COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

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The Foundations: Logic and Proofs

Part-2

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*Because learning changes everything.**

Outline

- Propositional Logic
  - The Language of Propositions.
  - Applications.
  - Logical Equivalences.
- Predicate Logic
  - The Language of Quantifiers.
  - Logical Equivalences.
  - Nested Quantifiers.
- Proofs
  - Rules of Inference.
  - Proof Methods.
  - Proof Strategy.

*Mc Graw Hill

Section Summary: Propositional Logic

- Propositions.
- Connectives.
  - Negation.
  - Conjunction.
  - Disjunction.
  - Implication; contrapositive, inverse, converse.
  - Biconditional.
- Truth Tables.

$(p\rightarrow q)\wedge(q\rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
TRUE	TRUE	TRUE	TRUE	
TRUE	FALSE	FALSE	TRUE	
FALSE	TRUE	TRUE	FALSE	
FALSE	FALSE	TRUE	TRUE	

$(p \rightarrow q) \wedge (q \rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	TRUE	FALSE
FALSE	TRUE	TRUE	FALSE	FALSE
FALSE	FALSE	TRUE	TRUE	TRUE

$(p \rightarrow q) \wedge (q \rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
TRUE	TRUE	TRUE	TRUE	TRUE	
TRUE	FALSE	FALSE	TRUE	FALSE	
FALSE	TRUE	TRUE	FALSE	FALSE	
FALSE	FALSE	TRUE	TRUE	TRUE	

$(p \rightarrow q) \wedge (q \rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
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TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
FALSE	TRUE	TRUE	FALSE	FALSE	FALSE
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE

$(p \rightarrow q) \wedge (q \rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$	$((p \rightarrow q) \wedge (q \rightarrow p)) \leftrightarrow (p \leftrightarrow q)$
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	
TRUE	FALSE	FALSE	TRUE	FALSE	FALSE	
FALSE	TRUE	TRUE	FALSE	FALSE	FALSE	
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE	

$(p \rightarrow q) \wedge (q \rightarrow p)$: Truth Table

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$	$((p \rightarrow q) \wedge (q \rightarrow p)) \leftrightarrow (p \leftrightarrow q)$
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	TRUE	FALSE	FALSE	TRUE
FALSE	TRUE	TRUE	FALSE	FALSE	FALSE	TRUE
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE	TRUE

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to

$(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Section:
Applications of
Propositional Logic

Translating English Sentences

Steps to convert an English sentence to a statement in propositional logic:

- Identify atomic propositions and represent using propositional variables.
- Determine appropriate logical connectives.

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's.
- q: I go to the country.
- r: I will go shopping.

- If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

One Solution: Let a, c, and f represent respectively “You can access the internet from campus,” “You are a computer science major,” and “You are a freshman.”

$$a \rightarrow (c \vee \neg f)$$

System Specifications

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

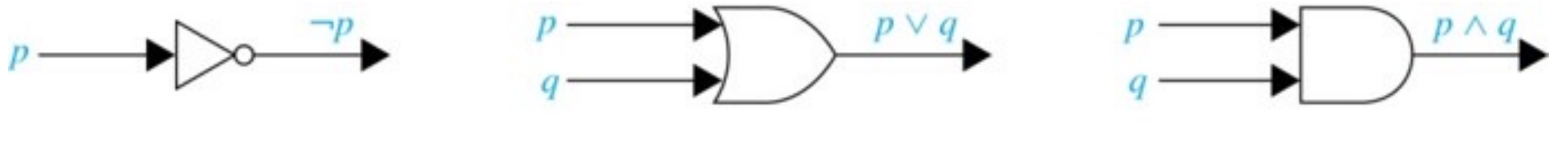
Remember; p **implies** q
Means q **when** p

