COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

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Basic Structures: Sets and Functions

Part-2

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Outline

Sets

- The Language of Sets.
- Set Operations.
- Set Identities.

Functions

- Types of Functions.
- Operations on Functions.
- Computability.

*Mc Graw Hill

Section Summary 1

Definition of sets.

Describing Sets.

- Roster Method.
- Set-Builder Notation.

Some Important Sets in Mathematics.

Empty Set and Universal Set.

Subsets and Set Equality.

Cardinality of Sets.

Tuples.

Cartesian Product.

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x\left(x \in A \rightarrow x \in B\right)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \not\subseteq B$, find an element $x \in A$ with $x \notin B$.

(Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x \left(x \in A \leftrightarrow x \in B \right)$$

Using logical equivalences, we have that $A = B$ iff

$$\forall x \left[\left(x \in A \rightarrow x \in B \right) \wedge \left(x \in B \rightarrow x \in A \right) \right]$$

This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

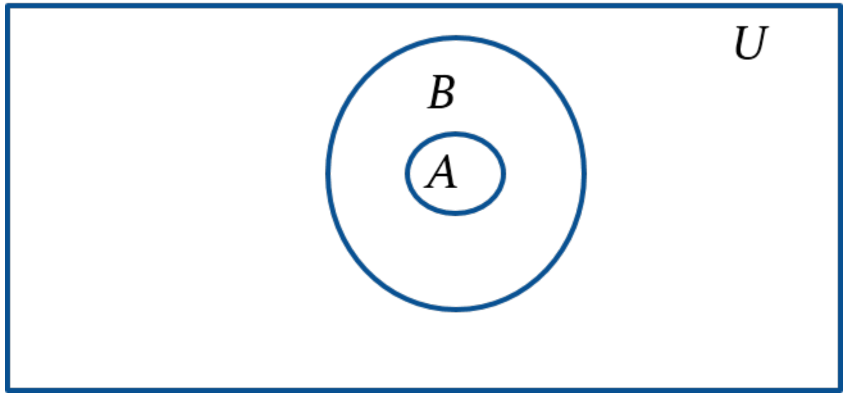
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

If $A \subset B$, then

$$\forall x\left(x \in A \rightarrow x \in B\right) \wedge \exists x\left(x \in B \wedge x \notin A\right)$$ is true.

Venn Diagram



Set Cardinality

Definition: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise, it is *infinite*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted *P*(*A*), is called the *power set* of *A*.

Example: If $A = \{a,b\}$ then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n.

Example:

Set A = {1,2,3}

Subsets of set A = {}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}

Power set P(A) = { {}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }

Tuples

Definition: The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

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PYTHON EXAMPLE:
mytuple = ("apple", "banana", "cherry", "apple", "cherry")
print(mytuple)          # OUTPUT:

