Discrete Mathematics and Its Applications

2.3 Functions

Wenjing Li
wjli@bupt.edu.cn
School of Computer Science
Beijing University of Posts & Telecommunications

Introduction of Function

- From calculus, you are familiar with the concept of a real-valued function *f*, which assigns to each number $x \in R$ a particular value $y=f(x)$, where $y \in R$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of any set to elements of any set. (Also known as a *map*)

Formal Definition

- **Definition:**
  - For any sets A, B, we say that a *function f from (or "mapping") A to B* (f:$A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
  - 1-ary function, functions of n arguments: relations (ch. 6).
- **Notation:**
  - Let A and B be *nonempty* sets. A *function* (*mapping*, *transformation*) f from A to B, denoted $f : A \rightarrow B$, is a subset of $A \times B$ such that

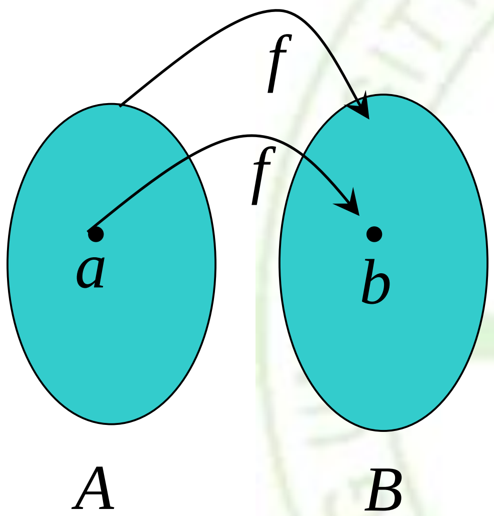
    $$\forall x[x \in A \rightarrow \exists y[y \in B \wedge \langle x, y\rangle \in f\,]]$$

    and

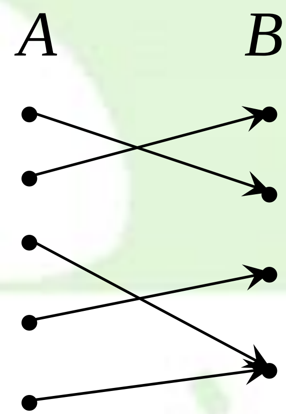
    $$[\langle x, y_1\rangle \in f \wedge \langle x, y_2\rangle \in f\,] \rightarrow y_1 = y_2$$

Graphical Representations

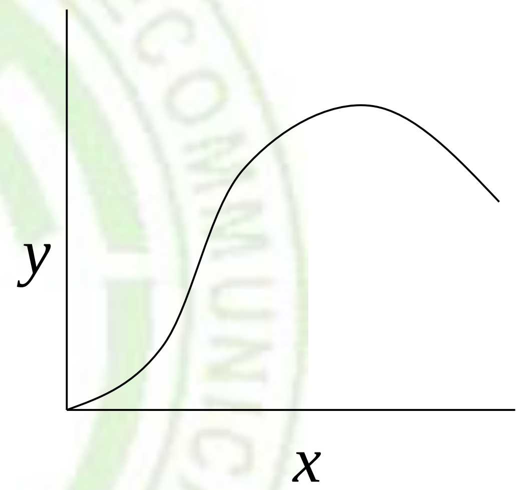
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph

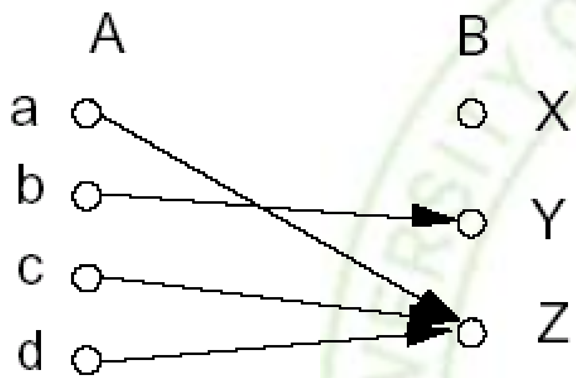


Plot

FUNCTION $f : A \rightarrow B$

- A is called the *domain*（域/定义域）
- B is called the *codomain* （陪域）.
- If $f(x) = y$
  - y is called the *image (像)* of x under f *(value)*
  - x is called a *preimage (源像)* of y *(argument)*
- The *range (值域)* of f is the set of all images of set A under f. It is denoted by $f(A)$.
- If S is a subset of domain A then
  - $\boldsymbol{f(S) = \{ f(s) \mid s \in S \}}$.

Example



- $f: A \rightarrow B$
- $f(a) = Z$, $f(b) = Y$, $f(c) = Z$, $f(d) = Z$
- the domain of f is $A = \{a, b, c, d\}$
- the codomain is $B = \{X, Y, Z\}$
- the image of d is Z
- the preimage of Y is b
- the preimages of Z are a, c and d
- $f(A) = \{Y, Z\}$ (range)
- $f(\{c,d\}) = \{Z\}$

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

- **Examples:**
  - Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
  - At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown!.
  - Suppose the grades turn out all As and Bs.
  - Then the range of *f* is {A,B}, but its codomain is {A,B,C,D,E}.

