Kramers-Wannier and Jordan-Wigner dualities in the transfer matrix approach to the two-dimensional Ising model

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Introduction

> "Of all the systems in statistical mechanics on which exact calculations have been performed, the two-dimensional Ising model is not only the most thoroughly investigated; it is also the richest and most profound." — B.M. McCoy and T.T. Wu [9]

Historical introduction

Any ferromagnetic material abruptly loses its magnetization at a specific temperature, a phenomenon known as the Curie phase transition. A natural question to ask is whether we can devise a statistical model that describes the Curie transition. There are two factors that complicate this business. Firstly, any statistical model describing such a transition should, at least, be infinite in two-directions. Secondly, the second-order nature of the Curie transition makes approximate methods rather ineffective.

To obtain a reliable description of second-order transitions, we should therefore analyze a two-dimensional infinite model analytically. The Ising model, invented by Lenz and first studied by Ising, is a simple model of a ferromagnetic material. In addition, Peierls showed that the two-dimensional (square) Ising model must have a Curie transition. The combination of its simplicity and its phase transition created an interest in the analytical analysis of the square Ising model.

Historically, the first exact result of the square Ising model was obtained by Kramers and Wannier [5] with the transfer matrix method. Kramers and Wannier located the critical temperature of the phase transition using a symmetry of the model now known as "Kramers-Wannier duality". Inspired by this result, Onsager computed the free energy [12] using the same transfer-matrix method, thereby describing all non-magnetic behavior analytically. Onsager's derivations were simplified by Kaufman [4] and Schultz et al. [13] using the same transfer matrix method. Other methods have been used as well, of which the "combinatorial" and the "commuting transfer matrices" solutions are most notable.

Because of its simplicity, the square Ising model is not a physically realistic model of a ferromagnetic material. Nevertheless, the existence of analytic results causes it to be "the basis of much of our theoretical understanding of

phase transitions" in general [9]. Remarkably, phase transitions have universal properties which are shared by large numbers of physical systems, which explains why we can learn about phase transitions in general by considering only the Ising model.

Since Onsager's description of the non-magnetic behavior, the most important result has been Yang's calculation of the spontaneous magnetization, i.e. the magnetization of the ferromagnetic material without the presence of an external magnetic field. The problem of the full magnetic behavior of the two-dimensional model, as well as the problem of the three-dimensional model, remain unsolved to this day.

Outline

In this thesis, we try to clarify why the free energy of the square Ising model can be obtained. To do this, we will take a somewhat historic approach, by using the transfer matrix method. After defining the square Ising model in Chapter 1, we will therefore derive the transfer matrix in Chapter 2. Subsequently, we will derive Kramers-Wannier duality and obtain the critical temperature in Chapter 3. Lastly, in Chapter 4, we will show how to use the "Jordan-Wigner transformation" to fermionic operators, first used by Schultz et al. [13], to reduce the problem significantly.

In our presentation of Kramers-Wannier duality and the Jordan-Wigner transformation, it will become clear that these two are in fact quite similar transformations.

In terms of style, we try to emphasize the physical interpretation and the mathematical structure of the steps in the solution, rather than the computational details. At the beginning of each chapter or appendix, we cite the sources on which our presentation is based, and what is actually our own work.

Chapter 1

The Hamiltonian

In this chapter, we will define the Ising model, that is, give its basic variables and its Hamiltonian. We will start with a general definition and after that we introduce the one- and two-dimensional special cases.

This chapter is based on a book by Simon [14].

1.1 The Ising model on a general graph

We want to define a spin model with interactions between the spins, so in our model we have to specify three things: the nature of the spins, the locations of the spins, and the nature of the interactions.

The basic constituent of the Ising model is the spin variable. Each spin variable can only take the values $+1$ (spin up) or -1 (spin down). To describe how many spins there are and which of these spins interact, we place the spins on a finite undirected graph $G = (\Lambda, E)$. Here Λ is the set labeling the spins, i.e. for any label $i \in \Lambda$ there is a spin variable σ_i which can take the values ± 1. Now by a spin configuration σ we mean a specification of the values of all these spin variables. This is simply a map $\sigma : \Lambda \to \{\pm 1\}$, so there are $2^{|\Lambda|}$ different spin configurations.

The set E, on the other hand, determines which spins interact with each other through the exchange interaction. Suppose we have a pair of interacting spins $\{i, j\} \in E$, then the exchange interaction is given by $-k_{\{i,j\}}\sigma_i\sigma_j$. The factor $k_{\{i,j\}} \in \mathbb{R}_{>0}$ determines the strength of the interaction, and can depend on the pair $\{i, j\}$. In our examples, however, the strength of the interaction will only have one or two different values.

We can place the model in an external magnetic field. Then in addition to interacting with other spins, a spin can interact with this magnetic field. This interaction is given by $-b\sigma_i$, where we write b for the product of the magnetic moment and the magnetic field.

Adding all these interactions gives us the total Hamiltonian $\mathcal{H}$ of a spin configuration σ:

$$\mathcal{H}(\sigma) = -\sum_{\{i,j\}\in E} k_{\{i,j\}}\sigma_i\sigma_j - \sum_i b\sigma_i. \tag{1.1}$$

Now we put the Ising model in a heat bath of a fixed inverse temperature $\beta = 1/kT$. This means we are using the canonical ensemble formalism. Therefore, to find the thermodynamic properties of the Ising model, we should find the (canonical) partition function Z. The partition function is given by a sum over all possible configurations σ. The summand depends on β and the Hamiltonian $\mathcal{H}$ of a configuration:

$$Z = \sum_\sigma \exp(-\beta\mathcal{H}(\sigma)). \tag{1.2}$$

Introducing the notations $K_{\{i,j\}} \equiv \beta k_{\{i,j\}}$ and $B \equiv \beta b$, we get the following expression for the partition function Z:

$$Z = \sum_\sigma \exp\left(\sum_{\{i,j\}\in E} K_{\{i,j\}}\sigma_i\sigma_j + \sum_i B\sigma_i\right) \tag{1.3}$$

Using the partition function Z, we find the free energy of the lattice by $F = -kT\log Z$. Actually we are interested in the behaviour of the system in the thermodynamic limit. In this infinite-size limit, the free energy may diverge. Therefore, we study the average free energy f instead, i.e. the free energy F divided by the number of spins:

$$f = -\frac{kT}{|\Lambda|}\log Z \tag{1.4}$$

Because finite systems do not exhibit phase transitions, we must study an infinite Ising lattice to study phase transitions. Since analyzing infinite lattices is very difficult, we start by analyzing finite lattices that resemble the infinite lattice. Then we compute the free energy per spin f (1.4) for these finite lattices. In the end, we take the limit of these finite lattices to the infinite lattice, and finally analyze the resulting expression f.

1.2 The one-dimensional Ising model

To become more concrete, in this section and the next we will introduce two explicit examples of Ising models. Because we have already defined the Ising model on a general graph/lattice, defining these is easy and is mostly a matter of introducing notation. The one-dimensional case is not very interesting by itself, since it has no phase transition. It only serves as a warm-up to the two-dimensional case, both here and in Chapter 2.

The lattice of the one-dimensional Ising model is simply a chain of length N. The last spin interacts with the first spin, so actually the lattice is circular. There is only one interaction direction, namely the direction along the chain,

and the interaction factor we will call k. Then the Hamiltonian of the one-dimensional Ising model is given by

$$\mathcal{H}(\sigma) = -\sum_{n=1}^{N}(k\sigma_n\sigma_{n+1} + b\sigma_n). \tag{1.5}$$

In this equation, σ_N indeed interacts with the first spin $\sigma_{N+1} \equiv \sigma_1$. We could also have taken free boundary conditions where the last spin and the first spin only interact with one other spin, in this case the first sum should run up to $N-1$ instead of to N. The periodic boundary conditions will, however, prove to be essential in the transfer matrix formalism of Chapter 2.

The partition function (using again $K \equiv \beta k$ and $B \equiv \beta b$) is now given by

$$Z = \sum_{\sigma} \exp\left(\sum_{n=1}^{N}(K\sigma_n\sigma_{n+1} + B\sigma_n)\right). \tag{1.6}$$

1.3 The square Ising model

This example of a two-dimensional Ising model is also the simplest and the most widely investigated. In fact, the square Ising model (which we will define shortly), is often referred to as "the two-dimensional Ising model" or sometimes even as only "the Ising model".

The lattice is rectangular and consists of squares: M in the vertical direction and N in the horizontal direction, hence there are MN spin variables in total. We will label the spin variables $\sigma_{i,j}$ in the matrix fashion by their row i and column j. There are only two interaction factors: the horizontal interaction factor k_1 and the vertical interaction factor k_2. Using this notation, the Hamiltonian of the square Ising model is given by

$$\mathcal{H}(\sigma) = -\sum_{m=1}^{M}\sum_{n=1}^{N}(k_1\sigma_{m,n}\sigma_{m,n+1} + k_2\sigma_{m,n}\sigma_{m+1,n} + b\sigma_{m,n}). \tag{1.7}$$

Similar to the one-dimensional case, we impose periodic boundary conditions. This means that the last row interacts with the first row, and the last column with the first column. To be precise, in (1.7), one should read $\sigma_{m,N+1} = \sigma_{m,1}$ and $\sigma_{M+1,n} = \sigma_{1,n}$.

Chapter 2

The transfer matrix

The one and two-dimensional lattices defined in Chapter 1 are highly regular. We could have built up the models by joining small models together. In this chapter, we will show how we can exploit this feature to reduce the computation of the partition function Z to the diagonalization of a certain matrix. This matrix is called the transfer matrix.

The content of Sections 2.1 and 2.2 is already apparent in the article by Kramers and Wannier [5]. The transformation of Section 2.3 is due to Onsager [12]. Our presentation is based on [3–5, 11–15].

2.1 Solving the one-dimensional Ising model

We try to build up the one-dimensional Ising model by adding one spin at a time. To do this, we investigate what happens to the Hamiltonian (1.5) if we add one spin to a chain of $n < N$ spins. The Hamiltonian is increased by two terms: the interaction $-k\sigma_n\sigma_{n+1}$ between the last two spins, and the interaction $-b\sigma_{n+1}$ of the last spin with the magnetic field. So the total increase of the Hamiltonian only depends on these last two spins and is given by

$$E(\sigma_n, \sigma_{n+1}) = -k\sigma_n\sigma_{n+1} - b\sigma_{n+1}. \tag{2.1}$$

Using this notation, we can rewrite the Hamiltonian as

$$\mathcal{H}(\sigma) = \sum_{n=1}^{N} E(\sigma_n, \sigma_{n+1}). \tag{2.2}$$

What happens to the partition function if we add one spin? The probability terms $\exp(-\beta\mathcal{H})$ are multiplied by a factor

$$V(\sigma_n, \sigma_{n+1}) \equiv \exp(-\beta E(\sigma_n, \sigma_{n+1})) \tag{2.3}$$

that only depends on the spins σ_n and σ_{n+1}.

To interpret this factor, consider again the chain of n spins. Then we define $P_n(s)$ as the probability that $\sigma_n = s$, where $s \in \{\pm 1\}$. Likewise, for a chain of $n+1$ spins, we define $P_{n+1}(s')$ as the probability that $\sigma_{n+1} = s'$, where $s' \in \{\pm 1\}$. Then if we fix $\sigma_n = s$, we can compute $P_{n+1}(s')$ as

$$P_{n+1}(s') \propto V(s, s').$$

Therefore if we do not fix σ_n, we get

$$P_{n+1}(s') \propto \sum_{s=\pm 1} V(s, s') P_n(s)$$

where we recognize the structure of a matrix multiplication. So V can be interpreted as a matrix that transfers the probability vector $P_n(s)$ to $P_{n+1}(s')$, which explains the name "transfer matrix".

