COT 3100C: INTRODUCTION TO DISCRETE STRUCTURES

SUMMER 2024

Basic Structures: Sets and Functions

Part-3

Mesut Ozdag, Ph.D.
mesut.ozdag@ucf.edu

Outline

Sets

- The Language of Sets.
- Set Operations.
- Set Identities.

Functions

- Types of Functions.
- Operations on Functions.
- Computability.

*Mc Graw Hill

Section

Functions

Section Summary

Definition of a Function.

- Domain, Codomain.
- Image, Preimage.

Injection, Surjection, Bijection.

Inverse Function.

Function Composition.

Graphing Functions.

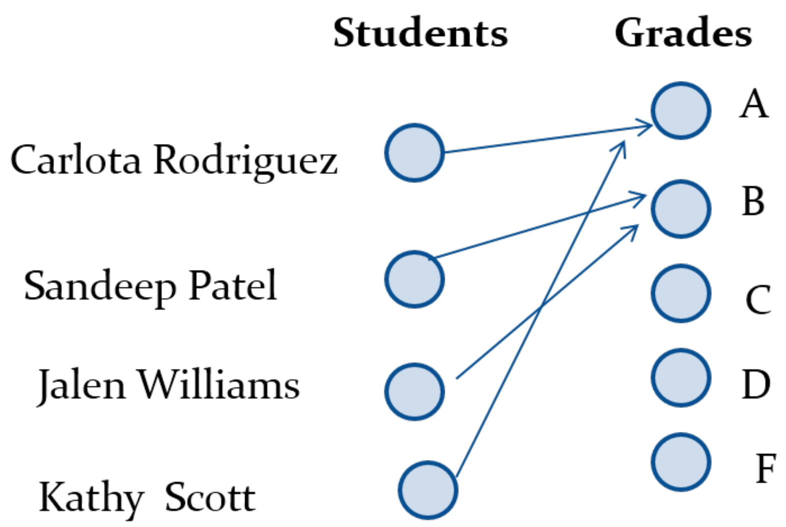
Floor, Ceiling, Factorial.

Partial Functions.

Functions [1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Assignment of Grades in a Discrete Structures Class.

Functions$_2$

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$. ***f(a)=b.***

$$\forall x \left[x \in A \rightarrow \exists y \left[y \in B \wedge (x, y) \in f \right] \right]$$

and

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \rightarrow y_1 = y_2 \right]$$

Functions[3]

Given a function f: $A \rightarrow B$:

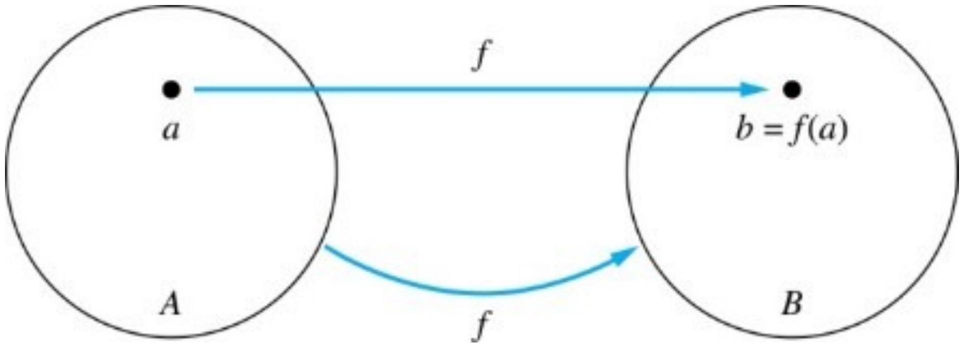
We say f *maps* A to B *or* f is a *mapping* from A to B.

A is called the ***domain*** of f.

B is called the ***codomain*** of f.

- If $f(a) = b$,
  - then b is called the ***image*** of a under f.
  - a is called the ***preimage*** of b.
- The ***range*** of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are ***equal*** when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

The Function f Maps A to B.