Special types of Functions

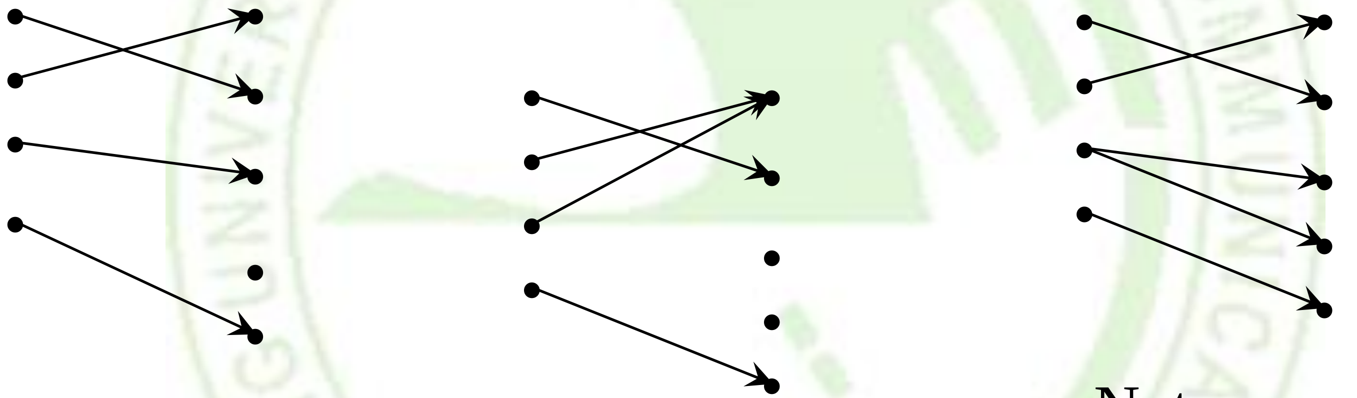
- **Injections（单射）**
  - *One-to-one* （一对一）
- **Surjections （满射）**
  - *onto* （上的）
- **Bijections （双射）**
  - *One-to-one* and *onto*
  - *One-to-one correspondence* （一一对应）

1. One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image. Formally:
  - given f:$A \rightarrow B$, "f is injective" $:\equiv (\neg\exists a, b$: $a \neq b \wedge f(a)=f(b))$.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?
- Memory jogger: Each element of the domain is <u>injected</u> into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

1. One-to-One Functions

- **illustration：**
  - Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one　　Not one-to-one　　Not even a function!

1. One-to-One Functions

- **Sufficient conditions for 1-1 functions**
- For functions f over numbers, we say:
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;

Sufficient Conditions

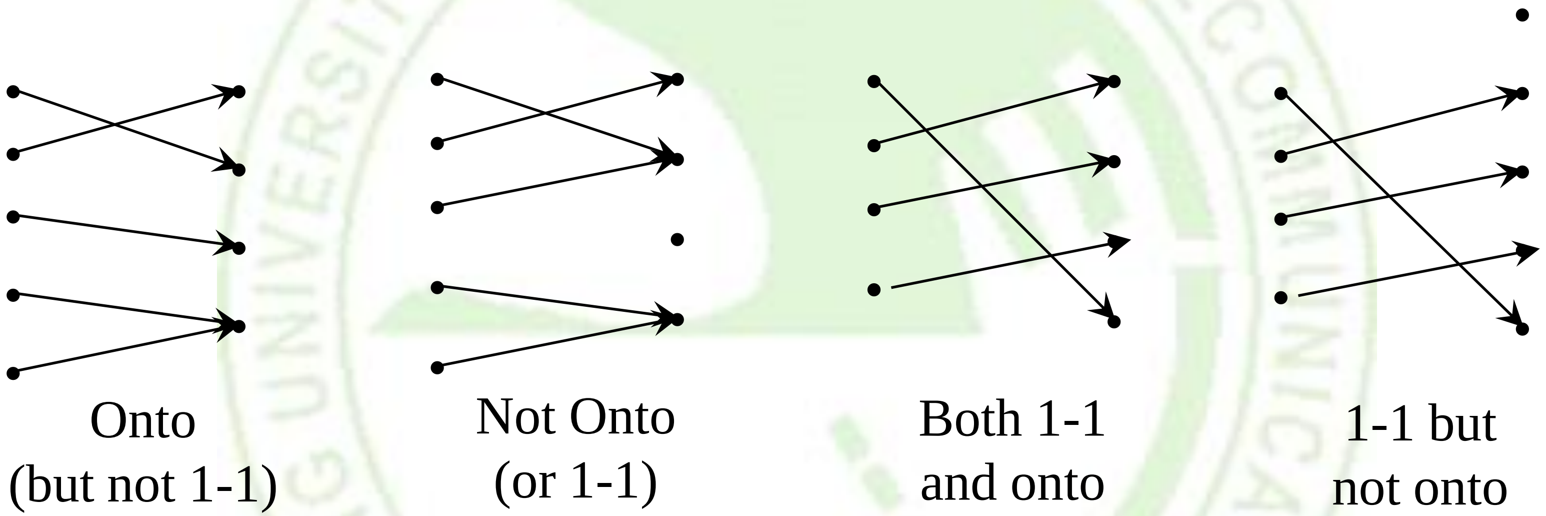
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true. E.g.* $1/x$