To formally use this idea, we expand the exponential in (1.6):

$$Z = \sum_{\sigma_1, \ldots, \sigma_N} V(\sigma_1, \sigma_2) V(\sigma_2, \sigma_3) \ldots V(\sigma_N, \sigma_1). \tag{2.4}$$

Again we recognize the structure of a matrix multiplication: Single out, for example, the summation over σ_2. We see that

$$\sum_{\sigma_2} V(\sigma_1, \sigma_2) V(\sigma_2, \sigma_3) \text{“=”} V^2(\sigma_1, \sigma_3)$$

so the summation over a spin variable is replaced by a matrix multiplication. To remove the apostrophes from the equality sign, we formally define the 2×2 transfer matrix V by its matrix elements

$$\langle s | V | s' \rangle \equiv V(s, s').$$

Then we can rewrite (2.4) further as

$$Z = \sum_{\sigma_1, \ldots, \sigma_N} \langle \sigma_1 | V | \sigma_2 \rangle \langle \sigma_2 | V | \sigma_3 \rangle \ldots \langle \sigma_N | V | \sigma_1 \rangle \tag{2.5}$$

$$= \sum_{\sigma_1} \langle \sigma_1 | V^N | \sigma_1 \rangle = \text{Tr } V^N = \lambda_1^N + \lambda_2^M. \tag{2.6}$$

Here Tr V^N denotes the trace of the 2×2 matrix V^N, and λ_1, λ_2 denote the eigenvalues of V. At this point we see that we indeed needed periodic boundary conditions: For free boundary conditions the factor $\langle \sigma_N | V | \sigma_1 \rangle$ is not present. So we see we have reduced the computation of the partition function Z to the diagonalization of the 2×2 matrix V. This only involves solving a quadratic equation.

The result of this calculation (which we omit), is that the average free energy of the one-dimensional Ising model is given by

$$f(\beta) = -k - \frac{1}{\beta} \log \left(\cosh(\beta b) + \sqrt{\sinh^2(\beta b) + \exp(-4\beta k)} \right).$$

Because the functions $\cosh, \sinh, \exp$ are analytic on $\mathbb{R}$, and the logarithm and the power functions are analytic on $\mathbb{R}_{>0}$, we see that the average free energy $f(\beta)$ is an analytic function of the temperature β with domain $\beta \in \mathbb{R}_{>0}$. In particular, the one-dimensional Ising model does not have a phase transition for any finite temperature $\beta_c > 0$.

2.2 The square transfer matrix

Now we try to use the same idea in two dimensions: building up the square lattice step by step to simplify the Hamiltonian. Instead of adding one spin at a time, we build up the lattice one column at a time. The state of the nth column can be summarized in a vector $\sigma_n = (\sigma_{1,n}, \ldots, \sigma_{M,n})$. What happens with the Hamiltonian if we add one such column? It is increased by three terms: the interaction between the last two columns $E_1(\sigma_n, \sigma_{n+1})$, the interactions within the last column $E_2(\sigma_{n+1})$, and the interaction of the last column with the magnetic field $E_3(\sigma_{n+1})$. To be precise, the increase in the Hamiltonian can be written as a sum of three terms

$$E_1(\sigma_n, \sigma_{n+1}) = -\sum_{m=1}^{M} k_1 \sigma_{m,n} \sigma_{m,n+1}, \tag{2.7a}$$

$$E_2(\sigma_{n+1}) = -\sum_{m=1}^{M} k_2 \sigma_{m,n+1} \sigma_{m+1,n+1}, \tag{2.7b}$$

$$E_3(\sigma_{n+1}) = -\sum_{m=1}^{M} b \sigma_{m,n+1}. \tag{2.7c}$$

So the total increase of the Hamiltonian

$$E(\sigma_n, \sigma_{n+1}) = E_1(\sigma_n, \sigma_{n+1}) + E_2(\sigma_{n+1}) + E_3(\sigma_{n+1})$$

only depends on the column states σ_n and σ_{n+1}. This is the same situation as we encountered in the one-dimensional case (see (2.1)), but now we will also try to keep the different terms E_1, E_2 and E_3 separate. Physically this corresponds to building up the lattice in even smaller steps: first adding the horizontal interaction between the two columns, then the vertical interaction of the column and lastly the magnetic interaction of the column.

Similar to (2.3), we define the transfer matrices by their entries as multiplicative factors in the probabilities $\exp(-\beta \mathcal{H})$:

$$\langle \tau | V_1' | \tau' \rangle \equiv \exp(-\beta E_1(\tau, \tau')) \tag{2.8a}$$

$$\langle \tau | V_2 | \tau' \rangle \equiv \delta_{\tau,\tau'} \exp(-\beta E_2(\tau')) \tag{2.8b}$$

$$\langle \tau | V_3 | \tau' \rangle \equiv \delta_{\tau,\tau'} \exp(-\beta E_3(\tau')) \tag{2.8c}$$

This should be read as follows: The variable τ is the state of a column, so it can take 2^M different values. Thus the matrices V_1', V_2 and V_3 are all $2^M \times 2^M$

matrices. The factor $\langle\sigma_n|V_2|\sigma_{n+1}\rangle$, for example, is the multiplicative factor that corresponds to adding the vertical interaction of the column.

Following approximately the same steps as in the one-dimensional case, we obtain the transfer matrix for the square Ising model. Again, this can be interpreted as a matrix that transfers the probability vectors of one column to the next.

Theorem 2.2.1. *The partition function Z of the square Ising model is given by*

$$Z = \text{Tr } V^N \tag{2.9}$$

where the transfer matrix V is defined by $V \equiv V_1' V_2 V_3$.

Proof. We first rewrite the Hamiltonian as

$$\mathcal{H}(\sigma) = \sum_{n=1}^{N} (E_1(\sigma_n, \sigma_{n+1}) + E_2(\sigma_{n+1}) + E_3(\sigma_{n+1})).$$

Inserting this in (1.2) and expanding the exponential, we get

$$Z = \sum_{\sigma_1,\ldots,\sigma_N} \prod_{n=1}^{N} \exp(-\beta E_1(\sigma_n, \sigma_{n+1})) \exp(-\beta E_2(\sigma_{n+1})) \exp(-\beta E_3(\sigma_{n+1})).$$

We introduce $2N$ delta functions and sums over their variables to rewrite Z as

$$\begin{aligned} Z = \sum_{\sigma_1,\ldots,\sigma_N} \sum_{\sigma_1',\ldots,\sigma_N'} \sum_{\sigma_1'',\ldots,\sigma_N''} \prod_{n=1}^{N} & \exp(-\beta E_1(\sigma_n, \sigma_n')) \\ & \times \delta_{\sigma_n',\sigma_n''} \exp(-\beta E_2(\sigma_n'')) \delta_{\sigma_n'',\sigma_{n+1}} \exp(-\beta E_3(\sigma_{n+1})). \end{aligned}$$

Now we recognize the matrix product as

$$\begin{aligned} Z &= \sum_{\sigma_1,\ldots,\sigma_N} \sum_{\sigma_1',\ldots,\sigma_N'} \sum_{\sigma_1'',\ldots,\sigma_N''} \prod_{n=1}^{N} \langle\sigma_n|V_1'|\sigma_n'\rangle\langle\sigma_n'|V_2|\sigma_n''\rangle\langle\sigma_n''|V_3|\sigma_{n+1}\rangle \\ &= \sum_{\sigma_1,\ldots,\sigma_N} \prod_{n=1}^{N} \langle\sigma_n|V_1'V_2V_3|\sigma_{n+1}\rangle \\ &= \sum_{\sigma_1} \langle\sigma_1|(V_1'V_2V_3)^N|\sigma_1\rangle \\ &= \text{Tr } (V_1'V_2V_3)^N \end{aligned}$$

□

Only the largest eigenvalue matters

We have seen that we can find a transfer matrix for the square Ising model as well. However, the transfer matrix $V_1'V_2V_3$ is no longer a simple 2×2 matrix,

but it has increased in size to a $2^M \times 2^M$ matrix. This complicates the problem because the diagonalization becomes much harder.

Suppose we are able to find the 2^M eigenvalues of this transfer matrix $V_1' V_2 V_3$, let's call them $\lambda_1, \ldots, \lambda_{2^M}$. Then the partition function is equal to

$$Z = \sum_{i=1}^{2^M} \lambda_i^N. \tag{2.10}$$

This still is an unwieldy expression for the partition function, since it contains 2^M terms. In fact, we are only interested in the average free energy in the thermodynamic limit, and for this, only the largest eigenvalue needs to be determined.

Proposition 2.2.2. *Suppose all eigenvalues of the transfer matrix are positive and* $\log(\lambda_{max}(M))/M$ *converges for* $M \to \infty$*. Then the average free energy* f *of the square Ising model in the thermodynamic limit* $M, N \to \infty$ *is given by*

$$f = -kT \lim_{M \to \infty} \log(\lambda_{max}(M))/M \tag{2.11}$$

where $\lambda_{max}(M)$ *denotes the largest eigenvalue of the transfer matrix* $V_1' V_2 V_3$*.*

Proof. Using (2.10) for Z and the assumption that all eigenvalues are positive, we obtain

$$(\lambda_{max}(M))^N \leq Z \leq 2^M (\lambda_{max}(M))^N.$$

Taking the logarithm and dividing by MN, we get

$$\log(\lambda_{max}(M))/M \leq \log(Z)/MN \leq \log(\lambda_{max}(M))/M + \log(2)/N.$$

Using the assumption that $\log(\lambda_{max}(M))/M$ converges as $M \to \infty$, and recalling the definition of the average free energy (1.4), we obtain the result by taking the limit $M, N \to \infty$. □

It will turn out that indeed the assumptions of Proposition 2.2.2 hold, although we will not prove it in this thesis.

2.3 The spin transfer matrix

To facilitate the actual diagonalization of the transfer matrix $V_1' V_2 V_3$, we will describe the matrices V_1', V_2 and V_3 in another, more convenient basis. The matrices V_1', V_2 and V_3 are all $2^M \times 2^M$ matrices, i.e. they act on $\mathbb{C}^{(2^M)}$. Mathematically, we know that $\mathbb{C}^{(2^M)} \cong \otimes^M (\mathbb{C}^2)$ where $\otimes$ denotes the tensor product. A convenient basis of the 2×2 matrices is given by

$$I_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \tau^x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices τ^a where $a = x, y, z$ are known as the Pauli spin matrices: These describe the spin of one spin-1/2 particle. Using this basis of the matrices acting on $\mathbb{C}^2$, we can also give a convenient basis for the matrices acting on $\otimes^M(\mathbb{C}^2)$. This basis consists of all products of the following matrices:

$$I \equiv I_2 \otimes \cdots \otimes I_2 \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2 \tag{2.12a}$$
$$\tau_m^x \equiv I_2 \otimes \cdots \otimes I_2 \otimes \tau^x \otimes I_2 \otimes \cdots \otimes I_2 \tag{2.12b}$$
$$\tau_m^y \equiv I_2 \otimes \cdots \otimes I_2 \otimes \tau^y \otimes I_2 \otimes \cdots \otimes I_2 \tag{2.12c}$$
$$\tau_m^z \equiv I_2 \otimes \cdots \otimes I_2 \otimes \tau^z \otimes I_2 \otimes \cdots \otimes I_2. \tag{2.12d}$$

So τ_m^a acts on the m'th component only, and contains M factors in total. These matrices satisfy the commutator/anti-commutator relations

$$[\tau_m^a, \tau_{m'}^b] = 2i\delta_{m,m'}\epsilon_{abc}\tau_{m'}^c \tag{2.13a}$$
$$\{\tau_m^a, \tau_m^b\} = 2\delta_{a,b}I \tag{2.13b}$$

where $a, b, c \in \{x, y, z\}$, ϵ_{abc} denotes the Levi-Civita symbol and Einstein summation convention is used.

In general, we will call a set of matrices, labeled by $a \in \{x, y, z\}$ and $m \in \{1, \ldots, M\}$, that satisfies the commutator/anti-commutator relations (2.13), a set of *spin matrices*. This is because they describe the spins of M spin-1/2 particles.

Now we can rewrite the matrices V_1', V_2 and V_3 using the spin matrices (2.12). We also write the matrices as exponentials to conveniently describe products. Carrying this out, we obtain the following result.

Proposition 2.3.1. *The transfer matrices V_1', V_2 and V_3 can be rewritten as*

$$V_1' = (2\sinh(2K_1))^{M/2}V_1 \tag{2.14a}$$
$$V_1 \equiv \prod_{m=1}^{M} \exp(K_1^*\tau_m^x) \tag{2.14b}$$
$$V_2 = \prod_{m=1}^{M} \exp(K_2\tau_m^z\tau_{m+1}^z) \tag{2.14c}$$
$$V_3 = \prod_{m=1}^{M} \exp(B\tau_m^z) \tag{2.14d}$$

where K_1^* is defined by

$$K_1^* \equiv \operatorname{artanh}(\exp(-2K_1)). \tag{2.15}$$

In the following proof, we will also see how the Pauli spin matrices τ_m^a arise physically. The Pauli matrix τ_m^z is related to the m'th spin of a single column, while τ_m^x is related to the disorder between the m'th spins of neighboring columns. This is non-trivial: In Chapter 1, the Ising spins were defined as (classical) numbers ± 1, but in the transfer matrix, they have become non-commuting

operators. This suggests a correspondence with a quantum mechanical model, but we will not go into this so-called "quantum Ising chain".

