COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

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Basic Structures: Sets and Functions

Part-3

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Section Summary

Definition of a Function.

- Domain, Codomain.
- Image, Preimage.

Injection, Surjection, Bijection.

Inverse Function.

Function Composition.

Graphing Functions.

Floor, Ceiling, Factorial.

Partial Functions.

Composition Questions[1]

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$
$$f \circ g(b) = f(g(b)) = f(c) = 1.$$
$$f \circ g(c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions 2

Example 2: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

Solution:

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Question

Let A, $B \subseteq \mathbb{R}$, where A=$\{x|\ x^2$-$8x \leq$ -$15\}$ and B=$\{x|\ x^2 - 2x \leq 3\}$. Determine the sets $A \cup B$ and $A \cap B$.

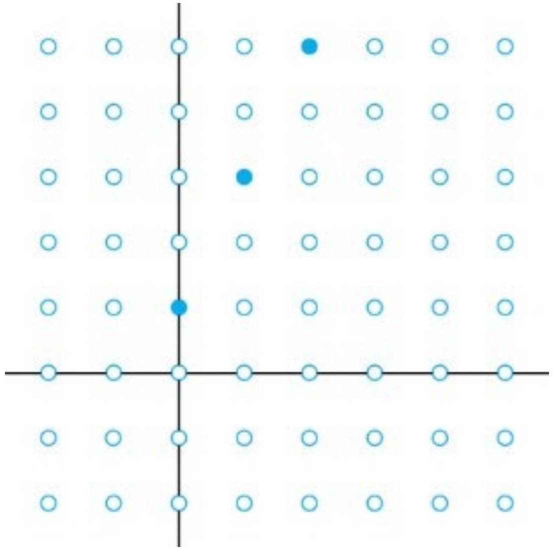
Identity Function

Definition: Let A be a set. The identity function on A is the function $\iota_A : A \to A$, where $\iota_A(x) = x$ for all $x \in A$.

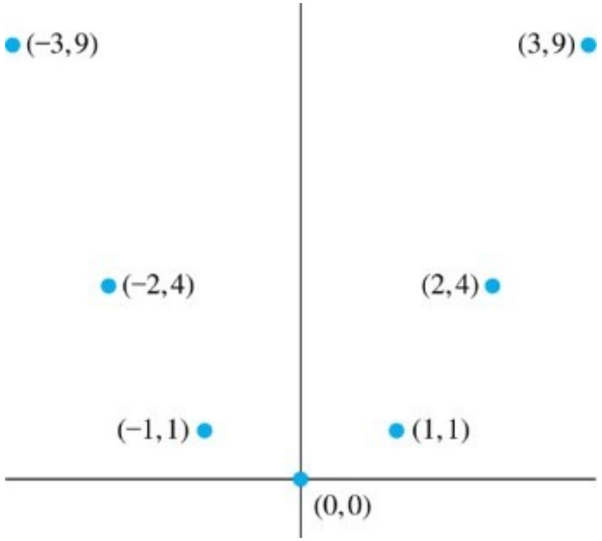
- The identity function ι_A is the function that assigns each element to itself.
- The function ι_A is one-to-one and onto, so it is a bijection.
- When the composition of a function and its inverse is formed, in either order, an identity function is obtained.
  - $f^{-1}(b) = a$ when $f(a) = b$
  - $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$
  - $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$
  - Consequently, $\mathbf{f^{-1} \circ f = \iota_A}$ and $\mathbf{f \circ f^{-1} = \iota_B}$; therefore, $\mathbf{(f^{-1})^{-1} = f}$.

Graphs of Functions

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a)=b\}$.



