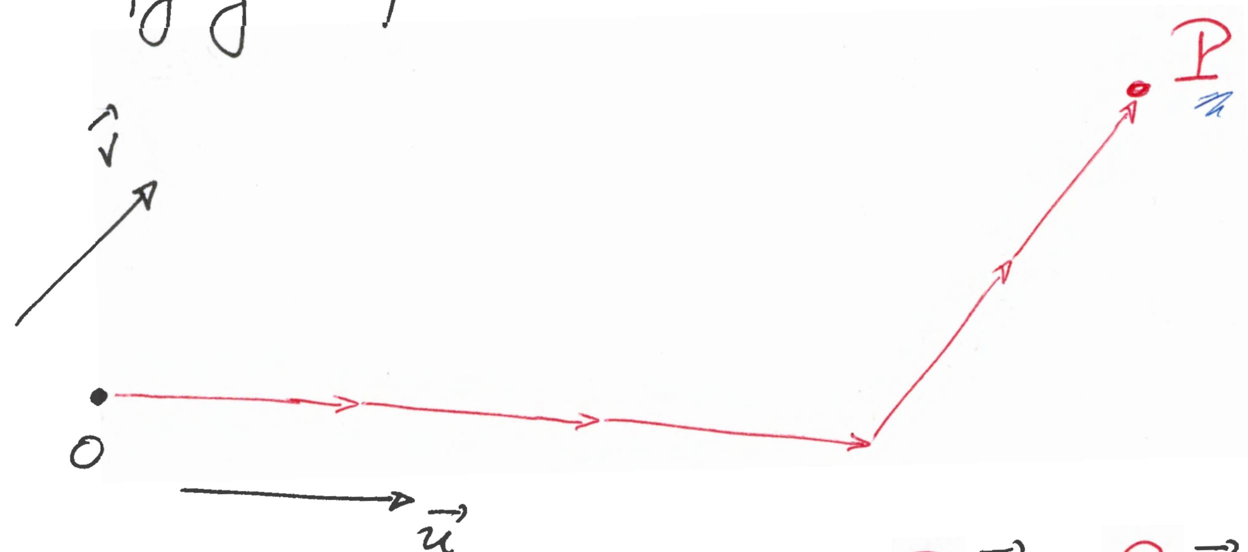
$\boxed{\text{Notably absent from discussion thus far ... points (or vectors) as lists of numbers.}}$

Coordinate systems A way to describe points & vectors as lists of numbers (and manipulate them using algebra, instead of geometry).

Coordinate system ingredients:

- A reference point O (origin)
- 2 "basis vectors" (in 2D) $\vec{u}, \vec{v}$
  (will be 3 in 3D)
- Let's just assume that $\vec{u}, \vec{v}$ are <u>not</u> parallel.

Identifying a <u>point</u>



$$P = O + 3\vec{u} + 2\vec{v}$$

→ For every point there will be one, and only one way to write it as $P = O + a \cdot \vec{u} + b \cdot \vec{v}$
(trust me)

→ (How to compute a & b? Wait until next lecture when we talk about projections).

→ Once $O, \vec{u}$ & $\vec{v}$ are agreed upon we can represent P using a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$

→ Vectors represented similarly: then $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ "represented by"
if $\vec{w} = a\vec{u} + b\vec{v}$

Why mess up with geometry & algebra?

⇝ Operations carry over.

$$\left.\begin{array}{l} \vec{u} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \\ \vec{v} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \end{array}\right\} \quad \vec{u}+\vec{v} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix} \text{ (represents vector)}$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P + \vec{u} = \begin{bmatrix} x + a_1 \\ y + a_2 \end{bmatrix} \text{ (represents point)}$$

$$\gamma \cdot \vec{u} = \begin{bmatrix} \gamma a_1 \\ \gamma a_2 \end{bmatrix}$$