In the language of Proposition 2.3.1, the partition function (2.9) is given by

$$Z = (2\sinh(2K_1))^{MN/2}\mathrm{Tr}(V_1V_2V_3)^N. \tag{2.16}$$

Proof (of Proposition 2.3.1). We first prove (2.14c) and (2.14d). Take a column τ, so τ_m is the m'th spin of the column τ. By definitions (2.7b), (2.7c) and (2.8b), (2.8c), V_2 and V_3 are diagonal and the diagonal elements are given by

$$\langle\tau_1,\dots,\tau_M|V_2|\tau_1,\dots,\tau_M\rangle = \prod_{m=1}^{M}\exp(K_2\tau_m\tau_{m+1}), \tag{2.17a}$$

$$\langle\tau_1,\dots,\tau_M|V_3|\tau_1,\dots,\tau_M\rangle = \prod_{m=1}^{M}\exp(B\tau_m). \tag{2.17b}$$

So V_2 and V_3 depend only on one column τ. The Pauli matrix τ_m^z (2.12d) measures the spin τ_m so

$$\begin{aligned}\langle\tau_1,\dots,\tau_M|\tau_m^z\tau_{m+1}^z|\tau_1,\dots,\tau_M\rangle &= \tau_m\tau_{m+1},\\ \langle\tau_1,\dots,\tau_M|\tau_m^z|\tau_1,\dots,\tau_M\rangle &= \tau_m.\end{aligned}$$

These also imply

$$\begin{aligned}\langle\tau_1,\dots,\tau_M|\exp(K_2\tau_m^z\tau_{m+1}^z)|\tau_1,\dots,\tau_M\rangle &= \exp(K_2\tau_m\tau_{m+1}),\\ \langle\tau_1,\dots,\tau_M|\exp(B\tau_m^z)|\tau_1,\dots,\tau_M\rangle &= \exp(B\tau_m).\end{aligned}$$

Combined with (2.17a) and (2.17b), these prove (2.14c) and (2.14d).

Now we prove (2.14a). Take neighboring columns τ and τ' of the lattice. By definitions (2.7a) and (2.8a), the matrix elements of V_1' are given by

$$\langle\tau_1,\dots,\tau_M|V_1'|\tau_1',\dots,\tau_M'\rangle = \prod_{m=1}^{M}\exp(K_1\tau_m\tau_m'). \tag{2.18}$$

Because V_1' implements the horizontal interaction, it depends on both τ and τ' (see (2.18)). In other words, it maps the state space of one column to the state space of the other. To describe V_1' with τ_m^x (2.12a), we should therefore also take τ_m^x from one state space to the other. This means that

$$\tau_m^x|\tau_1',\dots,\tau_m',\dots,\tau_M'\rangle = |\tau_1,\dots,-\tau_m,\dots,\tau_M\rangle.$$

Here we see how τ_m^x is related to the spins τ_m and τ_m': It measures the amount of disorder between the neighboring spins. If they are opposite, the matrix element of τ_m^x is 1, if they are alike, it is 0.

Now we look at the factor $\exp(K_1\tau_m\tau_m')$ in (2.18) and re-express it in terms of τ_m^x. This is dealt with in the lemma below. □

Lemma 2.3.2. *Define K_1^* as in* (2.15)*. Then the matrix elements of* $\exp(K_1^*\tau_m^x)$ *are given by*

$$(2\sinh(2K_1))^{\frac{1}{2}}\langle\tau_1,\dots,\tau_M|\exp(K_1^*\tau_m^x)|\tau_1',\dots,\tau_M'\rangle = \exp(K_1\tau_m\tau_m').$$

Furthermore, the map $\mathbb{R}_{>0}\to\mathbb{R}_{>0}$, $K\mapsto K^*$ is a decreasing bijection and satisfies

$$K^* = \frac{1}{2}\log\coth K \tag{2.19a}$$

$$\sinh(2K)\sinh(2K^*) = 1, \tag{2.19b}$$

$$(K^*)^* = K \tag{2.19c}$$

$$K^* = K \iff K = \frac{1}{2}\log(1+\sqrt{2}). \tag{2.19d}$$

So the map $K\mapsto K^*$ relates low couplings to high couplings, and keeps $K=\frac{1}{2}\log(1+\sqrt{2})$ fixed. This will be important in the next chapter.

Idea of proof. An explicit matrix representation of the factor $\exp(K_1\tau_m\tau_m')$ is

$$\begin{pmatrix} e^{K_1} & e^{-K_1} \\ e^{-K_1} & e^{K_1}\end{pmatrix} = e^{K_1}I + e^{-K_1}\tau_m^x = e^{K_1}(I+e^{-2K_1}\tau_m^x). \tag{2.20}$$

Because $(\tau_m^x)^2 = I$, it satisfies, for any $\alpha\in\mathbb{R}$:

$$\exp(\alpha\tau_m^x) = \cosh(\alpha)I + \sinh(\alpha)\tau_m^x = \cosh(\alpha)(I+\tanh(\alpha)\tau_m^x) \tag{2.21}$$

which can be seen by taking series expansions. Then we see where K_1^* comes from: To connect (2.20) with (2.21), we want to have the equality

$$I + e^{-2K_1}\tau_m^x = I + \tanh(\alpha)\tau_m^x.$$

Therefore, we have to find $\alpha = K_1^*$ such that $\tanh(K_1^*) = \exp(-2K_1)$. Taking this as our definition of K_1^*, yields the equality

$$\frac{\exp(K_1)}{\cosh(K_1^*)}\langle\tau_1,\dots,\tau_M|\exp(K_1^*\tau_m^x)|\tau_1',\dots,\tau_M'\rangle = \exp(K_1\tau_m\tau_m')$$

The rest of the statement follows from standard identities for hyperbolic functions and the definition (2.15). □

Chapter 3

Kramers-Wannier duality

A few years before Onsager's solution of the two-dimensional Ising model, Kramers and Wannier computed the critical temperature for the case $B = 0$, so without external magnetic field [5]. They located this point by using a symmetry of the matrices V_1 and V_2 known as Kramers-Wannier duality. We will derive Kramers-Wannier duality using the transfer matrix of Chapter 2. Before deriving it, however, we give a motivation. Furthermore, for the rest of this thesis, we set $B = 0$, i.e. we do not include an external magnetic field.

The duality of this chapter was first discovered by Kramers and Wannier [5], who also computed the critical temperature (Section 3.4). The proof of duality we present (Sections 3.1-3.3), however, is due to Onsager [12]. Section 3.2 is our elaboration of the discussion by Cobanera et al. [2]. Apart from these sources, our presentation is also based on Kaufman [4] and Mussardo [10].

3.1 Motivation of Kramers-Wannier duality

In Proposition 2.3.1, we have written the transfer matrix V as a product of a large number of matrices. These factors are in fact simple products of the 2×2 spin matrices (2.12). Therefore, a natural question to ask is whether these simple factors commute. If they did, we could diagonalize them independently, which would simplify the diagonalization of the total transfer matrix V enormously.

Using the commutation relations (2.13) of the spin matrices, we can summarize the commutation relations conveniently if we write the constituent factors of V_1 and V_2 in a cyclic row:

$$\tau_1^x, \tau_1^z\tau_2^z, \tau_2^x, \tau_2^z\tau_3^z, \dots, \tau_M^x, \tau_M^z\tau_1^z, (\tau_1^x, \dots). \tag{3.1}$$

Each of these factors (which we will call *bonds*) anti-commutes with its neighbors in the cyclic sequence, and commutes with all other bonds. So, unfortunately, there are still a lot of non-commuting matrices.

However, we do see an interesting feature of the sequence. The commutation relations are determined solely by which factors neighbor each other in the

sequence. So to describe the commutation relations, we can equally well shift the sequence and write

$$\tau_1^z\tau_2^z, \tau_2^x, \tau_2^z\tau_3^z, \dots, \tau_M^x, \tau_M^z\tau_1^z, \tau_1^x, (\tau_1^z\tau_2^z, \dots)$$

or reverse the sequence and write

$$\tau_M^z\tau_1^z, \tau_M^x, \dots, \tau_2^z\tau_3^z, \tau_2^x, \tau_1^z\tau_2^z, \tau_1^x, (\tau_M^z\tau_1^z, \dots).$$

Although this may seem to be artificial trickery, it reveals that both the map that shifts the sequence, i.e.

$$\tau_m^x \mapsto \tau_m^z\tau_{m+1}^z, \ \tau_m^z\tau_{m+1}^z \mapsto \tau_{m+1}^x \tag{3.2}$$

and the map that reverses the sequence, i.e.

$$\tau_m^x \mapsto \tau_{r(m)}^z\tau_{r(m)+1}^z, \ \tau_m^z\tau_{m+1}^z \mapsto \tau_{r(m)}^x$$

preserve the commutation relations between the bonds. Here $r(m) = M + 1 - m$ is the index reversal map that for example interchanges the index 1 with the index M. Therefore, we might think that these maps extend to an automorphism of the matrix algebra. For these cyclic boundary conditions, this is not the case. For example, if (3.2) was be an automorphism, it would map

$$I = (\tau_1^z\tau_2^z)(\tau_2^z\tau_3^z)\dots(\tau_M^z\tau_1^z) \mapsto \tau_1^x \dots \tau_M^x$$

but an automorphism should map the identity to itself. To fix this problem, which arises from the cylic boundary conditions, we will choose different boundary conditions.

3.2 The Kramers-Wannier automorphism

For this chapter only, we choose different boundary conditions, such that V_2 is given by

$$V_2 = \exp(K_2\tau_M^z) \prod_{m=1}^{M-1} \exp(K_2\tau_m^z\tau_{m+1}^z) \tag{3.3}$$

so the factor $\exp(K_2\tau_M^z\tau_{M+1}^z)$ is replaced by $\exp(K_2\tau_M^z)$. This could be described as taking "half-open" boundary conditions instead of periodic boundary conditions, in the vertical direction. The matrix V_1 is unchanged.

The sequence that now describes the commutation relations (so replacing (3.1)), is

$$\tau_1^x, \tau_1^z\tau_2^z, \tau_2^x, \tau_2^z\tau_3^z, \dots\dots, \tau_M^x, \tau_M^z.$$

We can no longer shift the sequence, but we can still reverse it, to obtain

$$\tau_M^z, \tau_M^x, \dots\dots, \tau_2^z\tau_3^z, \tau_2^x, \tau_1^z\tau_2^z, \tau_1^x.$$

which describes exactly the same commutation rules. Comparing these two sequences suggests defining the duality map Φ_d on the bonds by

$$\Phi_d(\tau_1^x) \equiv \tau_M^z, \tag{3.4a}$$
$$\Phi_d(\tau_m^x) \equiv \tau_{r(m)}^z \tau_{r(m)+1}^z, \qquad m \neq 1, \tag{3.4b}$$
$$\Phi_d(\tau_m^z \tau_{m+1}^z) \equiv \tau_{r(m)}^x, \qquad m \neq M, \tag{3.4c}$$
$$\Phi_d(\tau_M^z) \equiv \tau_1^x, \tag{3.4d}$$

where $r(m) = M + 1 - m$ is again the index reversal map. Then the duality map preserves the commutation rules.

The sequence of bonds is actually a generating set for the complete matrix algebra, so if the duality map can be extended to an algebra homomorphism, it is completely determined by this definition (3.4). A way to check whether this extension is possible, is to check whether the commutation relations (2.13) are preserved. Therefore, we compute the so-called *dual variables*. These are

$$\mu_m^x \equiv \Phi_d(\tau_m^x), \tag{3.5a}$$
$$\mu_m^y \equiv \Phi_d(\tau_m^y), \tag{3.5b}$$
$$\mu_m^z \equiv \Phi_d(\tau_m^z). \tag{3.5c}$$

By the definition of the duality map (3.4), we already know μ_m^x. However, we still need to compute μ_m^y and μ_m^z. This is done in the following lemma.