2. Onto (Surjective) Functions

- **Definition:**
  - A function $f{:}A\rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b\in B, \exists a\in A{:}\ f(a)=b$).
  - Think: An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
  - *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

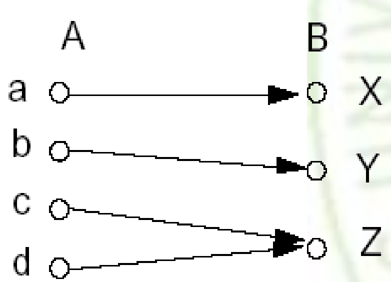
2. Onto (Surjective) Functions

- **Illustration:**
  - Some functions that are, or are not, *onto* their codomains:

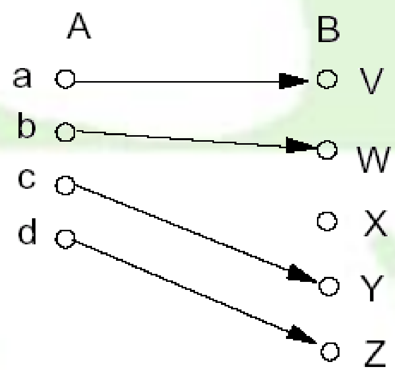


3. Bijection Functions

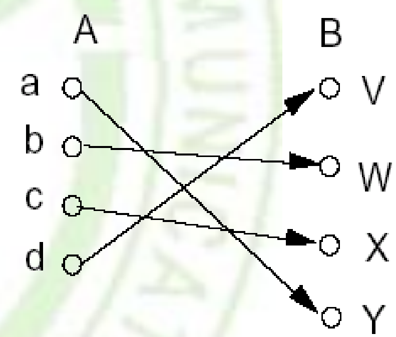
- **Definition:**
  - A function *f* is said to be *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- **Example:**



Surjection but not an injection



Injection but not a surjection



Surjection and an injection, hence a bijection

If two finite sets can be placed into 1-1 correspondence, then they have the same size

Examples

- Let
  - $A = B = \mathrm{R}$
- Determine which are injections, surjections, bijections:
  - $f(x) = x$,
  - $f(x) = x^2$,
  - $f(x) = x^3$,
  - $f(x) = x + sin(x)$,
  - $f(x) = | x |$

Properties of functions

- Function Operations
- The Identity Function
- Inverse Functions
- Composition of Functions
- Graphs of Functions

Function Operations

- If $\bullet$ ("dot") is any operator over set B, then we can extend $\bullet$ to also denote an operator over functions $f:A\to B$.
- *E.g.*: Given functions $f, g:A\to B$, we define $(f \bullet g):A\to B$ to be the function defined by:

$$\forall a\in A,\ \ (f \bullet g)(a) = f(a)\bullet g(a).$$

Function Operations

- **Examples:**
  - +(plus), ×(times) are binary operators.
  - We can also add and multiply functions f,g: R→R:

    $(f+g):R \to R$, *where* $(f+g)(x)= f(x)+g(x)$.

    $(f\times g):R \to R$, *where* $(f\times g)(x)= f(x)\times g(x)$.

- **Examples:**
  - $f(x)=x^2,\ g(x)=2x+1$
  - $(f+g)(x)=f(x)+ g(x)= x^2+2x+1$
  - $(f\times g)(x)=f(x)\times g(x)= x^2\ (2x+1)$

The Identity Function

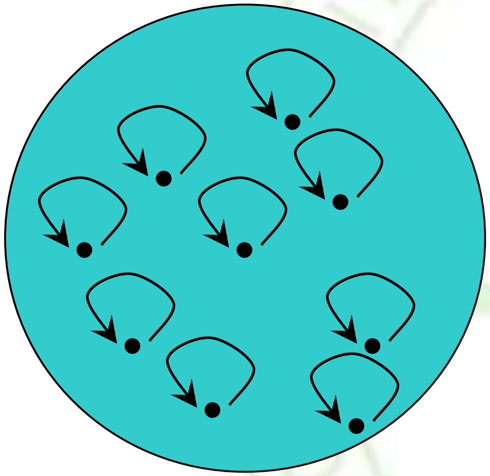
- **Definition:**
  - For any domain A, the *identity function* $I{:}A \rightarrow A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a)=a$.

