1. Determine whether $f_1(x) = x^2$, $f_2(x) = 1 - x^2$, and $f_3(x) = 2 + x^2$ are linearly independent.

Exam 2 Review

$$W = \begin{vmatrix} x^2 & 1-x^2 & 2+x^2 \\ 2x & -2x & 2x \\ 2 & -2 & 2 \end{vmatrix}$$

$$= x^2(0) - (1-x^2)(0) + (2+x^2)(0)$$

$$= 0 \longrightarrow \text{Nothing}$$

So we let $c_1 x^2 + c_2(1-x^2) + c_3(2+x^2) = 0$

$$\longrightarrow x^2(c_1 - c_2 + c_3) + (c_2 + 2c_3) = 0$$

$$\longrightarrow \begin{cases} c_1 - c_2 + c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases}$$

So let $c_2 = 2$, $c_3 = -1$, $c_1 = 3$

$\longrightarrow$ linear dependence

2: Find a second solution to $y'' + 4y = 0$ if $y_1 = \cos 2x$.

Using the reduction of order formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= \cos 2x \int \frac{e^{-\int 0\,dx}}{\cos^2 2x}\,dx$$

$$= \cos 2x \int \sec^2 2x\,dx$$

$$= \frac{1}{2}\cos 2x \tan 2x$$

$$= \frac{1}{2}\sin 2x$$

3: Find a second solution to

$$xy'' - 2(x+1)y' + (x+2)y = 0 \text{ if } y_1 = e^x.$$

$$y_2 = e^x \int \frac{e^{\int (2+\frac{2}{x})dx}}{e^{2x}} dx$$

$$= e^x \int \frac{e^{2x+2\ln x}}{e^{2x}} dx$$

$$= e^x \int \frac{\cancel{e^{2x}} \cdot e^{2\ln x}}{\cancel{e^{2x}}} dx$$

$$= e^x \int x^2 dx$$

$$= \frac{e^x x^3}{3}$$

4: Solve $y'' - 2y' - 2y = 0$

So $m^2 - 2m - 2 = 0$

$\longrightarrow \quad m = \dfrac{2 \pm \sqrt{4+8}}{2}$

$\longrightarrow \quad m = \dfrac{2 \pm 2\sqrt{3}}{2}$

$\longrightarrow \quad m = 1 \pm \sqrt{3}$

$\longrightarrow \quad y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$

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5. Solve $y''' + 10y'' + 25y' = 0$

So $m^3 + 10m^2 + 25m = 0$

$\longrightarrow \quad m(m^2 + 10m + 25) = 0$

$\longrightarrow \quad m = 0, \quad m = -5$ (repeated)

$\longrightarrow \quad y = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}$

6: Solve $y''' - 5y'' + 6y' = 2\sin x + 8$

So $m^3 - 5m^2 + 6m = 0$

$\longrightarrow m(m^2 - 5m + 6) = 0$

$\longrightarrow m(m-2)(m-3) = 0$

$\longrightarrow m = 0, 2, 3$

$\longrightarrow y_c = c_1 + c_2 e^{2x} + c_3 e^{3x}$

Now let $y_p = Ax + B\cos x + C\sin x$

$\longrightarrow y_p' = A - B\sin x + C\cos x$

$\longrightarrow y_p'' = -B\cos x - C\sin x$

$\longrightarrow y_p''' = B\sin x - C\cos x$

So substituting we get

$$B\sin x - C\cos x + 5B\cos x + 5C\sin x + 6A - 6B\sin x + 6C\cos x = 2\sin x + 8$$

$\longrightarrow 6A = 8$ $\begin{cases} 5C - 5B = 2 \\ 5C + 5B = 0 \end{cases}$ $\longrightarrow 10C = 2$

$\longrightarrow A = 4/3$ $\longrightarrow C = 1/5 \longrightarrow B = -1/5$

Hence the final answer is

$$y = c_1 + c_2 e^{2x} + c_3 e^{3x} + \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x$$

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7. Solve $y'' - y = x + \sin x$ given that $y(0) = 2,\ y'(0) = 3$.

So $m^2 - 1 = 0$

$\longrightarrow m = \pm 1$

$\longrightarrow y_c = c_1 e^{x} + c_2 e^{-x}$

Now let $y_p = Ax + B + C\sin x + D\cos x$

$\longrightarrow y_p' = A + C\cos x - D\sin x$

$\longrightarrow y_p'' = -C\sin x - D\cos x$

Substituting we get

$$-C\sin x - D\cos x - Ax - B - C\sin x - D\cos x = x + \sin x$$

$\longrightarrow B = 0,\ A = -1,\ -2C = 1,\ -2D = 0$

$C = -1/2 \qquad D = 0$

So the general solution is

$$y = c_1 e^{x} + c_2 e^{-x} - x - \tfrac{1}{2}\sin x$$

Finally

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = c_1 - c_2 - 1 - \tfrac{1}{2} = 3 \longrightarrow c_1 - c_2 = \frac{9}{2}$$

