EEN020 Assignment 1

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1 Theoretical Exercise 1

In order to calculate the 2D Cartesian coordinates of the homogenous points we do the following:

$$x_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x_1 = \begin{pmatrix} \frac{x}{z} \\ \frac{y}{z} \end{pmatrix} \in \mathbb{R}^2 \quad \text{for } z \neq 0 \tag{1}$$

Below are the calculations of 2D Cartesian coordinates for x_1, x_2 and x_3.

$$x_1 = \begin{pmatrix} -9 \\ 3 \\ 3 \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \in \mathbb{R}^2 \tag{2}$$

$$x_2 = \begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x_2 = \begin{pmatrix} -7 \\ 5 \end{pmatrix} \in \mathbb{R}^2 \tag{3}$$

$$x_3 = \begin{pmatrix} -2\lambda \\ 25\lambda \\ 5\lambda \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x_3 = \begin{pmatrix} \frac{-2}{5} \\ 5 \end{pmatrix} \in \mathbb{R}^2 \tag{4}$$

The interpretation of the point x_4 is that it is a point at infinity and it do not have a 2D Cartesian coordinate representation.

$$x_4 = \begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix} \tag{5}$$

No, x_4 is not the same point as x_5 but it is also a point at infinity.

$$x_5 = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \tag{6}$$

2 Computer Exercises 1

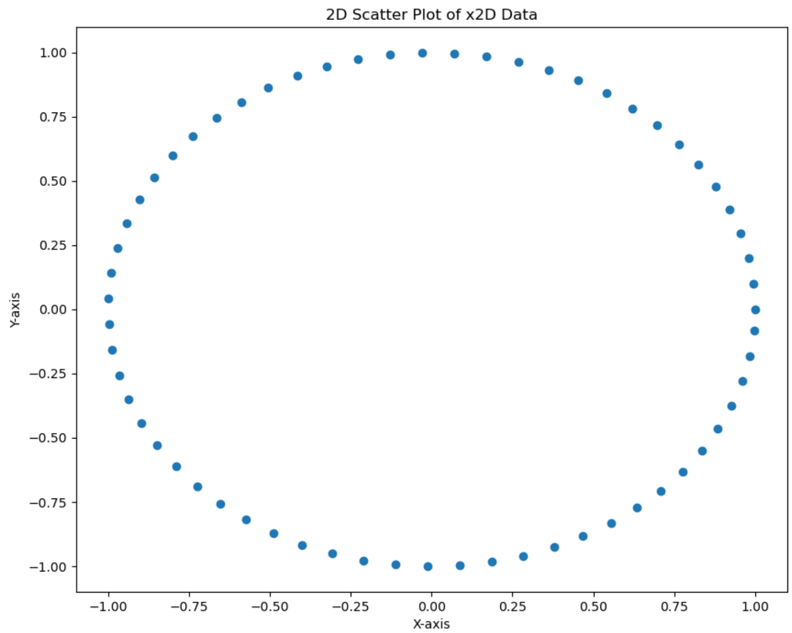

Figure 1: Computer Exercise 1: 2D plot computer exercises 1.

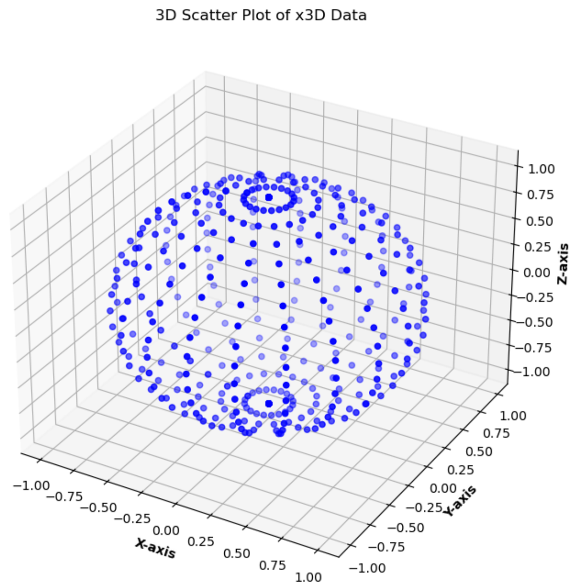

Figure 2: Computer Exercise 1: 3D plot computer exercises 1.

