1

Forward Kinematics

Last time: Coordinate frames, $\underline{R}$, $\underline{H}$.

Def: $H_1^0 = \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 000 & 1 \end{array}\right]$; $p^0 = H_1^0 p^1$.

Note: since $R \in SO(3)$, $H^{-1} = \left[\begin{array}{c|c} R^T & -R^T d \\ \hline 000 & 1 \end{array}\right]$

because $R^{-1} = R^T$; e.g. $\underline{R_0^1 = R_1^{0T}}$.

and $d_0^1 = -(R_0^1 d_1^0)$. $\Rightarrow$ opposite offset "vector directions".

Manipulator: set of links connected by joints.

Simple joints are revolute (rotational) or prismatic (translational).
$\rightarrow$ angular motion $\qquad$ $\rightarrow$ linear motion

An n-joint manipulator has (n+1) links.

We number the joints 1-n ; we number links 0-n, starting from the base.

Def: ith joint variable $q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic.} \end{cases}$

A coordinate frame is rigidly attached to each link (i) and moves with that link.
$(x_i\, y_i\, z_i\, O_i)$

We can then define a kinematic chain such that each link frame i is related to link frame $i-1$ as:

$$A_i = H_i^{i-1} = \left[\begin{array}{c|c} R_i^{i-1} & O_i^{i-1} \\ \hline 0 & 1 \end{array}\right] = A_i(q_i)$$

(2) Then the overall transformation for forward kinematics is:

$$H = H_n^0 = \left[\begin{array}{c|c} R_n^0 & O_n^0 \\ \hline O & 1 \end{array}\right]$$

and this H is computed from the "chain"

$$H_n^0 = T_n^0 = A_1(q_1)\, A_2(q_2) \ldots A_n(q_n).$$

Denavit-Hartenberg (D-H) convention for forward kinematics

Each A_i is a product of 4 basic transformations;

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

$$= \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ d_i \end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \begin{matrix} a_i \\ 0 \\ 0 \end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$A_i = \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$\theta_i, a_i, d_i, \alpha_i \Rightarrow$ D-H parameters describing the geometry of link i.

(3) D-H offers 4 parameters rather than 6 because of how we place the origin and axes of frame i w/r/t frame $i-1$. <u>D-H construction properties</u>:

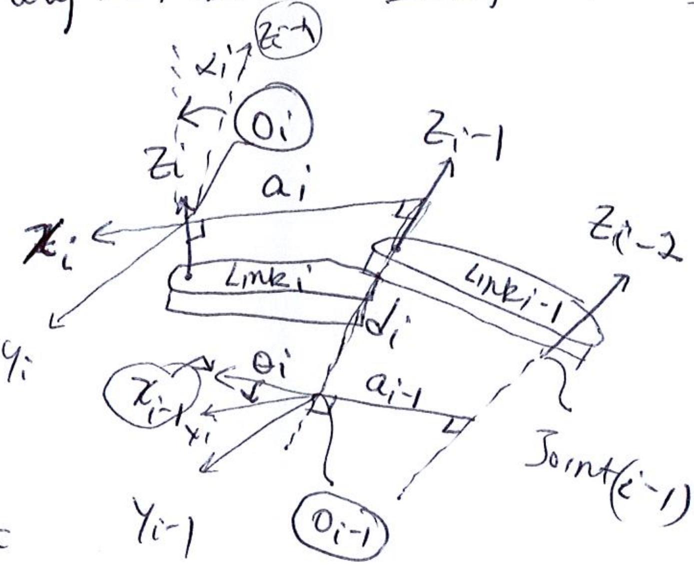
(DH1) Axis x_1 is perpendicular to axis z_0. (for example)

(DH2) Axis x_1 intersects axis z_0.

Coordinate frame assignment.

(1) Select <u>base frame</u>. (Arbitrary but define as <u>simply</u> as possible) (frame 0)

(2) Define axes & o's. 3 cases. Each discussed below.



Case

(1) <u>z_{i-1}, z_i not coplanar</u> $\Rightarrow$ define unique shortest segment from z_{i-1} to z_i as <u>x_i axis</u>. o_i is point where <u>x_i</u>, <u>z_i</u> intersect. <u>y_i</u> completes right-handed frame.