Lemma 3.2.1. *Assume Φ_d extends to an algebra homomorphism. Then the disorder variables μ_m^y and μ_m^z must be given by*

$$\mu_m^y = \frac{1}{2i}[\mu_m^z, \mu_m^x], \tag{3.6}$$
$$\mu_m^z = \tau_1^x \tau_2^x \dots \tau_{r(m)}^x. \tag{3.7}$$

Proof. We first prove the second equality for μ_m^z. This proof is by induction on m. The induction basis $m = 1$ is the definition of μ_1^z. For the induction step, assume (3.7) for $m = m' < M$. From (3.4c) and the assumption that Φ_d is a homomorphism we see that

$$\mu_{m'}^z \mu_{m'+1}^z = \Phi_d(\tau_{m'}^z)\Phi_d(\tau_{m'+1}^z) = \Phi_d(\tau_{m'}^z \tau_{m'+1}^z) = \tau_{r(m')}^x. \tag{3.8}$$

On the other hand, using (3.5c) we obtain

$$(\mu_{m'}^z)^2 = \Phi_d(\tau_{m'}^z)\Phi_d(\tau_{m'}^z) = \Phi_d(\tau_{m'}^z \tau_{m'}^z) = \Phi_d(I) = I.$$

Using this and multiplying (3.8) from the left with $\mu_{m'}^z$ we see

$$\mu_{m'+1}^z = (\mu_{m'}^z)^2 \mu_{m'+1}^z = \mu_{m'}^z \tau_{r(m')}^x.$$

Filling in the induction hypothesis (3.7) for $m = m'$ we see that (3.7) also holds for $m = m' + 1$.

Now we prove the equation for μ_m^y. This uses the commutator (2.13):

$$\mu_m^y = \Phi_d(\tau_m^y) = \Phi_d(\frac{1}{2i}[\tau_m^z, \tau_m^x]) = \frac{1}{2i}[\Phi_d(\tau_m^z), \Phi_d(\tau_m^x)] = \frac{1}{2i}[\mu_m^z, \mu_m^x]. \qquad \square$$

Now definitions (3.4), (3.5) and Lemma 3.2.1 imply that the disorder variables and the duality map must be given by

$$\mu_1^x = \Phi_d(\tau_1^x) = \tau_M^z, \tag{3.9a}$$
$$\mu_m^x = \Phi_d(\tau_m^x) = \tau_{r(m)}^z \tau_{r(m)+1}^z, \qquad m \neq 1, \tag{3.9b}$$
$$\mu_m^z = \Phi_d(\tau_1^z) = \tau_1^x \tau_2^x \dots \tau_{r(m)}^x, \tag{3.9c}$$
$$\mu_m^y = \Phi_d(\tau_m^y) = \frac{1}{2i}[\mu_m^z, \mu_m^x]. \tag{3.9d}$$

If the duality map indeed extends to an algebra isomorphism, the spin variables τ_m^a should be mapped to spin variables μ_m^a. With our definition (3.9), this is indeed the case.

Lemma 3.2.2. *The disorder variables* (3.9) *form a set of spin variables, i.e. they are self-adjoint matrices that satisfy the commutator/anti-commutator relations for $m, m' = 1, 2, \dots, M$:*

$$[\mu_m^a, \mu_{m'}^b] = 2i\delta_{m,m'}\epsilon_{abc}\mu_m^c \tag{3.10a}$$
$$\{\mu_m^a, \mu_m^b\} = 2\delta_{a,b} I. \tag{3.10b}$$

Proof. The proof is merely a computation and only uses the definitions (3.9), the commutator/anti-commutator relations (2.13) for τ_m^a, and the Jacobi identity.

To illustrate, we prove a hard case, which also illustrates why we had to be careful at the boundary. Let $m \neq m'$, we compute $[\mu_m^x, \mu_{m'}^z]$. Distinguish $m = 1$ and $m \neq 1$. If $m = 1$, then

$$[\mu_1^x, \mu_{m'}^z] \overset{(3.9)}{=} [\tau_M^z, \tau_1^x \tau_2^x \dots \tau_{r(m')}^x] \overset{(2.13)}{=} 0$$

where the last equality follows from the fact that $r(m') < M$ so all the factors act on different components. Note that the naive definition $\mu_1^x = \tau_M^z \tau_1^z$ (instead of our definition (3.9a)) would have spoiled this property.

If $m \neq 1$, then

$$[\mu_m^x, \mu_{m'}^z] \overset{(3.9)}{=} [\tau_{r(m)}^z \tau_{r(m)+1}^z, \tau_1^x \tau_2^x \dots \tau_{r(m')}^x].$$

We do another case distinction: If $m < m'$, then $r(m) > r(m')$ and again all factors act on different components, so we are done. If $m > m'$, we must do more work:

$$\begin{aligned}[\tau_{r(m)}^z \tau_{r(m)+1}^z, \tau_1^x \tau_2^x \dots \tau_{r(m')}^x] \\ = \tau_1^x \tau_2^x \dots \tau_{r(m)-1}^x [\tau_{r(m)}^z \tau_{r(m)+1}^z, \tau_{r(m)}^x \tau_{r(m)+1}^x] \tau_{r(m)+2}^x \dots \tau_{r(m')}^x\end{aligned} \tag{3.11}$$

considering again the components the factors act on. Now, using that $\tau_{r(m)}^z$ and $\tau_{r(m)}^x$ anti-commute, we see that

$$\begin{aligned}\tau_{r(m)}^z \tau_{r(m)+1}^z \tau_{r(m)}^x \tau_{r(m)+1}^x &= \tau_{r(m)}^z \tau_{r(m)}^x \tau_{r(m)+1}^z \tau_{r(m)+1}^x \\ &= (-\tau_{r(m)}^x \tau_{r(m)}^z)(-\tau_{r(m)+1}^x \tau_{r(m)+1}^z) \\ &= \tau_{r(m)}^x \tau_{r(m)}^z \tau_{r(m)+1}^x \tau_{r(m)+1}^z.\end{aligned}$$

So the commutator $[\tau^z_{r(m)}\tau^z_{r(m)+1}, \tau^x_r(m)\tau^x_{r(m)+1}]$ in (3.11) is zero and we are done. □

Now we can prove that the duality map Φ_d defined on the bonds by (3.4), extends to a C^*-algebra automorphism.

Theorem 3.2.3. *There is a unique C^*-algebra homomorphism $\Phi_d : \mathcal{B}(\otimes^M(\mathbb{C}^2)) \to \mathcal{B}(\otimes^M(\mathbb{C}^2))$ that satisfies* (3.4)*. Because Φ_d is its own inverse, it is also an automorphism and thus an inner automorphism.*

The statement that Φ_d is an inner automorphism just means that there exists $\phi_d \in \mathcal{B}(\otimes^M(\mathbb{C}^2))$ such that for all $a \in \mathcal{B}((\otimes^M(\mathbb{C}^2))$

$$\Phi_d(a) = \phi_d a \phi_d^{-1}. \tag{3.12}$$

Proof. Since the bonds form a generating set for $\mathcal{B}(\otimes^M(\mathbb{C}^2))$, and (3.4) fixes the map on these bonds, uniqueness is easy.

For existence, we need to do more work. Define the map Φ_d on the operators τ^a_m by (3.9). By Lemma 3.2.2, these form a set of spin variables. So Φ_d maps a set of spin variables to a set of spin variables, i.e. it preserves the spin commutation relations. In Appendix A, it will be proven that this alone implies that Φ_d extends to a C^*-algebra endomorphism of $\mathcal{B}(\otimes^M(\mathbb{C}^2))$. Now using the fact that Φ_d is an algebra homomorphism, it is easy to check that the definition (3.9) implies that Φ_d satisfies (3.4).

To prove that $\Phi_d^2 = I$, we only have to look at its action on the bonds, since these form a generating set. Because Φ_d satisfies (3.4), it just reverses the order of the bonds. Applying this reversal twice keeps the bonds invariant, i.e. $\Phi_d^2 = I$ on the bonds, so also on $\mathcal{B}(\otimes^M(\mathbb{C}^2))$.

Finally, as a corollary to the Skolem-Noether theorem, every algebra automorphism of a a central simple algebra is an inner automorphism [7]. Since $\mathcal{B}(\otimes^M(\mathbb{C}^2)) \cong \mathcal{B}(\mathbb{C}^{2^M})$ is such a central simple algebra [7], we obtain the result. □

3.3 The dual transfer matrix

From the defining properties of the dual variables (3.4), we can already see that the description in terms of the dual variables is very similar to the description in terms of the original variables. To be specific, the general terms in the transfer matrices V_1 and V_2 transform as

$$\begin{aligned} \Phi_d(K_1^*\tau^x_m) &= K_1^*\tau^z_{r(m)}\tau^z_{r(m)+1}, & m \neq 1, \\ \Phi_d(K_2\tau^z_m\tau^z_{m+1}) &= K_2\tau^x_{r(m)} & m \neq M. \end{aligned}$$

We see that the roles of V_1 and V_2 are exactly reversed! We will give an interpretation of this in Section 3.5, but for now we just note that the duality map not only reverses V_1 and V_2, but it also changes the couplings.

To discuss this more easily, we use the notation

$$V_1(K_1) \equiv \prod_{m=1}^{M} \exp(K_1^* \tau_m^x) \tag{3.13}$$

$$V_2(K_2) \equiv \exp(K_2 \tau_M^z) \prod_{m=1}^{M-1} \exp(K_2 \tau_m^z \tau_{m+1}^z) \tag{3.14}$$

to show that V_1 and V_2 depend on the couplings K_1 and K_2.

Theorem 3.3.1 (Kramers-Wannier duality)**.** *The square Ising model with $B = 0$ is self-dual: The duality map Φ_d is an inner C^*-algebra automorphism of $\mathcal{B}(\otimes^M \mathbb{C}^2)$ that preserves the structure of the transfer matrix in the sense that*

$$\Phi_d(V_1(K_1)) = V_2(K_1^*), \tag{3.15}$$

$$\Phi_d(V_2(K_2)) = V_1(K_2^*), \tag{3.16}$$

$$\Phi_d(V_1(K_1)V_2(K_2)) = V_2(K_1^*)V_1(K_2^*). \tag{3.17}$$

Proof. The statement that Φ_d is an inner algebra automorphism is just a restatement of Theorem 3.2.3. We actually only need to prove the first equality. The second follows from the fact that $\Phi_d^2 = I$, and the third follows automatically from the first two.

The computation is as follows:

$$\begin{aligned}
\Phi_d(V_1(K_1)) &= \Phi_d\left(\prod_{m=1}^{M} \exp(K_1^* \tau_m^x)\right) \\
&= \Phi_d\left(\exp(K_1^* \tau_1^x) \prod_{m=2}^{M} \exp(K_1^* \tau_m^x)\right) \\
&= \exp(K_1^* \Phi_d(\tau_1^x)) \prod_{m=2}^{M} \exp(K_1^* \Phi_d(\tau_m^x)) \\
&= \exp(K_1^* \tau_M^z) \prod_{m=2}^{M} \exp(K_1^* \tau_{r(m)}^z \tau_{r(m)+1}^z)) \\
&= \exp(K_1^* \tau_M^z) \prod_{m=1}^{M-1} \exp(K_1^* \tau_m^z \tau_{m+1}^z)) \\
&= V_2(K_1^*),
\end{aligned}$$

where we first split up the product, then use that Φ_d is an algebra homomorphism, use the definition (3.4) of Φ_d, and lastly reorder the product. □

Kramers-Wanniers duality has the following consequence for the average free energy. We use the notations $f = f(K_1, K_2)$ for the average free energy and $Z = Z(K_1, K_2)$ for the partition function to show that they are functions of the horizontal coupling K_1 and the vertical coupling K_2.

Corollary 3.3.2. *The average free energy $f(K_1, K_2)$ is related to $f(K_2^*, K_1^*)$ by*

$$f(K_1, K_2) = f(K_2^*, K_1^*) - \frac{1}{2\beta}\log(\sinh(K_1)\sinh(K_2)). \tag{3.19}$$

Note that the duality is independent of the dimensions of the lattice, and the duality is only exact because of the non-standard boundary conditions (3.3).

