Lecture 1

What is a differential equation?

$$x^2 + 5x + 4 = 0 \leftarrow \text{Unknown is } x$$

$$y'' + 5y' + 4y = 0 \leftarrow \text{Unknown is } y = f(x)$$

Def. An equation containing the derivatives of one or more dependent variables with respect to one or more of the independent variables is said to be a differential equation.

There are two main types of differential equation, ordinary and partial.

An ordinary differential equation (ODE) contains regular derivatives (from calculus 1).

$$\frac{dy}{dx} + 5y = e^x \qquad \text{[First-order]}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \qquad \text{[Second-order]}$$

$$\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} = 0 \qquad \text{[Third-order]}$$

A partial differential equation (PDE) contains partial derivatives (from calculus 3).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0 \qquad \text{[Second-order]}$$

Another way to classify ODEs is by linearity.

A linear ODE has the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x)$$

coefficients are functions of x

derivatives not raised to a power

function of x

Ex. $y'' - 2y' + y = 0$ [Linear]

$y''' + xy' - 5y = e^x$ [Linear]

Ex. $y'' + \sin y = 0$ [Non-linear]

$y^{(4)} + y^2 = 0$ [Non-linear]

We use ODEs to model real-life situations where an object is changing in some way.

Ex Population Dynamics (Thomas Malthus, 1798)

The rate of change of a population depends on the population at the current time.

$$\underbrace{\frac{dP}{dt}}_{\substack{\text{rate at which}\\ \text{population is}\\ \text{changing}}} \propto \underbrace{P}_{\substack{\text{population}\\ \text{now}}} \qquad \text{or} \qquad \frac{dP}{dt} = \underbrace{k}_{\substack{\text{proportionality}\\ \text{constant}\\ (k>0)}} P$$

An analogous equation represents radioactive decay.

$$\underbrace{\frac{dA}{dt}}_{\text{rate of decay}} = \underbrace{k}_{(k<0)} \underbrace{A}_{\text{current amount}}$$

Ex. Newton's Law of Cooling (~1670)

This states that the rate at which an object cools is proportional to the difference between the temperature of the object and the ambient (room) temperature.

$$\frac{dT \leftarrow \text{temp.}}{dt \leftarrow \text{time}} = k\,(\underbrace{T}_{\text{temp. of object}} - T_m) \leftarrow \text{temperature of medium}$$

Ex. Disease Spread

The rate at which a disease spreads is proportional to the number of people infected and the number who have not been exposed to the disease.

$$\frac{dx}{dt} = kxy$$

rate of infection   infected people   not infected

The next thing to concern us is how to solve ODEs.
This will take the entire course to answer!
We start to do this on Wednesday.
Today the solution will be provided, and we will verify it.

Ex. Verify that $y = \frac{x^4}{16}$ is a solution to the ODE

$$\frac{dy}{dx} = xy^{1/2}.$$

Left side: $\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$

Right side: $xy^{1/2} = x \cdot \frac{x^2}{4} = \frac{x^3}{4}$, as required.

Ex. Verify that $y = xe^x$ is a solution to the ODE

$$y'' - 2y' + y = 0$$

Calculating the derivatives of y we get

$$y' = xe^x + e^x$$

$$y'' = xe^x + e^x + e^x = xe^x + 2e^x$$

So the left side of the ODE is equal to

$$\cancel{xe^x} + \cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{xe^x}$$

$$= 0 \text{, as required.}$$

Hence $y = xe^x$ is a solution.

Ex. Verify that the implicit solution $x^2+y^2=25$ solves the ODE $\frac{dy}{dx} = \frac{-x}{y}$

Differentiating the solution implicitly we get

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ we get $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

as required.

Ex. Verify that $x_1 = c_1 \cos 4t$ and $x_2 = c_2 \sin 4t$ are both solutions of the ODE $x'' + 16x = 0$.

Checking x_1 : $x_1'' = -16c_1 \cos 4t$

So $x_1'' + 16x_1 = -16c_1 \cos 4t + 16c_1 \cos 4t = 0$

as required.

Checking x_2: $x_2'' = -16c_2 \sin 4t$

So $x_2'' + 16x_2 = -16c_2 \sin 4t + 16c_2 \sin 4t = 0$

as required.

The solutions in the last example contained constants. We can often find the value of the constants when given <u>initial conditions</u>.

These are known as IVPs (initial value problems)

Ex. Find ~~the~~ two non-trivial solutions of $x'' + 16x = 0$ if we know $x\left(\frac{\pi}{2}\right) = -2$ and $x'\left(\frac{\pi}{2}\right) = 1$ respectively.

We know $x_1 = c_1 \cos 4t$ --- ① and $x_2 = c_2 \sin 4t$ --- ②

are solutions.

We are given that $x\left(\frac{\pi}{2}\right) = -2$, so substituting into ① we get

$$-2 = c_1 \cos 4\left(\frac{\pi}{2}\right)$$

$$\rightarrow \underline{-2 = c_1}$$

We are also given that $x'\left(\frac{\pi}{2}\right) = 1$, so substituting into ② we get

$$1 = 4c_2 \cos 4\left(\frac{\pi}{2}\right)$$

$$\rightarrow 1 = 4c_2$$

$$\rightarrow \underline{c_2 = 1/4}$$

Thus two non-trivial solutions are $-2\cos 4t$ and $\frac{1}{4}\sin 4t$.