COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

SUMMER 2024

The Foundations: Logic and Proofs

Part-3

Mesut Ozdag, Ph.D.
mesut.ozdag@ucf.edu

*Because learning changes everything.**

Outline

- Propositional Logic
  - The Language of Propositions.
  - Applications.
  - Logical Equivalences.
- Predicate Logic
  - The Language of Quantifiers.
  - Logical Equivalences.
  - Nested Quantifiers.
- Proofs
  - Rules of Inference.
  - Proof Methods.
  - Proof Strategy.

*Mc Graw Hill

Section:
Predicate Logic

Propositional Logic Not Enough

If we have:

"All men are mortal."

"Socrates is a man."

Does it follow that "Socrates is mortal?"

Can't be represented in propositional logic.

- Inferences.

Introducing Predicate Logic

Predicate logic uses the following new features:

- Variables: x, y, z.
- Predicates: $P(x)$, $M(x)$.
- Quantifiers (*to be covered in a few slides*)

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x)$.
- Variables can be replaced by elements from their *domain*.

Predicates

Example: The statement “x is greater than 3”.

- the variable, x.
- the predicate, “is greater than 3”.

Denote the statement “x is greater than 3” by P(x):

The statement P(x) is also said to be the value of the propositional function P at x.

Question: When does the statement P(x) become a proposition?

Propositional Functions

Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).

For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

$P(-3)$ is false.

$P(0)$ is false.

$P(3)$ is true.

Often the domain is denoted by U. So, in this example U is the integers.

Examples of Propositional Functions

Let “$x + y = z$” be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Now let “$x - y = z$” be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions

If $P(x)$ denotes "$x > 0$,"

find these truth values:

$P(3) \lor P(-1)$

Solution: T

$P(3) \land P(-1)$

Solution: F

$P(3) \rightarrow P(-1)$

Solution: F

$P(3) \rightarrow \neg P(-1)$

Solution: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \land P(y)$

$P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier,* **"**For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is?
   False.

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is? **True.**

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is? **False.**

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.

2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is?

   False.

3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is?
   True.

Quantifiers

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Thinking about Quantifiers

When the domain is finite, we can think of quantification as looping through the elements of the domain.

To evaluate $\forall x\, P(x)$, loop through all x in the domain.

- If at every step P(x) is true, then $\forall x\, P(x)$ is true.
- If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.

To evaluate $\exists x\, P(x)$, loop through all x in the domain.

- If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
- If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.

Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\, P(x) \vee Q(x)$ means $\left(\forall x\, P(x)\right) \vee Q(x)$

$\forall x\, \left(P(x) \vee Q(x)\right)$ means something different.

Example

- Let P (x) be the statement “x + 1 > x.” What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution:

Because P(x) is true for all real numbers x, the quantification $\forall x\, P(x)$ is **true**.

Example

- What is the truth value of ∃x P(x), where P(x) is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4},

the proposition ∃x P(x) is the same as the disjunction:

P(1) ∨ P(2) ∨ P(3) ∨ P(4).

Because P(4), which is the statement "$4^2 > 10$," is true, it follows that ∃x P(x) is **true.**

Translating from English to Logic[1]

Example 1: Translate the following sentence into predicate logic:
"Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution.1: If *U* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as

$$\forall x\, J(x).$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is in this class" and translate as

$$\forall x\left(S(x) \rightarrow J(x)\right).$$

$\forall x\ \left(S(x) \wedge J(x)\right)$ is not correct. Why not?

Translating from English to Logic$_2$

Example 2: Translate the following sentence into predicate logic:
"Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\ \left(S(x) \wedge J(x)\right)$$

$\exists x\left(S(x) \rightarrow J(x)\right)$ is not correct. Why not?

Negating Quantified Expressions 1

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here J(x) is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java."

This implies that "There is a student in your class who has not taken Java."

$\neg\forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent.

Negating Quantified Expressions 2

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where J(x) is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java."

This implies that "Every student in this class has not taken Java."

$\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent.

De Morgan's Laws for Quantifiers

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

$$\exists x \left(S(x) \wedge M(x) \right)$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

$$\forall x \left(S(x) \rightarrow \left(M(x) \vee C(x) \right) \right)$$

Section

Nested Quantifiers

Nested Quantifiers

Necessary to express the meaning of sentence.

Example: "Every real number has an inverse" is $\forall x \exists y\ (x + y = 0)$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\ Q(x)$ where $Q(x)$ is
$\exists y\ P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

Nested Loops

To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :

- At each step, loop through the values for y.
- If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.

To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:

- At each step, loop through the values for y.
- The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
- If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\, \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers 1

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer : False

2. $\forall x \exists y P(x,y)$

   Answer : True

3. $\exists x \forall y P(x,y)$

   Answer : True

4. $\exists x \exists y P(x,y)$

   Answer : True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y.

Translating Nested Quantifiers into English

Example: Translate the statement

$$\forall x\Big(C(X)\vee\exists y\big(C(y)\wedge F(X,y)\big)\Big)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \left((x > 0) \land (y > 0) \right) \rightarrow (x + y > 0)$$

where the domain of both variables consists of all integers.

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement

"There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a)\right)$$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed a slide back:

$$\exists w \forall a \exists f\left(P(w,f) \wedge Q(f,a)\right)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\left(P(w,f) \wedge Q(f,a)\right)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\left(P(w,f) \wedge Q(f,a)\right)$
2. $\forall w \neg\forall a \exists f\left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg\exists f\left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg\left(P(w,f) \wedge Q(f,a)\right)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f\left(\neg P(w,f) \vee \neg Q(f,a)\right)$ by De Morgan's for $\wedge$.

Part 3: Translate the result back into English.

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."

Proofs

Section:

Rules of Inference

Arguments in Propositional Logic

- An *argument* in propositional logic is a sequence of propositions.
  - *Premises + conclusion.*
  - *valid* if the premises imply the conclusion.

- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

  $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Inference rules are all simple argument forms that will be used to construct more complex *argument forms*.

The Argument

We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{c} \forall x\left(Man(x) \rightarrow Mortal(x)\right) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

Valid argument.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rules of Inference for Propositional Logic: Modus Ponens (law of detachment)

Rule of Inference:

$$\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore q \end{array}$$

Corresponding Tautology:

$((p \rightarrow q) \land p) \rightarrow q$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore, I will study discrete math."

Modus Tollens

Rule of Inference:

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore, it is not snowing."

MESUT OZDAG, PH.D.
MESUT.OZDAG@UCF.EDU

UCF