- **Examples:**
  - Some identity functions you've seen:

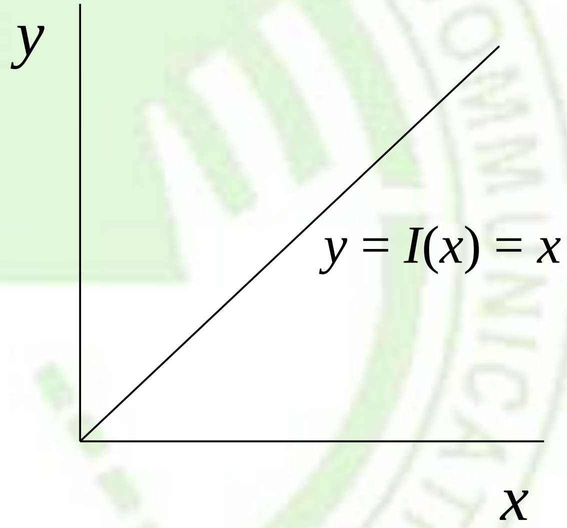
    $+0$, $\cdot 1$, $\wedge \mathbf{T}$, $\vee \mathbf{F}$, $\cup \varnothing$, $\cap U$.
  - **Note** that the identity function is always both one-to-one and onto (bijective).

The Identity Function

- **Illustration**
  - The identity function:



Domain and range

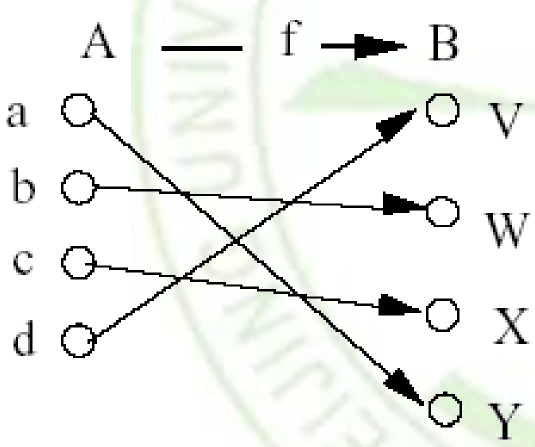


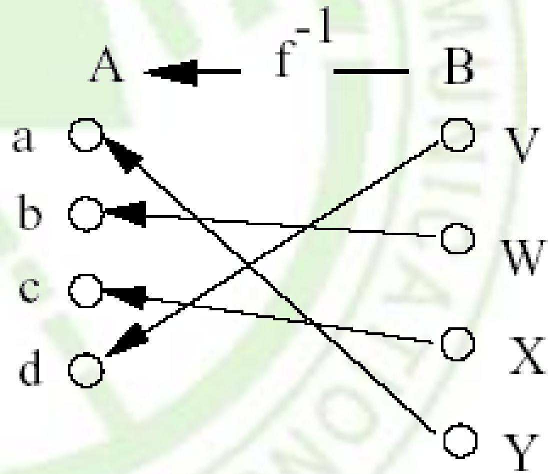
Inverse Functions

- **Definition:**
  - Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as

    $f^{-1}(y) = x$ ***iff*** $f(x) = y$

    $$\boldsymbol{f^{-1}(f(x))=x}$$
  - f is said to be ***invertible*(可逆的)**





Note: No inverse exists unless f is a bijection.

Inverse Functions

- **Notation:**
  - If S is a subset of domain A, then
    - $f(S) = \{f(x) \mid x \in S\}$
  - Let S be a subset of codomain B, then $f^{-1}(S) = ?$

$$f^{-1}(S) = \{x \mid f(x) \in S\}$$

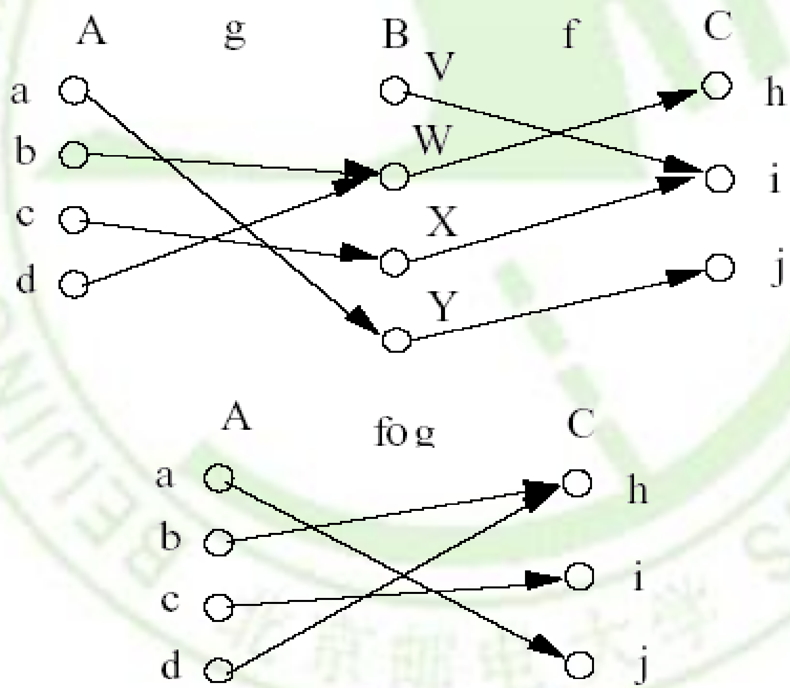
逆函数的值域 $f^{-1}(S)$为x的集合

x满足条件为：x的像 $f(x)$属于值域S

$f^{-1}(f(x))=x$

Composition of Functions

- **Definition:**
  - Let $g: A \rightarrow B$, $f: B \rightarrow C$.
  - The *composition of f with g*, denoted $f \circ g$, is the function from A to C defined by $f \circ g(x) = f(g(x))$



Composition of Functions

- **Example:**
  - If
    - $f(x) = x^2$
    - $g(x) = 2x + 1$
  - Then
    - $f \circ g(x) = f(g(x)) = (2x+1)^2$
    - $g \circ f(x) = g(f(x)) = 2x^2 + 1$

The commutative law does not hold for the composition of functions.