Graph of $f(x) = 2x + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

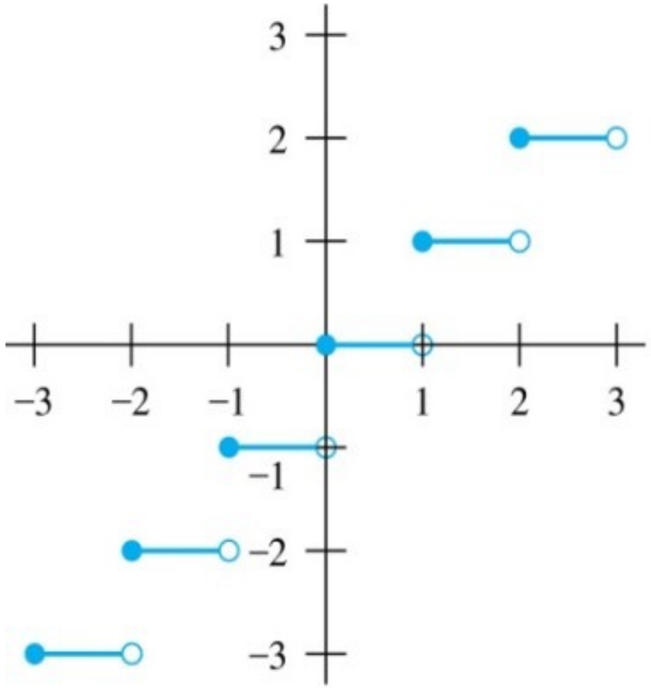
The ***floor*** function, denoted $f(x) = \lfloor x \rfloor$
is the largest integer less than or equal to *x*.

The ***ceiling*** function, denoted $f(x) = \lceil x \rceil$
is the smallest integer greater than or equal to *x*.

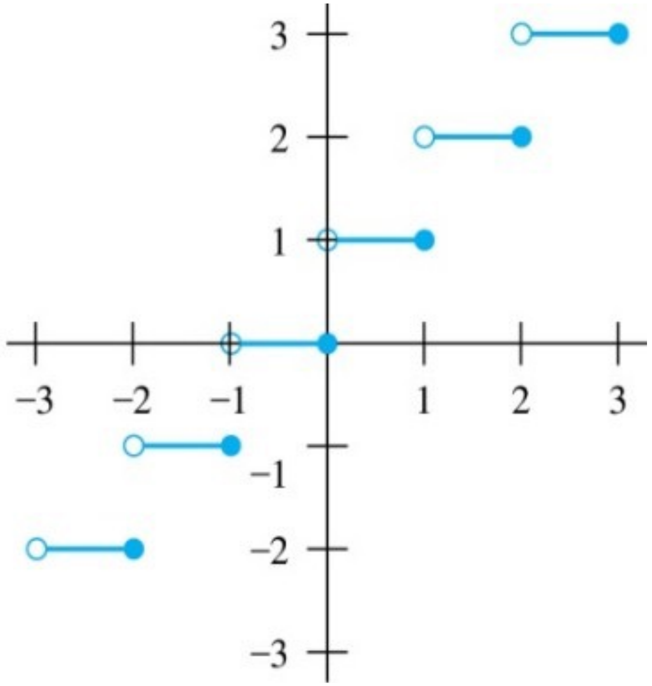
Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions[1]



$$f(x) = \lfloor x \rfloor$$



$$f(x) = \lceil x \rceil$$

Graph of Floor and Ceiling Functions

Floor and Ceiling Functions$_2$

(n is an integer, x is a real number)
(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$ (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$ (1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ (1d) $\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Proof

Prove $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ using a direct proof.

Solution:

Suppose that $\lfloor x \rfloor = m$, where m is a positive integer.

1. $m \leq x < m+1$, property (1a),
2. $m+n \leq x+n < m+n+1$, adding n to all quantities,
3. $\lfloor x+n \rfloor = m+n$, property (1a),
4. $m + n = \lfloor x \rfloor + n.$, adding n to both quantities $\lfloor x \rfloor = m$
5. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, from 3 and 4.

Proving Properties of Functions

Example: Prove that x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $0 <= \varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $1/2 <= \varepsilon < 1$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial Function

Definition: $f : \mathbf{N} \to \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first *n* positive integers when *n* is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n,$$

$$f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

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