Questions

$f(a) = ?$ ***z***

The image of d is ? ***z***

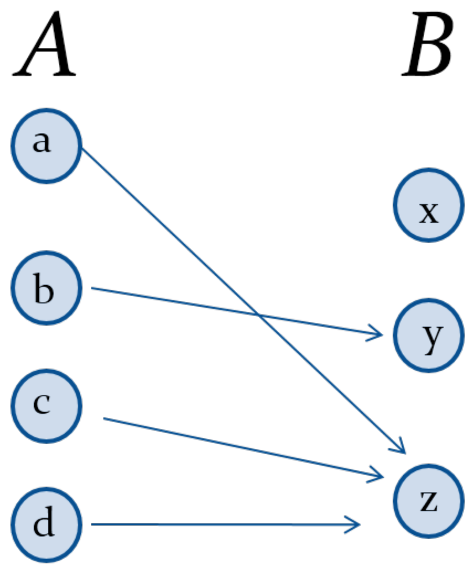
The domain of f is ? ***A***

The codomain of f is ? ***B***

The preimage of y is ? ***b***

$f(A) = ?$ $\{y,z\}$

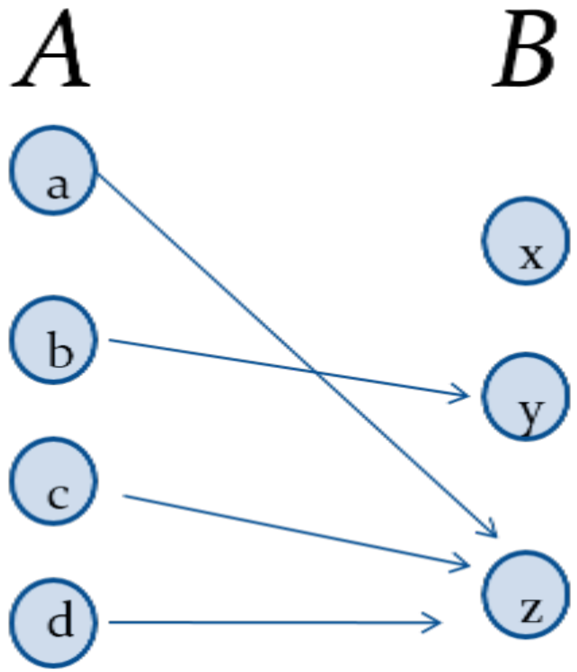
The preimage(s) of z is (are) ? $\{a,c,d\}$



Question on Functions and Sets

Definition: If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{f(s) \mid s \in S\}$$



$f\{a,b,c,\}$ is ? $\{y,z\}$

$f\{c,d\}$ is ? $\{z\}$

Functions

Definition: Let f_1 and f_2 be functions from A to R. Then f_1+f_2 and f1.f2 are also functions from A to R defined for all $x \in A$ by

- $(f_1+f_2)(x) = f_1(x)+f_2(x)$
- $(f_1.f_2)(x) = f_1(x).f_2(x)$

Example: Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1.f_2$?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

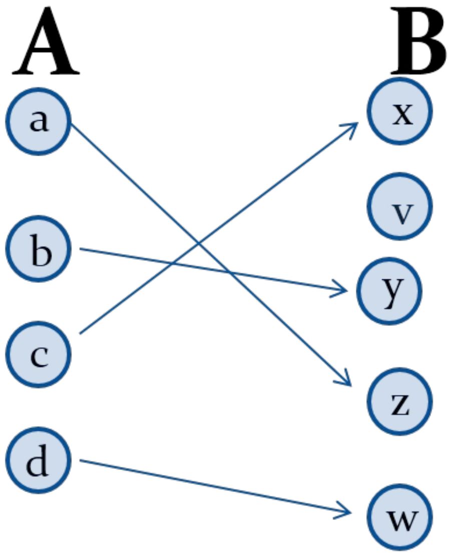
Injections

Definition: A function f is said to be ***one-to-one***, or ***injective***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

A function is said to be an ***injection*** if it is one-to-one.

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

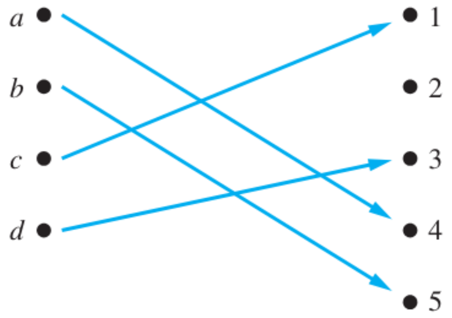
$$\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$



Injection - Examples

- Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with f(a)=4, f(b)=5, f(c)=1, and f(d)=3 is one-to-one.