Proof. Using (2.16), we first write down $Z(K_1, K_2)$ and $Z(K_2^*, K_1^*)$ for $B = 0$:

$$Z(K_1, K_2) = (2\sinh(2K_1))^{MN/2}\mathrm{Tr}(V_1(K_1)V_2(K_2))^N \tag{3.20a}$$
$$Z(K_2^*, K_1^*) = (2\sinh(2K_2^*))^{MN/2}\mathrm{Tr}(V_1(K_2^*)V_2(K_1^*))^N \tag{3.20b}$$

Using the cyclicity of the trace and Theorem 3.3.1, we rewrite

$$\begin{aligned} Z(K_2^*, K_1^*) &= (2\sinh(2K_2^*))^{MN/2}\mathrm{Tr}(V_2(K_1^*)V_1(K_2^*))^N \\ &= (2\sinh(2K_2^*))^{MN/2}\mathrm{Tr}(\Phi_d(V_1(K_1)V_2(K_2)))^N \\ &= (2\sinh(2K_2^*))^{MN/2}\mathrm{Tr}(\phi_d V_1(K_1)V_2(K_2)\phi_d^{-1})^N \\ &= (2\sinh(2K_2^*))^{MN/2}\mathrm{Tr}(V_1(K_1)V_2(K_2))^N \end{aligned}$$

Comparing this with (3.20a) and using (2.19b), we obtain

$$Z(K_1, K_2) = (\sinh(K_1)\sinh(K_2))^{MN/2} Z(K_2^*, K_1^*).$$

With the definition (1.4) of the average free energy, we obtain the result. □

3.4 The critical temperature

We derived Kramers-Wannier duality with the non-standard boundary conditions (3.3). In the thermodynamic limit, it seems reasonable to suspect that these boundary conditions do not matter, since the bulk is much larger than the boundary. Thus in the thermodynamic limit, any square Ising model, using any boundary conditions, is self-dual, at least in the sense of Corollary 3.3.2.

Now we can finally derive the critical temperature. We *assume* that there is a phase transition, and moreover that there is only one critical temperature.

Corollary 3.4.1. *Suppose that, for given interaction factors k_1 and k_2, the square Ising model with $B = 0$ has precisely one critical temperature β_c. Then at this critical temperature, $K_1 = \beta_c k_1$ and $K_2 = \beta_c k_2$ must satisfy*

$$K_1^* = K_2.$$

For equal interaction factors $k_1 = k_2 = k$, we find the critical temperature

$$\beta_c = \frac{1}{2k}\log(1 + \sqrt{2}).$$

Proof. Let us take equal interaction factors $k_1 = k_2 = k$. Since there is a phase transition, the average free energy is non-analytic at (K, K) where $K \equiv \beta_c k$. From Theorem 3.3.2, the average free energy is also non-analytic at (K^*, K^*). If $K^* \neq K$, we can define $\beta_c^* \neq \beta_c$ such that $K^* = \beta_c^* k$. Since the average free energy is non-analytic at $(\beta_c^* k, \beta_c^* k)$, there must be a phase transition at the temperature $\beta_c^* \neq \beta_c$.

So, if the critical temperature β_c is unique, we must have $K^* = K$. Using (2.19d), we obtain the result. □

For unequal interaction factors $k_1 \neq k_2$, the idea is the same, but the proof is more difficult. Instead of considering one critical point, we must consider a curve γ of critical points. This γ must either lie on, under, or above the curve $K_1^* = K_2$. If γ does not lie on the curve $K_1^* = K_2$, then by Corollary 3.3.2, there is a second curve of critical points. If we again assume that there is precisely one critical temperature for each pair of interaction factors, then there can be only one such curve γ. This forces γ to lie on the curve $K_1^* = K_2$.

3.5 Interpretation of Kramers-Wannier duality

In the proof of Proposition 2.3.1, we have seen the relation between the Pauli matrices τ_m^a and the classical spins σ_{mn}: The matrix τ_m^x represents horizontal "disorder", while τ_m^z "measures" a spin σ_{mn}. In Lemma 2.3.2, we see, on the other hand, that there is a replacement

$$K_1 \tau_m \tau_m' \to K_1^* \tau_m^x$$

i.e. horizontal correlation at a coupling K_1 is replaced by the horizontal "disorder" τ_m^x, but then at a coupling K_1^*. From the same lemma we know that the map $K \mapsto K^*$ relates low couplings to high couplings. So the replacement $K_1 \tau_m \tau_m' \to K_1^* \tau_m^x$ suggests, intuitively, that measuring order at low couplings is the same as measuring disorder at high couplings.

An interpretation of Kramers-Wannier duality is that the disorder should also be described by spin variables. The correlation of these "disorder variables" μ_m^z should then be the horizontal disorder, so

$$\tau_m^x = \mu_{r(m)}^z \mu_{r(m)+1}^z.$$

Then the intuition "measuring order at low couplings is the same as measuring disorder at high couplings" is described by the replacement

$$K_1 \tau_m \tau_m' \to K_1^* \mu_{r(m)}^z \mu_{r(m)+1}^z.$$

Similarly, the "order variables" τ_m^z have a correlation that is described by the "disorder" of the disorder variables, so

$$\tau_m^z \tau_{m+1}^z = \mu_{r(m)}^x.$$

These relations between the order variables and the disorder variables are part of the defining properties of the duality map (3.4).

Chapter 4

Jordan-Wigner duality

In this chapter we will give an argument to show that the partition function of the square Ising model with $B = 0$ can be obtained. To be precise, we reduce the diagonalization of the $2^M \times 2^M$ transfer matrix V to the independent diagonalization of 4×4 matrices. We do not, however, explicitly perform this diagonalization.

Looking at the transfer matrices V_1 and V_2 of Proposition 2.3.1, we see that they have a cyclic structure: Due to the periodic boundary conditions, the transfer matrix is invariant under a constant vertical translation $m \mapsto m + k$ of the lattice. To exploit this feature, we want to perform a Fourier transformation of the variables τ_m^a. In terms of the Fourier transformed spin variables, the transfer matrix gets a much simpler structure than it already has. The problem is, however, that such a Fourier transformation does not preserve the commutation relations (2.13) between the spin matrices. Although the transfer matrix has a simple structure, none of the constituent matrices commute!

Therefore we are looking for a transformation to commutation relations that are preserved under a Fourier transformation. To do this, we need to transform the spin operators to a new set of operators, while at the same time preserving the structure of the transfer matrix. In the previous chapter, we have seen an example of this: Kramers-Wannier duality. In fact, the transformation of this chapter, Jordan-Wigner duality, will be very similar.

Schultz et al. [13] discovered how to simplify Onsager's computation of the partition function [12] using the Jordan-Wigner transformation. Our definition of the CAR (Section 4.1) is an adaptation of lecture notes by Landsman [6]. Sections 4.2 and 4.3 form our elaboration of the discussion by Cobanera et al. [2]. The subsequent computations (Sections 4.4 and 4.5) follow the presentation of Schultz et al. [13], with some details added. Lastly, our presentation is influenced by [6, 10, 14, 16].

4.1 Canonical Anticommutation Relations

The transformed variables are fermionic variables, which are elements of the algebra of Canonical Anticommutation Relations (CAR). So we first need to introduce the CAR. For our purposes, the following *finite-dimensional* definitions will suffice.

Let H and F be two finite-dimensional Hilbert spaces, and $\hat{c} : H \to \mathcal{B}(F)$ be an anti-linear map. Then the map $\hat{c}^* : H \to \mathcal{B}(F)$ defined by $\hat{c}^*(h) = \hat{c}(h)^*$ is a linear map, where $\hat{c}(h)^*$ denotes the adjoint of $\hat{c}(h)$. Now we say that $\hat{c}$ satisfies the canonical anticommutation relations (CAR) if it satisfies, for all $h, h' \in H$:

$$\{\hat{c}(h), \hat{c}^*(h')\} = \langle h, h' \rangle_H I_F \tag{4.1a}$$

$$\{\hat{c}(h), \hat{c}(h')\} = 0 \tag{4.1b}$$

where $\langle h, h' \rangle_H$ is the inner product on H and I_K is the identity on F. Note our inner product is linear in the second variable.

If such a map exist, then we define the algebra $CAR(H, F, \hat{c})$ of Canonical Anticommutation Relations over H on F as the C^*-algebra $C^*(\hat{c}(H))$ generated by the set $\hat{c}(H)$. This simply means we take the algebra generated by all elements $\hat{c}(h)$ and $\hat{c}^*(h)$.

We want $CAR(H, F, \hat{c})$ to be undisturbed by unitary transformations, such as the Fourier transformation, which is dealt with in the following theorem.

Theorem 4.1.1. *Let $u_H : H_1 \to H_2$, $u_F : F_1 \to F_2$ be unitary transformations, and suppose both $\hat{c} : H_1 \to \mathcal{B}(F_1)$ and $\hat{d} : H_2 \to \mathcal{B}(F_2)$ satisfy the CAR* (4.1)*. Then the map $\hat{c}(h) \mapsto \hat{d}(u_H h)$ defines a unique C^*-algebra isomorphism between $CAR(H_1, F_1, \hat{c})$ and $CAR(H_2, F_2, \hat{d})$.*

Proof. See Appendix A. □

This theorem implies that any two maps $\hat{c}, \hat{d} : H \to F$ satisfying the CAR define the same C^*-algebra $CAR(H, F, \hat{c}) \cong CAR(H, F, \hat{d})$. This allows us to define $CAR(H, F)$ without reference to the map $\hat{c}$, provided such a map exists.

Corollary 4.1.2. *Let $u_H : H_1 \to H_2$, $u_F : F_1 \to F_2$ be unitary transformations, and suppose $\hat{c} : H_1 \to \mathcal{B}(F_1)$ satisfies the CAR. If we define the map $\hat{d} : H_2 \to \mathcal{B}(F_2)$ by*

$$\hat{d}(h) \equiv u_F \hat{c}(u_H^* h) u_F^*$$

then $\hat{d}$ also satisfies the CAR (4.1)*. Therefore, $CAR(H_1, F_1) \cong CAR(H_2, F_2)$.*

Proof. We prove (4.1a), the other is simpler. Substituting the definition of $\hat{d}$ into (4.1a) yields

$$\begin{aligned}
\{\hat{d}(h), \hat{d}^*(h')\} &= \{\hat{d}(h), \hat{d}(h')^*\} \\
&= u_F \{\hat{c}(u^* h), \hat{c}(u^* h')^*\} u_F^* \\
&= u_F \{\hat{c}(u^* h), \hat{c}^*(u^* h')\} u_F^* \\
&= u_F \langle u^* h, u^* h' \rangle_{H_1} I_K u_F^* \\
&= \langle h, h' \rangle_{H_2} I_K.
\end{aligned}$$

The isomorphism between the CARs is a consequence of Theorem 4.1.1. □

Now we have not given an example of a $CAR(H, F)$ yet. The most natural example is given by $H = \mathbb{C}^M$ and $F = F_-(H)$, where $F_-(H)$ is the fermionic Fock space of dimension 2^M [6]. In this example, $\hat{c}^+(h) \equiv \hat{c}^*(h)$ creates a fermion with one-particle wave function h, while $\hat{c}^-(h) \equiv \hat{c}(h)$ annihilates a fermion with one-particle wave function h.

If we also choose an orthonormal basis $\{h_1, \ldots, h_M\}$ of H, we define

$$\hat{c}_m^\pm \equiv \hat{c}^\pm(h_m). \tag{4.2}$$

Clearly $\{\hat{c}_m^-\}$ is a linear basis of $\hat{c^-}(H)$, so $\{\hat{c}_m^-\}$ contains all information about $\hat{c}$. In fact, $\{\hat{c}_m^-\}$ satisfies the canonical anticommutation relations

$$\{\hat{c}_m^-, \hat{c}_{m'}^+\} = \delta_{m,m'} I_F, \tag{4.3a}$$

$$\{\hat{c}_m^-, \hat{c}_{m'}^-\} = 0. \tag{4.3b}$$

if and only if $\hat{c}$ satisfies the CAR (4.1). So if this is easier, it suffices to find operators $\{\hat{c}_m^-\}$ that satisfy (4.3).

