Solutions to the Problems in Lab 2

Problem 1. Prove or disprove the following equivalences. To prove, use a series of logical equivalences and the fundamental equivalences (laws/rules/etc.) that we have seen in the course. To disprove, give truth assignments to the variables that make one side true and the other side false.

Note: You can use a truth table to check and/or verify your finding but showing only the truth table is not a correct solution when the equivalence holds.

- $(p \to q) \to r \equiv p \to (q \to r)$
- $(p \to q) \to r \equiv (\neg p \to r) \land (q \to r)$

The first equivalence is not correct. When p and r are both false, the left side will evaluate to false, but the right side evaluates to true. [We found one counter example, and we are done.]

For the second, maybe after trying to make the two sides different with a few example values, we can begin to suspect that the equivalence holds. We can try starting with one of the expressions and reaching the other one. At each step, we are using a fundamental equivalence we know (see course notes and/or textbook).

$$\begin{aligned}
(p \to q) \to r &\equiv \neg(p \to q) \lor r && \text{We used the equivalence } a \to b \equiv \neg a \lor b. \\
&\equiv \neg(\neg p \lor q) \lor r && \text{Applied the same equivalence again.} \\
&\equiv (\neg\neg p \land \neg q) \lor r && \text{De Morgan's law} \\
&\equiv (p \land \neg q) \lor r && \text{Double negation law} \\
&\equiv (p \lor r) \land (\neg q \lor r) && \text{Distributive law} \\
&\equiv (p \lor r) \land (q \to r) && \text{Used } a \to b \equiv \neg a \lor b \text{ again, right to left} \\
&\equiv (\neg\neg p \lor r) \land (q \to r) && \text{Double negation} \\
&\equiv (\neg p \to r) \land (q \to r) && \text{Used } a \to b \equiv \neg a \lor b \text{ again, right to left}
\end{aligned}$$

Since we were able to reach from one expression to the other one with solid steps, we showed that they are equivalent.

We can double check with truth tables

p	q	r	$p \to q$	$(p \to q) \to r$		$q \to r$	$p \to (q \to r)$
T	T	T	T	T		T	T
T	T	F	T	F		F	F
T	F	T	F	T		T	T
T	F	F	F	T		T	T
F	T	T	T	T		T	T
F	T	F	T	F		F	T
F	F	T	T	T		T	T
F	F	F	T	F		T	T

p	q	r	$\neg p$	$\neg p \to r$	$q \to r$	$(\neg p \to r) \land (q \to r)$
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Problem 2. Let $P(m, n)$ be "n is greater than or equal to m" where the domain (universe of discourse) is the set of nonnegative integers. What are the truth values of $\exists n \forall m\, P(m, n)$ and $\forall m \exists n\, P(m, n)$?

Note that the universe is nonnegative integers, so $m, n \in \{0,1,2, \ldots\}$.

Let's consider the first one, $\exists n \forall m\, P(m, n)$.
This says that there is at least one nonnegative integer that is greater than or equal to all the nonnegative integers!
But we know that since nonnegative integers keep increasing infinitely, no such "largest" nonnegative integer exists.
In other words, the opposite of this statement is true which can be expressed as $\forall n \exists m (n < m)$. In English, this says for all nonnegative integers, there is at least one nonnegative integer that is greater.
Therefore, $\exists n \forall m\, P(m, n)$ is false.

Now, the second one: $\forall m \exists n\, P(m, n)$. This says something like the first one's opposite that we mentioned above with mainly just a small difference.
We can read this one as "for all nonnegative integers, there is at least one nonnegative integer that is greater or equal". Since every nonnegative integer has another nonnegative integer that is greater or equal, $\forall m \exists n\, P(m, n)$ is true.

Problem 3. A stamp collector wants to include in her collection exactly one stamp from each country of Africa. If $P(s)$ means that they have a stamp s in their collection, $Q(s, c)$ means that stamp s was issued by country c, the domain for s is all stamps, and the domain for c is all countries of Africa, express the statement that the collection satisfies the requirement.

This is a bit tricky as it is because it says "exactly one stamp". Let's relax that constraint first and try to write for "at least one stamp from each country. This shouldn't be too difficult.

$$\forall c \exists s \big(P(s) \land Q(s, c)\big)$$

To ensure that there is exactly one, we must make sure that the stamp in the collection is unique.
We can do this by adding another quantifier over all stamps (like another loop that iterates over stamps) and make sure that no two stamps (s1 and s2) from the same country are in the collection unless they are the same stamp (s1=s2). Here is how it looks

$$\forall c \exists s_1 \forall s_2 \Big(\big(P(s_2) \land Q(s_2, c)\big) \leftrightarrow (s_2 = s_1)\Big)$$