Yes



- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

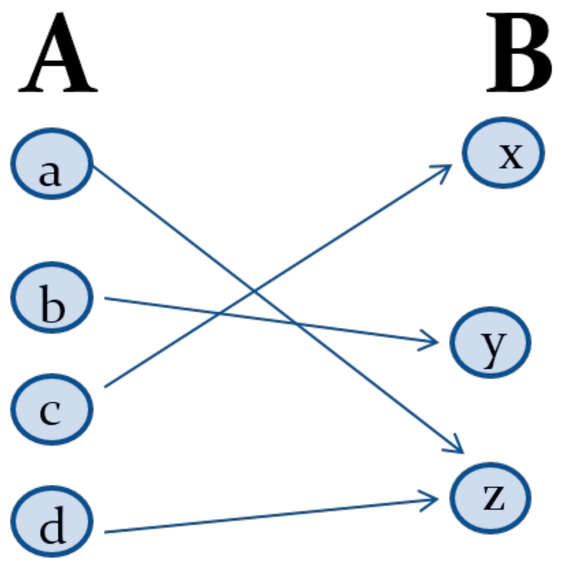
No

e.g., $f(1) = f(-1) = 1$, but $1 = -1$

Surjections

Definition: A function f from A to B is called ***onto*** or ***surjective***, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

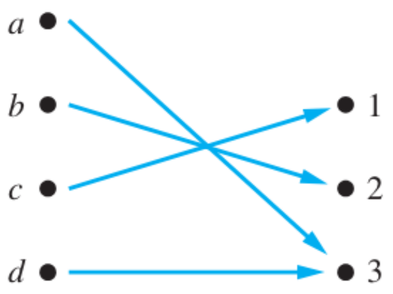
- A function f is called a *surjection* if it is *onto*.
- f's range and codomain are equal.
- $\forall y \exists x (f(x) = y)$



Surjection - Examples

- Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a)=3, f(b)=2, f(c)=1, and f(d)=3. Is f an onto function?

Yes



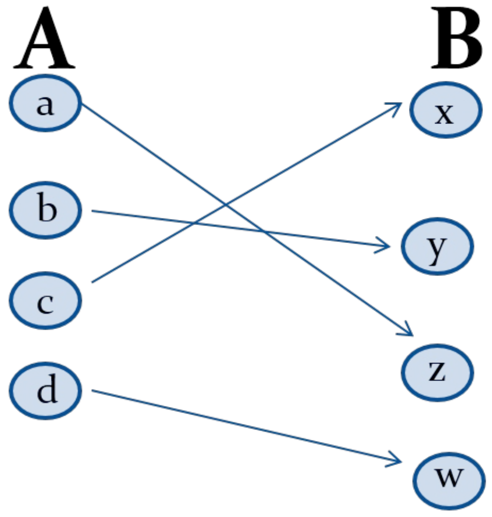
- Is the function $f(x)=x^2$ from the set of integers to the set of integers onto?

No

e.g., no integer x with $x^2 = -1$

Bijections

Definition: A function f is a ***one-to-one correspondence***, or a ***bijection***, if it is both one-to-one and onto (surjective and injective).



Example: Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f (a) = 4, f (b) = 2, f (c) = 1, and f (d) = 3. Is f a bijection?

Yes

Showing that *f* is one-to-one or onto

Suppose that $f : A \to B$.

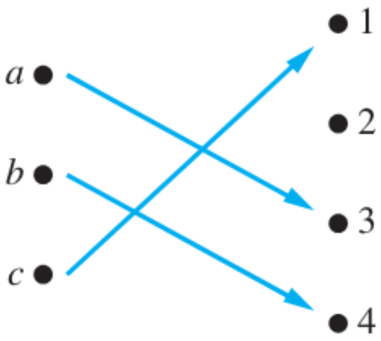
To show that f is injective Show that if $f(x)=f(y)$ for arbitrary $x,y \in A$, then $x=y$.

To show that f is not injective Find particular elements $x,y \in A$ such that $x \neq y$ and $f(x)=f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x)=y$.

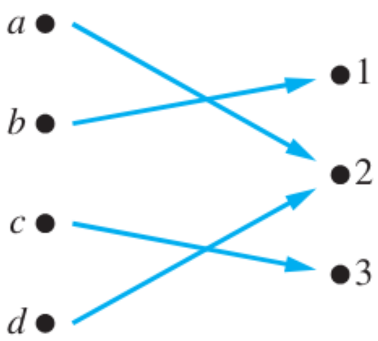
To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