Corollary 4.1.3. *We have the isomorphisms*

$$F_-(\mathbb{C}^M) \cong \otimes^M \mathbb{C}^2 \qquad \textit{as Hilbert spaces,}$$

$$CAR(\mathbb{C}^M, F_-) \cong CAR(\mathbb{C}^M, \otimes^M \mathbb{C}^2) = \mathcal{B}(\otimes^M \mathbb{C}^2) \qquad \textit{as } C^*\textit{-algebras.}$$

Proof. The first statement follows from the fact that both Hilbert spaces have dimension 2^M. The second isomorphism follows immediately from the first, using Corollary 4.1.2. The last equality follows from Schur's Lemma, but we will prove it by other means in the following sections. □

4.2 Motivation of Jordan-Wigner duality

Now why are these fermionic variables relevant for our problem? Firstly, Corollary 4.1.3 shows us that we can use a C^*-algebra isomorphism to transform our transfer matrix to an operator in $CAR(\mathbb{C}^M, F_-)$. Then, because of Theorem 4.1.2, we can finally apply a Fourier transformation! However, we should first construct an explicit C^*-algebra isomorphism Φ_d between $CAR(\mathbb{C}^M, F_-)$ and $\mathcal{B}(\otimes^M \mathbb{C}^2)$.

To start with, we do this for $M = 1$. This isomorphism Φ_d is completely determined by the choice of $\phi \equiv \Phi_d(\hat{c}(1))$. From the CAR, we infer that the 2×2 matrix ϕ should satisfy

$$\{\phi, \phi^*\} = I_2$$
$$\{\phi, \phi\} = 0.$$

From these properties we see that ϕ must be one of the spin raising/lowering operators $\tau^\pm$ (or any unitary conjugate). For future convenience, we choose the

spin raising/lowering operators $\tau^{\pm}$ with respect to the x-axis, which we define by

$$\tau^{\pm} \equiv \frac{1}{2}(\tau^z \mp i\tau^y).$$

Then it is easily checked that $\phi = \tau^-$ defines the sought C^*-algebra isomorphism for the case $M = 1$.

To construct the isomorphism in the general case $M \neq 1$, we will follow a similar strategy as we used to construct the Kramers-Wannier duality automorphism: We look for an isomorphism that preserves the structure of the transfer matrix. Specifically, the isomorphism should preserve the commutation relations between the bonds. This section and the next form an elaboration of the discussion by Cobanera et al. [2].

To start with, we rewrite these bonds in terms of the spin raising/lowering operators

$$\tau_m^{\pm} \equiv \frac{1}{2}(\tau_m^z \mp i\tau_m^y) \tag{4.5}$$

that were hinted by the case $M = 1$. From the commutation relations of the spin matrices (2.13), we can compute the commutation relations

$$[\tau_m^+, \tau_{m'}^{\pm}] = [\tau_m^-, \tau_{m'}^{\pm}] = 0, \qquad m \neq m', \tag{4.6a}$$

$$\{\tau_m^+, \tau_m^+\} = \{\tau_m^-, \tau_m^-\} = 0, \tag{4.6b}$$

$$\{\tau_m^+, \tau_m^-\} = I, \tag{4.6c}$$

Rewriting the spin matrices τ_m^a gives

$$\tau_m^x = 2(\tau_m^+\tau_m^- - 1/2) \tag{4.7a}$$

$$\tau_m^y = i(\tau_m^+ - \tau_m^-) \tag{4.7b}$$

$$\tau_m^z = \tau_m^+ + \tau_m^-. \tag{4.7c}$$

Substituting these expressions into (3.1) and expanding the products, gives us the *bonds*

$$\tau_1^+\tau_1^-, \tau_1^+\tau_2^{\pm}, \tau_2^+\tau_2^-, \tau_2^+\tau_3^{\pm}, \dots, \tau_M^+\tau_M^-, \tau_M^+\tau_1^{\pm}, (\tau_1^+\tau_1^-, \dots). \tag{4.8}$$

The commutation rules between these bonds are not as simple as those of (3.1). To be specific, bonds only fail to commute if they have overlapping indices.

Comparing the sequence (4.8) with the same sequence with fermionic operators

$$\hat{c}_1^+\hat{c}_1, \hat{c}_1^+\hat{c}_2^{\pm}, \hat{c}_2^+\hat{c}_2, \hat{c}_2^+\hat{c}_3^{\pm}, \dots, \hat{c}_M^+\hat{c}_M, \hat{c}_M^+\hat{c}_1^{\pm}, (\hat{c}_1^+\hat{c}_1, \dots),$$

reveals the same commutation structure! Again, bonds only fail to commute if they have overlapping indices. This shows that the map

$$\tau_m^+\tau_m^- \mapsto \hat{c}_m^+\hat{c}_m^-, \quad \tau_m^+\tau_{m+1}^{\pm} \mapsto \hat{c}_m^+\hat{c}_{m+1}^{\pm} \tag{4.9}$$

preserves all commuting bonds. This argument shows that the transition to fermionic variables could truly be practical. Most importantly, if the transition satisfies (4.9), then all the commuting bonds are preserved.

4.3 The Jordan-Wigner isomorphism

Motivated by the previous section, we define the map $\Phi_d : B \to CAR(\mathbb{C}^M, F_-)$, where B is the set $B \equiv \{\tau_1^-, \tau_1^+\tau_2^-, \tau_2^+\tau_3^-, \ldots, \tau_{M-1}^+\tau_M^-\}$ by

$$\Phi_d(\tau_1^-) \equiv \hat{c}_1^-, \tag{4.10a}$$

$$\Phi_d(\tau_m^+\tau_{m+1}^-) \equiv \hat{c}_m^+\hat{c}_{m+1}^-, \qquad m \neq M. \tag{4.10b}$$

The first equality is motivated by the case $M = 1$, while the second equality is motivated by the bonds of (4.9). Because

$$\tau_{m+1}^- = \tau_{m+1}^-\{\tau_m^+, \tau_m^-\} = \{\tau_m^+\tau_{m+1}^-, \tau_m^-\}, \tag{4.11}$$

the set B generates all elements $\tau_m^\pm$, and via (4.7), it generates the spin matrices τ_m^a. So B is a generating basis for the whole C^*-algebra $\mathcal{B}(\otimes^M\mathbb{C}^2)$. Therefore, if Φ_d extends to a homomorphism, it is uniquely determined by the definition (4.10).

Lemma 4.3.1. *Assume that Φ_d extends to a C^*-homomorphism $\mathcal{B}(\otimes^M\mathbb{C}^2) \to CAR(\mathbb{C}^M, F_-)$. Then $\Phi_d(\tau_m^\pm)$ must be given by*

$$\Phi_d(\tau_m^\pm) = \left(\prod_{j=1}^{m-1} \exp(i\pi\hat{c}_j^+\hat{c}_j^-)\right)\hat{c}_m^\pm. \tag{4.12}$$

Proof. The proof is very similar to the proof of Lemma 3.2.1, and by induction on m. The induction basis $m = 1$ follows directly from the definition (4.10), together with the small computation

$$\Phi_d(\tau_1^+) = \Phi_d((\tau_1^-)^*) = (\Phi_d(\tau_1^-))^* = (\hat{c}_1^-)^* = (\hat{c}_1^+). \tag{4.13}$$

For the induction step, assume (4.12) for $m < M$. From the definition (4.10), the equality (4.11) and the assumption that Φ_d is a homomorphism we obtain

$$\begin{aligned}
\Phi_d(\tau_{m+1}^-) &= \Phi_d(\{\tau_m^+\tau_{m+1}^-, \tau_m^-\}) \\
&= \{\Phi_d(\tau_m^+\tau_{m+1}^-), \Phi_d(\tau_m^-)\} \\
&= \{\hat{c}_m^+\hat{c}_{m+1}^-, \exp(i\pi\hat{c}_1^+\hat{c}_1^-)\ldots\exp(i\pi\hat{c}_{m-1}^+\hat{c}_{m-1}^-)\hat{c}_m^-\} \\
&= \exp(i\pi\hat{c}_1^+\hat{c}_1^-)\ldots\exp(i\pi\hat{c}_{m-1}^+\hat{c}_{m-1}^-)\{\hat{c}_m^+\hat{c}_{m+1}^-, \hat{c}_m^-\} \\
&= \exp(i\pi\hat{c}_1^+\hat{c}_1^-)\ldots\exp(i\pi\hat{c}_m^+\hat{c}_m^-)\hat{c}_{m+1}^-.
\end{aligned}$$

In the last step we use the properties

$$\begin{aligned}
\{\hat{c}_m^+\hat{c}_{m+1}^-, \hat{c}_m^-\} &= [\hat{c}_m^-, \hat{c}_m^+]\hat{c}_{m+1}^- \\
[\hat{c}_m^-, \hat{c}_m^+] &= I - 2\hat{c}_m^+\hat{c}_m^- = \exp(i\pi\hat{c}_m^+\hat{c}_m^-)
\end{aligned}$$

which can be verified using the commutation rules (4.1) and the power series expansion of the exponential.

A computation similar to (4.13) shows that also

$$\Phi_d(\tau^-_{m+1}) = \exp(i\pi\hat{c}^+_1\hat{c}^-_1)\ldots\exp(i\pi\hat{c}^+_m\hat{c}^-_m)\hat{c}^-_{m+1},$$

which completes the proof. □

Lemma 4.3.2. *The dual variables $\Phi_d(\tau^\pm_m)$ of (4.12) are a set of spin variables, i.e. they satisfy $\Phi_d(\tau^\pm_m)^* = \Phi_d(\tau^\mp_m)$ and they satisfy the commutation relations (4.6).*

Proof. The proof is merely a computation, which uses only the fermionic commutation relations (4.1). To facilitate the proof, it is handy to first check the following three identities:

$$\begin{aligned}(\exp(i\pi\hat{c}^+_j\hat{c}^-_j))^2 &= I,\\ \{\exp(i\pi\hat{c}^+_j\hat{c}^-_j), \hat{c}^\pm_j\} &= 0,\\ [\exp(i\pi\hat{c}^+_j\hat{c}^-_j), \exp(i\pi\hat{c}^+_{j'}\hat{c}^-_{j'})] &= 0 \qquad \text{for } j \neq j'.\end{aligned}$$ □

Using this last lemma, we can prove that the duality map Φ_d defined on the generating set B by (4.10), extends to a C^*-algebra isomorphism.

Theorem 4.3.3 (Jordan-Wigner transformation)**.** *There exists a unique C^*-algebra isomorphism $\Phi_d : \mathcal{B}(\otimes^M\mathbb{C}^2) \to CAR(\mathbb{C}^M, F_-)$ that satisfies (4.10). Therefore, this isomorphism Φ_d also satisfies (4.12).*

Proof. The proof is, again, very similar to the proof of Theorem 3.2.3. Since the set B is a generating set for $\mathcal{B}(\otimes^M\mathbb{C}^2)$, uniqueness is automatic.

For existence, we define the map Φ_d on the operators $\tau^\pm_m$ by (4.12). By Lemma 4.3.2, the operators $\Phi_d(\tau^\pm_m)$ also satisfy the CAR (4.1). By Theorem A.1, this implies that Φ_d extends uniquely to a C^*-algebra isomorphism from the C^*-algebra $C^*(\{\tau^\pm_m\})$ to the C^*-algebra $C^*(\{\ \Phi_d(\tau^\pm_m)\})$. Since the former is equal to $\mathcal{B}(\otimes^M\mathbb{C}^2)$ and the latter is equal to $CAR(\mathbb{C}^M, F_-)$, we are nearly done. Lastly, it is easy to show that Φ_d indeed satisifes (4.10), using the fact that Φ_d is a homomorphism.

The last statement just restates Lemma 4.3.1. □

4.4 The fermionic transfer matrix

To map the transfer matrix to the fermionic algebra, we first rewrite it in terms of the spin raising/lowering operators $\tau^\pm_m$. Using Proposition 2.3.1 and (4.7), this results in the expressions

$$V_1 = \prod_{m=1}^{M} \exp(2K^*_1(\tau^+_m\tau^-_m - 1/2)) \tag{4.14a}$$

$$V_2 = \prod_{m=1}^{M} \exp(K_2(\tau^+_m + \tau^-_m)(\tau^+_{m+1} + \tau^-_{m+1})). \tag{4.14b}$$

Proposition 4.4.1. *The Jordan-Wigner transformation Φ_d transforms the transfer matrices V_1 and V_2 as*

$$\tilde{V_1} \equiv \Phi_d(V_1) = \prod_{m=1}^{M} \exp(2K_1^*(\hat{c}_m^+\hat{c}_m^- - 1/2)) \tag{4.15a}$$

$$\tilde{V_2} \equiv \Phi_d(V_2) = \left(\prod_{m=1}^{M-1} \exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) \times \exp\left(-K_2(\hat{c}_M^+ - \hat{c}_M^-)(\hat{c}_1^+ + \hat{c}_1^-)(-1)^{\hat{N}}\right). \tag{4.15b}$$

In these expressions, we use the notations

$$\hat{N} \equiv \sum_{m=1}^{M} \hat{c}_m^+\hat{c}_m^- \tag{4.16a}$$

$$(-1)^{\hat{N}} \equiv \exp(i\pi\hat{N}). \tag{4.16b}$$

The operator $\hat{N}$ is called the total number operator, because it counts the total number of fermions. In this light, the operator $(-1)^{\hat{N}}$ determines the evenness $(+1)$ or oddness (-1) of the total number of fermions.

