Hodge Theory : Final Exam

Bojin Han*

July 19, 2024

Instruction: *You are allowed to use the course materials and all references during the exam. Discussion with other students is also allowed; however, the writing up of your solutions should be done by yourself. Please submit your solutions digitally via email (M.Shen@uva.nl) before the deadline* ***31 July, 23:00*** *(Beijing time).*

Problem 1.

(1) Compute $\mathrm{H}^p(\mathbb{P}^2_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^2}(d))$ using Čech cohomology, where $d \in \mathbb{Z}$.

(2) Compute $\dim_{\mathbb{C}} \mathrm{H}^q(\mathbb{P}^n_{\mathbb{C}}, \Omega^p_{\mathbb{P}^n/\mathbb{C}})$, for all $p, q \geq 0$.

(3) Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth hypersurface of degree d. Compute the dimension of $\mathrm{H}^i(X, \mathscr{O}_X)$ for $i = 0, 1, \ldots, n-1$.

Solution.

(1) We have[1]

$$\mathrm{H}^p(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(d)) = \begin{cases} (\mathbb{C}[T_0, \ldots, T_n])_d & \text{if} \quad p = 0,\ d \geq 0; \\ \left(\frac{1}{T_0 \ldots T_n}\mathbb{C}[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} \quad p = n,\ d < 0; \\ 0 & \text{if} \quad p, n, d \text{ not as above.} \end{cases}$$

* * *

Firstly, we use the standard affine covering of $\mathbb{P}^n_{\mathbb{C}}$ as $\mathscr{U} : \mathbb{P}^n_{\mathbb{C}} = \bigcup D_+(T_i)$. As shown in class, the intersections of $D_+(T_i)$ are still affine spaces, therefore, we may use the ordered Čech complex.

Using the usual ordering $\{0, \ldots, n\}$, the complex has terms

$$\check{\mathscr{C}}^p_{ord}(\mathscr{U}, \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(d)) = \bigoplus_{i_0 < \ldots < i_p} \left(\mathbb{C}\left[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}\right]\right)_d$$

with $\partial(s)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ as the boundary map. Whence

$$\check{\mathscr{C}}^0_{ord} \to \ldots \to \check{\mathscr{C}}^p_{ord} \xrightarrow{\partial} \check{\mathscr{C}}^{p+1}_{ord} \to \ldots.$$

* * *

We may use vector $\alpha \in \mathbb{Z}^{n+1}$ to denote a multiindex, and $T^\alpha := T_0^{\alpha_0} T_2^{\alpha_2} \ldots T_n^{\alpha_n}$. Taking $N(\alpha) := \{i \in \{0, \ldots, n\} \mid \alpha_i < 0\}$, $\sum \alpha := \sum_{i=1}^n \alpha_i$. The complex thus has the decomposition

*PB22000258, william_han@mail.ustc.edu.cn

[1] See also [Har13, Theorem 5.1], [Ara12, Theorem 16.2.1] or [Sta18, Tag 01XT]

as follows

$$\check{\mathscr{C}}^p_{ord}(\mathscr{U}, \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(d)) = \bigoplus_{i_0<\ldots<i_p} \left(\mathbb{C}\left[T_0, \ldots, T_n, \frac{1}{T_{i_0}\ldots T_{i_p}}\right]\right)_d$$
$$= \bigoplus_{i_0<\ldots<i_p, N(\alpha)\subset\{0,\ldots,n\}, \sum\alpha=d} \mathbb{C}T^\alpha$$
$$=: \bigoplus_{i_0<\ldots<i_p, N(\alpha)\subset\{0,\ldots,n\}, \sum\alpha=d} \check{\mathscr{C}}^p(\alpha).$$

We hence need to calculate the cohomology of the decomposed pieces $\check{\mathscr{C}}^p(\alpha)$. The differential is the same, as

$$\check{\mathscr{C}}^0(\alpha) \to \ldots \to \check{\mathscr{C}}^p(\alpha) \xrightarrow{\partial} \check{\mathscr{C}}^{p+1}(\alpha) \ldots.$$

* * *

If $N(\alpha) = \{0, \ldots, n\}$, which implies $\sum\alpha \leq n$, the complex will be

$$0 \to \ldots \to 0 \to \check{\mathscr{C}}^n(\alpha) = \mathbb{C}T^\alpha \to 0.$$

Therefore, the cohomology is

$$\mathrm{H}^p\left(\check{\mathscr{C}}^-(\alpha)\right) = \begin{cases} \mathbb{C}T^\alpha = \left(\frac{1}{T_0\ldots T_n}\mathbb{C}[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d, & p = n; \\ 0, & \text{else.} \end{cases}$$

Noticing we can just suppose $\sum\alpha = d < 0$, for $\left(\frac{1}{T_0\ldots T_n}\mathbb{C}[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d = 0$, if $d > -n$.

