<u>Definition</u>: A second-order linear ODE is said to be <u>homogeneous</u> if it can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad \ldots (1)$$

We say that a second-order linear ODE is <u>non-homogeneous</u> if it can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \quad \ldots (2)$$

It turns out that to solve (2) we must first solve (1).

<u>Note</u>: $y=0$ is always a trivial solution to (1)

We are not in a position yet to solve (1) and (2), but once we are, the following theorem will be useful.

Theorem: If y_1 and y_2 are solutions to ①, then any linear combination $(c_1 y_1 + c_2 y_2)$ will also be a solution.

Proof: Let y_1 and y_2 be solutions of ①. Let $y = c_1 y_1 + c_2 y_2$ be a linear combination.

Then

$$a_2(x)\left[c_1 y_1'' + c_2 y_2''\right] + a_1(x)\left[c_1 y_1' + c_2 y_2'\right] + a_0(x)\left[c_1 y_1 + c_2 y_2\right]$$

$$= c_1\left[a_2(x) y_1'' + a_1(x) y_1' + a_0(x) y_1\right] + c_2\left[a_2(x) y_2'' + a_1(x) y_2' + a_0(x) y_2\right]$$

$$= c_1(0) + c_2(0)$$ (by assumption since y_1 and y_2 are solutions to ①)

$= 0$, as required.

Ex. e^{2x} and e^{x} are solutions to the equation $y'' - 3y' + 2 = 0$.

Here $\boxed{y = c_1 e^{2x} + c_2 e^{x}}$ is also a solution.

General solution

<u>Checklist for Solving Homogeneous Equations</u>

i) Find two solutions y_1 and y_2 (see next week)

ii) Check that $W(y_1, y_2) \neq 0$.

iii) Form the general solution $y = c_1 y_1 + c_2 y_2$.

Ex. e^{3x} and e^{-3x} are solutions of $y'' - 9y = 0$.

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3-3 = -6 \neq 0.$$

Hence the general solution is $y = c_1 e^{3x} + c_2 e^{-3x}$.

Ex. $\cosh 2x$ and $\sinh 2x$ are solutions of $y'' - 4y = 0$

$$W(\cosh 2x, \sinh 2x) = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2\cosh^2 2x - 2\sinh^2 2x$$

$$= 2(\cosh^2 2x - \sinh^2 2x)$$

$$= 2 \neq 0$$

Hence the general solution is $y = c_1 \cosh 2x + c_2 \sinh 2x$.

Non-homogeneous equations

Definition: Any solution to ② which is free of c_1, c_2 etc is said to be a particular solution, denoted by y_p.

Ex. A particular solution to $y'' + 9y = 27$ is $y_p = 3$.

Theorem: If the general solution to ① is y_c, and ~~the~~ a particular solution to ② is y_p, then the general solution to ② is $y = y_c + y_p$.

Note: y_c is called the complementary solution.

Checklist for Solving Non-homogeneous Equations

i) Find the general solution to 1.

ii) Find a particular solution to 2.

iii) The answer is then the sum of the above two steps.

Ex A particular solution to $y''' - 6y'' + 11y' - 6y = 3x$ is $y_p = \frac{-11}{12} - \frac{x}{2}$. The general solution to the associated homogeneous equation is $c_1e^x + c_2e^{2x} + c_3e^{3x}$.

Therefore the general solution to the non-homogeneous equation is $y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{x}{2}$.

Reduction of order

Given any linear second-order ODE we can use a known solution to reduce the ODE down to first-order, and hence find a second solution using techniques from Chapter 2. Furthermore, if we utilize the algorithm below, the second solution will be linearly independent to the first.

i) Suppose y_1 is a solution. Let the second solution have the form $y = u(x)\,y_1$.

ii) Differentiate y twice and substitute into the original equation. Factor if possible.

iii) Let $w = u'$ (and hence $w' = u''$) to obtain a linear first-order equation.

iv) Solve it using the integrating factor $\mu = e^{\int p(x)\,dx}$.

v) Resubstitute and make judicious choices for the constants.

Ex. Find a second solution to $y'' - y' = 0$ if one solution is $y_1 = 1$.

Let $y = u$

$\rightarrow y' = u'$

$\rightarrow y'' = u''$

So we get $u'' - u' = 0$.

If we let $w = u'$ and $w' = u''$ we get

$$w' - w = 0$$

$$\frac{dw}{dx} - w = 0$$

Thus $\mu(x) = e^{\int -1\,dx} = e^{-x}$

Hence $e^{-x}\frac{dw}{dx} - e^{-x}w = 0$

$\longrightarrow \frac{d}{dx}\left[e^{-x}w\right] = 0$

$\longrightarrow e^{-x}w = c$

$\longrightarrow w = ce^{x}$

$\longrightarrow u' = ce^{x}$

$\longrightarrow u = c_0e^{x} + c_1$

$\longrightarrow y = c_0e^{x} + c_1$

Finally we let $c_0 = 1$ and $c_1 = 0$, and our second solution is $y = e^{x}$.

Ex: Find a second solution to $6y'' + y' - y = 0$
if one solution is $y_1 = e^{x/3}$.

Let $y = ue^{x/3}$.

$$\rightarrow y' = \frac{1}{3}e^{x/3}u + e^{x/3}u'$$

$$\rightarrow y'' = \frac{1}{3}e^{x/3}u' + \frac{1}{9}e^{x/3}u + e^{x/3}u'' + \frac{1}{3}e^{x/3}u'$$

$$= \frac{1}{9}e^{x/3}u + \frac{2}{3}e^{x/3}u' + e^{x/3}u''$$

Substituting we get

$$6\left(\frac{1}{9}\cancel{e^{x/3}}u + \frac{2}{3}\cancel{e^{x/3}}u' + \cancel{e^{x/3}}u''\right) + \left(\frac{1}{3}\cancel{e^{x/3}}u + \cancel{e^{x/3}}u'\right) - u\cancel{e^{x/3}} = 0$$

$$\rightarrow 6u'' + 5u' = 0$$

Now let $\omega = u'$ and $\omega' = u''$ to get

$$6\omega' + 5\omega = 0$$

$$\frac{d\omega}{dx} + \frac{5\omega}{6} = 0$$

$$\longrightarrow \mu(x) = e^{\int \frac{5}{6} dx} = e^{\frac{5x}{6}}$$

$$\longrightarrow e^{\frac{5x}{6}} \frac{d\omega}{dx} + e^{\frac{5x}{6}} \cdot \frac{5\omega}{6} = 0$$

$$\longrightarrow \frac{d}{dx}\left[e^{\frac{5x}{6}} \omega\right] = 0$$

$$\longrightarrow \omega e^{\frac{5x}{6}} = C$$

$$\longrightarrow \omega = Ce^{\frac{-5x}{6}}$$

$$\longrightarrow u' = Ce^{-\frac{5x}{6}}$$

$$\longrightarrow u = C_0 e^{-\frac{5x}{6}} + C_1$$

Finally, we choose $C_0 = 1$ and $C_1 = 0$ to get $u = e^{-5x/6}$, and so $y = e^{-5x/6} e^{x/3} = e^{-x/2}$.