Lecture 21

Ex. Solve the ODE

$$y'' + y = \sin 2t \quad \text{subject to } y(0) = 2,\ y'(0) = 1.$$

Using the theorem we get

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$$

$$\longrightarrow s^2 \mathcal{L}\{y\} - s y(0) - y'(0) + \mathcal{L}\{y\} = \frac{2}{s^2+4}$$

$$\longrightarrow s^2 \mathcal{L}\{y\} - 2s - 1 + \mathcal{L}\{y\} = \frac{2}{s^2+4}$$

$$\longrightarrow \mathcal{L}\{y\}(s^2+1) = \frac{2}{s^2+4} + 1 + 2s$$

$$\longrightarrow \mathcal{L}\{y\} = \frac{2 + (s^2+4) + 2s(s^2+4)}{(s^2+4)(s^2+1)}$$

$$\longrightarrow \mathcal{L}\{y\} = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)}$$

$$\longrightarrow \mathcal{L}\{y\} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$\longrightarrow \quad (As+B)(s^2+1)+(Cs+D)(s^2+4) = 2s^3+s^2+8s+6$

$\longrightarrow \quad s^3(A+C)+s^2(B+D)+s(A+4C)+(B+4D)$
$= 2s^3+s^2+8s+6$

$$\longrightarrow \quad \begin{cases} A+C=2 \\ B+D=1 \\ A+4C=8 \longrightarrow 3C=6 \longrightarrow C=2 \longrightarrow A=0 \\ B+4D=6 \longrightarrow 3D=5 \longrightarrow D=5/3 \longrightarrow B=-2/3 \end{cases}$$

$$\longrightarrow \quad \mathcal{L}\{y\} = \frac{-2/3}{s^2+4} + \frac{2s+5/3}{s^2+1}$$

$$\longrightarrow \quad \mathcal{L}\{y\} = \frac{-2/3}{s^2+4} + \frac{2s}{s^2+1} + \frac{5/3}{s^2+1}$$

$$\longrightarrow \quad y(t) = -\frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t$$

Ex: Solve the ODE

$y^{(4)} - y = 0$ subject to $y(0)=0$, $y'(0)=1$,
$y''(0)=0$, $y'''(0)=0$

$\longrightarrow \mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$

$\longrightarrow s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - \mathcal{L}\{y\} = 0$

$\longrightarrow \mathcal{L}\{y\}(s^4 - 1) = s^2$

$\longrightarrow \mathcal{L}\{y\} = \dfrac{s^2}{(s^2+1)(s^2-1)}$

$\longrightarrow \mathcal{L}\{y\} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2-1}$

$\longrightarrow (As+B)(s^2-1) + (Cs+D)(s^2+1) = s^2$

$\longrightarrow \quad s^3(A+C) + s^2(B+D) + s(-A+C) + (-B+D) = s^2$

$$\longrightarrow \quad \begin{cases} A+C = 0 \\ B+D = 1 \\ -A+C = 0 \longrightarrow A=0 \longrightarrow C=0 \\ -B+D = 0 \longrightarrow 2B=1 \longrightarrow B=1/2 \longrightarrow D=1/2 \end{cases}$$

$$\longrightarrow \quad \mathcal{L}\{y\} = \frac{1/2}{s^2+1} + \frac{1/2}{s^2-1}$$

$$\longrightarrow \quad y(t) = \frac{1}{2}\sin t + \frac{1}{2}\sinh t$$

Step Functions and Translation Theorems

In the previous lecture we needed to calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$, and you were told without reasoning that the answer was $\frac{1}{2}t^2e^{-2t}$. This is as a result of the following theorem.

<u>Theorem</u>: If $\mathcal{L}\{f(t)\} = F(s)$ exists when $s > a \geq 0$, and c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad \text{when } s > a+c$$

("Multiplication of $f(t)$ by an exponential function shifts the Laplace transform")

Conversely, if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$$

Proof: $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt$

$$= \int_0^\infty e^{-(s-c)t} f(t)\,dt$$

$$= F(s-c)\text{, as required.}$$

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*Ex. $\mathcal{L}\{t^{10}e^{-7t}\} = F(s+7)$

where $F(s) = \mathcal{L}\{t^{10}\} = \dfrac{10!}{s^{11}}$

So the answer is $F(s+7) = \dfrac{10!}{(s+7)^{11}}$

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*Ex. $\mathcal{L}\left\{\dfrac{\cosh t}{e^t}\right\} = F(s+1)$

where $F(s) = \mathcal{L}\{\cosh t\} = \dfrac{s}{s^2-1}$

So the answer is $F(s+1) = \dfrac{s+1}{(s+1)^2-1}$

* **Ex:** Calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s+2)^3}\right\}$$

So here $c=-2$ and by the theorem we get

$$f(t) = \frac{1}{2}t^2 e^{-2t}$$

* **Ex** Calculate $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}\right\}$

Completing the square we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$$

So here $c=2$ and by the theorem we get

$$f(t) = \sin t \, e^{2t}$$

*Ex: $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+6s+13}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$$

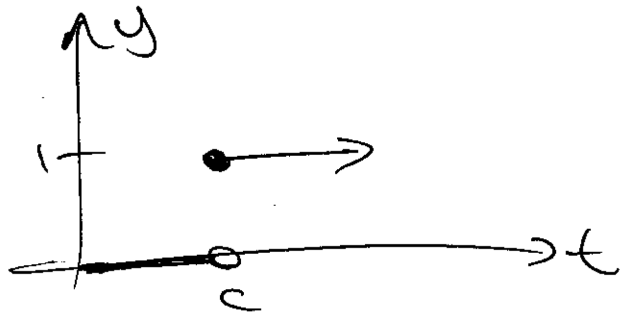
$$= \cos 2t \cdot e^{-3t} - \frac{3}{2}\sin 2t \cdot e^{-3t}$$

In engineering, step functions are often used to describe functions that are constant on a subset of the real line.