('apple', 'banana', 'cherry', 'apple', 'cherry')
```

Cartesian Product[1]

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by *A* × *B* is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example:

$$A = \{a,b\} \quad B = \{1,2,3\}$$

$$A \times B = \{(a,1),(a,2),(a,3),\ (b,1),(b,2),(b,3)\}$$

Definition: A subset *R* of the Cartesian product *A* × *B* is called a *relation* from the set A to the set B.

Example

Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution:

(a, b), where a is a student at the university and b is a course offered at the university.

All possible enrollments of students in courses at the university.

Cartesian Product$_2$

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}, B = \{1,2\}$ and $C = \{0,1,2\}$

Solution:

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: What is the truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" ?

$$\{-1,1\}$$

Section

Set Operations

Section Summary[2]

Set Operations.

- Union.
- Intersection.
- Complementation.
- Difference.

More on Set Cardinality.

Set Identities.

Proving Identities.

Membership Tables.

Union

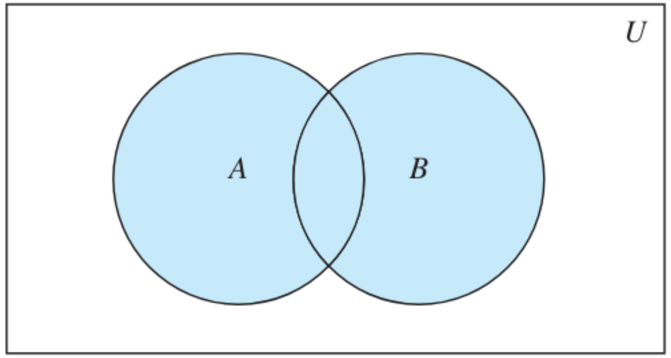
Definition: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x \mid x \in A \wedge x \in B\}.$$

Note if the intersection is empty, then A and B are said to be *disjoint*.

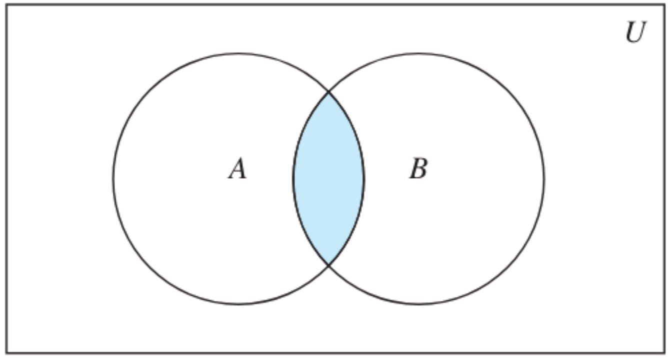
Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

Solution: $\{3\}$

Example: What is? $\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

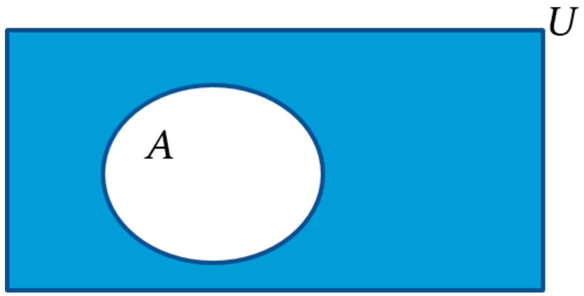
$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement



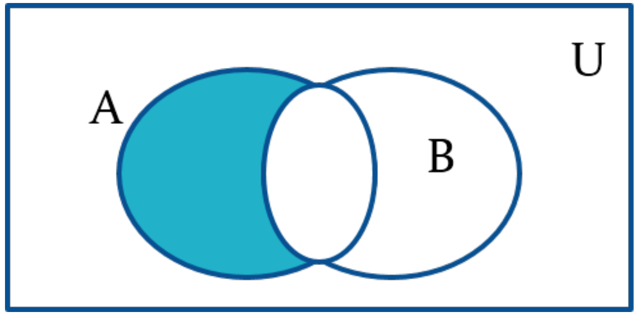
Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, or A\B
is the set containing the elements of A that are not in B. The difference of A and B is also called:

the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors:

Add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Example: A= {1, 3, 5} and B = {1, 2, 3}. $|A \cup B| = ?$

Solution:

$|A \cup B| = |A| + |B| - |A \cap B| = 3 + 3 - 2 = 4$

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\overline{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\overline{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set $(A - B) \cup (B - A)$

Example:

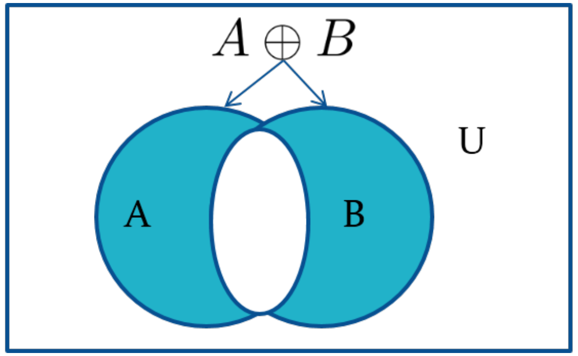
$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

Solution: $\{1,2,3,6,7,8\}$

Venn Diagram



Set Identities[1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities $_2$

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities[3]

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. *Membership Tables*: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of De Morgan Law [1]

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

Continued on the next slides…

Proof of De Morgan Law₂

The following steps are to show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg\left(\left(x \in A\right) \wedge \left(x \in B\right)\right)$	by defn. of intersection
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Continued on the next slides…

Proof of De Morgan Law[3]

The following steps are to show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \quad \left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: De Morgan Law

$\overline{A \cap B}$	$= x \in \overline{A \cap B}$	by defn. of complement
	$= \{x \mid \neg(x \in (A \cap B))\}$	by defn. of does not belong symbol
	$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by defn. of intersection
	$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by 1st De Morgan law for Prop Logic
	$= \{x \mid x \notin A \vee x \notin B\}$	by defn. of not belong symbol
	$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by defn. of complement
	$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by defn. of union
	$= \overline{A} \cup \overline{B}$	by meaning of notation

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	*B*	*C*	*B∩C*	*A∪ (B∩C)*	*A∪B*	*A∪C*	*(A∪B) ∩ (A∪C)*
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Example

Let A, B, and C be sets.

Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Solution: We have

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}
$$

Generalized Union and Intersection Operators

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

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