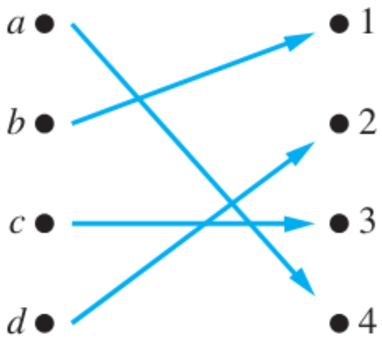
Examples



One-to-one, not onto

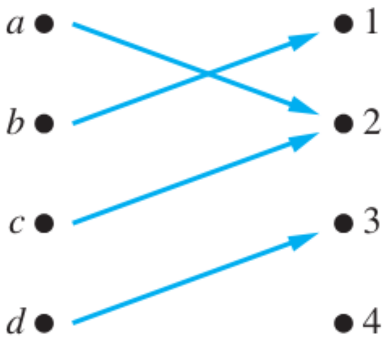
Onto, not one-to-one

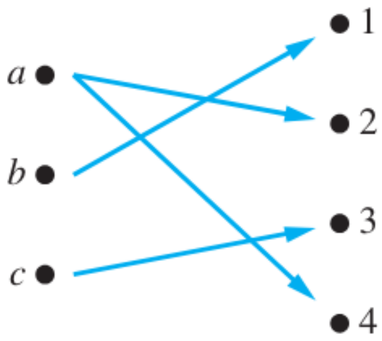




One-to-one, and onto

Neither one-to-one nor onto





Not a function

Inverse Functions [1]

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

- No inverse exists unless *f* is a bijection. Why?

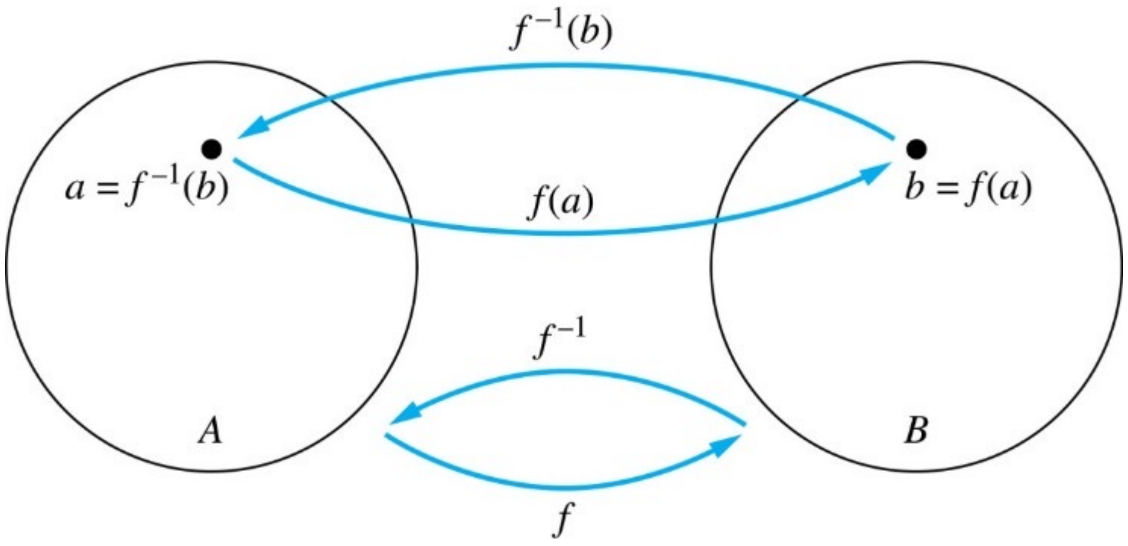
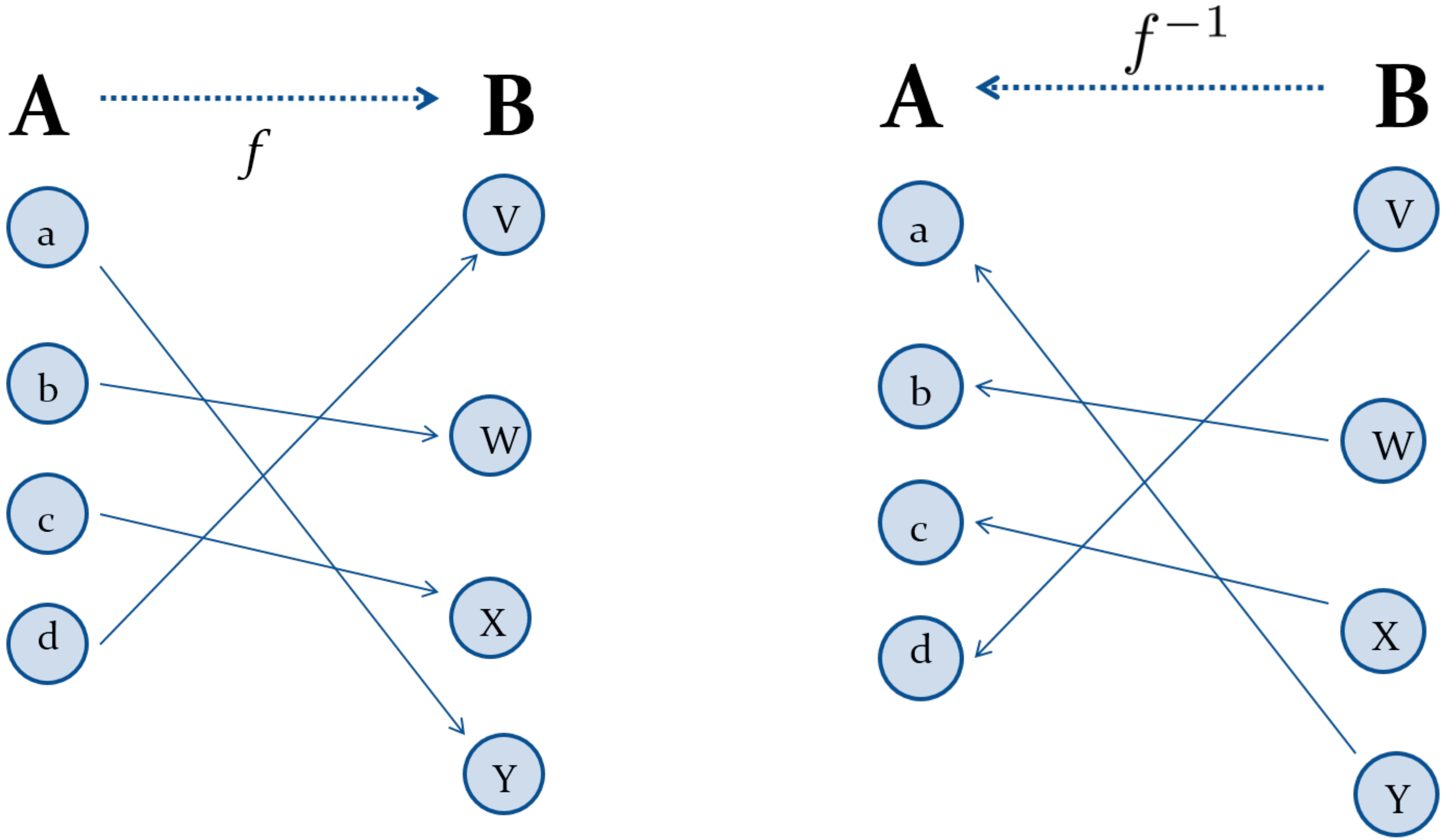