3 Theoretical Exercise 2

Compute the intersection of lines:

$$l_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad l_2 = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \tag{7}$$

The line equation for a point $\alpha \sim (x, y, z)$ in $\mathbb{P}^2$ is:

$$ax + by + cy = 0 \tag{8}$$

Using the line equation (8) we can formulate the following two equations in order to find the intersection point between the two lines, where $\alpha = (x, y, z)$.

$$l_1^T \alpha = 0 \tag{9}$$

$$l_2^T \alpha = 0 \tag{10}$$

From the equations above we get:

$$x - 2y + z = 0 \tag{11}$$

$$5x + 3y + z = 0 \tag{12}$$

By subtracting (11) from (12) we get:

$$x = \frac{-5}{4}y \tag{13}$$

And by substituting x into (11) we get:

$$z = \frac{13}{4}y \tag{14}$$

Let $y = t$ then we get the following intersection point:

$$(x, y, z) = (\frac{-5}{4}t, t, \frac{13}{4}t) \tag{15}$$

If we let $t = 4$ we get $(x, y, z) = (-5, 4, 13)$ and in Cartesian 2D coordinates we get the point $(\frac{-5}{13}, \frac{4}{13}) \in \mathbb{R}^2$.

Using the same logic the intersection in $\mathbb{P}^2$ for the lines:

$$l_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad l_4 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \tag{16}$$

$$l_3^T \alpha = 0 \tag{17}$$

$$l_4^T \alpha = 0 \tag{18}$$

From the equations above we get:

$$-2x + z = 0 \tag{19}$$

$$3x + 2z = 0 \tag{20}$$

By subtracting (19) from (20) we get:

$$z = -5x \tag{21}$$

Then substitute z in (20) and we get:

$$x = 0 \tag{22}$$

Then will also $z = 0$. Let $y = t$ then we get that the intersection points is $(x, y, z) = (0, t, 0)$. This means that the corresponding point in 2D is at infinity and this in turn means that the two lines are parallel and therefore don't have a finite intersection point in $\mathbb{R}^2$.

The line that goes through the points:

$$l_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad l_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{23}$$

Can be calculated by the following equations:

$$l^T \alpha_1 = 0 \tag{24}$$

$$l^T \alpha_2 = 0 \tag{25}$$

This will result in the same calculations as the first sub-question in theoretical exercises 2 but for the line equations coefficients instead of the point (x,y,z) values. So $(a, b, c) = (\frac{-5}{4}t, t, \frac{13}{4}t)$. If we let $t = 4$ we get the following line:

$$l = \begin{pmatrix} -5 \\ 4 \\ 13 \end{pmatrix} \tag{26}$$

4 Theoretical Exercise 3

The intersection point of l_1 and l_2: $\begin{pmatrix} -5 \\ 4 \\ 13 \end{pmatrix}$.

Definition of nullspace:

$$N(A) = \{x \in \mathbb{R}^n; Ax = 0\}$$

For all x for which the matrix multiplication Mx gives zero vector as result x is the nullspace of M.

Lets check this for our intersection point:

$$N(M) = \begin{pmatrix} 5 & 3 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have verified above that the intersection point of l_1 and l_2 is in the nullspace of M.

Its only the intersection point:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ t \\ 13t \end{pmatrix}$$

that satisfy the nullspace definition. We can see that we can choose different values of $t \in \mathbb{R}\backslash\{0\}$ and scale the intersection point and still satisfy the nullspace definition. But this is still the intersection point.