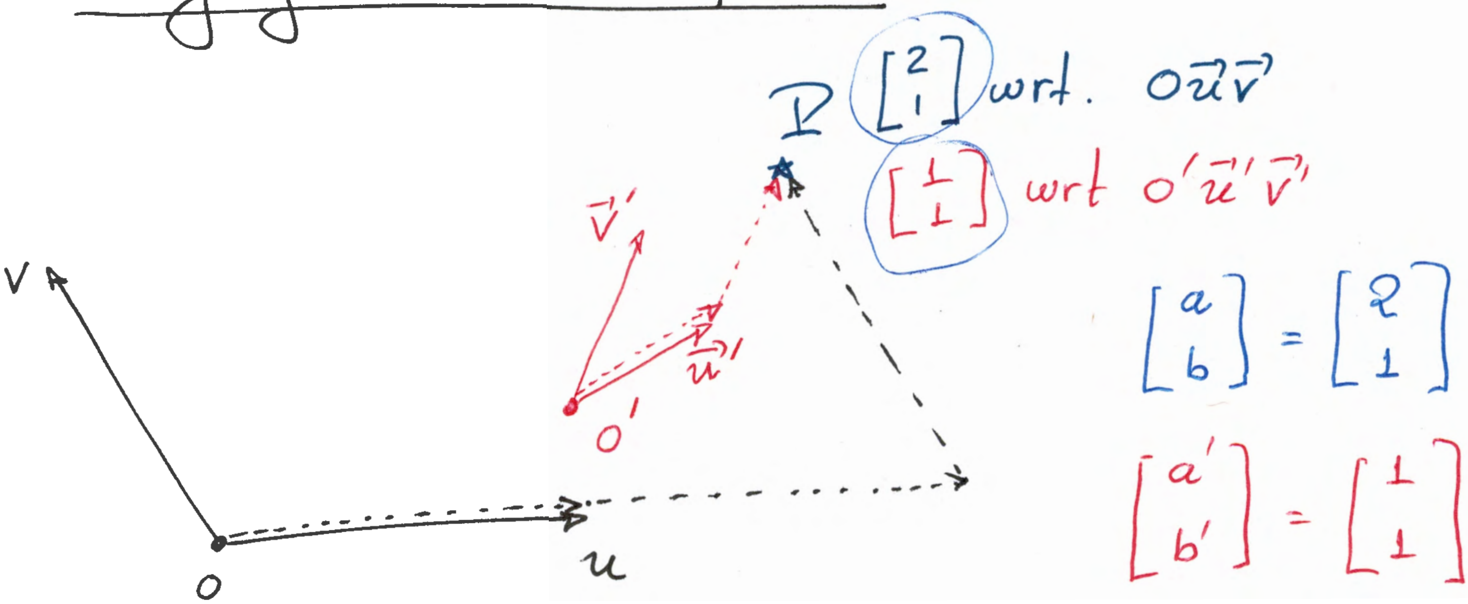
(Cute detail) since $\vec{u} = 1\vec{u} + 0\vec{v}$ $\quad \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v} = 0\vec{u} + 1\vec{v}$ $\quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(we often say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the basis vectors)

<u>Caution</u> We can manipulate (add, scale, etc) points/vectors w/o a coordinate system, but the vector notation only makes sense after $O, \vec{u}, \vec{v}$ have been agreed upon!

Changing Coordinate systems :



How do we translate between $\begin{bmatrix}a\\b\end{bmatrix}$ & $\begin{bmatrix}a'\\b'\end{bmatrix}$?

→ Let $O' = \begin{bmatrix}o_1'\\o_2'\end{bmatrix}$ be the representation of O' w.r.t. $O(\vec{u},\vec{v})$

Likewise $\vec{u}' = \begin{bmatrix}u_1'\\u_2'\end{bmatrix}$ wrt. $O\vec{u}\vec{v}$

$\vec{v}' = \begin{bmatrix}v_1'\\v_2'\end{bmatrix}$

Then:

$$\vec{OP} = \vec{OO'} + a'\vec{u}' + b'\vec{v}'$$

$$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}o_1'\\o_2'\end{bmatrix} + a'\begin{bmatrix}u_1'\\u_2'\end{bmatrix} + b'\begin{bmatrix}v_1'\\v_2'\end{bmatrix}$$

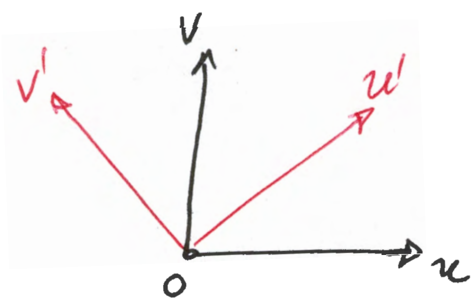
$$= \underbrace{\begin{bmatrix}o_1'\\o_2'\end{bmatrix}} + \underbrace{\begin{bmatrix}u_1' & v_1'\\u_2' & v_2'\end{bmatrix}}\begin{bmatrix}a'\\b'\end{bmatrix}$$

$$\begin{bmatrix}a\\b\end{bmatrix} = \vec{t} + M\begin{bmatrix}a'\\b'\end{bmatrix}$$

Translate $\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} a' \\ b' \end{bmatrix}$?