* * *

If $N(\alpha) = \varnothing$, we may consider the Čech complex of $\operatorname{Spec}\mathbb{C} =: \mathbb{A}_{\mathbb{C}}$, with covering of $n+1$ sheets $\mathscr{V} : \bigcup_{i\in\{0,\ldots,n+1\}} \mathbb{A}_{\mathbb{C}}$. We can show the cohomology of this covering is exactly the one of $\check{\mathscr{C}}^-(-)$ by sending the summand of $\check{\mathscr{C}}^p(\mathscr{V})_{i_0,\ldots i_p}$ to certain $\check{\mathscr{C}}^p(\alpha)$[2]. Therefore, they have exactly the same cohomology as follows

$$\mathrm{H}^p\left(\check{\mathscr{C}}^-(\alpha)\right) = \begin{cases} \mathbb{C}T^\alpha, & p = 0; \\ 0, & \text{else.} \end{cases}$$

Notice that this case implies $\sum\alpha = d \geq 0$.

* * *

If $N(\alpha)$ is neither empty nor equal to $\{0, \ldots, n\}$, we may show this complex is acyclic; hence the cohomology is zero. We will show it by defining a homotopy between identity and zero as follows[3]. Take $i \notin N(\alpha)$

$$h(s)_{i_0,\ldots,i_p} = \begin{cases} 0 & p \notin \{0, \ldots, n-1\}; \\ 0 & i \in \{i_0, \ldots, i_p\}; \\ s_{i,i_0,\ldots,i_p} & i < i_0; \\ (-1)^q s_{i_0,\ldots,i_{q-1},i,i_q,\ldots,i_p} & i_{q-1} < i < i_q; \\ (-1)^{p+1} s_{i_0,\ldots,i_p,i} & i_p < i. \end{cases}$$

One can show it's a homotopy between id and 0.

[2]This is a bit ambiguous, for closer discussion, see the section Čech cohomology of [Sta18]

[3]Very classic proof with topological intuition.

$$\begin{tikzcd}
\check{\mathscr{C}}^{p-1}(\alpha) \arrow[r, "\partial"] & \check{\mathscr{C}}^{p}(\alpha) \arrow[r, "\partial"] \arrow[dl, dotted, "h_p"'] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "0"] & \check{\mathscr{C}}^{p+1}(\alpha) \arrow[dl, dotted, "h_{p+1}"'] \\
\check{\mathscr{C}}^{p-1}(\alpha) \arrow[r, "\partial"] & \check{\mathscr{C}}^{p}(\alpha) \arrow[r, "\partial"] & \check{\mathscr{C}}^{p+1}(\alpha)
\end{tikzcd}$$

* * *

Therefore, the only two contributors of the cohomology are as above.

(2) Use the Euler sequence with maximal wedge product[4]

$$0 \to \Omega^p_{\mathbb{P}} \to \mathscr{O}(-p)^{\binom{n+1}{p}} \to \Omega^{p-1}_{\mathbb{P}}.$$

Hence, using 1, we have $\mathrm{H}^q\left(\mathbb{P}^n_{\mathbb{C}}, \Omega^p\right) = \mathrm{H}^{q-1}\left(\mathbb{P}^n_{\mathbb{C}}, \Omega^{p-1}\right)$. Thus use 1 again,

$$\mathrm{H}^q\left(\mathbb{P}^n_{\mathbb{C}}, \Omega^p_{\mathbb{P}}\right) = \begin{cases} \mathbb{C}, & p = q \leq n \\ 0, & \text{otherwise} \end{cases}$$

(3) We have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-d) \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_X \to 0$$

where the left map is multiplication by f. Then one can take cohomology of this to compute $\mathrm{H}^i(X, \mathscr{O}_X)$. We get

$$\ldots \to \mathrm{H}^i(\mathscr{O}_{\mathbb{P}^n}(-d)) \to \mathrm{H}^i(\mathscr{O}_{\mathbb{P}^n}) \to \mathrm{H}^i(\mathscr{O}_X) \to \mathrm{H}^{i+1}(\mathscr{O}_{\mathbb{P}^n}(-d)) \to \ldots$$

If $0 < i < n$, then by 1.1, we have $\dim \mathrm{H}^i(\mathscr{O}_{\mathbb{P}^n}(l)) = 0$, so we get $\dim \mathrm{H}^i(\mathscr{O}_X(k)) = 0$ for $0 < i < n-1$. It remains to compute $\mathrm{H}^0(\mathscr{O}_X)$ and $\mathrm{H}^{n-1}(\mathscr{O}_X)$.