5 Computer Exercise 2

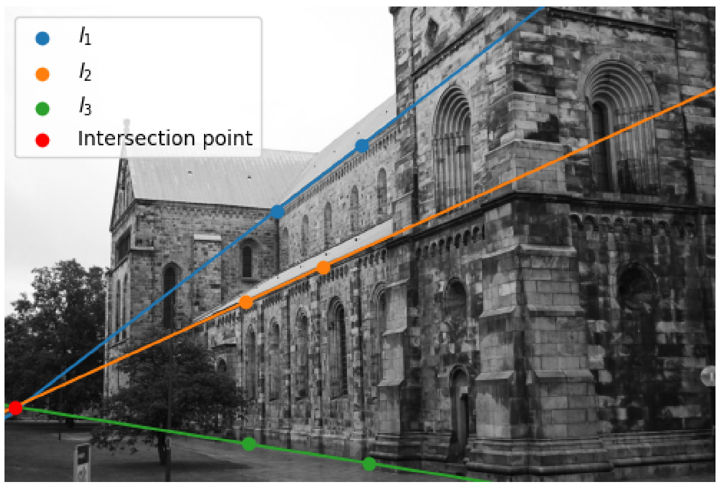

Figure 3: Computer Exercise 2: Plotting of lines.

By examining the lines l_1, l_2, and l_3, they appear to be parallel in 3D, which makes sense, as those parts of the building would be parallel in reality.

For the three lines to be perfectly parallel in 3D, they would need to intersect at a single point. In our case, this is not true: lines l_2 and l_3 intersect at a point, while line l_1 is approximately $d \approx 8.27$ units away from this intersection point. This indicates that the three lines are not perfectly parallel in 3D.

Although the distance $d \approx 8.27$ is not close to zero, it is relatively small on the scale of the picture and close to the intersection of l_2 and l_3. Thus, on the picture scale, it appears that all three lines almost intersect at a single point, supporting the impression that they are parallel in 3D. One possible reason they do not intersect exactly at one point could be minor inaccuracies in the data points, resulting in slight errors preventing perfect alignment.

6 Theoretical Exercise 4

Given the projective transformation matrix:

$$H = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate $y_1 \sim Hx_1$ and $y_2 \sim Hx_2$ where:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We get:

$$y_1 \sim Hx_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y_2 \sim Hx_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Then we compute the line l_1 by the following:

$$l_1 = x_1 \times x_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

And compute the line l_2 by following:

$$l_2 = y_1 \times y_2 = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

In order to compute $(H^{-1})^T l_1$ we must first compute H^{-1}.

$$H^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$(H^{-1})^T l_1 = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

We can see that:

$$l_2 = 2(H^{-1})^T l_1$$

So we can see that $l_2 \sim (H^{-1})^T l_1$.

7 Theoretical Exercise 5

Given that x belongs to $l1$:

$$l_1^T x = 0$$

And that y belongs to:

$$l_2^T = 0$$

The matrix transpose rule:

$$(AB)^T = B^T A^T$$

We get the following proof:

$$l_1^T x = l_1^T H^{-1} H x = (H^{-T} l_1)^T H x \sim l_2^T y$$

8 Theoretical Exercise 6

a) H_1, H_2, H_3, H_4
b) H_2, H_3
c) H_3
d) H_3
e) H_3
f) H_1, H_2, H_3, H_4
g) H_2, H_3

9 Theoretical Exercise 7

The projections of the points X_1, X_2 and X_3:

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

is the following:

$$x_1 \sim PX_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$x_2 \sim PX_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$x_3 \sim PX_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Inhomogeneous coordinates of $x_2 =$ infinite. Which means that the projection x_2 of X_2 is that it projects to an infinite point in the image plane.

We know that the projection $x \sim PX$. The camera center C in the world frame is the point that maps to the origin in the camera's coordinate frame.
C is a 3D coordinate hence:

$$C_* = \begin{pmatrix} C \\ 1 \end{pmatrix}$$

In other words, it is the world point for which the projection yields a 0 vector in the image, meaning that $0 \sim PC_*$.

