

$$1, 1$$

$$a \xrightarrow{ab} b$$

$$ac \downarrow$$

$$c$$

$$\text{If } \text{Task}(ab) = \text{Task}(ac)$$

Then: $ab = ac$ by Task Reduction Uniqueness

Else:

$$a \xrightarrow{ab} b$$

$$\downarrow ac \quad \downarrow ac \text{ by Commutativity}$$

$$c \xrightarrow{ab} d$$

$$X, Y = Y, X$$

by renaming

$X+1, Y$ with IHs X, Y and $1, Y$

$$a \xrightarrow{ab^x} b \xrightarrow{bb'} b'$$

$$ac^y \downarrow \quad \downarrow \quad \downarrow$$

$$c \xrightarrow{\quad} d \xrightarrow{\quad} d'$$

$$\text{IH: } \forall X', Y'. X' \leq X \wedge Y' \leq Y \Rightarrow$$

$$\forall a, b, c \in S. a \xrightarrow{ab^x} b \Rightarrow \exists d \in S. b \xrightarrow{bd^*} d \wedge c \xrightarrow{cd^*} d$$

$$ac^y \downarrow \quad \downarrow$$

$$c \quad \quad \quad$$

$$\wedge |bd^*| \leq X'$$

$$\wedge |cd^*| \leq Y'$$

$$\text{IH: } X, Y \quad a \xrightarrow{ab^x} b$$

$$ac^y \downarrow \quad \downarrow bd^*$$

$$c \xrightarrow{cd^*} d$$

$$\text{IH: } 1, Y \quad b \xrightarrow{bb'} b'$$

$$bd^* \downarrow \quad \downarrow b'd'^*$$

$$d \xrightarrow{dd'^*} d'$$

$$\Rightarrow$$

$$a \xrightarrow{ab^x} b \xrightarrow{\quad} b'$$

$$ac^y \downarrow \quad \downarrow bd^* \quad \downarrow b'd'^*$$

$$c \xrightarrow{\quad} d \xrightarrow{\quad} d'$$

$$cd'^* \quad \quad \quad$$

$$\text{with } |cd'^*| = |cd^*| + |dd'^*| \leq |ab^x| + |bb'| = X+1$$

$$|b'd'^*| \leq |bd^*| \leq |ac^y| = Y$$

□

Induction completeness:

X, Y reached by $1, 1 \xrightarrow[(x,y) \rightarrow (x+1,y)]{(Y-1) \text{ times}} Y, 1 \xrightarrow[\text{symm}]{X-1 \text{ times}} 1, Y \longrightarrow X, Y$
for $X, Y \geq 1$

Task Reduction Uniqueness

Proof by induction on term syntax:

Always exactly one rule applicable

Commutativity:

$$H, \{T_1, T_2\} \wp TS \leadsto H_1, \{T_1', T_2\} \wp TS \leadsto H_{12}, \{T_1', T_2''\} \wp TS$$

$$H, \{T_1, T_2\} \wp TS \leadsto H_2, \{T_1, T_2'\} \wp TS \leadsto H_{21}, \{T_1'', T_2'\} \wp TS$$

$$\text{Show: } T_1' = T_1'' \quad T_2' = T_2'' \quad H_{12} = H_{21}$$

$$\forall o \in H. \text{accRoot}(o, T_1) \Rightarrow H(o) = H_2(o)$$

$$\forall o \in H. \text{accRoot}(o, T_2) \Rightarrow H(o) = H_1(o)$$

Edge case: one task finishing