Homogeneous Equations

Lecture 3

Definition: A function $f(x,y)$ is said to be **homogeneous** if $f(tx,ty) = t^n f(x,y)$ for some value of n, which we call the **degree**.

Ex If $f(x,y) = x^2 - 3xy + 5y^2$

Then
$$f(tx,ty) = (tx)^2 - 3(tx)(ty) + 5(ty)^2$$
$$= t^2x^2 - 3t^2xy + 5t^2y^2$$
$$= t^2(x^2 - 3xy + 5y^2)$$
$$= t^2 f(x,y)$$

So f is homogeneous of degree 2.

Note: It is no coincidence that each of the terms of f has degree 2.

Ex. $f(x,y) = x^2y + 2xy^2 + \frac{y^4}{x}$

is homogeneous of degree 3.

When f is not a polynomial we must use the definition.

Ex. $f(x,y) = \sin \frac{x}{x+y}$

So $f(tx, ty) = \sin \frac{tx}{tx+ty}$

$= \sin \frac{x}{x+y}$

So f is homogeneous of degree 0

Ex: $f(x,y) = \dfrac{\ln x^3}{\ln y^3}$

So $f(tx,ty) = \dfrac{\ln (tx)^3}{\ln (ty)^3}$

$= \dfrac{3\ln(tx)}{3\ln(ty)}$

$= \dfrac{\ln(tx)}{\ln(ty)}$

Thus f is not homogeneous.

Definition: An ODE is said to be homogeneous if it can be written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

where M and N are homogeneous of the same degree.

To solve an equation of this type we reduce it to a separable equation by making the substitution

$$y = ux \longrightarrow dy = u\,dx + x\,du$$

or

$$x = vy \longrightarrow dx = v\,dy + y\,dv$$

depending on whether N or M is simpler, respectively.

Ex. Solve $(x^2+y^2)\,dx + (x^2-xy)\,dy = 0$.

$$\underset{M}{\underset{\parallel}{(x^2+y^2)}}\,dx + \underset{N}{\underset{\parallel}{(x^2-xy)}}\,dy = 0$$

M and N are both HOD 2, and since neither is simpler, so let $y = ux \longrightarrow dy = u\,dx + x\,du$

Rewriting the ODE we get

$$(x^2+u^2x^2)\,dx + (x^2-ux^2)(u\,dx + x\,du) = 0$$

$$x^2\,dx + \cancel{u^2x^2\,dx} + x^2u\,dx + x^3\,du - \cancel{u^2x^2\,dx} - ux^3\,du = 0$$

$$(x^2+x^2u)\,dx = (ux^3-x^3)\,du$$

$$x^2(1+u)\,dx = x^3(u-1)\,du$$

$$\int \frac{dx}{x} = \int \frac{u-1}{1+u}\,du + C$$

$$\ln|x| = \int \left(1 - \frac{2}{1+u}\right) du + C$$

$$\ln|x| = u - 2\ln|1+u| + C$$

$$\begin{array}{r|l} & 1 \\ \hline 1+u & u-1 \\ & u+1 \\ \hline & -2 \end{array}$$

$$\ln|x| = \frac{y}{x} - 2\ln\left|1+\frac{y}{x}\right| + c$$

$$\ln|x| + \ln\left|\frac{x+y}{x}\right|^2 = \frac{y}{x} + c$$

$$x\left(\frac{x+y}{x}\right)^2 = c\,e^{\frac{y}{x}}$$

$$\boxed{(x+y)^2 = c\,x\,e^{\frac{y}{x}}}$$

Ex Solve $-y\,dx + (x+\sqrt{xy})\,dy = 0$

This is HOD 1.

M is simpler, so we let $x = vy \longrightarrow dx = v\,dy + y\,dv$

So rewriting we get

$$-y(v\,dy + y\,dv) + (vy + \sqrt{vy^2})\,dy = 0$$

$$-\cancel{yv\,dy} - y^2\,dv + \cancel{vy\,dy} + |y|\sqrt{v}\,dy = 0$$

$$|y|\sqrt{v}\,dy = y^2\,dv$$

$$\int \frac{dy}{|y|} = \int \frac{dv}{\sqrt{v}} + c$$

$$\ln|y| = 2\sqrt{v} + c$$

$$\ln|y| = 2\sqrt{\frac{x}{y}} + c$$

$$\sqrt{y}\,\ln|y| = 2\sqrt{x} + c\sqrt{y}$$

$$\sqrt{y}\,(\ln|y| - c) = 2\sqrt{x}$$

$$\boxed{y\,(\ln|y| - c)^2 = 4x}$$

<u>Exact Equations</u>

If we have a differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ which is <u>not</u> homogeneous, the next thing to check is whether it is exact.