* * *

Since $\mathrm{H}^0\left(X, \mathscr{O}_X(-d)\right) = \mathrm{H}^1\left(X, \mathscr{O}_X(-d)\right) = 0$, We have

$$0 \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}) \to \mathrm{H}^0(\mathscr{O}_X) \to 0$$

so $\dim \mathrm{H}^0(\mathscr{O}_X) = 1$.

Moreover, we have

$$0 \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}) \to \mathrm{H}^0(\mathscr{O}_X) \to 0$$

so $\mathrm{H}^0(\mathscr{O}_X) = 1$. But twisting will give

$$0 \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}(-d+k)) \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}(k)) \to \mathrm{H}^0(\mathscr{O}_X(k)) \to 0$$

So for $k < d$, we have $\mathrm{H}^0(\mathscr{O}_X(k)) \simeq \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}(k))$.

* * *

To compute $\dim \mathrm{H}^{n-1}(X, \mathscr{O}_X)$, we can use Serre duality, $\dim \mathrm{H}^{n-1}(X, \mathscr{O}_X) = \dim \mathrm{H}^0(X, \mathscr{O}_X \otimes \omega_X) = \dim \mathrm{H}^0(X, \mathscr{O}_X \otimes \omega_X)$, and then adjunction formula,

$$\begin{aligned}
\dim \mathrm{H}^0(X, \mathscr{O}_X \otimes \omega_X) &= \dim \mathrm{H}^0(X, \mathscr{O}_X \otimes \mathscr{O}(-n-1+d)) \\
&= \dim \mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^n}(-n-1+d)) \\
&= C^n_{d-1} = \binom{d-1}{n},
\end{aligned}$$

which has been calculated in 1.1[5].

[4] For detailed calculation, see [Ara12, Proposition 17.1.4].

[5] If $d-1 < 0$ or $n > d-1$, it's defined to be zero.

* * *

The isomorphism $\mathrm{H}^0(X, \mathscr{O}_X \otimes \mathscr{O}(-n-1+d)) = \mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^n}(-n-1+d))$ is nothing but the similar proof as above.[6]

Problem 2. Let $X \subseteq \mathbb{P}^4_{\mathbb{C}}$ be a smooth hypersurface of degree 3.

(1) Compute the Betti numbers $b_i = \dim_{\mathbb{Q}} \mathrm{H}^i(X, \mathbb{Q})$.

(2) Show that the Hodge structure of $\mathrm{H}^3(X, \mathbb{Z})$ is given by a Hodge decomposition of the form

$$\mathrm{H}^3(X, \mathbb{Z}) \otimes \mathbb{C} = \mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{2,1}(X).$$

(3) Let $\Lambda = \mathrm{H}^3(X, \mathbb{Z})$ and $V = \mathrm{H}^{1,2}(X)$. Show that the inclusion $\Lambda \subset V$ induced by

$$\mathrm{H}^3(X, \mathbb{Z}) \subset \mathrm{H}^3(X, \mathbb{C}) \longrightarrow \mathrm{H}^{1,2}(X)$$

defines a complex torus $A = V/\Lambda$. Show that A can be polarized via the intersection bilinear form (with the correct sign) on $\mathrm{H}^3(X, \mathbb{Z})$.

(4) Now assume that $Y \subseteq \mathbb{P}^4_{\mathbb{C}}$ is a smooth projective hypersurface of degree 5. Let $F^\bullet \mathrm{H}^3(X, \mathbb{C})$ be the Hodge filtration and let $V = \mathrm{H}^3(X, \mathbb{C})/F^2\mathrm{H}^3(X, \mathbb{C})$. The natural map $\Lambda := \mathrm{H}^3(X, \mathbb{Z}) \longrightarrow V$ gives rise to a lattice in V. What goes wrong if one uses the same method as above to polarize V/Λ?

Solution.

