Problem 3. Disprove the following statement about sets A, B, C and D by finding a counterexample.

If $A \subseteq C \cap D$ and $B \subseteq C \cup D$, then either $B - A \subseteq C$ or $B - A \subseteq D$

Explicitly state the elements in sets A, B, C, D and B – A in your counterexample.

Here is one counterexample:

$A = \{\}$ $\quad B = \{1,2\}$ $\quad C = \{1\}$ $\quad D = \{2\}$

Trivially, $A \subseteq C \cap D$, since the empty set is a subset of all sets.

$B = \{1,2\}$ and $C \cup D = \{1,2\}$, thus, $B \subseteq C \cup D$.

But $B - A = \{1,2\}$, thus $B - A \nsubseteq C$ and $B - A \nsubseteq D$.

Problem 4. Prove or disprove the following for arbitrary sets A, B and C.

If $B \subseteq C$, then $B - A \subseteq C - A$.

The assertion is true.

We will assume that $B \subseteq C$ is true and prove that $B - A \subseteq C - A$ must be true under this assumption (direct proof).

To prove $B - A \subseteq C - A$, we start with an arbitrarily chosen element $x \in B - A$ and we aim to show that $x \in C - A$ must be true.

By the definition of the set difference, from $x \in B - A$, we have $x \in B \wedge x \notin A$.

Since $x \in B$ and $B \subseteq C$, by the definition of the subset, it follows that $x \in C$.

Since $x \in C$ and $x \notin A$, by the definition of set the difference again, it follows that $x \in C - A$, as desired.

Problem 3. Find the inverse of the function $f: \mathbb{R} \to \mathbb{R}^+$ defined as below.

$$f(x) = e^{2x+5}$$

Also show that $f \circ f^{-1}$ equals the identity function ($f(x) = x$) on set $\mathbb{R}^+$.

If $f(x) = y = e^{2x+5}$, then $\ln y = 2x + 5$

$$\Rightarrow 2x = (\ln y) - 5$$

$$\Rightarrow x = \frac{(\ln y) - 5}{2}. \text{ So, } f^{-1}(y) = \frac{(\ln y) - 5}{2}.$$

Now, looking at the composite function,

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{(\ln x) - 5}{2}\right)$$

$$= e^{2\left(\frac{(\ln x)-5}{2}\right)+5} = e^{(\ln x)-5+5} = e^{\ln x} = x,$$

we note that it equals the identity function $f(x) = x$.

Problem 4. Let $f, g, h: \mathbb{R} \to \mathbb{R}$ be the following functions.

$$f(x) = x^2 \qquad g(x) = x + 5 \qquad h(x) = \sqrt{x^2 + 2}$$

a) Using these example functions, check if the composite function $h \circ (g \circ f)$ is the same as the composite function $(h \circ g) \circ f$.

b) If you found them to be the same, do you think that this is a general result? In other words, is function composition associative?

a) $\left(h \circ (g \circ f)\right)(x) = h\left((g \circ f)(x)\right) = h\left(g\left(f(x)\right)\right) = h\left(g(x^2)\right) = h(x^2 + 5)$

$$= \sqrt{(x^2 + 5)^2 + 2} = \sqrt{x^4 + 10x^2 + 27}$$

For the other one, $\left((h \circ g) \circ f\right)(x) = (h \circ g)\left(f(x)\right) = (h \circ g)(x^2) = h\left(g(x^2)\right) = h(x^2 + 5)$ and the rest is same as above. So, $h \circ (g \circ f) = (h \circ g) \circ f$

b) Yes, this is a general result, and it can be shown that function composition is associative.