Need to ask how O, u, v are represented in the system $O'\vec{u}'\vec{v}'$ instead!

Special case:
- $O' = O$
- $\cancel{\vec{u} \perp \vec{v}}$
- $\vec{u}' \perp \vec{v}'$
- $\text{length}(\cancel{u}, \cancel{v}, u', v') = 1$



$$\begin{bmatrix} a \\ b \end{bmatrix} = M \begin{bmatrix} a' \\ b' \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} a' \\ b' \end{bmatrix} = M^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$$

<u>Just in this case</u> $M^{-1} = M^{T}$

$$\rightarrow \begin{bmatrix} a' \\ b' \end{bmatrix} = M^{T} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} u_1' & u_2' \\ v_1' & v_2' \end{bmatrix}$$

Such basis vectors u', v' are called orthonormal $\Rightarrow$ extra convenience in inverting translation

Changes of coordinates as transforms:

$$\vec{\omega} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \boxed{\text{function}} \longrightarrow \omega' = \begin{bmatrix} a' \\ b' \end{bmatrix} = f(\omega)$$

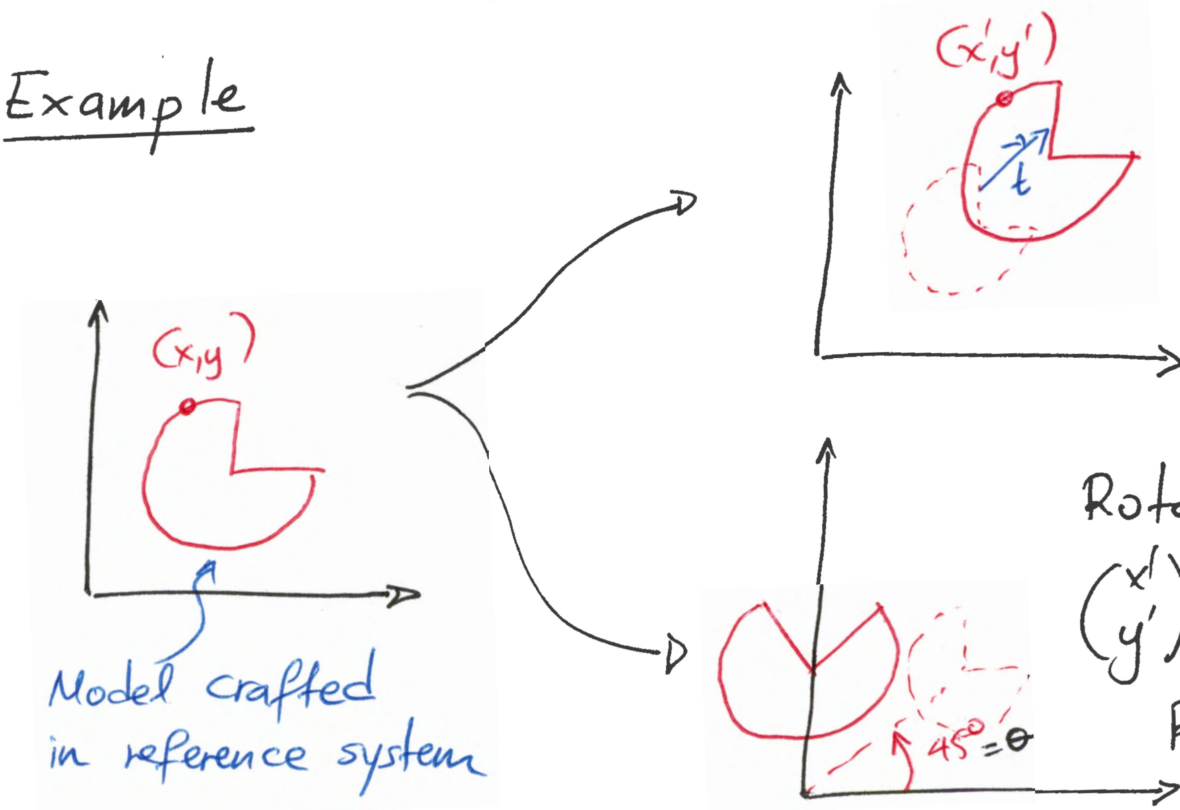
in our case $f(\vec{\omega}) = \vec{t} + A\vec{\omega}$

