If we adjust ⊛ to allow for annual investments, of a constant amount k, then

$$\frac{dS}{dt} = rS + k$$

$$\Rightarrow \quad \frac{dS}{dt} - rS = k$$

This is linear, with $\mu(x) = e^{\int -r\,dt} = e^{-rt}$

So $$e^{-rt}\frac{dS}{dt} - rSe^{-rt} = ke^{-rt}$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{-rt}S\right] = ke^{-rt}$$

$$\Rightarrow \quad e^{-rt}S = -\frac{k}{r}e^{-rt} + C$$

$$\Rightarrow \quad \boxed{S = -\frac{k}{r} + Ce^{rt}}$$

Now if $S(0) = S_0$, then $C = S_0 + \frac{k}{r}$

and hence $\boxed{S = -\frac{k}{r} + \left(S_0 + \frac{k}{r}\right)e^{rt}}$

Ex/ If at age 25 you deposit \$5000 in an account with $r = 0.1$, and deposit \$500 every month (so $k = 6000$) until you are 60, then

$$S(35) = \frac{-6000}{0.1} + \left(5000 + \frac{6000}{0.1}\right)e^{(0.1)(35)}$$

$$= \$2,092,504$$

Ex. We know from Lecture 1 that Newton's Law of Cooling tells us that

$$\frac{dT}{dt} = k(T - T_m)$$

lecture 6

where T is the temperature of the object and T_m is the ambient temperature. If a cake leaves an oven at 300° F and cools to 200° F after 3 minutes, how long will it take to reach 75° if room temperature is 70° F?

$$\frac{dT}{dt} = k(T-70)$$

$$\Rightarrow \int \frac{dT}{T-70} = \int k\, dt + C$$

$$\Rightarrow \ln(T-70) = kt + C$$

$$\Rightarrow \quad T - 70 = Ce^{kt}$$

$$\Rightarrow \quad \boxed{T = 70 + Ce^{kt}}$$

Now $T(0) = 300$, implying $C = 230$

$$\Rightarrow \quad \boxed{T = 70 + 230e^{kt}}$$

Also $T(3) = 200$, implying $k \cong -0.19$

$$\Rightarrow \quad \boxed{T = 70 + 230\, e^{-0.19t}}$$

Finally, we set $T = 75$ and solve for t, to get

$$\boxed{t \cong 20 \text{ minutes}}$$

We now look at two examples of non-linear modeling involving first-order ODEs.

Ex. (Logistic Growth)

Populations should grow in proportion to their size. However, they cannot keep growing indefinitely (resources will run out). So we need to modify the equation $\frac{dy}{dt} = ry$ ... (*), where y is the population and r is the constant growth rate.

Now (*) has the solution $y = y_0 e^{rt}$.

If we suppose that the environment can sustain no more than k individuals then we must modify (*) so that $\frac{dy}{dt} = 0$ as $y \to k$. So we write

$$\boxed{\frac{dy}{dt} = ry\left(1 - \frac{y}{k}\right)} \quad \text{... (**)}$$

This is called the logistic equation.

To solve (**) we separate the variables and use partial fractions

$$\frac{dy}{y\left(1-\frac{y}{K}\right)} = r\,dt$$

We set $\frac{A}{y} + \frac{B}{1-\frac{y}{K}} = \frac{1}{y\left(1-\frac{y}{K}\right)}$ --- (***)

$$\Longrightarrow A\left(1-\frac{y}{K}\right) + By = 1$$

Letting $y=0$ we get $A=1$

Letting $y=K$ we get $B=1/K$

Substituting into (***) we get

$$\int \frac{1}{y}\,dy + \int \frac{1/K}{1-y/K} = \int r\,dt + C$$

$$\ln y - \ln\left(1-\frac{y}{K}\right) = rt + C$$

$$\ln \frac{y}{1-y/k} = rt + c$$

$$\frac{y}{1-y/k} = Ce^{rt}$$

$$\Rightarrow \frac{1-\frac{y}{k}}{y} = Ce^{-rt}$$

$$\Rightarrow \frac{1}{y} - \frac{1}{k} = Ce^{-rt}$$

$$\Rightarrow \frac{1}{y} = Ce^{-rt} + \frac{1}{k}$$

$$\Rightarrow \boxed{y = \frac{1}{Ce^{-rt} + \frac{1}{k}}}$$

Now $y(0) = y_0$, implying

$$y_0 = \frac{1}{C + \frac{1}{k}}$$

$$\Rightarrow \frac{1}{y_0} = C + \frac{1}{k}$$

$$\Rightarrow \quad C = \frac{1}{y_0} - \frac{1}{k}$$

Thus $$y = \frac{1}{\left(\frac{1}{y_0} - \frac{1}{k}\right)e^{-rt} + \frac{1}{k}}$$

Multiplying by $y_0 k$ we get $$\boxed{y = \frac{y_0 k}{y_0 + (k - y_0)e^{-rt}}}$$ See pg. 83

This is called the logistic function.

Ex. Suppose there are 100 people to begin with on a small island, and that $k = 1000$. Suppose that in 20 years there is a 50% increase in the population. This implies $r = 1 - \sqrt[20]{1.5} \approx 0.0205$.

So $y = \dfrac{100 \cdot 1000}{100 + (900)e^{-0.0205t}}$

Thus after 100 years, $y = \dfrac{100000}{100 + 900e^{-2.05}}$

$y \sim 463$

Note: After 300 years $y \sim 981$

Ex. (Escape velocity)

We know from physics that the gravitational force acting on a body is inversely proportional to the square of the distance from the center of the earth. So $F = \dfrac{-k}{(R+x)^2} \quad \ldots (*)$

where k is a constant, x is the distance of the body above the earth's surface, and R is the radius of the earth. We also know that $F(0) = -Mg$, where g is the gravitational constant. Hence $k = MgR^2$. Now, since $F = Ma$, we can write ⊛ as

$$\cancel{M}\frac{dv}{dt} = \frac{-\cancel{M}gR^2}{(R+x)^2} \quad \text{where } v(0) = v_0.$$

Now using the Chain Rule we can write

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$
$$= \frac{dv}{dx} \cdot v$$

Thus our equation becomes

$$v\frac{dv}{dx} = \frac{-gR^2}{(R+x)^2}$$

This is separable, so we can write

$$\int v \, dv = \int \frac{-gR^2}{(R+x)^2} \, dx + c$$

$$\Longrightarrow \frac{v^2}{2} = \frac{gR^2}{R+x} + c \quad \ldots \; ** $$

When $x=0$, $v=v_0$, so it follows that

$$c = \frac{v_0^2}{2} - gR$$

Hence we can write ** as

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + \frac{v_0^2}{2} - gR$$

$$\Longrightarrow \boxed{v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}}$$

where we take $v>0$ if the object is rising and $v<0$ if it is falling back to earth.

To determine the maximum altitude we set $v = 0$ and solve for x, to get

$$x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$$

Now if we solve this equation for v_0 we get

$$v_0 = \sqrt{\frac{2gR\, x_{max}}{R + x_{max}}}$$

Finally, to find the escape velocity we let $x_{max} \to \infty$, to get

$$\boxed{v_{escape} = \sqrt{2gR}}$$

<u>Note</u>: In reality $v_{escape} \approx 6.9$ miles/second

<u>Note</u>: On the moon $v_{escape} \approx 1.5$ miles/second