FIGURE: The Function f^{-1} is the inverse of Function f

Inverse Functions 2



Questions[1]

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1)=c$, $f^{-1}(2)=a$, and $f^{-1}(3)=b$.

Questions$_2$

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence.

Format: f(x) = y; therefore, f(x)=y=x+1.

$f^{-1}(y) = x$, so, rewrite x in terms of y.

x = y-1

$$f^{-1}(y) = y - 1.$$

Questions$_3$

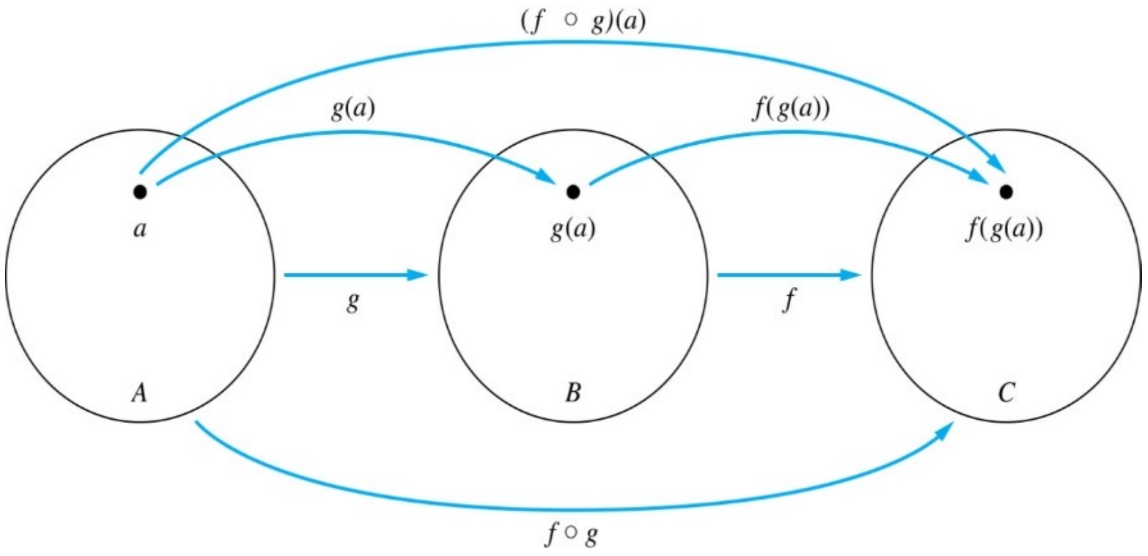
Example 3: Let *f*: **R** → **R** be such that $f(x) = x^2$
Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one. $f(-2) = f(2) = 4$.

