Exam 1 Review (Assume throughout that $x, y > 0$)

1. $\dfrac{dy}{dx} = \dfrac{1}{y-x} \longrightarrow \dfrac{dx}{dy} = y - x$

$[\Rightarrow (y-x)\,dy = dx] \longrightarrow \dfrac{dx}{dy} + x = y$ This is linear in x

2. $\dfrac{dy}{dx} = \dfrac{x-y}{x} = 1 - \dfrac{y}{x} \longrightarrow \dfrac{dy}{dx} + \dfrac{1}{x}y = 1$

This is linear, exact, and homogeneous.

$\longrightarrow \mu(x) = e^{\int \frac{1}{x}dx} = x$

$\longrightarrow x\dfrac{dy}{dx} + y = x$

$\longrightarrow \dfrac{d}{dx}[xy] = x$

$\longrightarrow xy = \dfrac{x^2}{2} + C$

$\longrightarrow \boxed{y = \dfrac{x}{2} + \dfrac{C}{x}}$

3. $\dfrac{dy}{dx} = \dfrac{1}{x(x-y)} \longrightarrow \dfrac{dx}{dy} = x^2 - xy$

This is Bernoulli in x.
Impossible to solve using
the regular technique.

4. $\dfrac{dy}{dx} = \dfrac{y^2+y}{x^2+x}$

This is separable

$$\int \frac{dy}{y^2+y} = \int \frac{dx}{x^2+x} + C$$

$$\int \frac{dy}{y(y+1)} = \int \frac{dx}{x(x+1)} + C$$

We use partial fractions to integrate both sides.

5. $\dfrac{dy}{dx} = 4+5y+y^2$

This is separable

$$\int \frac{dy}{4+5y+y^2} = \int 1\,dx + C$$

$$\int \frac{dy}{(y+4)(y+1)} = \int 1\,dx + C \quad \text{---} \circledast$$

Now $\dfrac{A}{y+4} + \dfrac{B}{y+1} = \dfrac{1}{(y+4)(y+1)}$

$\longrightarrow A(y+1)+B(y+4) = 1$

Let $y=-4$ to get $-3A=1 \rightarrow A=-1/3$

Let $y=-1$ to get $3B=1 \rightarrow B=1/3$

So $\circledast$ becomes

$$\int \frac{-1/3}{y+4}\,dy + \int \frac{1/3\,dy}{y+1} = \int 1\,dx + C$$

$$\longrightarrow -\frac{1}{3}\ln(y+4) + \frac{1}{3}\ln(y+1) = x + C$$

$$\longrightarrow \quad -\ln(y+4) + \ln(y+1) = 3x + C$$

$$\longrightarrow \quad \ln \frac{y+1}{y+4} = 3x + C$$

$$\longrightarrow \quad \frac{y+1}{y+4} = Ce^{3x}$$

$$\longrightarrow \quad y+1 = yCe^{3x} + 4Ce^{3x}$$

$$\longrightarrow \quad y(1-Ce^{3x}) = 4Ce^{3x} - 1$$

$$\longrightarrow \quad \boxed{y = \frac{4Ce^{3x} - 1}{1 - Ce^{3x}}}$$

6. $y\,dx = (y - xy^2)\,dy$

$$\cancel{\frac{dy}{dx} = \frac{y}{y - xy^2}}$$

$$\frac{dx}{dy} = \frac{y - xy^2}{y} \longrightarrow \frac{dx}{dy} = 1 - xy$$

$$\longrightarrow \quad \frac{dx}{dy} + xy = 1$$

This is linear in x.

7: $x\dfrac{dy}{dx} = ye^{x/y} - x$

$\rightarrow x\,dy = (ye^{x/y} - x)\,dx$

$\rightarrow (ye^{x/y} - x)\,dx - x\,dy = 0$

this is homogeneous of degree 1.

So we would let $y = ux$, but the calculations are challenging.

8. $xyy' + y^2 = 2x$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{2}{y}$$

this is a Bernoulli equation with $n = -1$.

So we let $w = y^{1-(-1)} = y^2 \quad [\Rightarrow y = \sqrt{w}]$

$$\frac{dw}{dx} = 2y\frac{dy}{dx} \quad \left[\Rightarrow \frac{dy}{dx} = \frac{dw}{dx}\cdot\frac{1}{2y}\right]$$

Rewriting we get

$$\frac{dw}{dx} \cdot \frac{1}{2y} + \frac{\sqrt{w}}{x} = \frac{2}{\sqrt{w}}$$

$$\longrightarrow \frac{dw}{dx} \cdot \frac{1}{2\sqrt{w}} + \frac{\sqrt{w}}{x} = \frac{2}{\sqrt{w}}$$

$$\longrightarrow \frac{dw}{dx} + \frac{2w}{x} = 4$$

This is now linear, so we calculate

$$\mu(x) = e^{\int \frac{2}{x} dx} = x^2$$

Thus $x^2 \frac{dw}{dx} + 2wx = 4x^2$

$$\longrightarrow \frac{d}{dx}\left[x^2 w\right] = 4x^2$$

$$\longrightarrow x^2 w = \frac{4x^3}{3} + C$$

$$\longrightarrow \boxed{y^2 = \frac{4x}{3} + \frac{C}{x^2}}$$

9. $2xyy' + y^2 = 2x^2$

This is homogeneous of degree 2.

It is also Bernoulli. $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y}$

It is also exact.

10. $y\,dx + x\,dy = 0$

This is separable, homogeneous, exact, linear in y, and linear in x.

11. $\left(x^2 + \frac{2y}{x}\right)dx = (3 - \ln x^2)\,dy$

This is exact (it is also linear in y) if we write

$$\left(x^2 + \frac{2y}{x}\right)dx + (\ln x^2 - 3)\,dy = 0$$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ since both are $\frac{2}{x}$.

We let $\frac{\partial f}{\partial x} = x^2 + \frac{2y}{x}$

$\longrightarrow f(x,y) = \frac{x^3}{3} + 2y \ln x + g(y)$

$\longrightarrow \frac{\partial f}{\partial y} = 2 \ln x + g'(y)$

$\longrightarrow g'(y) = -3$

$\longrightarrow g(y) = -3y + c$

$\longrightarrow \boxed{\frac{x^3}{3} + 2y \ln x - 3y = c}$

12. $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1$

This is homogeneous of degree 0.

$\longrightarrow \frac{dy}{dx} = \frac{x^2 + y^2 + xy}{yx}$

$\longrightarrow (x^2 + y^2 + xy)\,dx - xy\,dy = 0$

Let $y = ux \longrightarrow dy = u\,dx + x\,du$

So rewriting we get

$$(x^2 + u^2x^2 + ux^2)\,dx - ux^2(u\,dx + x\,du) = 0$$

$$x^2\,dx + \cancel{u^2x^2\,dx} + ux^2\,dx - \cancel{u^2x^2\,dx} - ux^3\,du = 0$$

$$x^2(1+u)\,dx = ux^3\,du$$

$$\frac{dx}{x} = \frac{u\,du}{1+u}$$

$$\ln x = \int\left(1 - \frac{1}{1+u}\right)du \qquad u+1\,\overline{)\,u\,}\ \ 1,\quad \frac{u+1}{-1}$$

$$\ln x = u - \ln(1+u) + c$$

$$\ln x = \frac{y}{x} - \ln\left(1+\frac{y}{x}\right) + c$$

$$\boxed{x = c\,e^{\frac{y}{x} - \ln\left(1+\frac{y}{x}\right)}}$$