Composition of Functions

- Let
  - $f: A \to B$
  - $g: B \to C$
  - $h: C \to D$
- Then
  - $\boldsymbol{(h \circ g) \circ f = h \circ (g \circ f)}$

The associative law holds for the composition of functions.

Composition of Functions

- **Exercise: Let $A = B = C = \mathbf{R}$**
  - $f : A \to B,\ g: B \to C$
  - $f(a)=a-1$
  - $g(b)=b^2$.
- **Find**
  - $(f \circ g)(2)$
  - $(g \circ f)(2)$
  - $(g \circ f)(x)$
  - $(f \circ g)(x)$
  - $(g \circ g)(y)$
  - $(f \circ f)(y)$

Composition of Functions

- **Theorem:**
- Let $f:A\rightarrow B$ be any function. Then
  - $1_B \circ f = f$
  - $f \circ 1_A = f$

- **Proof**
  - For all a in Domain(f)
    - $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, so $1_B \circ f = f$.
    - $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, so $f \circ 1_A = f$.

Composition of Functions

- **Theorem:**
- If f is a one-to-one correspondence between A and B, then
  - $f^{-1} \circ f = 1_A$
  - $f \circ f^{-1} = 1_B$
- **Proof**

  $f^{-1}(f(x))=x$

  - For all a in A,
    - $1_A(a)=a=f^{-1}(f(a))=(f^{-1} \circ f)(a)$, Thus $1_A=f^{-1} \circ f$

  $f(f^{-1}(y))=y$

  - For all b in B,
    - $1_B(b)=b=f(f^{-1}(b))=(f \circ f^{-1})(b)$, Thus $1_B=f \circ f^{-1}$
- **Note:** $b=f(a)$ is equivalent to the $a=f^{-1}(b)$

Graphs of Functions

- We can represent a function f:$A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair (a,b).
  - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

The function's *graph*.

Graphs of Functions

- **Example 26**
- **Example 27**

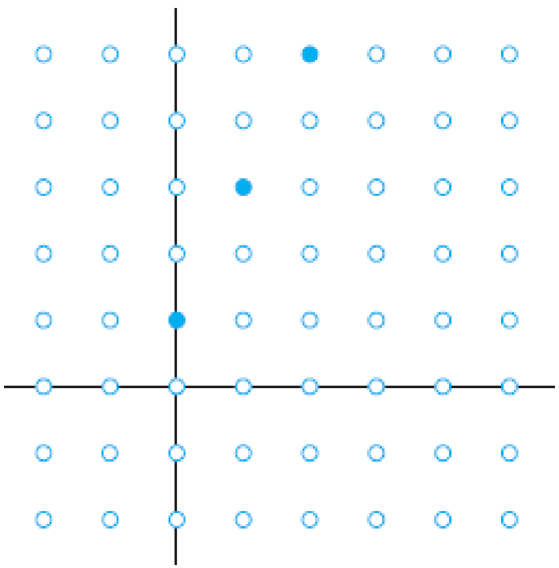


FIGURE 8 The Graph of $f(n) = 2n + 1$ from $\mathbf{Z}$ to $\mathbf{Z}$.

FIGURE 9 The Graph of $f(x) = x^2$ from $\mathbf{Z}$ to $\mathbf{Z}$.

Some Important functions

- Floor and ceiling
- Factorial function
- Characteristic function
- Mod-n function
- Partial function

Floor and ceiling function

- The *floor* function, denoted $f(x) = \lfloor x \rfloor$ or $f(x)$ = floor(x), is the largest integer less than or equal to x.
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$ or $f(x)$ = ceiling(x), is the smallest integer greater than or equal to x.
- Examples: $\lfloor 3.5 \rfloor = 3$, $\lceil 3.5 \rceil = 4$.
  - Note: the floor function is equivalent to truncation for positive numbers.