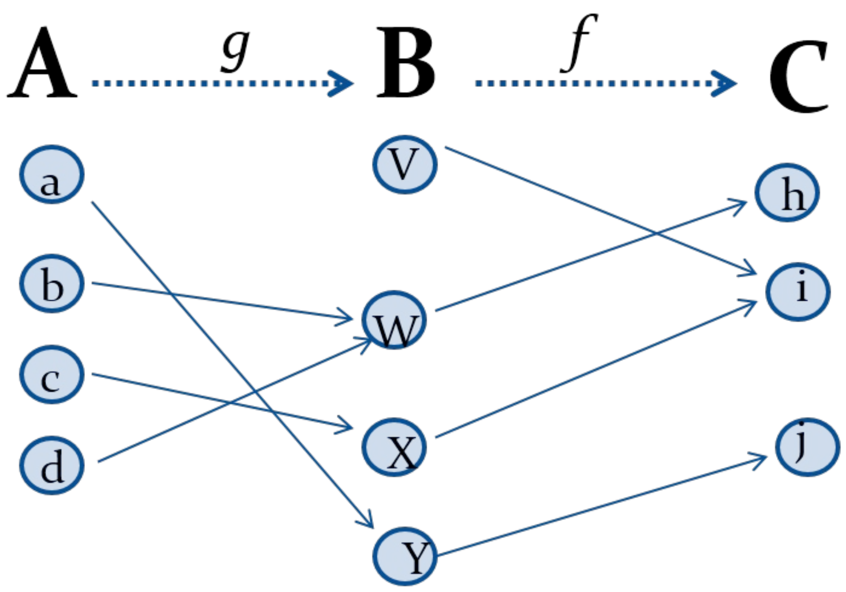
Composition[1]

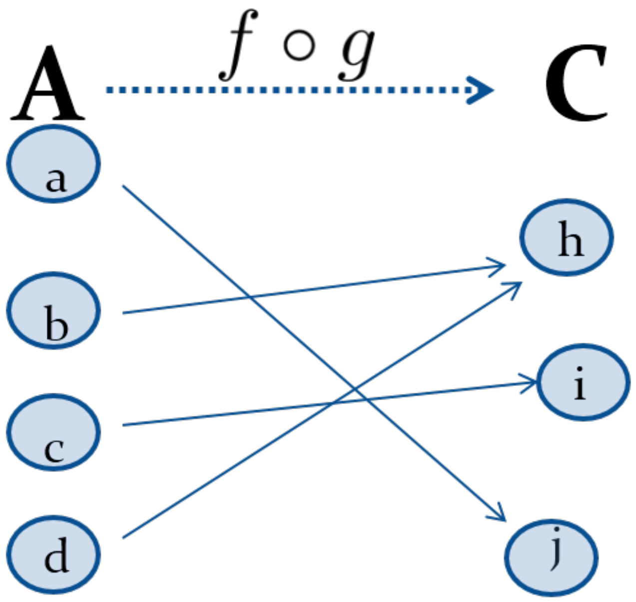
Definition: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$(f \circ g)(a) = f(g(a))$$



Composition ₂





Composition 3

Example 1: $f(x)=x^2$ and $g(x)=2x+1$.

$f \circ g = ?$

$g \circ f = ?$

$$f(x)=x^2 \text{ and } g(x)=2x+1,$$

then

$$f(g(x))=(2x+1)^2$$

and

$$g(f(x))=2x^2+1$$

MESUT OZDAG, PH.D.
MESUT.OZDAG@UCF.EDU

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