Hence $c_1 = \frac{13}{4}$ and $c_2 = \frac{-5}{4}$

and the answer follows

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8. Solve $y'' - 2y' + 2y = e^{x}\tan x$

So $m^2 - 2m + 2 = 0$

$$\longrightarrow m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\longrightarrow m = \frac{2 \pm 2i}{2}$$

$$\longrightarrow m = 1 \pm i$$

$\longrightarrow y_c = e^x(c_1 \sin x + c_2 \cos x)$

Now $W = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix} = e^x \sin x\,(\cancel{e^x \cos x} - e^x \sin x) - e^x \cos x\,(\cancel{e^x \sin x} + e^x \cos x)$

$= -e^{2x}(\sin^2 x + \cos^2 x)$

$= -e^{2x}$

Thus $u_1' = \dfrac{-e^x \cos x \cdot e^x \tan x}{-e^{2x}} = \cos x \tan x = \sin x$

and $u_2' = \dfrac{e^x \sin x \cdot e^x \tan x}{-e^{2x}} = -\sin x \tan x = \dfrac{-\sin^2 x}{\cos x}$

$= \dfrac{\cos^2 x - 1}{\cos x}$

$= \cos x - \sec x$

Hence $u_1 = -\cos x$ and $u_2 = \sin x - \ln|\sec x + \tan x|$

Thus $y = e^x(c_1 \sin x + c_2 \cos x)$

$$- \cancel{\underbrace{e^x \sin x}_{y_1}} \underbrace{\cos x}_{u_1} + e^x \cos x \left(\cancel{\sin x} - \ln|\sec x + \tan x|\right)$$

$$\boxed{y = e^x(c_1 \sin x + c_2 \cos x) - e^x \cos x \ln|\sec x + \tan x|}$$

9. A 12lb weight stretches a spring 2ft. The weight is released from a point 1ft below equilibrium with upward velocity 4ft/sec. Find the position function and calculate the first time the weight returns to the equilibrium position.

$k = \frac{12}{2} = 6$, $M = \frac{12}{32} = \frac{3}{8}$

So $y'' + \underbrace{16}_{\frac{k}{M}} y = 0$ with $y(0) = 1$, $y'(0) = -4$

thus $m^2+16=0 \rightarrow m=\pm 4i$

So $y = c_1 \cos 4t + c_2 \sin 4t$

Now $y(0) = c_1 = 1$

and $y'(0) = 4c_2 = -4 \rightarrow c_2 = -1$

Hence $\boxed{y = \cos 4t - \sin 4t}$

Finally, set $y=0$ to get

$\tan 4t = 1$

$\rightarrow \quad 4t = \pi/4$

$\rightarrow \quad t = \pi/16$

10. A force of 2lbs stretches a spring 1ft. With one end fixed, an 8lb weight is attached to a system lying on a table that imparts damping 3/2 times the instantaneous velocity. Find the equation of motion if the weight is initially displaced from rest 4 inches above equilibrium.

$k = \frac{2}{1} = 2$, $m = \frac{8}{32} = \frac{1}{4}$, $\beta = \frac{3}{2}$. $\left[\frac{\beta}{m} = 6, \frac{k}{m} = 8\right]$

So $y'' + 6y' + 8y = 0$; $x(0) = -\frac{1}{3}$, $x'(0) = 0$

$\rightarrow m^2 + 6m + 8 = 0$

$\rightarrow m = -4, -2$

$\rightarrow y = c_1 e^{-4t} + c_2 e^{-2t}$

Finally $y(0) = -1/3 \rightarrow c_1 + c_2 = -1/3$

$y'(0) = 0 \rightarrow -4c_1 - 2c_2 = 0$

$\rightarrow \begin{array}{l} 2c_1 + 2c_2 = -2/3 \\ -4c_1 - 2c_2 = 0 \end{array}$

$\rightarrow -2c_1 = -2/3$

$\rightarrow c_1 = 1/3,\ c_2 = -2/3$

The result follows.

11. A 4 lb weight is suspended from a spring with constant 3 lb/ft. The system is immersed in a fluid with damping equal to the velocity. When $t=0$ an external force of $f(t)=e^{-t}$ is applied. Determine $y(t)$ if the system is initially displaced from rest 2ft below equilibrium.

$k=3, \quad m=\frac{4}{32}=\frac{1}{8}, \quad \beta=1 \quad \left[\frac{\beta}{m}=8, \ \frac{k}{m}=24\right]$

So $y''+8y'+24y=8e^{-t}$

Thus $m^2+8m+24=0$

$\longrightarrow m=-4\pm 2\sqrt{2}i$

$\longrightarrow y_c=e^{-4t}(c_1\cos 2\sqrt{2}t+c_2\sin 2\sqrt{2}t)$

Now $y_p=Ae^{-t}$. Differentiate twice and substitute to get $A=8/17$

Finally $y(0)=2, \ y'(0)=0 \longrightarrow c_1=\frac{26}{17}, \ c_2=\frac{28\sqrt{2}}{17}$