Note that the transformed transfer matrices (4.15) are indeed very similar in structure to the original transfer matrices (4.14), except for the end factor involving $(-1)^{\hat{N}}$. This was of course also the motivation (4.9) of the Jordan-Wigner transformation.

Proof (of Proposition 4.4.1). To compute $\Phi_d(V_1)$ and $\Phi_d(V_2)$, we first compute Φ_d for the simple constituent matrices, using (4.10) and (4.12). This gives

$$\Phi_d(\tau_m^+\tau_m^-) = \hat{c}_m^+\hat{c}_m^-, \tag{4.17a}$$

$$\Phi_d(\tau_m^+\tau_{m+1}^\pm) = \hat{c}_m^+\hat{c}_{m+1}^\pm \qquad \text{for } m \neq M, \tag{4.17b}$$

$$\Phi_d(\tau_m^-\tau_{m+1}^\pm) = -\hat{c}_m^-\hat{c}_{m+1}^\pm \qquad \text{for } m \neq M. \tag{4.17c}$$

For $m = M$ we obtain

$$\Phi_d(\tau_M^+\tau_1^\pm) = -\hat{c}_M^+\hat{c}_1^\pm(-1)^{\hat{N}} \neq \hat{c}_M^+\hat{c}_1^\pm, \tag{4.18a}$$

$$\Phi_d(\tau_M^-\tau_1^\pm) = \ \ \hat{c}_M^+\hat{c}_1^\pm(-1)^{\hat{N}} \neq -\hat{c}_M^+\hat{c}_1^\pm. \tag{4.18b}$$

Filling in these results and using the homomorphism property of Φ_d completes the proof. We show this for V_1, the computation of V_2 is more lengthy but not

more difficult.

$$\begin{aligned}\Phi_d(V_1) &= \Phi_d\left(\prod_{m=1}^{M}\exp(2K_1^*(\tau_m^+\tau_m^- - 1/2))\right)\\ &= \prod_{m=1}^{M}\exp(2K_1^*(\Phi_d(\tau_m^+\tau_m^-) - 1/2))\\ &= \prod_{m=1}^{M}\exp(2K_1^*(\hat{c}_m^+\hat{c}_m^- - 1/2)).\end{aligned}$$ □

We performed the Jordan-Wigner transformation in order to be able to do a Fourier transformation. In (4.15b), however, we see that we have lost the translational invariance, because of the end factor. Therefore, we need to get rid of this end factor. To do this, we note that $\tilde{V_1}$ and $\tilde{V_2}$ only contain even products of fermion operators, so they conserve the evenness or oddness of the total number of particles. More formally,

$$[(-1)^{\hat{N}}, \tilde{V_1}] = [(-1)^{\hat{N}}, \tilde{V_2}] = 0.$$

This means that $(-1)^{\hat{N}}$ and the transfer matrix $\tilde{V_1}\tilde{V_2}$ can be diagonalized simultaneously, i.e. we can consider simultaneous eigenvectors. The operator $(-1)^{\hat{N}}$ has eigenvalues ± 1, so we can separate the eigenvectors of $\tilde{V_1}\tilde{V_2}$ into two groups: the "even" eigenvectors v^+ satisfying $(-1)^{\hat{N}}v^+ = v^+$ and the "odd" eigenvectors v^- satisfying $(-1)^{\hat{N}}v^- = -v^-$. This means we can replace $(-1)^{\hat{N}}$ by ± 1. More precisely, this means the following.

Lemma 4.4.2. *If v^+ is an even vector, then we have $\tilde{V_2}\ v^+ = \tilde{V_2}^{+}v^+$, where*

$$\tilde{V_2}^{+} \equiv \left(\prod_{m=1}^{M}\exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) \tag{4.19}$$

and we have the anti-cyclic definitions

$$\hat{c}_{M+1}^+ \equiv -\hat{c}_1^+ \text{ and } \hat{c}_{M+1}^- \equiv -\hat{c}_1^-. \tag{4.20}$$

On the other hand, if v^- is an odd vector, then we have $\tilde{V_2}\ v^- = \tilde{V_2}^{-}v^-$, where

$$\tilde{V_2}^{-} \equiv \left(\prod_{m=1}^{M}\exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) \tag{4.21}$$

but now with the cyclic definitions

$$\hat{c}_{M+1}^+ \equiv \hat{c}_1^+ \text{ and } \hat{c}_{M+1}^- \equiv \hat{c}_1^-. \tag{4.22}$$

Proof. We prove the first statement, the second is completely analogous. Let v^+ be an even vector. We show that $\tilde{V_2}v^+ = \tilde{V_2}^+ v^+$. Firstly, we define for convenience the operator

$$\hat{A} \equiv K_2(\hat{c}_M^+ - \hat{c}_M^-)(\hat{c}_1^+ + \hat{c}_1^-). \tag{4.23}$$

Because $\hat{A}$ involves only even products of fermion operators, it commutes with $(-1)^{\hat{N}}$. Therefore we have

$$\begin{aligned}
\exp(-\hat{A}(-1)^{\hat{N}})v^+ &= \sum_{j=0}^{\infty} \frac{1}{j!}(-\hat{A}(-1)^{\hat{N}})^j v^+ \\
&= \sum_{j=0}^{\infty} \frac{1}{j!}(-\hat{A})^j((-1)^{\hat{N}})^j v^+ \\
&= \sum_{j=0}^{\infty} \frac{1}{j!}(-\hat{A})^j v^+ \\
&= \exp(-\hat{A})v^+
\end{aligned}$$

where we used that v^+ is an even vector in the penultimate step. With the definition (4.20), we clearly have

$$-\hat{A} = K_2(\hat{c}_M^+ - \hat{c}_M^-)(\hat{c}_{M+1}^+ + \hat{c}_{M+1}^-)$$

Using this, we obtain

$$\begin{aligned}
\tilde{V_2}\ v^+ &= \left(\prod_{m=1}^{M-1} \exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) \exp(-\hat{A}(-1)^{\hat{N}})v^+ \\
&= \left(\prod_{m=1}^{M-1} \exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) \exp(-\hat{A})v^+ \\
&= \left(\prod_{m=1}^{M} \exp(K_2(\hat{c}_m^+ - \hat{c}_m^-)(\hat{c}_{m+1}^+ + \hat{c}_{m+1}^-))\right) v^+ \\
&= \tilde{V_2}^+ v^+.
\end{aligned}$$

□

Lemma 4.4.2 shows us that it suffices to find the eigenvectors of $\tilde{V}^\pm \equiv \tilde{V_1}\tilde{V_2}^\pm$. We are looking for the even eigenvectors of $\tilde{V}^+$ and the odd eigenvectors of $\tilde{V}^-$. Thus our task is to diagonalize $\tilde{V}^\pm$.

4.5 The Fourier transformed transfer matrix

Now that we have restored the translational invariance of the transfer matrix, it is time to apply the promised Fourier transformation. This still requires some

care, because $\tilde{V}^+$ involves the anti-cyclic condition (4.20) while $\tilde{V}^-$ involves the cyclic condition (4.22).

To exploit the (anti-)cyclic conditions on $\hat{c}^\pm_{M+1}$, we go to a new set of "running wave operators" $\{\hat{d}^\pm_q\}$ defined by

$$\hat{d}^\pm_q \equiv M^{-\frac{1}{2}} e^{\pm i\phi} \sum_{m=1}^{M} \hat{c}^\pm_m e^{\pm iqm}. \tag{4.24}$$

To exploit the anti-cyclic conditions, we take

$$q \equiv l = \pm\pi/M, \pm 3\pi/M, \ldots, \pm(M-1)\pi/M \tag{4.25}$$

and to exploit the cyclic conditions, we take

$$q \equiv k = 0, \pm 2\pi/M, \pm 4\pi/M, \ldots, \pm(M-2)\pi/M, \pi. \tag{4.26}$$

In these formulas the implicit assumption is that M is even. This assumption does not matter, since in the end, we are interested only in the limit $M \to \infty$. The definition (4.24) implies that the inverse transformation is given by

$$\hat{c}^\pm_m = M^{-\frac{1}{2}} e^{\mp i\phi} \sum_q \hat{d}^\pm_q e^{\mp iqm}. \tag{4.27}$$

The mathematical justification of the Fourier transformation (4.24)-(4.27) uses Corollary 4.1.2, and is given in detail in Appendix B.

In the following theorem, one should read $q = l$ for $\tilde{V}^+$ and $q = k$ for $\tilde{V}^-$. In addition, it turns out to be most convenient to set the phase $\phi = \pi/4$.

Theorem 4.5.1. *In terms of the running wave operators* (4.24)*, the transfer matrix $\tilde{V}^\pm$ is given by a product of M commuting operators V_q:*

$$\tilde{V}^\pm = \prod_{0 \le q \le \pi} V_q. \tag{4.28}$$

The operators V_q for $q \neq 0$ or π are given explicitly by

$$V_q = V_{1q} V_{2q}, \tag{4.29a}$$

$$V_{1q} = \exp(-2K_1^*(\hat{d}^+_q \hat{d}^-_q + \hat{d}^+_{-q}\hat{d}^-_{-q} - 1), \tag{4.29b}$$

$$V_{2q} = \exp(2K_2[\cos(q)(\hat{d}^+_q \hat{d}^-_q + \hat{d}^+_{-q}\hat{d}^-_{-q}) + \sin(q)(\hat{d}^-_q \hat{d}^-_{-q} + \hat{d}^+_{-q}\hat{d}^+_q)]) \tag{4.29c}$$

For $q = 0$ or π they are given by

$$V_0 = \exp(-2(K_1^* - K_2)(\hat{d}^+_0 \hat{d}^-_0 - 1/2)) \tag{4.30a}$$

$$V_\pi = \exp(-2(K_1^* + K_2)(\hat{d}^+_\pi \hat{d}^-_\pi - 1/2)). \tag{4.30b}$$

Idea of proof. To start with, all the factors in $\tilde{V}_1$ commute and all the factors in $\tilde{V}_2^\pm$ commute, so we can write the product of exponentials as an exponential

of a sum. Therefore, we need to transform operators of the type $\sum_m \hat{c}_m^+ \hat{c}_m^-$ and $\sum_m \hat{c}_m^\pm \hat{c}_{m+1}^\pm$. These are expressed in terms of the running wave operators using the inverse transformation (4.27) and Lemma B.1 in Appendix B. To obtain more convenient expressions, it is handy to take the terms with q and $-q$ together, but to avoid double counting, V_0 and V_π should be kept separate.

Finally, if $q \neq q'$, then V_q and $V_{q'}$ involve only fermion operators of different indices. These anticommute, so that the bilinear products involved actually commute. □

Because the factors V_q commute, they can be diagonalized simultaneously and independently. From (4.29), we see that we have reduced the problem to the diagonalization of rather simple 4×4 matrices.

The explicit diagonalization can be found in the article by Schultz et al. [13].

Appendix A

Constructing isomorphisms of spin/fermionic algebras

In this appendix, we will show an easy way to construct C^*-algebra isomorphisms between C^*-algebras generated by either spin or fermion operators. We will prove the theorem in one go for both the spin raising/lowering operators and the fermion creation/annihilation operators. Therefore, we will denote them both by $f_m^\pm$ where $m = 1, \ldots, M$. It is understood that $f_m^\pm$ refers to either the spin operators, in which case they satisfy the commutation relations (4.6), or to the fermion operators, in which case they satisfy the commutation relations (4.1).