Application:
Truncation used in computer network and communication network

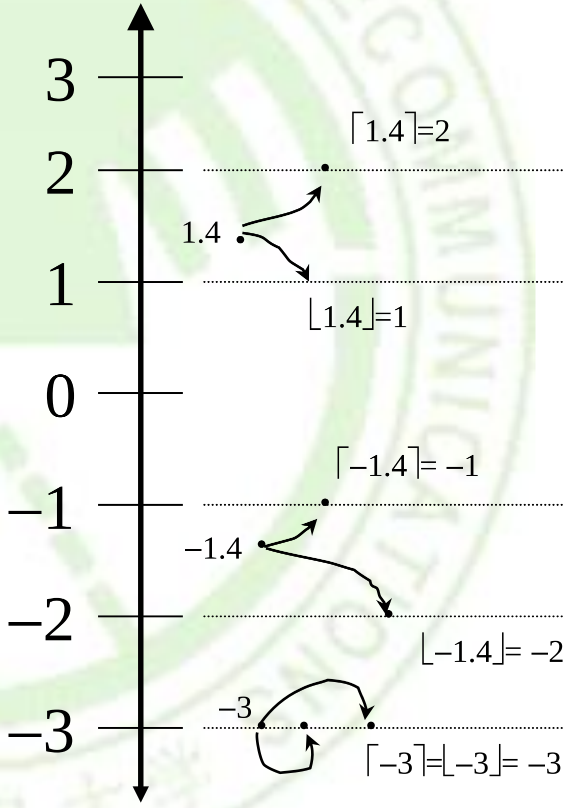
Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”
- Note that if $x \in \mathbf{Z}$,

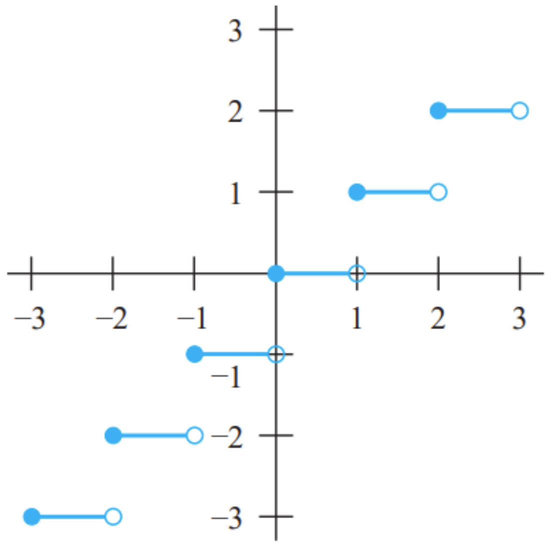
$$\lfloor x \rfloor = \lceil x \rceil = x.$$

- Note that if $x \notin Z$,

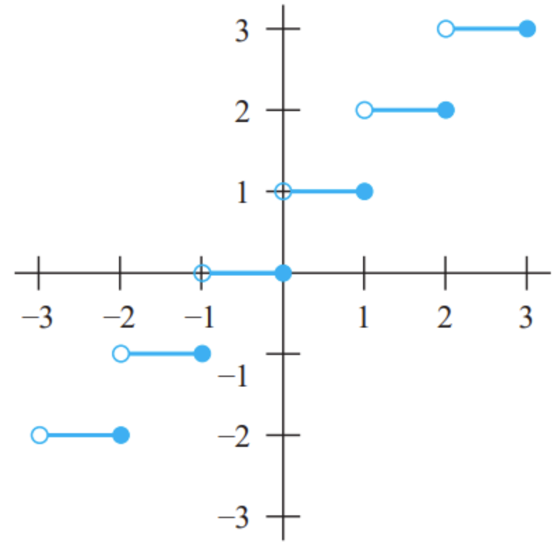
$$\lfloor -x \rfloor \neq -\lfloor x \rfloor \ \& \ \lceil -x \rceil \neq -\lceil x \rceil$$



