Lecture 9/11

1) $\frac{dy}{dx} = \frac{1}{y-x}$

$\longrightarrow \frac{dx}{dy} = y - x$

$\longrightarrow \frac{dx}{dy} + x = y$ This is linear in x

2) $\frac{dy}{dx} = \frac{x-y}{x}$

$\longrightarrow \frac{dy}{dx} = 1 - \frac{y}{x}$

$\longrightarrow \frac{dy}{dx} + \frac{y}{x} = 1$

Here $\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Thus $x\frac{dx}{dx} + y = x$

$\longrightarrow \frac{d}{dx}[xy] = x$

$\to$ So $xy = \frac{x^2}{2} + C$

$\to y = \frac{x}{2} + \frac{C}{x}$

3) $\frac{dy}{dx} = \frac{1}{x(x-y)}$

$\longrightarrow \frac{dx}{dy} = x(x-y)$

$\to \frac{dx}{dy} = x^2 - xy$

$\to \frac{dx}{dy} + xy = x^2$ This is Bernoulli (in x)

4) $\frac{dy}{dx} = \frac{y^2+y}{x^2+x}$

$\to \frac{dy}{y^2+y} = \frac{dx}{x^2+x}$ This is separable

$\to \frac{dy}{y(y+1)} = \frac{dx}{x(x+1)}$ Use partial fractions

5) $\frac{dy}{dx} = 4 + 5y + y^2$

$\to \frac{dy}{dx} = (y+4)(y+1)$

$\to \frac{dy}{(y+4)(y+1)} = dx$

$\longrightarrow \int \frac{dy}{(y+4)(y+1)} = \int dx + C$

$\longrightarrow \int \left(\frac{-1/3}{y+4} + \frac{1/3}{y+1}\right) dy = x + C$

$\longrightarrow -\frac{1}{3}\ln|y+4| + \frac{1}{3}\ln|y+1| = x + C$

$\to \frac{1}{3}\ln\left|\frac{y+1}{y+4}\right| = x + C$

$\to \ln\left|\frac{y+1}{y+4}\right| = 3x + C$

$\to \frac{y+1}{y+4} = Ce^{3x}$

6) $y\,dx = (y - xy^2)\,dy$

$\longrightarrow dx = (1 - xy)\,dy$

$\to \frac{dx}{dy} = 1 - xy$

$\to \frac{dx}{dy} + xy = 1$ This is linear in x

7) $x\frac{dx}{dx} = ye^{x/y} - x$

$\longrightarrow x\,dy = (ye^{x/y})\,dx$

$\to (ye^{x/y} - x)\,dx - x\,dy = 0$ This is homogeneous of degree 1

$\to$ Since N is simpler, we let $y = ux$

8) $xyy' + y^2 = 2x$

$\longrightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{2}{y}$ This is a Bernoulli Equation with $n=-1$

$\longrightarrow$ So we let $w = y^2$ $[\to y = w^{1/2}]$

$\to \frac{dw}{dx} = 2y\frac{dy}{dx}$ $\left[\to \frac{dy}{dx} = \frac{1}{2y}\frac{dw}{dx}\right]$

$\to$ The ODE becomes $\frac{1}{2y}\frac{dw}{dx} + \frac{w^{1/2}}{x} = 2w^{-1/2}$

$\to \frac{1}{2w^{1/2}}\frac{dw}{dx} + \frac{w^{1/2}}{x} = 2w^{-1/2}$

$(\times 2w^{1/2})$

$\longrightarrow \frac{dw}{dx} + \frac{2w}{x} = 4$ This is linear

$\to \mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\to$ Multiplying, we get $x^2\frac{dw}{dx} + 2wx = 4x^2$

$\to \frac{d}{dx}[x^2 w] = 4x^2$

$\to x^2 w = \frac{4}{3}x^3 + C$

$\to x^2y^2 = \frac{4}{3}x^3 + C$

$\to$ So $y^2 = \frac{4}{3}x + \frac{C}{x^2}$

9) $2xyy' + y^2 = 2x^2$

$\longrightarrow 2xy\frac{dy}{dx} + y^2 = 2x^2$

$\to 2xy\,dy + y^2\,dx = 2x^2\,dx$

$\to (y^2 - 2x^2)\,dx + (2xy)\,dy = 0$

$\hookrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$

$\to$ Let $\frac{\partial f}{\partial x} = y^2 - 2x^2$ ①

and $\frac{\partial f}{\partial y} = 2xy$ ②

$\to$ Integrating ①, we get

$f(x,y) = y^2x - \frac{2}{3}x^3 + g(y)$

$\to \frac{\partial f}{\partial y} = 2xy + g'(y)$

$\to g'(y) = 0$ $\to g(y) = C$

So

$y^2x - \frac{2}{3}x^3 = C$

10) $y\,dx + x\,dy = 0$

This can be done using any valid technique

11) $\left(x^2 + \frac{2y}{x}\right)dx = (3 - \ln x^2)\,dy$

$\to \left(x^2 + \frac{2y}{x}\right)dx + (\ln x^2 - 3)\,dy = 0$

$\hookrightarrow$ This is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{2}{x}$

$\to$ Let $\frac{\partial f}{\partial x} = x^2 + \frac{2y}{x}$ ①

and $\frac{\partial f}{\partial y} = 2\ln x - 3$ ②

$\to$ Integrating ①, we get