(1) By Lefschetz hyperplane theorem, we may deduce $\mathrm{H}^i(X, \mathbb{Q}) \simeq \mathrm{H}^i\left(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}\right)$ for all $i \neq 3$. Using cell complex, we may see

$$\dim \mathrm{H}^i\left(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}\right) = \begin{cases} 1, & i \leq 6 \text{ is even;} \\ 0, & i \leq 6 \text{ is odd;} \\ 0, & i > 6; \end{cases}$$

Therefore,

$$b_i = \begin{cases} 1, & i \neq 3 \quad \wedge \quad i \leq 6 \text{ is even;} \\ 0, & i \neq 3 \quad \wedge \quad i \leq 6 \text{ is odd;} \\ 0, & i \neq 3 \quad \wedge \quad i > 6; \\ ?, & i = 3 \end{cases}$$

We only need to deduce b_3 as follows.

* * *

As is known in our class[7], the Euler characteristic number of the complex projective hypersurface $c_3(X) = \chi(X) = \sum_i (-1)^i b_i$; and[8] $c(X) = \frac{(1+h)^{4+1}}{1+3h}$. Consequently,

$$c_3(X) = \left(\frac{(1+h)^{4+1}}{1+3h}\right)_{\deg=3} = \left((1+5h+10h^2+10h^3)(1-3h+9h^2-27h^3)\right)_{\deg=3} = -2h^3.$$

Using $h^3 = \deg X = 3$, we deduce $\chi(X) = c_3(X) = -6$. Thus $b_3 = 4 - c_3 = 4 + 6 = 10$[9].

[6]Another proof may be the remark after [Ara12, Corollary 17.3.3]

[7]Theorem 4.5.12

[8]Example 4.5.10, (4)

[9]Moreover, the middle hodge numbers are 0 5 5 0.

(2) We only need to show that $\mathrm{H}^{3,0}(X) = \mathrm{H}^3(X, \mathscr{O}_X) = 0$, which has been shown in problem 1.3.

(3) Since $\dim_{\mathbb{R}} \mathrm{H}^3(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = b_3 = 10 = h^{1,2} + h^{2,1} = 2h^{1,2} = \dim_{\mathbb{R}} \mathrm{H}^{1,2}(X)$. By using $\bar{\alpha}^{p,q} = \alpha^{q,p}$, we may see $\mathrm{H}^3(X, \mathbb{Z})$ can be included in $\mathrm{H}^{1,2}(X)$. Therefore, $\mathrm{H}^3(X, \mathbb{Z})$ is necessarily a lattice.

* * *

Following the example 3.3.6, one wants to show there exists a positive definite Hermitian form $H\colon \mathrm{H}^{1,2}(X) \times \mathrm{H}^{1,2}(X) \to \mathbb{C}$, such that the imaginary part of H restricted to $\mathrm{H}^3(X, \mathbb{Z}) \times \mathrm{H}^3(X, \mathbb{Z})$ is precisely

$$E\colon \mathrm{H}^3(X, \mathbb{Z}) \times \mathrm{H}^3(X, \mathbb{Z}) \to \mathbb{Z}\colon (\alpha, \beta) \mapsto \int_X \alpha \cup \beta.$$

Taking $\omega, \tau \in \mathrm{H}^{1,2}(X)$, $H(\omega, \tau) := 2\mathrm{i} \int_X \omega \wedge \bar{\tau}$ is indeed a positive definite Hermitian form on $\mathrm{H}^{1,2}(X)$, since there locally exist $f_I \in \mathscr{C}^\infty$, such that $\omega = \sum f_I \mathrm{d}z^{I_1} \wedge \mathrm{d}\bar{z}^{I_2}$. Whence $\int_X \omega \wedge \bar{\omega} = \int_X \sum |f_I|^2 (-1) \mathrm{d}z^{I_1} \wedge \mathrm{d}\bar{z}^{I_1} \wedge z^{I_2} \mathrm{d}\bar{z}^{I_2} = -(-2\mathrm{i})^3 \int_X \sum |f_I|^2 = -8\mathrm{i} \int_X \sum |f_I|^2$.

* * *

Now let $\alpha = \alpha^{1,2} + \alpha^{2,1}, \beta = \beta^{1,2} + \beta^{2,1} \in \mathrm{H}^3(X, \mathbb{Z})$. We have

$$\begin{aligned}
\Im H\left(\alpha^{1,2}, \beta^{1,2}\right) &= \Im 2\mathrm{i} \int_X \alpha^{1,2} \wedge \overline{\beta^{1,2}} \\
&= \frac{1}{2\mathrm{i}} \left(2\mathrm{i} \int_X \alpha^{1,2} \wedge \beta^{2,1} + 2\mathrm{i} \int_X \alpha^{2,1} \wedge \beta^{1,2} \right) \\
&= \int_X \alpha^{1,2} \wedge \beta^{2,1} + \int_X \alpha^{2,1} \wedge \beta^{1,2}.
\end{aligned}$$

Meanwhile, since $E(\alpha, \beta) = \int_X \alpha \wedge \beta = \int_X \alpha^{1,2} \wedge \beta^{2,1} + \int_X \alpha^{2,1} \wedge \beta^{1,2}$, we deduce A can be polarized.