(2) <u>z_{i-1} parallel to z_i</u>: choose o_i anywhere on z_i; extend x_i perpendicular to pass through O_i. <u>y_i</u> completes RHS. (<u>typical</u>: choose o_i s.t. normal passes through O_{i-1} $\Rightarrow d_i = 0$).

(3) <u>z_{i-1} intersects z_i</u>: <u>x_i</u> is normal to the z_{i-1}-z_i plane. o_i typically chosen at intersection point. <u>$a_i = 0$</u>.

Inverse Kinematics

Given a 4x4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

find one or more solutions of the equation

$$T_n^0(q_1, q_2, \ldots, q_n) = H$$

where

$$T_n^0(q_1, q_2, \ldots, q_n) = A_1(q_1) \ldots A_n(q_n).$$

The equation for $T_n^0 = H$ represents
<u>12</u> <u>nonlinear</u> <u>equations</u> $T_{ij}(q_1, \ldots, q_n) = h_{ij}$,
$i = 1,2,3,\ j = 1,2,3,4.$

Inverse kinematics is <u>typically</u> more difficult than forward kinematics. We <u>could</u> use a numerical solver (e.g. in Matlab) to solve these equations, but this would be <u>slow</u> embedded in a control loop and would offer little intuition. Also, in cases with multiple solutions, a numerical solver would find <u>only</u> <u>one</u> with sensitivity to the initial guess.

Some <u>manipulators</u> can be designed with

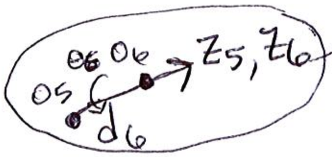
<u>Kinematic decoupling</u>: inverse position kinematics
v. inverse orientation ".

(2) Suppose a revolute arm has 6 DOF. The first three position the "origin" of a "wrist frame" and the last three specify its orientation. (wrist is where z_3, z_4, z_5 intersect!)

Let $R = R_6^0(q_1, \ldots, q_6)$, $o = o_6^0(o_1, \ldots, o_6)$.

Let o_c be the wrist ~~frame~~ (sometimes, not always, $o_c = o_3$).

Then $o = o_c + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so $\boxed{o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$



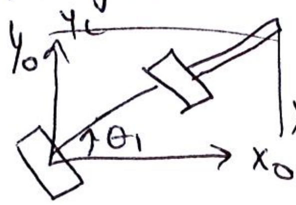
Now note that $R = R_3^0 R_6^3$.

R_3^0 is computed from $o_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

We can also use geometry! The "articulated manipulator" shown in Spong Sec 3.3.4 provides good insight on solving for inverse kinematics geometrically

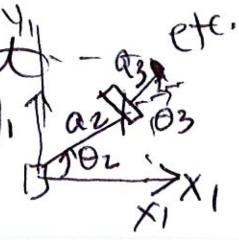
(1) Project wrist center onto x_0-y_0 plane



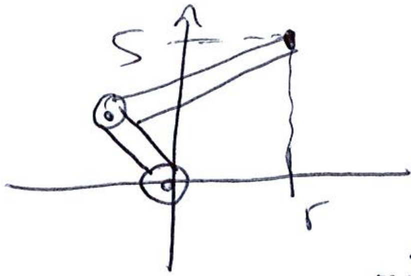
$\Rightarrow \theta_1 = \text{Atan2}(x_c, y_c)$

use atan2 to assure correct quadrant

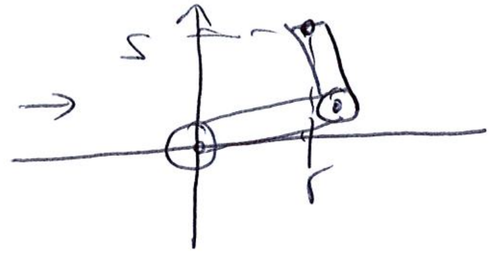
Spong shows a case with offsets. Without offsets, one can then look @ planar manipulator