Plots with floor/ceiling



Floor Function

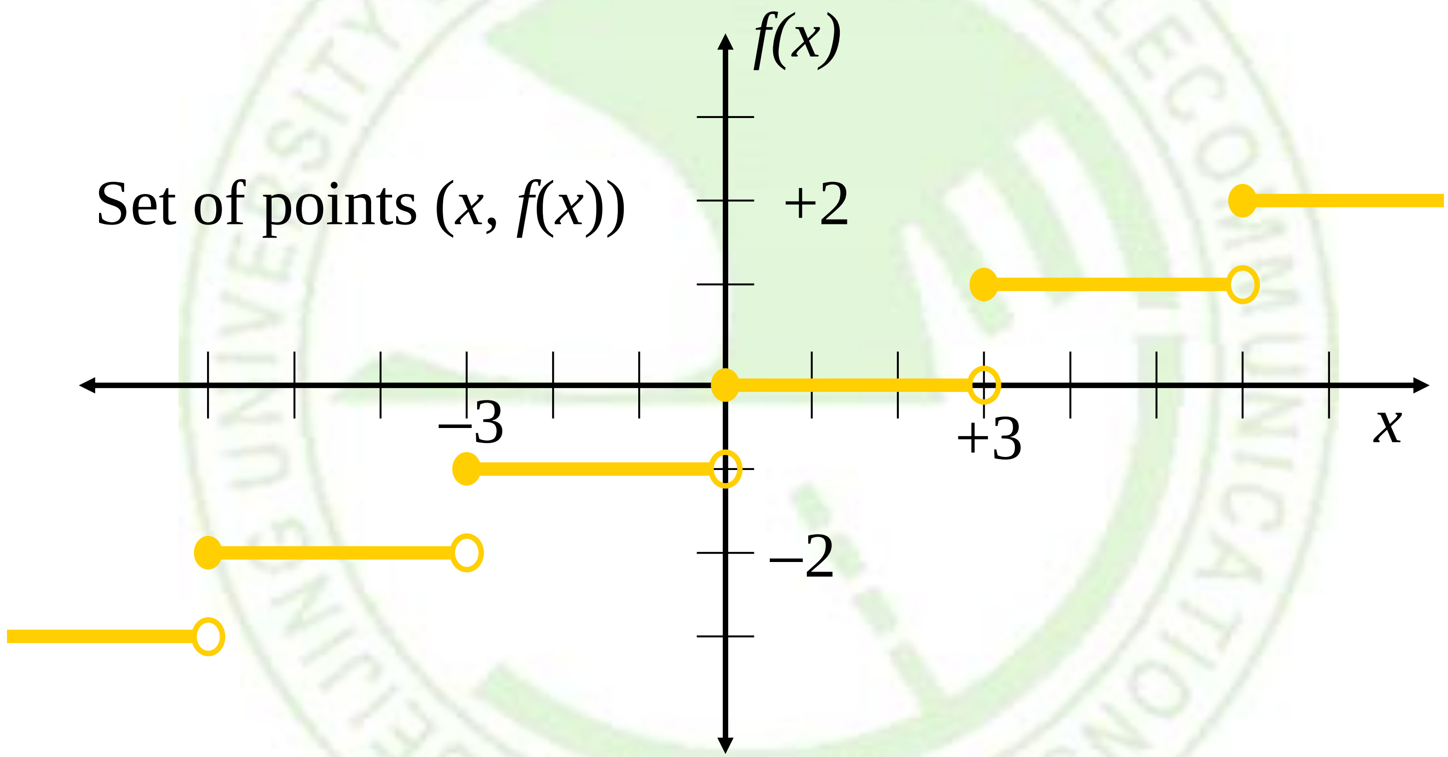


Ceiling Function

- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.
- For $f(x)=\lfloor x \rfloor$, $f(x)$=a when x in [a, a+1).
- For $f(x)=\lceil x \rceil$, $f(x)$=a+1 when x in (a, a+1].

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Properties of Floor and Ceiling

- **Table 1**

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Properties of Floor and Ceiling

- **Example 31**
  - Prove that if x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{2} \rfloor.$$

- **Proof:** ***Let $x=n+\varepsilon$ ($0\le \varepsilon<1$)***

1) Case 1: $0\le \varepsilon<1/2$, $\lfloor 2x\rfloor=\lfloor 2n+2\varepsilon\rfloor=2n$

$\lfloor x\rfloor=\lfloor n+\varepsilon\rfloor=n$, $\lfloor x+1/2\rfloor=\lfloor n+\varepsilon+1/2\rfloor=n$

show $\lfloor 2x\rfloor=\lfloor x\rfloor+\lfloor x+1/2\rfloor$

2) Case 2: $1/2\le \varepsilon<1$, $\lfloor 2x\rfloor=\lfloor 2n+2\varepsilon\rfloor=2n+1$

$\lfloor x\rfloor=\lfloor n+\varepsilon\rfloor=n$, $\lfloor x+1/2\rfloor=\lfloor n+\varepsilon+1/2\rfloor=n+1$

show $\lfloor 2x\rfloor=\lfloor x\rfloor+\lfloor x+1/2\rfloor$

Factorial function

- **Definition:**

  f: N→ Z^+ , $f(n)=n!$

  n! grows extremely rapidly as n grows.

Stirling's formula: $n! \approx \sqrt{2\pi n}(n/e)^n$

$$\lim_{n\to+\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Characteristic Functions

- Let A be a subset of the universal set $U=\{u_1, u_2, u_3, \ldots, u_n\}$.
- The characteristic function of A is defined as a function from U to $\{0,1\}$ by the following:

$$f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$$

Characteristic Functions

- **Example**
  - if
    - $U=\{1, 2, 3, \ldots, 10\}$
    - $A=\{4, 7, 9\}$
  - then
    - $f_A(2)=0$,
    - $f_A(4)=1$,
    - $f_A(7)=1$,
    - $f_A(12)$ is undefined.
- Note: f_A is onto but not one to one

Mod-n Functions

- **Definition:**
  - Let f_n is a function from the nonnegative integers to the set {0, 1, 2, 3,…, n-1}.
  - For a fixed n, any nonnegative integer z can be written as

    $z=kn+r$ with $0 \le r < n$,
  - Then $f_n(z)=r$ or $r= z \bmod n$.
- **Example:**
  - *Let* $f_5(x)=y$ or $y= x \bmod 5$ *, then* $f_5(7)=?$, $f_5(9)=?$, $f_5(20)=?$
- **Note:** f_n is onto but not one to one.

Partial Functions

- **Definition:**
  - A ***partial function*** *f* from a set *A* to a set *B* is an assignment to each element *a* in *a subset of A, called the* ***domain of definition of f*** , of a unique element *b* in *B*.
  - The sets *A* and *B* are called the *domain* and *codomain of f* , *respectively*.
  - We say that *f* is *undefined* for elements in *A* that are not in the domain of definition of *f* .
  - When the *domain of definition of f* equals *A*, we say that *f* is a *total function*.