(4) In this case, we have $\dim \mathrm{H}^{3,0}(X) = \dim \mathrm{H}^3(X, \mathscr{O}_X) = \dim \mathrm{H}^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-4 - 1 + 5)) \neq 0$. Therefore, things turn difficult, since the Hermitian form H as below have different sign when calculation over $\mathrm{H}^{2,1}(X)$ and $\mathrm{H}^{3,0}(X)$. Thence, we cannot say H is positive definite.[10]

Problem 3. Let V be a complex vector space of dimension g and $\Lambda \subset V$ a lattice in V. Set $X = V/\Lambda$.

(1) Show that the following are equivalent to each other.
  (a) There is a polarization on the Hodge structure $V_{\mathbb{Z}} := \mathrm{H}^1(X, \mathbb{Z})$.
  (b) There is a positive definite Hermitian form $H : V \times V \longrightarrow \mathbb{C}$ whose imaginary part is integral on $\Lambda \times \Lambda$.
  (c) There is a divisor $\Theta \subset X$ which is ample.

(2) Assume that X is polarizable. Show that $X^\vee = \overline{V}^* / \Lambda^*$ is also polarizable.

(3) Assume that $\Lambda' \subset V$ is another lattice such that $\Lambda_{\mathbb{Q}} = \Lambda'_{\mathbb{Q}}$. Show that $X = V/\Lambda$ is algebraic if and only if $X' = V/\Lambda'$ is algebraic.

Solution.

(1) As a polarization on $V_{\mathbb{Z}}$ naturally gives out a positive definite Hermitian form H with $\Im H\colon \mathrm{H}^1(X, \mathbb{Z}) \times \mathrm{H}^1(X, \mathbb{Z}) \to \mathbb{Z}$[11], we deduce (a) $\implies$ (b).

[10] This variety is known as quintic threefold, with middle Hodge number 1 101 101 1.

[11] Here, $\Im$ is the symbol for taking the imaginary part.

* * *

Since $X = V/\Lambda$ is a complex torus, we have seen in theorem 3.3.3 that

(b) There is a positive definite Hermitian form $H : V \times V \longrightarrow \mathbb{C}$ whose imaginary part is integral on $\Lambda \times \Lambda$;

(b') Let α be a multiplicator of H, then the space of global sections of $\mathscr{L}(H,\alpha)^{\otimes n}$ defines an embedding $X \hookrightarrow \left(\mathscr{L}(H,\alpha)^{\otimes n}\right)^{\vee}$

are equivalent. Moreover, (b') equals to $\mathscr{L} := \mathscr{L}(H,\alpha)$ is an ample line bundle. When given a section α, we may choose $\Theta := \{\alpha = 0\}$ to be the divisor, which is definitely an ample divisor. Consequently, we deduce (b) $\implies$ (c).

* * *

If there exists an ample divisor X, then the associated line bundle L is ample. We may thus suppose there exist n such that $\varphi_{L^n} : X \to |L|^{\vee}$ is an embedding. Then by theorem 3.3.3 we may notice $c_1(L)$ is positive definite. Therefore, by definition there is a polarization. This gives out (c) $\implies$ (a).

(2) Let's first define the concept of isogeny. An isogeny of a complex torus X to a complex torus X' is by definition a surjective homomorphism $X \to X'$ with finite kernel. Obviously, a homomorphism $X \to X'$ is an isogeny if and only if it is surjective and $\dim X = \dim X'$. It is also worth noticing that if there exists an isogeny from X to Y, there also exist an isogeny from Y to X. So it makes sense to say X and Y are isogenous.

* * *

We may deduce an isomorphism between $X^{\vee}$ and $\mathrm{Pic}^0(X)$, for there exists a canonical homomorphism $\tau\colon \bar{V}^* \to \mathsf{Hom}(\Lambda, \mathbb{C}_1^*) \simeq \mathrm{Pic}^0 : l \mapsto \exp(2\pi\mathrm{i}(l \mid -))$ with kernel Λ^*.[12] Given a line bundle L over X, we may define the homomorphism $\phi_L\colon X \to X^{\vee}\colon x \mapsto t_x^* L \otimes L^{-1}$, where t_x is the translation map $p \mapsto x + p$. This is well-defined, recalling Pic^0 is the kernel of c_1.