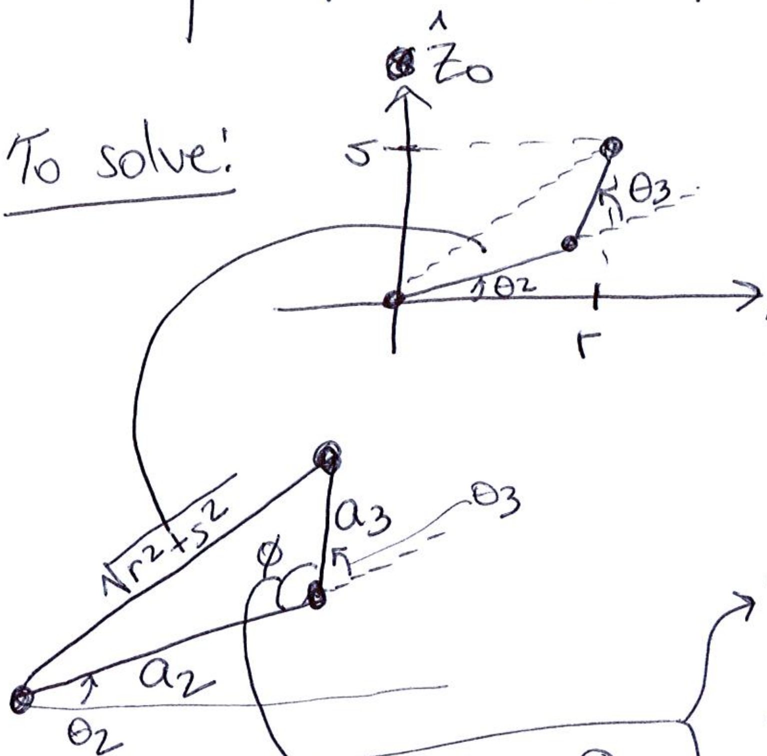
③ Left configuration   Right configuration



← two I.K. solutions to choose from! →



To solve!

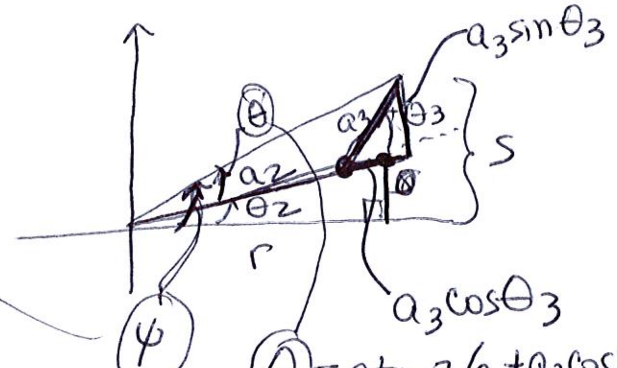


(Note $\hat{x}_1$ is $\hat{x}_0$ rotated θ_1 about $\hat{z}_0$)

$$\boxed{\phi = \pi - \theta_3}$$

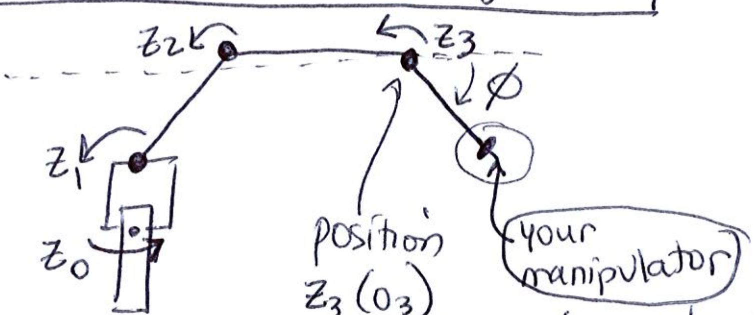
- Solve for ϕ from the Law of Cosines.
- Then solve for θ_3.
- Then solve for θ_2:

$$\boxed{\begin{aligned}\varphi &= \tan^{-1}\left(\frac{s}{r}\right)\\ &= \text{atan2}(r, s).\end{aligned}}$$



$\theta = \text{atan2}(a_2 + a_3\cos\theta_3,\ a_3\sin\theta_3)$

Diagram in the assignment:



s.t. rotating ϕ about z3 gives the illustrated "wrist orientation" (w/r/t the horizontal plane.)