Chapter 3 – Second-Order Differential Equations

(lecture 7)

In this chapter we look at linear second-order ODEs of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0 y = g(x)$$

Our goal is to find two linearly independent solutions to the equation, say $y_1(x)$ and $y_2(x)$. The overall solution is then just a linear combination of y_1 and y_2, so we write

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x)}$$

Often we will be given two <u>initial</u> <u>conditions</u> of the form $y(x_0) = y_0$, $y'(x_0) = y_0'$, which allow us to solve for c_1 and c_2.

Or we might be given two <u>boundary</u> <u>conditions</u> of the form $y(a) = y_a$ and $y(b) = y_b$, which again allow us to solve for c_1 and c_2.

As at the start of the course, you may be asked to verify a given solution, or be asked to use the boundary values to find the unique value of c_1 and c_2.

Ex. If $f_1(x) = 1+x$ and $f_2(x) = x^3$, then

$$W = \begin{vmatrix} 1+x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^2 + 3x^3 - x^3$$
$$= 2x^3 + 3x^2$$
$$\neq 0$$

So f_1 and f_2 are linearly independent.

Ex. If $f_1(x) = e^x$, $f_2(x) = xe^x$, $f_3(x) = x^2e^x$, then

$$W = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & xe^x + e^x + e^x & x^2e^x + 2xe^x + 2xe^x + 2e^x \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2e^x + 4xe^x + 2e^x \end{vmatrix}$$

Ex Given that $y = c_1 e^{x} + c_2 e^{-x}$ is a solution to $y'' - y = 0$, find the values of c_1 and c_2 if $y(0) = 0$, $y(1) = 1$.

Substituting in we get

$$y(0) = c_1 + c_2 = 0 \quad \text{---} \ ①$$

$$y(1) = c_1 e + \frac{c_2}{e} = 1 \quad \text{---} \ ②$$

So ① implies $c_1 = -c_2$.

Hence $-c_2 e + \frac{c_2}{e} = 1$

$$c_2\left(\frac{1}{e} - e\right) = 1$$

$$c_2\left(\frac{1-e^2}{e}\right) = 1$$

$$\boxed{c_2 = \frac{e}{1-e^2}} \Longrightarrow \boxed{c_1 = \frac{-e}{1-e^2} = \frac{e}{e^2-1}}$$

Linear Independence

Definition: A set of functions are said to be **linearly dependent** if there exists non-zero constants $c_1, c_2, c_3, \dots, c_n$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \dots + c_n f_n(x) = 0$$

Definition: If the set of functions are not linearly dependent we say that they are **linearly independent**

— which means that the only solution to the above equation is $c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ \dots,\ c_n = 0$

Ex. x and $5x$ are linearly dependent

since $\underline{-5}\, x + \underline{1}\,(5x) = 0$

<u>Ex</u> $\sin^2 x$, $\cos^2 x$, and 1 are linearly dependent

since $\underline{\ 1\ } \cdot \sin^2 x + \underline{\ 1\ } \cdot \cos^2 x + \underline{-1} \cdot 1 = 0$

<u>Ex.</u> x and x^2 are linearly independent

since $\underline{\qquad} x + \underline{\qquad} x^2 = 0$
has no solution other than the trivial one.

<u>Ex</u> $\sin x$ and $\cos x$ are linearly independent.

<u>Ex</u> 1, x, and e^x are linearly independent.

It is often very difficult to determine linear independence just by looking at the function, or solving the equation, and hence we often appeal to the following theorem.

Theorem: Suppose $f_1, f_2, f_3, \ldots, f_n$ are all at least $(n-1)$ times differentiable and the determinant

$$W = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is non-zero, then the set of function is linearly independent

Note: We call this matrix the Wronskian.

Note: If $W = 0$ we conclude nothing, and must appeal to the definition.

Note:

$$\boxed{\begin{array}{ll} W \neq 0 \Rightarrow \text{Independence} & W = 0 \Rightarrow \text{Nothing} \\ \text{Dependence} \Rightarrow W = 0 & \text{Independence} \Rightarrow \text{Nothing} \end{array}}$$

$$= e^{x}\left(\cancel{x^3e^{2x}} + \cancel{4x^2e^{2x}} + 2xe^{2x} + x^2e^{2x} + \cancel{4xe^{2x}} + 2e^{2x} - \cancel{x^3e^{2x}} - \cancel{2x^2e^{2x}} - \cancel{2x^2e^{2x}} - \cancel{4xe^{2x}}\right)$$

$$- xe^{x}\left(\cancel{x^2e^{2x}} + 4xe^{2x} + 2e^{2x} - \cancel{x^2e^{2x}} - 2xe^{2x}\right)$$

$$+ x^2e^{x}\left(\cancel{xe^{2x}} + 2e^{2x} - \cancel{xe^{2x}} - e^{2x}\right)$$

$$= e^{3x}\left(\cancel{2x} + \cancel{x^2} + 2 - \cancel{4x^2} - \cancel{2x} + \cancel{2x^2} + \cancel{2x^2} - \cancel{x^2}\right)$$

$$= 2e^{3x}$$

$\neq 0$ Hence f_1, f_2, f_3 are linearly independent.