Actually, ϕ_L is isogeny iff the Hermitian form $c_1(L)$ is nondegenerate. Thence, $c_1(L)$ is nondegenerate iff $\ker \phi_L$ is finite. For calculation reasons, we may notice that the following diagram is commute.

$$\begin{tikzcd} V \arrow[r, "\phi_H"'] \arrow[d] & \bar{V}^* \arrow[d] \\ X \arrow[r, "\tau\circ\phi_L"] & X^{\vee} \end{tikzcd}$$

where $\phi_H\colon V \to \bar{V}^*\colon v \mapsto H(v,-)$, H is the Hermitian form such that $\Im H = c_1(L)$.

* * *

If a complex torus is isogenous to a complex Abelian variety, then this complex torus is a complex Abelian variety. The proof is simply noticing if $f\colon Y \to X$ is isogeny, then we may transplant the polarization L over X to f^*L over Y, since $\ker f$ is finite and $\ker \phi_{f^*L} = \ker$. Therefore, if X is an Abelian variety, $X^{\vee}$ is the same.[13]

(3) We only need to show there exists a polarization over X'. We may take $E \in \bigwedge \Lambda^*$ to be an alternating bilinear form, H to be a positive definite Hermitian form as a polarization

[12] We use Theorem 3.2.15 (the Appell–Humbert) here.

[13] Some parts of this solution can be found at [LB13, Section 2.4 and 4.1].

of X. Since $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}$, we may choose the bases of Λ and Λ' as $\{e_1, \ldots, e_g\}$ and $\{e'_1, \ldots, e'_g\}$. and a diagonal matrix $M = \operatorname{diag}\left(\left|\frac{e_1}{e'_1}\right|, \ldots, \left|\frac{e_g}{e'_g}\right|\right)$. Then we may define E' as an alternating bilinear form as $E'(-,-) := E(M-, M-)$. It's easy to see that with $H'(M-, M-)$ is a polarization.

Problem 4. Let $S \subseteq \mathbb{P}^4_{\mathbb{C}}$ be the intersection of a hyperplane and a degree 4 hypersurface. Assume that S is smooth. Let X be the blow-up of $\mathbb{P}^4_{\mathbb{C}}$ with center S. Compute all the Hodge numbers $h^{p,q}(X)$.

Solution.

Above all, we know that $\dim_{\mathbb{C}} X = 4$, therefore $\dim \mathrm{H}^p(X, \mathbb{C}) = 0$, for all $p > 8$. And, in these circumstances, we have isomorphism not only on $\mathrm{H}^i(X, \mathbb{Z})$, but also on the Hodge classes $\mathrm{H}^{p,q}(X)$[14], following we use $\simeq$ in the means of isomorphism of Hodge structure.

We have the following diagram.

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "\eta"] & X \arrow[d, "\rho"] \\ S \arrow[r, hook, "i"'] & \mathbb{P}^4_{\mathbb{C}} \end{tikzcd}$$

By using the blow up formula (Proposition 4.5.7 and [Voi03, Theorem 1.29]), we have

$$\mathrm{H}^k(X) = \mathrm{H}^k\left(\mathbb{P}^4_{\mathbb{C}}\right) \oplus j_*\Big(\eta^* \mathrm{H}^{k-2}(S) \oplus \xi\eta^* \mathrm{H}^{k-4}(S) \oplus \ldots \oplus \xi^{c-1}\eta^* \mathrm{H}^{k-2c}(S)\Big),$$
$$\mathrm{H}^{p,q}(X) = \mathrm{H}^{p,q}\left(\mathbb{P}^4_{\mathbb{C}}\right) \oplus j_*\big(\eta^* \mathrm{H}^{p-1,q-1}(S) \oplus \xi\eta^* \mathrm{H}^{p-2,q-2}(S) \oplus \ldots \oplus \xi^{c-1}\eta^* \mathrm{H}^{p-c,q-c}(S)\big).$$

where ξ is $c_1(\mathcal{O}_E(1)) \in \mathrm{H}^2(E, \mathbb{Z})$ and $c = \dim \mathbb{P}^4_{\mathbb{C}} - \dim S - 1 = 1$.

