COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

SUMMER 2024

The Foundations: Logic and Proofs

Part-1

Mesut Ozdag, Ph.D.
mesut.ozdag@ucf.edu

*Because learning changes everything.**

Outline

- Propositional Logic
  - The Language of Propositions.
  - Applications.
  - Logical Equivalences.
- Predicate Logic
  - The Language of Quantifiers.
  - Logical Equivalences.
  - Nested Quantifiers.
- Proofs
  - Rules of Inference.
  - Proof Methods.
  - Proof Strategy.

*Mc Graw Hill

Propositional Logic Summary

- The Language of Propositions
  - Connectives.
  - Truth Values.
  - Truth Tables.
- Applications
- Logical Equivalences
  - Important Equivalences.
  - Showing Equivalence.

Section Summary: Propositional Logic

- Propositions.
- Connectives.
  - Negation.
  - Conjunction.
  - Disjunction.
  - Implication; contrapositive, inverse, converse.
  - Biconditional.
- Truth Tables.

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:

a) The Moon is made of green cheese.
b) Trenton is the capital of New Jersey.
c) Toronto is the capital of Canada.
d) $1 + 0 = 1$
e) $0 + 0 = 2$

- Examples that are not propositions.

f) Sit down!
g) What time is it?
h) $x + 1 = 2$
i) $x + y = z$

Propositional Logic

Constructing Propositions

Propositional Variables: $p, q, r, s, \ldots$

The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.

Compound Propositions; constructed from logical connectives and other propositions.

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Compound Propositions: Negation

The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Example: If p denotes "The earth is round.", then $\neg p$ denotes:

"It is not the case that the earth is round,"
or more simply "The earth is not round."

Example: Negation

Find the negation of the proposition "Alice's smartphone has at least 64GB of memory." and express this in simple English.

Solution:

"It is not the case that Alice's smartphone has at least 64GB of memory."

"Alice's smartphone does not have at least 64GB of memory."

"Alice's smartphone has less than 64GB of memory."

Conjunction

The *conjunction* of propositions p and q is denoted by $p \land q$ and has this truth table:

p	q	$P \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \land q$ denotes:

"I am at home **and** it is raining."

Disjunction

The *disjunction* of propositions *p* and *q* is denoted by $p \vee q$ and has this truth table:

p	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \vee q$ denotes:

"I am at home or it is raining."

The Connective Or in English

In English “or” has two distinct meanings.

- “Inclusive Or” - In the sentence “Students who have taken CS1 or MAC2311 may take this class,” we assume that students need to have taken one of the prerequisites but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
- “Exclusive Or” - When reading the sentence “Soup or salad comes with this entrée,” we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$:

p	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."

In $p \rightarrow q$: p is the *hypothesis* (*antecedent* or *premise*), and q is the *conclusion* (or *consequence*).

Understanding Implication 1

In $p \to q$ there does not need to be any connection between the antecedent or the consequent. The “meaning” of $p \to q$ depends only on the truth values of p and q.

These implications are perfectly fine but would not be used in ordinary English.

- “If the moon is made of green cheese, then I have more money than Bill Gates. ”
- “If the moon is made of green cheese, then I’m on welfare.”
- “If 1 + 1 = 3, then your grandma wears combat boots.”

Understanding Implication $_2$

One way to view the logical conditional is to think of an obligation or contract.

- “If I am elected, then I will lower taxes.”
- “If you get 100% on the final, then you will get an A.”

Different Ways of Expressing $p \rightarrow q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

Example: Implication

Let p be the statement "Maria learns discrete structures." and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.

Solution:

"If Maria learns discrete structures, then she will find a good job."

"Maria will find a good job when she learns discrete structures."

"For Maria to get a good job, it is sufficient for her to learn discrete structures."

"Maria will find a good job unless she does not learn discrete structures."

Converse, Contrapositive, and Inverse

From $p \rightarrow q$ we can form new conditional statements.

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$.
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$.
- $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$.

Example: Find the converse, inverse, and contrapositive of "The home team wins whenever it is raining."

Solution: (Remember "q whenever p" means $p \rightarrow q$)

"If it is raining, then the home team wins."

converse: "If the home team wins, then it is raining."

inverse: "If it is not raining, then the home team does not win."

contrapositive: "If the home team does not win, then it is not raining."

Biconditional

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes:

"I am at home if and only if it is raining."

Expressing the Biconditional

Some alternative ways "*p* if and only if *q*" is expressed in English:

- ***p* is necessary and sufficient for *q*.**
- **if *p* then *q*, and conversely.**
- ***p* iff *q*.**

$p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

Truth Tables For Compound Propositions

Construction of a truth table:

Rows.

Need a row for every possible combination of values for the atomic propositions.

Columns.

Need a column for the compound proposition.

Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

- This includes the atomic propositions.

Example Truth Table

Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Example Truth Table

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Equivalent Propositions

Two propositions are *equivalent* if they always have the same truth value.

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

How many rows are there in a truth table with *n* propositional variables?

Solution: 2^n

Note that this means that with n propositional variables, we can construct 2^n distinct (not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Section:
Applications of
Propositional Logic

Translating English Sentences

Steps to convert an English sentence to a statement in propositional logic:

- Identify atomic propositions and represent using propositional variables.
- Determine appropriate logical connectives.

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's.
- q: I go to the country.
- r: I will go shopping.

- If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

One Solution: Let a, c, and f represent respectively “You can access the internet from campus,” “You are a computer science major,” and “You are a freshman.”

$$a \to (c \vee \neg f)$$

System Specifications

System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

$$q \rightarrow \neg p$$

MESUT OZDAG, PH.D.
MESUT.OZDAG@UCF.EDU