This appendix is entirely our own development. However, as the claims seem to be quite standard, we do not expect the results to be new. For example, we have found the fermionic case as a claim in an article by Araki [1].

Theorem A.1. *Consider a map Φ defined on $f_m^\pm$ by*

$$\Phi(f_m^\pm) \equiv g_m^\pm.$$

Assume that the operators $g_m^\pm$ satisfy the same (spin/fermionic) commutation relations as the $f_m^\pm$. Then Φ extends uniquely to a C^*-algebra isomorphism between the C^*-algebra $C^*(\{f_m^\pm\})$ generated by $\{f_m^\pm\}$ and the C^*-algebra $C^*(\{g_m^\pm\})$ generated by $\{g_m^\pm\}$.

Our proof relies on the concept of (extended) normal ordering. By this, we mean that the finite set of products

$$B = \left\{ (f_M^+)^{n_M^+} \ldots (f_1^+)^{n_1^+} (f_1^-)^{n_1^-} \ldots (f_M^-)^{n_M^-} \mid n_m^\pm \in \{0,1\} \right\}$$

is a linear basis for $C^*(\{f_m^\pm\})$. Here we use explicitly that $(f_m^\pm)^2 = 0$ to restrict the values of $n_m^\pm$ to $\{0,1\}$ only. In addition, we use that M is finite. Similarly, the finite set of products

$$\tilde{B} = \left\{ (g_M^+)^{n_M^+} \ldots (g_1^+)^{n_1^+} (g_1^-)^{n_1^-} \ldots (g_M^-)^{n_M^-} \mid n_m^\pm \in \{0,1\} \right\}$$

is a linear basis for $C^*(\{g_m^\pm\})$.

To illustrate that these are indeed linear bases, we give a simple example. We show that $f_1^- f_2^+ f_1^+$ can be written in this basis:

$$\begin{aligned} f_1^- f_2^+ f_1^+ &= \pm f_2^+ f_1^- f_1^+ \\ &= \mp f_2^+ (f_1^+ f_1^- + I) \\ &= \mp f_2^+ f_1^+ f_1^- + f_2^+ \end{aligned}$$

where the upper sign refers to the spin case and the lower sign to the fermion case. So we see that $f_1^- f_2^+ f_1^+$ can be written in the basis B, just by using the commutation relations.

Proof (Of Theorem A.1). We first extend Φ to the basis B. Because we want Φ to be a homomorphism, we are forced to define

$$\begin{aligned} \Phi\left((f_M^+)^{n_M^+} \dots (f_1^+)^{n_1^+} (f_1^-)^{n_1^-} \dots (f_M^-)^{n_M^-}\right) \\ \equiv (g_M^+)^{n_M^+} \dots (g_1^+)^{n_1^+} (g_1^-)^{n_1^-} \dots (g_M^-)^{n_M^-}. \end{aligned}$$

So we see that Φ sends the linear basis B to the linear basis $\tilde{B}$, so it extends uniquely to a linear isomorphism $C^*(\{f_m^\pm\}) \to C^*(\{g_m^\pm\})$.

To prove that Φ is a C^*-algebra isomorphism, we take two arbitrary elements v, v' of the basis B, given by

$$\begin{aligned} v &= (f_M^+)^{n_M^+} \dots (f_1^+)^{n_1^+} (f_1^-)^{n_1^-} \dots (f_M^-)^{n_M^-} \\ v' &= (f_M^+)^{n_M'^+} \dots (f_1^+)^{n_1'^+} (f_1^-)^{n_1'^-} \dots (f_M^-)^{n_M'^-} \end{aligned}$$

Now we show that $\Phi(v^*) = \Phi(v)^*$ and $\Phi(vv') = \Phi(v)\Phi(v')$. Here we go:

$$\begin{aligned} \Phi(v^*) &= \Phi\left(\left((f_M^+)^{n_M^+} \dots (f_1^+)^{n_1^+} (f_1^-)^{n_1^-} \dots (f_M^-)^{n_M^-}\right)^*\right) \\ &= \Phi\left((f_M^+)^{n_M^-} \dots (f_1^+)^{n_1^-} (f_1^-)^{n_1^+} \dots (f_M^-)^{n_M^+}\right) \\ &= (g_M^+)^{n_M^-} \dots (g_1^+)^{n_1^-} (g_1^-)^{n_1^+} \dots (g_M^-)^{n_M^+} \\ &= \left((g_M^+)^{n_M^+} \dots (g_1^+)^{n_1^+} (g_1^-)^{n_1^-} \dots (g_M^-)^{n_M^-}\right)^* \\ &= \Phi(v)^* \end{aligned}$$

where we use that $(f_m^\pm)^* = f_m^\mp$ and $(g_m^\pm)^* = g_m^\mp$.

The proof of $\Phi(vv') = \Phi(v)\Phi(v')$ relies on the fact that the expansion of an element of $C^*(\{f_m^\pm\})$ in the basis B can be computed using only the commutation relations. Writing the bases as $B = \{v_i\}$ and $\tilde{B} = \{\Phi(v_i)\}$, this implies that

$$\begin{aligned} vv' &= \sum_i \alpha_i v_i \\ \Phi(v)\Phi(v') &= \sum_i \alpha_i \Phi(v_i) \end{aligned}$$

have exactly the same coefficients α_i when expanded in the bases $B, \tilde{B}$, respectively, because the commutation relations are exactly the same. Using this, we see that

$$\begin{aligned}\Phi(vv') &= \Phi\left(\sum_i \alpha_i v_i\right) \\ &= \sum_i \alpha_i \Phi(v_i) \\ &= \Phi(v)\Phi(v').\end{aligned}$$

This completes the proof. □

Now we show how we can apply this theorem to the spin variables τ_m^a that satisfy the commutation relations (2.13).

Corollary A.2. *Consider a map Φ defined on τ_m^a by*

$$\Phi(\tau_m^a) \equiv \mu_m^a.$$

Assume that the operators μ_m^a satisfy the same commutation relations (2.13) *as τ_m^a. Then Φ extends uniquely to a C^*-algebra isomorphism between the C^*-algebra $C^*(\{\tau_m^a\})$ generated by $\{\tau_m^a\}$ and the C^*-algebra $C^*(\{\mu_m^a\})$ generated by $\{\mu_m^a\}$.*

Proof. To use Theorem A.1, we go to the spin raising/lowering operators $\tau_m^\pm$ and $\mu_m^\pm$ defined by (4.5). Because both τ_m^a and μ_m^a satisfy the commutation relations (2.13), the spin raising/lowering operators automatically satisfy the commutation relations (4.6). We now extend Φ linearly to $\tau_m^\pm$, which forces it to satisfy

$$\Phi(\tau_m^\pm) = \mu_m^\pm.$$

Then we apply Theorem A.1 and we are done. □

Appendix B

Fermionic Fourier transformation

In this appendix, we show how to use Corollary 4.1.2 to justify the Fourier transformation (4.24)-(4.27). Because the Fourier transformation we use is not immediately a standard discrete Fourier transformation, we show in detail how this works.

The fermionic Fourier transformation is found in the article by Schultz et al. [13]. The (simple) justification presented in this appendix is our own development. However, again, as the claims seem to be quite standard, we do not expect the results to be new.

To use Corollary 4.1.2, we need to implement the Fourier transformation as a unitary transformation $H \to H$. Therefore, we use the definition (4.2) of $\{\hat{c}_m^{\pm}\}$ and the (anti-)cyclic conditions (4.20) and (4.22), to translate the (anti-)cyclic conditions on $\{\hat{c}_{M+1}^{\pm}\}$ into the (anti-)cyclic conditions on the orthonormal basis $\{h_1, \ldots, h_M\}$ of H:

$$\begin{aligned} &\hat{c}^{\pm}(h_{M+1}) = \hat{c}_{M+1}^{\pm} = -\hat{c}_1^{\pm} = -\hat{c}^{\pm}(h_1) && \text{in the anti-cyclic case,} \\ &\hat{c}^{\pm}(h_{M+1}) = \hat{c}^{\pm}(h_1) && \text{in the cyclic case.} \end{aligned}$$

These conditions are satisfied if we have the condition $h_{M+1} = -h_1$ in the anti-cyclic case and $h_{M+1} = h_1$ in the cyclic case. This suggests that we should define the "Fourier transformed basis" by

$$\tilde{h}_q \equiv M^{-\frac{1}{2}} \sum_{m=1}^{M} h_m e^{-iqm}, \tag{B.1}$$

where q takes the values (4.25) in the anti-cyclic case and (4.26) in the cyclic case. We repeat that we need to assume that M is even. Because M is even, the set $\{\tilde{h}_q\}$ contains precisely M vectors. So if the set $\{\tilde{h}_q\}$ is an orthonormal set, then it is automatically an orthonormal basis of $H = \mathbb{C}^M$. Therefore, we

compute

$$\begin{aligned}\langle \tilde{h}_q, \tilde{h}_{q'} \rangle &= M^{-1} \left\langle \sum_{m=1}^{M} h_m e^{-iqm}, \sum_{m'=1}^{M} h_m e^{-iq'm'} \right\rangle \\ &= M^{-1} \sum_{m=1}^{M} e^{-iqm} e^{iq'm} \\ &= \delta_{q,q'}.\end{aligned}$$

In the computation, we use that $\{h_m\}$ is an orthonormal set. In the last step, we use the lemma below.

Lemma B.1. *Suppose q, q' are two elements of either the set (4.22) or the set (4.25). Then*

$$\sum_{m=1}^{M} e^{-iqm} e^{iq'm} = M\delta_{q,q'}. \tag{B.2}$$

Proof. We first compute

$$\sum_{m=1}^{M} e^{-iqm} e^{iq'm} = \sum_{m=1}^{M} e^{i(q'-q)m} = \sum_{m=1}^{M} \zeta^m$$

where we have introduced $\zeta \equiv e^{i(q'-q)}$. If $q = q'$, then $\zeta = 1$ so that we already have $\sum_{m=1}^{M} e^{-iqm} e^{iqm} = M$. If $q \neq q'$, then $\zeta \neq 1$. For both sets (4.26) and (4.25), we see that $q - q'$ is an even multiple of $2\pi/M$, so that $\zeta^M = 1$. Therefore

$$\sum_{m=1}^{M} \zeta^m = 1 + \zeta + \zeta^2 + \cdots + \zeta^{M-1} = \frac{1 - \zeta^M}{1 - \zeta} = 0$$

which completes the proof. □

So we see that $\{\tilde{h}_q\}$ is indeed an orthonormal basis for H. This shows that the change of basis $\{h_m\} \to \{\tilde{h}_q\}$ is implementable by a unitary transformation. Using Lemma B.1, we can also compute the inverse transformation

$$\begin{aligned}h_m &= \sum_q \langle \tilde{h}_q, h_m \rangle \tilde{h}_q \\ &= \sum_q \left\langle \sum_{m'} h_{m'} e^{-iqm'}, h_m \right\rangle \tilde{h}_q \\ &= M^{-\frac{1}{2}} \sum_q \tilde{h}_q e^{iqm}.\end{aligned}$$

Now we use Corollary 4.1.2. Because the change of basis $\{h_m\} \to \{\tilde{h}_q\}$ is a unitary transformation, the "running wave variables" defined by

$$\hat{d}^-_q = \hat{d}^-(\tilde{h}_q) = \hat{c}^- \left(M^{-\frac{1}{2}} \sum_{m=1}^{M} h_m e^{iqm} \right) = M^{-\frac{1}{2}} \sum_{m=1}^{M} e^{-iqm} \hat{c}^-(h_m)$$

define the same CAR algebra. Except for a trivial phase factor, this definition of the running wave operators is precisely (4.24).

Finally, to show the connection of all this with the standard discrete Fourier transform, denote the j-th component of the basis vector h_m by h_m^j, we see that

$$h^j : \{1, \dots, M\} \to \mathbb{C}, \quad h^j(m) = h_m^j$$

is a map satisfying $h^j(M+1) = \mp h^j(1)$, depending on the (anti-)cyclic conditions. Therefore we can apply the discrete Fourier transform to the component maps h^j, which results in

$$\tilde{h}_q^j = M^{-\frac{1}{2}} \sum_{m=1}^{M} h_m^j e^{-iqm}$$
$$h_m^j = M^{-\frac{1}{2}} \sum_{q} \tilde{h}_q^j e^{iqm}.$$

This leads to the definition (B.1) as well.

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