Using $P = k[R|T]$, we have:

$$PC = K[R|t]C_* = 0$$

Multiplying with K^{-1} on both side we get:

$$[R|t]C_* = 0$$

Then we have:

$$[R|t] \begin{pmatrix} C \\ 1 \end{pmatrix} = 0$$

The equations can be written as:

$$RC + t = 0$$

Thus:

$$C = -R^{-1}t$$

The camera center C is:

$$C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The principal axis also called the viewing direction is:

$$axis = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

10 Computer Exercise 3

The camera centers c_1 and c_2 and principle axis a_1 and a_2 of $P1$ and $P2$.

$$c_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p_1 = \begin{pmatrix} 0.31292281 \\ 0.94608467 \\ 0.08368463 \end{pmatrix}$$

$$c_2 = \begin{pmatrix} 6.6352039 \\ 14.84597919 \\ -15.06911585 \end{pmatrix} \quad p_2 = \begin{pmatrix} 0.03186384 \\ 0.34016542 \\ 0.93982561 \end{pmatrix}$$

In figure 4 we can see that the camera view direction seems to match those in figure 5. So the results seems reasonable.

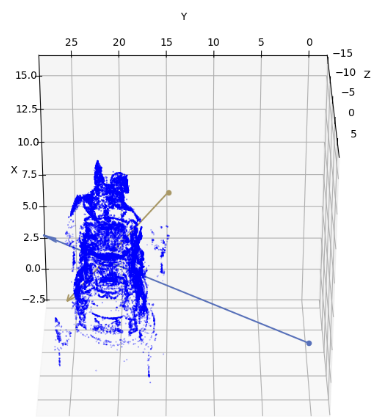

Figure 4: Computer Exercise 3

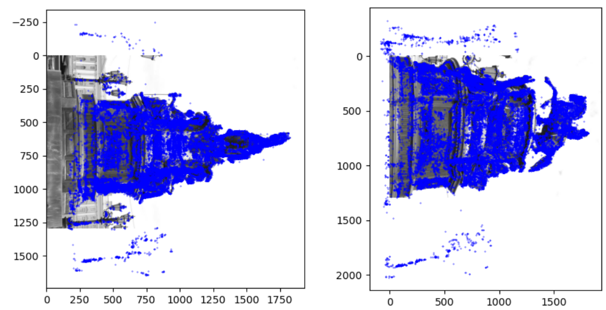

Figure 5: Computer Exercise 3

11 Theoretical Exercise 8

We have two calibrated camera pairs:

$$P_1 = \begin{pmatrix} I & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} R & t \end{pmatrix}$$

Where $x \in \mathbb{P}^2$ is the 2D projection in P_1 of the 3D point $U \in \mathbb{P}^3$ and we want to verify that:

$$U \sim \begin{pmatrix} x \\ s \end{pmatrix}$$

where $s \in \mathbb{R}$.

We have that:

$$P_1 U = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = Ix + 0 \cdot s = Ix = x \Rightarrow x \sim P_1 U$$

So it is verified that for any s the point of the form $U(s) = (x^T, s)^T$ project to x.
The collections of points of the form $U(s) = \begin{pmatrix} x & s \end{pmatrix}^T$ will represent lines in 3D:

$$U(s) = \begin{pmatrix} x \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

No, it is not possible to determine s from using only information from´P_1. This is because s gets cancelled out $(0 \cdot s)$ in the multiplication above.

U belongs to the plane:

$$\Pi = \begin{pmatrix} \pi \\ 1 \end{pmatrix}$$

Where $\pi \in \mathbb{R}^3$.

In order to compute an s that belongs to the plane we do the following calculation:

$$\Pi^T U(s) = 0$$

$$\begin{pmatrix} \pi^T & 1 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = 0$$

$$\pi^T x + s = 0$$

$$s = -\pi^T x$$

According to the hint in the assignment we should calculate $P_2U(s)$ for $s = -\pi^T x$.

$$y \sim P_2U = \begin{pmatrix} R & t \end{pmatrix} \begin{pmatrix} x \\ -\pi^T x \end{pmatrix} = Rx - t\pi^T x = R + t(-\pi^T x)$$

In order to show that H maps x to y we do the following:

$$H = \begin{pmatrix} R & -t\pi^T \end{pmatrix}$$

$$Hx = \begin{pmatrix} R & -t\pi^T \end{pmatrix} x = Rx - t\pi^T x = Rx + t(-\pi^T) \sim y$$

This shows that H maps x to y.