COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

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The Foundations: Logic and Proofs

Part-4

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*Because learning changes everything.**

Outline

- Propositional Logic
  - The Language of Propositions.
  - Applications.
  - Logical Equivalences.
- Predicate Logic
  - The Language of Quantifiers.
  - Logical Equivalences.
  - Nested Quantifiers.
- Proofs
  - Rules of Inference.
  - Proof Methods.
  - Proof Strategy.

*Mc Graw Hill

Proofs

Section:

Rules of Inference

Arguments in Propositional Logic

- An *argument* in propositional logic is a sequence of propositions.
  - *Premises + conclusion.*
  - *valid* if the premises imply the conclusion.

- If the premises $p_1, p_2, \ldots, p_n$ and the conclusion q valid, then

  $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is a tautology.

Inference rules are all simple argument forms that will be used to construct more complex *argument forms*.

The Argument

We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{l} \forall x\left(Man(x) \rightarrow Mortal(x)\right) \\ \underline{\qquad Man(Socrates) \qquad} \\ \therefore \quad Mortal(Socrates) \end{array}$$

Valid argument.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rules of Inference for Propositional Logic: Modus Ponens (law of detachment)

Rule of Inference:

$$\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore q \end{array}$$

Corresponding Tautology:

$((p \rightarrow q) \land p) \rightarrow q$

p	q	$(p \rightarrow q)$	$(p \rightarrow q) \land p$	$[(p \rightarrow q) \land p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example:

Let p be "It is snowing."

Let q be "I will study discrete Math."

"If it is snowing, then I will study discrete Math."

"It is snowing."

"Therefore, I will study discrete Math."

Modus Tollens

Rule of Inference:

$$
\begin{array}{l}
p \rightarrow q \\
\neg q \\
\hline
\therefore \neg p
\end{array}
$$

Corresponding Tautology:

$$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$$

Example:

Let p be "it is snowing."

Let q be "I will study discrete Math."

"If it is snowing, then I will study discrete Math."

"I will not study discrete Math."

"Therefore, it is not snowing."

Hypothetical Syllogism

Rule of Inference:

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let p be "it snows."

Let q be "I will study discrete Math."

Let r be "I will get an A."

"If it snows, then I will study discrete Math."

"If I study discrete Math, I will get an A."

"Therefore, if it snows, I will get an A."

Disjunctive Syllogism

Rule of Inference:

$$\begin{array}{l} \quad p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

Corresponding Tautology:

$$\left(\neg p \wedge \left(p \vee q\right)\right) \rightarrow q$$

Example:

Let p be "I will study discrete Math."

Let q be "I will study English literature."

"I will study discrete Math or I will study English literature."

"I will not study discrete Math."

"Therefore, I will study English literature."

Addition (Disjunctive Amplification)

Rule of Inference:

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let p be "I will study discrete Math."

Let q be "I will visit Las Vegas."

"I will study discrete Math."

"Therefore, I will study discrete Math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let *p* be "I will study discrete Math."

Let *q* be "I will study English literature."

"I will study discrete Math and English literature."

"Therefore, I will study discrete Math."

Conjunction

$$\begin{array}{c} p \\ \underline{\quad q \quad} \\ \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

Example:

Let p be "I will study discrete Math."

Let q be "I will study English literature."

"I will study discrete Math."

"I will study English literature."

"Therefore, I will study discrete Math and I will study English literature."

Resolution

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Resolution plays an important role in AI and is used in Prolog.

Corresponding Tautology:

$$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$$

Example:

Let *p* be "I will study discrete Math."

Let *r* be "I will study English literature."

Let q be "I will study databases."

"I will not study discrete Math or I will study English literature."

"I will study discrete Math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Using the Rules of Inference to Build Valid Arguments

A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

A valid argument takes the following form:

$$
\begin{array}{c}
S_1 \\
S_2 \\
. \\
. \\
. \\
S_n \\
\therefore C
\end{array}
$$

Valid Arguments

Example:

With these hypotheses (premises):

"It is not sunny this afternoon and it is colder than yesterday."

"We will go swimming only if it is sunny."

"If we do not go swimming, then we will take a canoe trip."

"If we take a canoe trip, then we will be home by sunset."

Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution:

1. Choose propositional variables:

p : "It is sunny this afternoon."
q : "It is colder than yesterday."
r : "We will go swimming."
s : "We will take a canoe trip."
t : "We will be home by sunset."

2. Translation into propositional logic:

Hypotheses: $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s, s \rightarrow t$

Conclusion: t

Continued on next slide →

Valid Arguments

3. Construct the Valid Argument.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

$\blacksquare$

Valid Arguments

Example: From the single proposition

$$p \wedge (p \to q)$$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \to q)$	Premise
2. p	Simplification using (1)
3. $p \to q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Example

Given the premises:

1. p
2. $p \rightarrow q$
3. $s \vee r$
4. $r \rightarrow \neg q$

Conclude: $s \vee t$

	p	Premise 1
$\therefore$ (1)	q	Modus Ponens, Premise 2
	$r \rightarrow \neg q$	Premise 4
$\therefore$	$q \rightarrow \neg r$	Contrapositive
$\therefore$ (2)	$\neg r$	Modus Ponens, (1)
	$s \vee r$	Premise 3
$\therefore$	s	Disjunctive Syllogism, (2)
$\therefore$	$s \vee t$	Disjunctive Amplification

Example

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," **and** "If I go to sleep early, then I will wake up feeling refreshed" **lead to the conclusion** "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution:

p: "You send me an e-mail message,"

q: "I will finish writing the program,"

r: "I will go to sleep early,"

s: "I will wake up feeling refreshed."

Premises: $p \to q$, $\neg p \to r$, and $r \to s$

Conclusion: $\neg q \to s$

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Handling Quantified Statements

Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:

- Rules of Inference for Propositional Logic.
- Rules of Inference for Quantified Statements (next slides).

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."

"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c)\ \text{for some element}\ c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."

"Therefore, someone got an A in the class."

Rules of Inference for Quantified Statements

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Returning to the Socrates Example

$$\begin{array}{l} \forall x\left(Man\left(x\right)\rightarrow Mortal\left(x\right)\right) \\ \quad Man\left(Socrates\right) \\ \hline \therefore \quad Mortal\left(Socrates\right) \end{array}$$

Solution for Socrates Example

$$\begin{array}{l} \forall x\left(Man(x) \rightarrow Mortal(x)\right) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

Valid Argument

Step	**Reason**
1. $\forall x\left(Man(x) \rightarrow Mortal(x)\right)$	Premise
2. $\text{Man}(Socrates) \rightarrow Mortal(Socrates)$	UI from (1)
3. $\text{Man}\ Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Example

Show that the premises "A student in this class has not read the book," **and** "Everyone in this class passed the first exam" **imply the conclusion** "Someone who passed the first exam has not read the book."

Solution:

$C(x)$ "x is in this class,"

$B(x)$ "x has read the book,"

$P(x)$ "x passed the first exam."

The premises: $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$.

The conclusion: $\exists x(P(x) \land \neg B(x))$.

Step	Reason
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

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