① Coordinate Systems, Homogeneous Transformations

**Goals:**
① Compute desired end effector position + orientation (or rigid body) for a task.
② Compute joint angles (for a manipulator) to realize the end effector's desired 6DOF state. (forward kinematics)
③ Specify a path from manipulator's <u>current</u> <u>state</u> to <u>desired state</u>. ⇒ Account for any obstacles to avoid collisions.

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manipulator tool tip frame T (fixed relative to last link frame n for an n-link manipulator)

base frame 0 (<u>fixed</u>)

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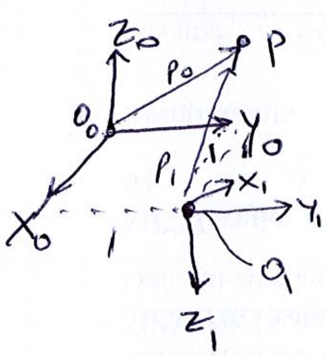
<u>From the beginning:</u>




**Positions**

* Point $p$ represented in frame 0 as $p^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (for example). (reference frame)

" " " " 1 as $p^1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

given indicated orientation and position of frame 1.

The origin of frame 1 relative to frame 0 is $O_1^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ while

" " " " relative to frame 1 is $O_0^1 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}$.

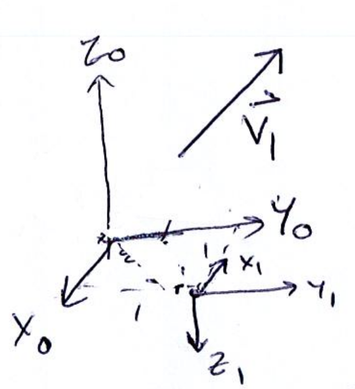
(2) A **point** is a location in space.

A **vector** has a direction and magnitude but is "free" (by default)
↳ no "root".

Vectors are also defined relative to a coordinate system

$$V_1^0$$

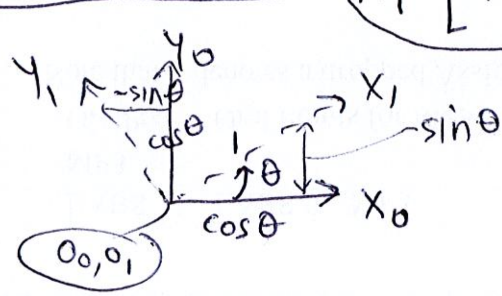
0 ← reference frame
1 ← vector ID




Suppose $V_1^0 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Then $V_1^1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

**Rotations:** To represent the relative position and orientation of one rigid body w/r/t another (or a fixed frame), we attach a coordinate system to the rigid body and to the other "reference" frame and specify their geometric relationship. Relative position of the two origins can be specified as above. We must also specify relative **orientation** of the two frames.

**Planar Rotation:** $R_1^0 = [x_1^0 \; y_1^0]$, $x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ ; $y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.




$$R_1^0 = \left[ \underbrace{x_1^0 \,\vdots\, y_1^0}_{} \right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

columns are unit vectors of frame 1 axes relative to frame 0.

3 We could also express:

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}.$$

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}.$$

Since dot product is commutative, e.g., $x_1 \cdot x_0 = x_0 \cdot x_1$,
we find that $R_0^1 = (R_1^0)^T$.

Geometrically, the orientation of frame 0 w/r/t frame 1
is the inverse of the " " 1 w/r/t " 0.

A rotation matrix is "orthonormal" →

$$(R_1^0)^T = (R_1^0)^{-1}.$$

We refer to $n \times n$ matrices as a special orthogonal group $SO(n)$. For any $R \in SO(n)$:

1. $R^T = R^{-1} \in SO(n)$.
2. Columns + rows of R are mutually orthogonal.
3. $\det(R) = 1$.

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The planar result generalizes to 3D rotation:

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}.$$

Let $c\theta = \cos\theta$,
$s\theta = \sin\theta$.

We can rotate about any single axis:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}, \quad R_{y,\theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}, \quad R_{z,\theta} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

④ A point defined in one coordinate frame can be expressed in another via the <u>rotational</u> transformation

$$p^0 = R_1^0 \, p^1 \quad \text{(assuming } o_0, o_1 \text{ coincident)}.$$

"Rotation <u>from</u> 1 to 0".

A matrix $R \in SO(2)$ or $R \in SO(3)$ can be interpreted as:

1. <u>Coordinate transformation</u> relating coordinates of $p$ in 2 diff. frames
2. <u>orientation</u> of one frame w/r/t another.
3. <u>Operator</u> taking a vector and rotating it to give a new vector in the same frame.

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Rotations can be <u>composed</u>. Suppose we have 3 frames and a point $p$.

$$p^0 = R_1^0 p^1, \quad p^1 = R_2^1 p^2, \quad p^2 = R_2^0 p^2.$$

we can also express

$$p^0 = R_1^0 \underbrace{R_2^1 p^2}_{p^1} \Rightarrow R_2^0 = R_1^0 R_2^1 .$$

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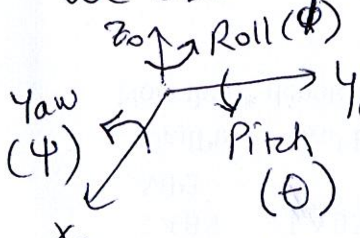
<u>Euler Angles</u> represent the sequential application of 3 <u>rotation operations</u> in 3-D.

We use the following convention:




Suppose the operations are applied in X-Y-Z order (first yaw, then pitch, then roll).

$$R = \underbrace{R_{x,\phi} R_{y,\theta} R_{x,\psi}}_{\text{see text}}$$

or compute result!

* Note: For quadrotor/Aero Yaw is about z, Roll is about x!!!

(5) A <u>rigid motion</u> is an ordered pair $(d, R)$ where $d \in \mathbb{R}^3$ and $R \in SO(3)$. Group of all rigid motions is Special Euclidean group $SE(3) = \mathbb{R}^3 \times SO(3)$.

pure translation <u>and</u> pure rotation.

$$p^0 = R_1^0 \, p^1 + d^0$$

We can represent a rigid motion (e.g. from one <u>frame</u> to <u>another frame</u>) in matrix form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \; d \in \mathbb{R}^3.$$

$H$ is a <u>homogeneous transformation</u> $\Rightarrow$ a matrix representation of a rigid motion.

Because $R \in SO(3)$ we can show that

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

We can augment coordinates with a 1 to support operations with matrix $H$:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \Rightarrow P^0 = H_1^0 P^1,$$

$$H_1^0 = \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 000 & 1 \end{array}\right]$$

$\Rightarrow$ Try simple exercises before Monday to gain intuition on use of $H$. $\leftarrow$ (Basis for forward kinematics!)