Laplace Transforms

(lecture 19)

Many practical problems involve a system acted on by a discontinuous or impulsive force. Laplace transforms are well suited to solve such problems, though they can also be applied to the types of equation we have already considered.

Definition: Let $f(t)$ be a function defined for $t \geq 0$, and suppose $f(t)$ is piecewise continuous (i.e. has a finite number of discontinuities) and bounded (does not become infinite). Then the Laplace transform is given by

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

Note: $\mathcal{L}\{f(t)\}$ is a function of s.

Ex $\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \lim_{A\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^A$$

$$= \lim_{A\to\infty}\frac{e^{-sA}}{-s} + \frac{1}{s}$$

$$= \boxed{\frac{1}{s}}\ ,\ s > 0$$

Ex $\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{-st}(e^{2t})\,dt$

$$= \int_0^\infty e^{t(2-s)}\,dt$$

$$= \left[\frac{e^{t(2-s)}}{2-s}\right]_0^\infty$$

$$= \lim_{A\to\infty}\left[\frac{e^{A(2-s)}}{2-s}\right] - \frac{1}{2-s}$$

$$= 0 - \frac{1}{2-s}\ ,\ s > 2 \qquad = \boxed{\frac{1}{s-2}}\ ,\ s > 2$$

Ex. Evaluate $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ e^{2t}, & t \ge 1 \end{cases}$

Here we split up the interval from 0 to ∞ as intervals from 0 to 1 and 1 to ∞ when using the definition.

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(1)\,dt + \int_1^\infty e^{-st}(e^{2t})\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{t(2-s)}}{2-s}\right]_1^\infty$$

$$= \left[\frac{e^{-s}}{-s} + \frac{1}{s}\right] + \lim_{A\to\infty}\left[\frac{e^{t(2-s)}}{2-s}\right]_1^A$$

$$= \boxed{\frac{-e^{-s}}{s} + \frac{1}{s} + \frac{e^{2-s}}{s-2}}, \quad s > 2$$

Note: It didn't occur in the previous example, but if the pieces "connect" at the endpoints, then we get cancellation of terms.

Ex: Evaluate $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$

Since $f(t) = k$ only at $t = 1$ and the function is zero thereafter, the definition gives us

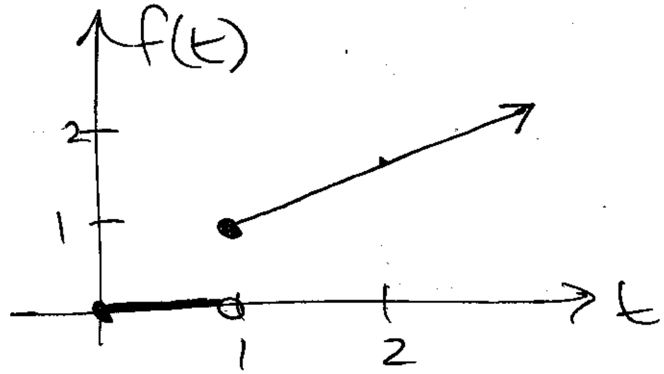
$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(1)\,dt + \int_1^1 e^{-st}(k)\,dt + \int_1^\infty e^{-st}(0)\,dt$$

$$= \int_0^1 e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1$$

$$= \boxed{\frac{1-e^{-s}}{s}}, \quad s > 0$$

Ex. Find $\mathcal{L}\{f(t)\}$ for the function below.



So $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & t \geq 1 \end{cases}$

Here $\mathcal{L}\{f(t)\} = 0 + \int_1^\infty e^{-st}(t)\,dt$

Here we use the by-parts method:

$u = t \qquad dv = e^{-st}\,dt$

$du = 1\,dt \qquad v = \dfrac{e^{-st}}{-s}$

So $\displaystyle\int_1^\infty e^{-st}(t)\,dt = \left[\frac{-te^{-st}}{s}\right]_1^\infty + \int_1^\infty \frac{e^{-st}}{s}\,dt$

$\displaystyle = \lim_{A\to\infty}\left[\frac{-te^{-st}}{s}\right]_1^A - \left[\frac{e^{-st}}{s^2}\right]_1^\infty$

$$= 0 + \frac{e^{-s}}{s} - \lim_{A \to \infty} \left[\frac{e^{-st}}{s^2} \right]_1^A$$

$$= \boxed{\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}} \quad , \quad s > 0$$

$$\boxed{\frac{e^{-s}}{s^2}(s+1)}$$

Ex Evaluate $\mathcal{L}\{f(t)\}$ if $f(t) = \sin 2t \; , \; t \geq 0$

Using the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} (\sin 2t) \, dt$$

Using the by-parts method:

$$u = e^{-st} \qquad dv = \sin 2t \, dt$$

$$du = -s e^{-st} \qquad v = -\tfrac{1}{2} \cos 2t$$

Hence $$\mathcal{L}\{f(t)\} = \left[\frac{-e^{-st} \cos 2t}{2} \right]_0^\infty - \int_0^\infty \frac{s}{2} e^{-st} \cos 2t \, dt$$

$$= \lim_{A\to\infty} \left[\frac{-e^{-st}\cos 2t}{2}\right]_0^A - \frac{s}{2}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$= \frac{1}{2} - \frac{s}{2}\int_0^\infty e^{-st}\cos 2t\,dt \quad \ldots \circledast$$

Using the by-parts method again:

$u = e^{-st}$ $\qquad dv = \cos 2t\,dt$

$du = -se^{-st}$ $\qquad v = \frac{1}{2}\sin 2t$

So we get

$$\int_0^\infty e^{-st}\cos 2t\,dt = \left[\frac{e^{-st}\sin 2t}{2}\right]_0^\infty + \int_0^\infty \frac{s}{2}e^{-st}\sin 2t\,dt$$

$$= \lim_{A\to\infty}\left[\frac{e^{-st}\sin 2t}{2}\right]_0^A + \frac{s}{2}\int_0^\infty e^{-st}\sin 2t$$

$$= (0-0) + \frac{s}{2}\int_0^\infty e^{-st}\sin 2t$$

Substituting into ⊛ we get

$$\int_0^\infty e^{-st}\sin 2t\,dt = \frac{1}{2} - \frac{s}{2}\left(\frac{s}{2}\int_0^\infty e^{-st}\sin 2t\,dt\right)$$

$$\longrightarrow \left(1+\frac{s^2}{4}\right)\int_0^\infty e^{-st}\sin 2t\,dt = \frac{1}{2}$$

$$\longrightarrow \int_0^\infty e^{-st}\sin 2t\,dt = \frac{\frac{2}{\cancel{4}}}{\frac{4+s^2}{\cancel{4}}} = \boxed{\frac{2}{4+s^2}}$$

Thankfully it is not necessary (after Friday) to do these calculations often, as we have the following table to help us.

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$

Another useful rule is the following:

<u>Theorem</u>: $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$

<u>Ex.</u> If $f(t) = 5e^{-2t} - 3\sin 4t \,, \; t \geq 0$

then $\mathcal{L}\{f(t)\} = \frac{5}{s+2} - \frac{12}{s^2+16} \,, \; s > 0$

Ex. $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$$= \frac{s^2+4-s^2}{2s(s^2+4)}$$

$$= \boxed{\frac{2}{s(s^2+4)}}$$