# Inverse Laplace Transforms

Lecture 20

We define inverse Laplace transforms as might be expected.

**Definition**: $\mathcal{L}^{-1}\{F(s)\}$ is the function whose Laplace transform is $F(s)$.

In the most basic cases, finding the inverse Laplace transform is simply a matter of using the table from the previous lecture, reading from right to left.

| $f(t)$ | $\mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin kt$ | $\frac{k}{s^2+k^2}$ |
| $\cos kt$ | $\frac{s}{s^2+k^2}$ |
| $\sinh kt$ | $\frac{k}{s^2-k^2}$ |
| $\cosh kt$ | $\frac{s}{s^2-k^2}$ |

Ex $\mathcal{L}^{-1}\left\{\frac{s}{s^2+64}\right\} = \cos 8t$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \overset{(*)}{=} \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{1}{2}t^2$

Note: Step (*) relies on the fact that $\mathcal{L}^{-1}$ is linear, so $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$

Ex. $\mathcal{L}^{-1}\left\{\frac{5}{s-6} + \frac{2}{s^2+1}\right\}$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 5e^{6t} + 2\sin t$$

Ex. $\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2+9}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

$$= 3\cos 3t + \frac{5}{3}\sin 3t$$

For more challenging problems we often use partial fractions, and consider three cases seen in Calculus II.

i) <u>Distinct</u> <u>Linear</u> <u>Terms</u>

<u>Ex</u> Calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s+5)(s-4)}\right\}$

So we let $\frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$

$$\longrightarrow \quad 1 = A(s-4) + B(s+5)$$

So when $s=-5$ we get $1 = -9A \longrightarrow A = -1/9$
and when $s=4$ we get $1 = 9B \longrightarrow B = 1/9$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{(s+5)(s-4)}\right\} = -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$

$$= -\frac{1}{9}e^{-5t} + \frac{1}{9}e^{4t}$$

(ii) Repeated Linear Terms

Ex. Calculate $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)^3}\right\}$

So we let $\frac{s+1}{s^2(s+2)^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{(s+2)^3}$

$$\longrightarrow \quad s+1 = As(s+2)^3 + B(s+2)^3 + Cs^2(s+2)^2 + Ds^2(s+2) + Es^2$$

So when $s=0$ we get $1 = 8B \longrightarrow B = 1/8$

and when $s=-2$ we get $-1 = 4E \longrightarrow E = -1/4$

Hence $s+1 = As(s+2)^3 + \frac{1}{8}(s+2)^3 + Cs^2(s+2)^2 + Ds^2(s+2) - \frac{1}{4}s^2$

$$= \underline{As^4} + \underline{6As^3} + 12As^2 + 8As + \frac{1}{8}s^3 + \frac{3s^2}{4} + \frac{3s}{2} + 1 + \underline{Cs^4} + \underline{4Cs^3} + 4Cs^2 + \underline{Ds^3} + 2Ds^2 - \frac{1}{4}s^2$$

Equating the coefficients of $s^4$, $s^3$ and $s^1$ we get

$$\begin{cases} 0 = A + C \\ 0 = 6A + \frac{1}{8} + 4C + D \\ 1 = 8A + \frac{3}{2} \longrightarrow A = -1/16 \longrightarrow C = 1/16 \\ \qquad\qquad\qquad\qquad\qquad\quad \longrightarrow D = 0 \end{cases}$$

So
$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)^3}\right\} = \frac{-1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$$

$$= -\frac{1}{16} + \frac{t}{8} + \frac{e^{-2t}}{16} - \frac{1}{8}t^2e^{-2t}$$

We cover this technique in the next lecture

### iii) Irreducible Quadratic Terms

Ex Calculate $\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2(s^2+1)}\right\}$

So we let $\dfrac{s-1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}$

$$\longrightarrow \quad s-1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

If we let $s=0$ we get $-1=B$

Hence $\quad s-1 = As^3 + As - s^2 - 1 + Cs^3 + Ds^2$

Equating the coefficients of $s^3$, $s^2$, and $s^1$ we get

$$\begin{cases} 0 = A+C & \longrightarrow C=-1 \\ 0 = -1+D & \longrightarrow D=1 \\ 1 = A & \end{cases}$$

So $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-s+1}{s^2+1}\right\}$

$= 1 - t - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= \underline{1 - t - \cos t + \sin t}$

We now look at how to use Laplace transforms to solve differential equations.

<u>Theorem</u>: If $f(t)$, $f'(t)$, and $f''(t)$ are continuous on $[0,\infty)$, then

i) $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

ii) $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

Proof : (i) $\mathcal{L}\{f'(t)\} = \int_0^\infty \overset{u}{e^{-st}}\, \overset{dv}{f'(t)}\, dt$

$$\overset{\text{(by-parts)}}{=} \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= \underline{-f(0) + s\,\mathcal{L}\{f(t)\}}, \text{ as required}$$

(ii) $\mathcal{L}\{f''(t)\} = \int_0^\infty \overset{u}{e^{-st}}\, \overset{dv}{f''(t)}\, dt$

$$= \left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\, dt$$

$$= -f'(0) + s\,\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s\left[s\,\mathcal{L}\{f(t)\} - f(0)\right]$$

$$= \underline{s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)},$$

as required

Ex: Solve $y'' - y' - 2y = 0$ subject to $y(0) = 1$,
$y'(0) = 0$.

We could solve this using techniques from Exam 2.

$$m^2 - m - 2 = 0 \longrightarrow (m-2)(m+1) = 0$$
$$\longrightarrow m = 2, -1$$
$$\longrightarrow y = c_1 e^{2t} + c_2 e^{-t}$$

Now
$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= 2c_1 - c_2 = 0 \end{aligned}\right\} \longrightarrow c_1 = 1/3,\ c_2 = 2/3$$

Hence $\boxed{y = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}}$

Now if we apply Laplace transforms to the original equation and apply linearity, we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

So by the theorem

$$s^2\mathcal{L}\{y\} - s\,y(0) - \cancel{y'(0)} - (s\mathcal{L}\{y\} - y(0)) - 2\mathcal{L}\{y\} = 0$$

$$\longrightarrow \quad s^2\mathcal{L}\{y\} - s - s\mathcal{L}\{y\} + 1 - 2\mathcal{L}\{y\} = 0$$

$$\longrightarrow \quad (s^2 - s - 2)\,\mathcal{L}\{y\} = s - 1$$

$$\longrightarrow \quad \mathcal{L}\{y\} = \frac{s-1}{(s-2)(s+1)}$$

Using partial fractions we can write this as

$$\mathcal{L}\{y\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$\longrightarrow \quad y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-}$$