Series Solutions Near an Ordinary Point

Lecture 16

Definition: Given a differential equation of the form

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

we say that $x = x_0$ is an *ordinary point* if $P(x)$ and $Q(x)$ are differentiable at x_0. Otherwise we say x_0 is a *singular point*.

Note: If $P(x)$ and $Q(x)$ are polynomials, exponential functions, or sine/cosine functions then every point is an ordinary point.

Ex. $y'' + (\ln x)\, y = 0$ has a singular point at $x = 0$.

Ex. $y'' + \dfrac{2x}{x^2-1}\, y' + \dfrac{6}{x^2-1}\, y = 0$ has singular points at $x = \pm 1$.

<u>Theorem</u>: If $x = x_0$ is an ordinary point of an ODE, then we can find two linearly independent power series solutions centered at $x = x_0$ of the form $\sum_{n=0}^{\infty} c_n (x - x_0)^n$.
The series will converge on an interval containing $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

<u>Note</u>: If the two solutions found are y_1 and y_2, then the general solution is $y = c_0 y_1 + c_1 y_2$.

<u>Ex.</u> Find a series solution of $y'' + y = 0$.
Every point is an ordinary point, so let

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \ldots \circledast \quad (\text{choosing } x_0 = 0).$$

$$\longrightarrow \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\longrightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

So substituting we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\longrightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + c_n \right] x^n = 0 + 0x + 0x^2 + \cdots$$

$$\longrightarrow (n+2)(n+1) c_{n+2} + c_n = 0$$

$$\longrightarrow \boxed{c_{n+2} = \frac{-c_n}{(n+2)(n+1)}} \quad n = 0, 1, 2, \ldots$$

This is our recurrence relation.

So $c_0 = c_0$ and $c_1 = c_1$

$\longrightarrow$ $c_2 = \frac{-c_0}{2}$ and $c_3 = \frac{-c_1}{6}$

$\longrightarrow$ $c_4 = \frac{c_0}{24}$ and $c_5 = \frac{c_1}{120}$

$\vdots$

$\longrightarrow$ $\boxed{c_{2n} = \frac{(-1)^n c_0}{(2n)!}}$ and $\boxed{c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}}$

Substituting back into ⊛ we get

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n c_1}{(2n+1)!} x^{2n+1}$$

$$\boxed{y = c_0 \cos x + c_1 \sin x}$$

<u>Ex.</u> Find a series solution of $y'' - xy = 0$.

Every point is an ordinary point, so we let

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\longrightarrow \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\longrightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

So substituting we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\longrightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

$$\longrightarrow \quad 2c_2 + \sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

$$\longrightarrow \quad \underline{\underline{2c_2}} + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - c_{n-1}\right]x^n = 0$$

So $c_2 = 0$ and $\boxed{c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}}$, $\underline{\underline{n = 1, 2, 3, \ldots}}$

Hence

$c_0 = c_0$

$c_1 = c_1$

$c_2 = 0$

$c_3 = \dfrac{c_0}{6}$

$c_4 = \dfrac{c_1}{12}$

$c_5 = 0$

$c_6 = \dfrac{c_3}{30} = \dfrac{c_0}{180}$

$c_7 = \dfrac{c_4}{42} = \dfrac{c_1}{504}$

$c_8 = 0$

$\vdots$

Thus one solution is

$$y_1 = c_0\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right)$$

and another is

$$y_2 = c_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right)$$

So the general solution is

$$y = c_0\left(1+\frac{x^3}{6}+\frac{x^6}{180}+\cdots\right)+c_1\left(x+\frac{x^4}{12}+\frac{x^7}{504}+\cdots\right)$$

Ex. Find a series solution of $y''+x^2y=0$.

Every point is an ordinary point, so we let

$$y=\sum_{n=0}^{\infty} c_n x^n$$

$$\longrightarrow y'=\sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\longrightarrow y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

So substituting we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=0}^{\infty} c_n x^{n+2}=0$$

$$\longrightarrow \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n+\sum_{n=2}^{\infty}c_{n-2}x^n=0$$

$$\longrightarrow 2c_2+6c_3x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)c_{n+2}+c_{n-2}\right]x^n=0$$

$\rightarrow c_2 = 0, \; c_3 = 0$, and

$$\boxed{c_{n+2} = \frac{-c_{n-2}}{(n+2)(n+1)}} \quad n = 2, 3, 4, \ldots$$

So $c_0 = c_0$

$c_1 = c_1$

$c_2 = 0$

$c_3 = 0$

$c_4 = \frac{-c_0}{12}$

$c_5 = \frac{-c_1}{20}$

$c_6 = 0$

$c_7 = 0$

$c_8 = \frac{-c_4}{56} = \frac{c_0}{672}$

$c_9 = \frac{-c_5}{72} = \frac{c_1}{1440} \quad \ldots$

thus one solution is

$$y_1 = c_0\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - \cdots\right)$$

and another solution is

$$y_2 = c_1\left(x - \frac{x^5}{20} + \frac{x^9}{1440} - \cdots\right)$$ thus the general

solution is $y = c_0\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - \cdots\right) + c_1\left(x - \frac{x^5}{20} + \frac{x^9}{1440} - \cdots\right)$

Ex: Find a series solution of

$$(x^2-1)y'' + xy' - y = 0$$

Zero is an ordinary point, so let

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\longrightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\longrightarrow y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

So substituting we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$+ \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\longrightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\longrightarrow -\underline{2c_2} - 6c_3 x + \cancel{c_1 x} - \underline{c_0} - \cancel{c_1 x}$$

$$+ \sum_{n=2}^{\infty} \left[n(n-1)c_n - (n+2)(n+1)c_{n+2} + nc_n - c_n \right] x^n = 0$$

Hence $-2c_2 - c_0 = 0$

and $-6c_3 = 0$

and $n(n-1)c_n - (n+2)(n+1)c_{n+2} + nc_n - c_n = 0$

$$\longrightarrow \quad c_{n+2} = \frac{n(n-1)c_n + nc_n - c_n}{(n+2)(n+1)}$$

$$\longrightarrow \quad c_{n+2} = \frac{n^2 c_n - \cancel{nc_n} + \cancel{nc_n} - c_n}{(n+2)(n+1)}$$

$$\longrightarrow \quad c_{n+2} = \frac{(n^2-1)c_n}{(n+2)(n+1)}$$

$$\longrightarrow \quad \boxed{c_{n+2} = \frac{(n-1)c_n}{n+2}} \quad n = 2, 3, 4, \ldots$$

So $c_0 = c_0$ $\qquad c_1 = c_1$

$c_2 = -\frac{c_0}{2}$ $\qquad c_3 = 0$

$c_4 = \frac{c_2}{4} = -\frac{c_0}{8}$ $\qquad c_5 = 0$

$c_6 = \frac{3c_4}{6} = -\frac{c_0}{16}$ $\qquad c_7 = 0$

$c_8 = \frac{5c_6}{8} = -\frac{5c_0}{128}$ $\qquad c_9 = 0$ $\quad \cdots$

Thus one solution is

$$y_1 = c_0\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \cdots\right)$$

and the other is

$$y_2 = c_1 x$$

Thus the general solution is $y_1 + y_2$.