Definition: The above equation is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Ex. $(5y - 2x)\,dx + (5x - 3y^2)\,dy = 0$

This is not homogeneous; however, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 5$.

To solve an exact equation we use the following algorithm (Note: You saw something similar in Calculus III).

i) Set $\frac{\partial f}{\partial x} = M$ ... ①

and $\frac{\partial f}{\partial y} = N$ ... ②

ii) Integrate ① with respect to x.

iii) Differentiate the result with respect to y and compare with ② to find the value of $g(y)$ by integrating.

iv) The solution is $f(x,y) = c$.

In the example above, we let

$$\frac{\partial f}{\partial x} = 5y - 2x \longrightarrow f(x,y) = 5xy - x^2 + g(y)$$

$$\longrightarrow f_y(x,y) = 5x + g'(y)$$

and $\frac{\partial f}{\partial y} = 5x - 3y^2$

$$\longrightarrow g'(y) = -3y^2$$

$$\longrightarrow g(y) = -y^3 + c$$

So the solution is

$$\boxed{5xy - x^2 - y^3 = c}$$

Ex. Solve $\left(1 + \ln x + \frac{y}{x}\right) dx = (1 - \ln x)\, dy$

So we write $\left(1 + \ln x + \frac{y}{x}\right) dx + (\ln x - 1)\, dy = 0$

This is exact since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{1}{x}$

So we let $\frac{\partial f}{\partial x} = 1 + \ln x + \frac{y}{x} \longrightarrow f(x,y) = x + \underbrace{x\ln x - x}_{\text{by parts}} + y\ln|x| + g(y)$

and $\frac{\partial f}{\partial y} = \ln x - 1$

$$\rightarrow f_y(x,y) = \ln|x| + g'(y)$$
$$\rightarrow g'(y) = -1$$
$$\rightarrow g(y) = -y + c$$

So the solution is $\boxed{x\ln x + y\ln|x| - y = c}$

Ex. Solve $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0$
subject to $y(0) = 1$.

So let $\frac{\partial f}{\partial x} = e^x + y \longrightarrow f(x,y) = e^x + xy + g(y)$

and $\frac{\partial f}{\partial y} = 2 + x + ye^y$

$$\rightarrow f_y(x,y) = x + g'(y)$$
$$\rightarrow g'(y) = 2 + ye^y$$
$$\rightarrow g(y) = 2y + \underbrace{ye^y - e^y}_{\text{by parts}} + c$$

So the solution is $\boxed{e^x + xy + 2y + ye^y - e^y = c}$

To find c, we let $y=1$ and $x=0$ to get

$$1+0+2+\cancel{e}-\cancel{e}=c$$

$$c=3$$

Thus $\boxed{e^x+xy+2y+ye^y-e^y=3}$

Sometimes we can convert an equation which is <u>not</u> exact to an exact equation by multiplying each term by an <u>integrating factor</u> $\mu(x,y)$.

<u>Ex.</u> Solve $(x+y)\,dx + x\ln x\,dy = 0$.

This is <u>not</u> exact since $\frac{\partial M}{\partial y}=1$ and $\frac{\partial N}{\partial x}=\underbrace{1+\ln x}_{\text{product rule}}$

However, if we multiply through by $\mu(x,y)=\frac{1}{x}$ (and assume $x,y>0$) then we get

$$\left(1+\frac{y}{x}\right)dx + \ln x \, dy = 0$$

This is now exact since $\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$

So we let

$$\frac{\partial f}{\partial x} = 1 + \frac{y}{x} \longrightarrow f(x,y) = x + y\ln x + g(y)$$

$$\longrightarrow f_y(x,y) = \ln x + g'(y)$$

and $\frac{\partial f}{\partial y} = \ln x$

$$\longrightarrow g'(y) = 0$$

$$\longrightarrow g(y) = c$$

Hence the solution is $\boxed{x + y\ln x = c}$