Accordingly $\mathrm{H}^{p,q}(X) = \mathrm{H}^{p,q}\left(\mathbb{P}^4_{\mathbb{C}}\right) \oplus j_*\big(\eta^* \mathrm{H}^{p-1,q-1}(S)\big)$, which means $h^{p,q}(X) = h^{p,q}\left(\mathbb{P}^4_{\mathbb{C}}\right) + h^{p-1,q-1}(S)$. Since $\dim_{\mathbb{C}} S = 2$, we only need to calculate the hodge number $h^{p,q}(S)$, with $p+q \leq 4$. By duality, one only need $p + q \leq 2$.

* * *

Supposing $S = H \cap Y$, with H the hyperplane and Y the hypersurface and $\dim Y = 3$, we may firstly calculate $\mathrm{H}^p(S)$ by using Lefschetz hyperplane theorem. Therefore, we have the following diamond

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ ? & & ? & & ? \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

We only need to calculate the middle Hodge number $h^{0,2}$, $h^{1,1}$ and $h^{2,0}$. Since, supposing $S \overset{j'}{\hookrightarrow} Y$, we have

$$0 \to \mathcal{O}_{\mathbb{P}}(-4) \restriction_Y \to \mathcal{O}_Y \to j'_*\mathcal{O}_S \to 0.$$

It induces the long exact sequence as follows:

[14] [Voi03, Theorem 1.29]

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\mathscr{O}_{\mathbb{P}}(-4)\restriction_Y)=0 \arrow[r] & \mathrm{H}^1(\mathscr{O}_Y) \arrow[r] & \mathrm{H}^1(j'_*\mathscr{O}_S) \arrow[r] & \bullet \\
\bullet \arrow[r] & \mathrm{H}^2(\mathscr{O}_{\mathbb{P}}(-4)\restriction_Y)=0 \arrow[r] & \mathrm{H}^2(\mathscr{O}_Y) \arrow[r] & \mathrm{H}^2(\mathscr{O}_Y) \arrow[r] & \bullet \\
\bullet \arrow[r] & \mathrm{H}^3(\mathscr{O}_{\mathbb{P}}(-4)\restriction_Y)=0 \arrow[r] & \mathrm{H}^3(\mathscr{O}_Y) \arrow[r] & \mathrm{H}^3(\mathscr{O}_Y) \arrow[r] & \cdots .
\end{tikzcd}$$

Whence $h^{0,2}(S) = h^{2,0}(S) = \dim \mathrm{H}^2(j'_*\mathscr{O}_S) = \dim \mathrm{H}^2(\mathscr{O}_Y) = 0$, by vanishing theorems. Thus $h^{1,1}(S) = b_2$, the Betti number. By using the similar argument as solution , say $\chi(S) = b_0 - b_1 + b_2 - b_3 + b_4$, we deduce $b_2 = \chi(S) - 2$. Hence, we would like to calculate $\chi(S) = c_2(S)$.

* * *

Moreover, as the method of normal bundle shown in class, we use

$$0 \to \mathscr{T}_S \to \mathscr{T}_{\mathbb{P}} \restriction_S \to \mathscr{N}_{S/\mathbb{P}} \to 0$$

and $\mathscr{N}_{S/\mathbb{P}} \simeq \mathscr{N}_{Y/\mathbb{P}} \oplus \mathscr{N}_{S/Y} \simeq \mathscr{O}_Y(4) \oplus \mathscr{O}_Y(1)$, to deduce the Chern class is nothing but

$$c(S) = \frac{c(\mathscr{T}_{\mathbb{P}})}{c(\mathscr{O}_Y(4) \oplus \mathscr{O}_Y(1))} = \frac{(1+h)^{4+1}}{(1+h)(1+4h)},$$

where $h \in \mathrm{H}^2(S, \mathbb{Z})$ is the restriction of the hyperplane class to S.

It follows that $c_2(S) = 6h^2 = 6 \times 4 = 24$. Thus $c_2(S) = 22$.

Whence the diamond of S is

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & 22 & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

* * *

Hence the diamond of X is

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 2 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 23 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 2 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Problem 5. Let $\tau_1 = i$ and $\tau_2 = \frac{-1+\sqrt{3}i}{2}$. Let $E_1 = E_{\tau_1}$ and $E_2 = E_{\tau_2}$.

(1) Compute explicitly the Mumford-Tate group of E_1.

(2) Find all homomorphisms $f: E_1 \longrightarrow E_2$.
(3) Show that there is a nonzero homomorphism between two CM elliptic curves if and only if their Mumford-Tate groups are isomorphic.