$$q \rightarrow \neg p$$

Logic Circuits

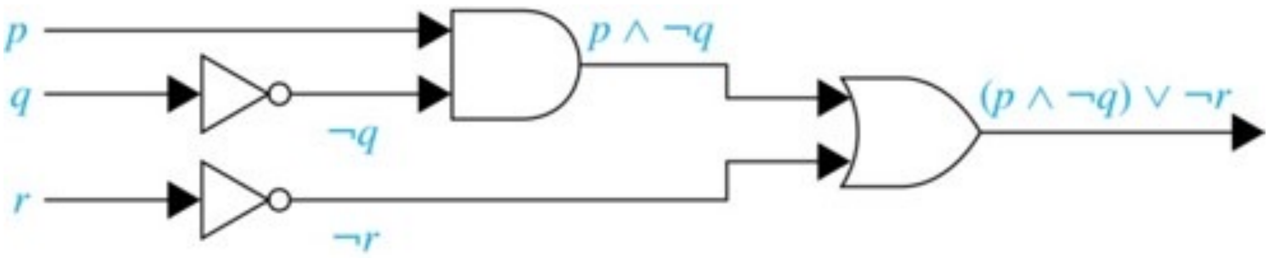
Electronic circuits; each input/output signal can be viewed as a 0 or 1.

- 0 represents **False**.
- 1 represents **True**.



Inverter OR gate AND gate

Example: Construct the circuit for (p and not q) or not r



Section:

Propositional Equivalences

Tautologies, Contradictions, and Contingencies

A *tautology* is a proposition which is always true.

- Example: $p \vee \neg p$.

A *contradiction* is a proposition which is always false.

- Example: $p \wedge \neg p$.

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences[1]

Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$

Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$

Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$

Double Negation Law: $\neg(\neg p) \equiv p$

Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

Key Logical Equivalences 2

Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$

Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$

$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$

Practice:

Show $(a \vee b) \wedge b$ is equivalent to b.

Practice

Show, using the laws of logic, that $[a \vee (b \wedge c) \vee (b \wedge \neg c)] \wedge [b \vee (c \wedge \neg c)] \leftrightarrow b$.

	$[a \vee (b \wedge c) \vee (b \wedge \neg c)] \wedge [b \vee (c \wedge \neg c)]$	
$\leftrightarrow$	$[a \vee (b \wedge c) \vee (b \wedge \neg c)] \wedge [b \vee \mathbf{F}]$	Negation
$\leftrightarrow$	$[a \vee (b \wedge c) \vee (b \wedge \neg c)] \wedge b$	Identity
$\leftrightarrow$	$\left[a \vee \left(b \wedge (c \vee \neg c)\right)\right] \wedge b$	Factor out b (reverse distribution)
$\leftrightarrow$	$[a \vee (b \wedge \mathbf{T})] \wedge b$	Negation
$\leftrightarrow$	$(a \vee b) \wedge b$	Identity
$\leftrightarrow$	b	Absorption

Q.E.D. symbol "■" (or "□") is a symbol used to denote the end of a proof, "Q.E.D." for the Latin phrase "quod erat demonstrandum".

More Logical Equivalences

Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing New Logical Equivalences

To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$$\begin{aligned} p \rightarrow q &\equiv \neg p \vee q \\ \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\ &\equiv p \wedge \neg q && \text{by the double negation law} \end{aligned}$$

The Implication Identity

$$(p \to q) \leftrightarrow (\neg p \lor q)$$

- "If it's raining, the ground outside is wet."
- "The ground outside is wet or it's not raining."
- "It's not raining or the ground outside is wet."

p	q	$\neg p$	$\neg p \lor q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Equivalence Proofs[1]

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
$\equiv (\neg p \wedge \neg q)$	By the identity law for **F**

Equivalence Proofs$_2$

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q) \qquad \text{by truth table for } \rightarrow$$

$$\equiv (\neg p \vee \neg q) \vee (p \vee q) \qquad \text{by the first De Morgan law}$$

$$\equiv (\neg p \vee p) \vee (\neg q \vee q) \qquad \text{by associative and commutative laws laws for disjunction}$$

$$\equiv T \vee T \qquad \text{by truth tables}$$

$$\equiv T \qquad \text{by the domination law}$$

Propositional Satisfiability

A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that makes it true.

When no such assignments exist, the compound proposition is *unsatisfiable*.

A compound proposition is unsatisfiable if and only if its negation is a tautology.

Questions on Propositional Satisfiability

Determine the satisfiability of the following compound propositions:

Example.1: $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

Example.2: $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

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