__Definition__: The unit step function (or Heaviside function) is defined by

$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \geq c \end{cases}$$

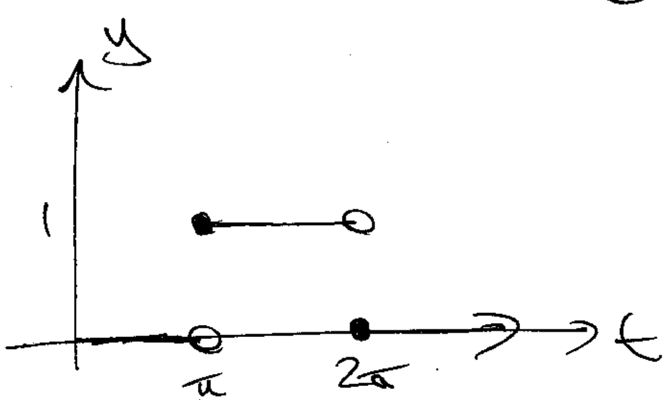
So the graph is



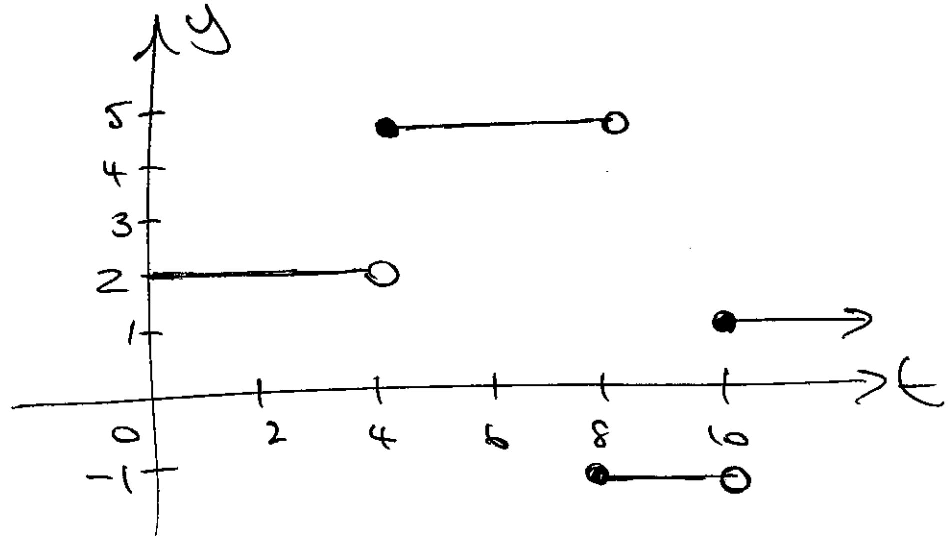
Ex Sketch the graph of $u_\pi(t) - u_{2\pi}(t)$, $t \geq 0$

$$u_\pi(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & t \geq \pi \end{cases} \qquad u_{2\pi}(t) = \begin{cases} 0, & 0 \leq t < 2\pi \\ 1, & t \geq 2\pi \end{cases}$$

So $$u_\pi(t) - u_{2\pi}(t) = \begin{cases} 0 & , 0 \leq t < \pi \\ 1 & , \pi \leq t < 2\pi \\ 0 & , t \geq 2\pi \end{cases}$$



* **Ex:** Describe the function below in terms of step functions



$$f(t) = 2 + 3u_4(t) - 6u_8(t) + 2u_{10}(t)$$

Jumps up 3 when $t=4$

jumps down 6 when $t=8$

jumps up 2 when $t=10$

Ex Describe the function below in terms of step functions

$$f(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ \sin t, & t \ge 2\pi \end{cases}$$

So $f(t) = \sin t \, u_{2\pi}(t)$

We can now state and prove the second translation theorem.

<u>Theorem</u>: If $\mathcal{L}\{f(t)\} = F(s)$ exists when $s > a \geq 0$ and c is a positive constant, then

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\, \mathcal{L}\{f(t)\}, \quad s > a.$$

("A translation of $f(t)$ by an amount c multiplies the Laplace transform by an exponential function").

Conversely, if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t)\, f(t-c)$$

<u>Proof</u>:
$$\mathcal{L}\{u_c(t)\, f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$
$$= \int_0^c e^{-st} u_c(t) f(t-c)\, dt + \int_c^\infty e^{-st} u_c(t) f(t-c)\, dt$$

$$= 0 + \int_c^\infty e^{-st} \underbrace{u_c(t)}_{=1} f(t-c)\, dt$$

We now let $u = t - c$, and hence $du = dt$, so the integral becomes

$$\int_0^\infty e^{-s(u+c)} f(u)\, du$$

$$= e^{-cs} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-cs} \mathcal{L}\{f(t)\}$$ since u is just a dummy variable.