<u>Why?</u>

* Systematic way to translate an <u>entire</u> object in a description relative to a new system
* We like to understand operations (or functions) by approximating them with lines

$\Rightarrow$ Transforms (functions) between descriptions make sense <u>independent</u> of coordinate changes, too.

<u>Example</u>



Translation

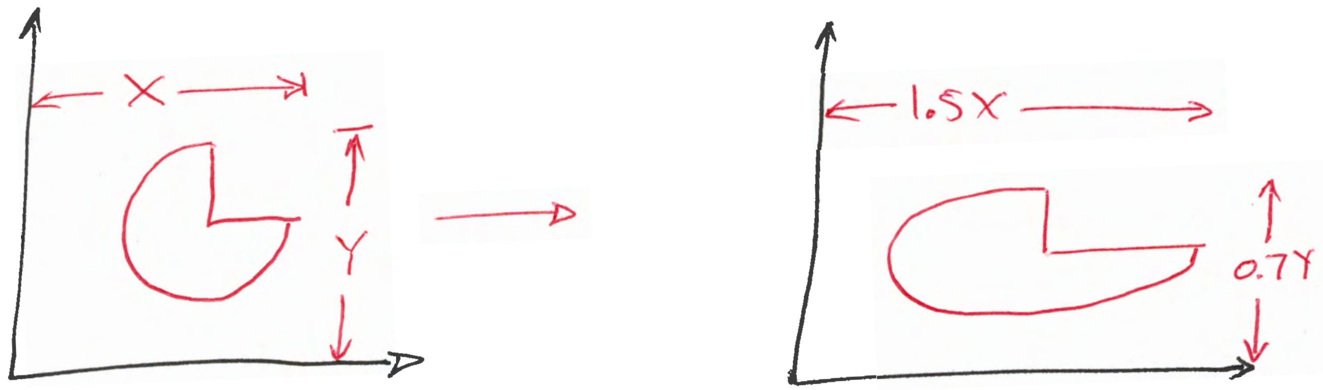
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = t' + I \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$

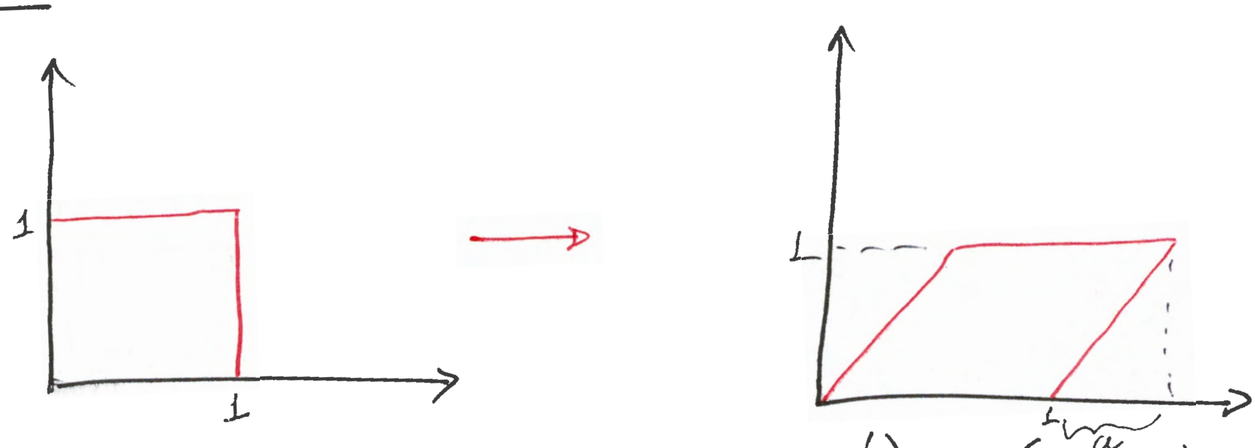
$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Scale



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1.5 & \\ & 0.7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Shear



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$