Solution.

(1) Since $i = \sqrt{-1}$ is imaginary quadratic, we have in subsection 5.2.5 the Mumford-Tate group of E_1 is $\mathbb{Q}(i)^\times = \mathrm{Spec}\left(\mathbb{Q}\left[x, y, \frac{1}{x^2-y^2}\right]\right)$.
(2) There exists a bijection between all homomorphisms f between E_1 and E_2 and all $\mathbb{C}-$linear endomorphism F of $\mathbb{C}$ with $F(\tau_1) \in \mathbb{Z}[\tau_2]$. It's the same as choosing an arbitrary complex number α such that $\alpha \in \mathbb{Z}[\tau_2]$ and $\alpha i \in \mathbb{Z}[\tau_2]$. As we know, there does not exist such a homomorphism.
(3) Take $E_3 := E_{\tau_3}$, $E_4 := E_{\tau_4}$ be two CM elliptic curves, with $\sqrt{-d_3} = \tau_3$ and $\sqrt{-d_4} = \tau_4$, where $d_3, d_4 \in \mathbb{Z}_{>0}$. As we showed in 1, we may deduce $\mathrm{MT}(E_3) = \mathbb{Q}(\tau_3)^\times$ and $\mathrm{MT}(E_4) = \mathbb{Q}(\tau_3)^\times$. As shown in 2, a homomorphism between E_3 and E_4 is equal to a complex number $\alpha \in \mathbb{Z}[\tau_4]$ such that $\alpha\tau_3 \in \mathbb{Z}[\tau_4]$. Moreover, in such case, α must be real or purely imaginary. We may suppose α is real. Then since $\alpha\tau_3 \in \mathbb{Z}\tau_4$, we have $\alpha \in \mathbb{Q}$.

* * *

Take $\tau_4/\tau_3 = \mu$. If such $\alpha \in \mathbb{R}$ exists, then $\sqrt{d_4/d_3} = \mu = \alpha \in \mathbb{Q}$ and we will have a homomorphism of group between $\mathbb{Z}[\tau_3]$ and $\mathbb{Z}[\tau_4]$.
We have the following commutative diagram,

$$y \longmapsto y' = \mu y$$

$$\begin{tikzcd}
\mathbb{Q}[x,y,z] \arrow[r, "t"] \arrow[d, "\mathrm{ev}_3"'] & \mathbb{Q}[x',y',z'] \arrow[d, "\mathrm{ev}_4"] \\
\mathbb{Q}\left[x, y, \frac{1}{x^2+(\tau_3 y)^2}\right] \arrow[r, dashed, "\exists"'] & \mathbb{Q}\left[x', y', \frac{1}{x'^2+(\tau_4 y')^2}\right]
\end{tikzcd}$$

Therefore there exists a homomorphism of $\mathbb{Q}-$group from E_4 to E_3, as the functority of Spec. By using μ^{-1}, we have the inverse homomorphism from E_3 to E_4.

* * *

If there is an isomorphism between two MT groups, it means the existence of ring homomorphism

$$t' : \mathbb{Q}\left[x, y, \frac{1}{x^2+(\tau_3 y)^2}\right] \to \mathbb{Q}\left[x', y', \frac{1}{x'^2+(\tau_4 y')^2}\right].$$

Considering the image of x and y under t', we have $\frac{1}{t'(x)-d_4 t'(y)^2} = \frac{1}{t'(x)-t'(\tau_3 y)^2}$. It follows that $-d_4 t'(y)^2 \equiv t'(\tau_3^2)t'(y)^2$. Since t' is isomorphism, $t'(1)^2\tau_3^2 = t'(\tau_3^2) \equiv -d_4 = \tau_4^2$. Therefore, $(\tau_4/\tau_3)^2 = t'(1)^2 \in \mathbb{Q}^2$. Take $t'(1) =: \alpha$, then, α will introduce a homomorphism of elliptic curves.

Thank you for reading.[15]

[15] and also for the splendid class !

Reference

[Ara12] Donu Arapura. Algebraic geometry over the complex numbers. Springer, 2012.

[Har13] Robin Hartshorne. Algebraic geometry, volume 52. Springer Science & Business Media, 2013.

[LB13] Herbert Lange and Christina Birkenhake. Complex abelian varieties, volume 302. Springer Science & Business Media, 2013.

[Sta18] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

[Voi03] Claire Voisin. Hodge Theory and Complex Algebraic Geometry II: Volume 2, volume 77. Cambridge University Press, 2003.