Partial Functions

- **Example 34**
  - The function $f : Z \rightarrow R$ where f(n) $=\sqrt{n}$ is a partial function from Z to R where the domain of definition is the set of nonnegative integers.
  - f is undefined for negative integers.

Review of 2.3 Functions

- **Function variables:** *f, g, h…*
- **Function Notation:** f:$A \rightarrow B$, $f(a)$, $f(A)$
- **Terms:**
  - *domain, codomain, image, preimage, range, one-to-one (injection), onto (surjection), one-to-one correspondence (bijection), strictly (in/de)creasing, identity, inverse, composite*
- **Function operations:** +, -, ×, I_A, f^{-1}, $\circ$
- **Special Functions:**
  - *Floor and ceiling, Factorial function, Characteristic function, Mod-n function, Partial function…*

Homework

- **§ 2.3**
  - 20, 24, 33, 66

2.4 Sequences and Summations

Wenjing Li

wjli@bupt.edu.cn

School of Computer Science

Beijing University of Posts & Telecommunications

Introduction

- A ***sequence*** or ***series*** is just like an ordered n-tuple, except:
  - Each element in the series has an associated *index* number.
  - A sequence or series may be *infinite*.
  - **Example:** A *string* is a sequence of *symbols* from some finite *alphabet*.

- A ***summation*** is a compact notation for the sum of all terms in a (possibly infinite) series.

SEQUENCES

- **Definition:**
  - A *sequence* is a function from a subset of the natural numbers (usually of the form {0, 1, 2, . . . } to a set S.
  - Note: the sets {0, 1, 2, 3, . . . , k} and {1, 2, 3, 4, . . . , k} are called *initial segments* of N.

- **Notation:**
  - if f is a function from {0, 1, 2, . . .} to S, we usually denote $f(i)$ by a_i and we write $\{a_0, a_1, a_2, a_3, \ldots\} = \{a_i\}_{i=0}^{k} = \{a_i\}_0^k$ where k is the upper limit (usually ∞).
  - And the a_i is called a ***term*** of the sequence.

Sequences

- **Example:**
  - Some authors write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.
  - Be careful: Our book often leaves the indices ambiguous.

- **An example of an infinite series:**
  - Consider the series $\{a_n\} = a_1, a_2, \ldots$, where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
  - Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$

- **Example with Repetitions:**
  - Like tuples, but unlike sets, a sequence may contain *repeated* instances of an element.
  - Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note that 0 is an index) where $b_n = (-1)^n$.
  - Thus, $\{b_n\} = 1, -1, 1, -1, \ldots$
    - Note repetitions!
  - This $\{b_n\}$ denotes an infinite sequence of 1's and –1's, *not* the 2-element set $\{1, -1\}$.

Sequences

- **Recognizing Sequences:**
  - Sometimes, you’re given the first few terms of a sequence, and you are asked to find the sequence’s generating function,
  - or a procedure to enumerate the sequence.

- **Examples:**
  - What’s the next number?
  - 1,2,3,4,… 5 (the 5th smallest number >0)
  - 1,3,5,7,9,… 11 (the 6th smallest odd number >0)
  - 2,3,5,7,11,... 13 (the 6th smallest prime number)

Sequences: Recurrence relations

- **Definition:**
  - A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
  - *A* sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
  - The *initial conditions* for a recursively defined sequence specify the terms that **precede** the first term where the recurrence relation takes effect.

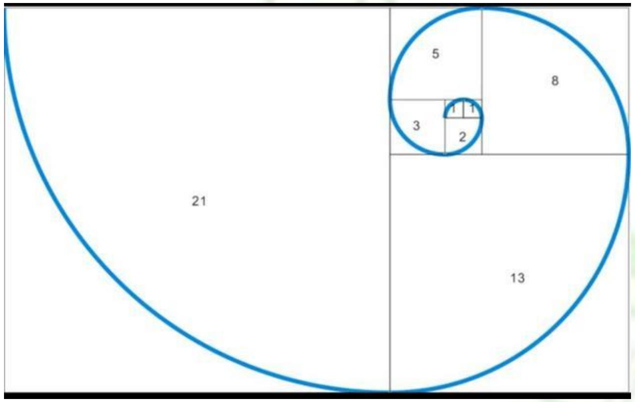
- **Example 5**
  - Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$,and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

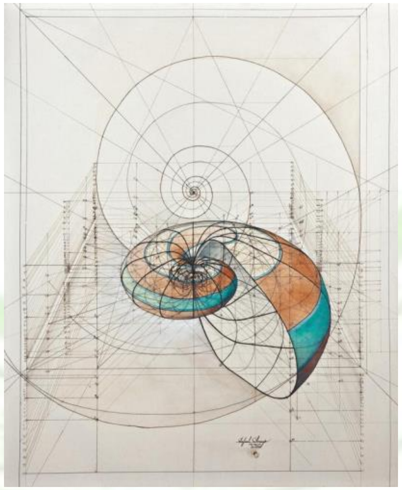
- **Example 6**
  - Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n =2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

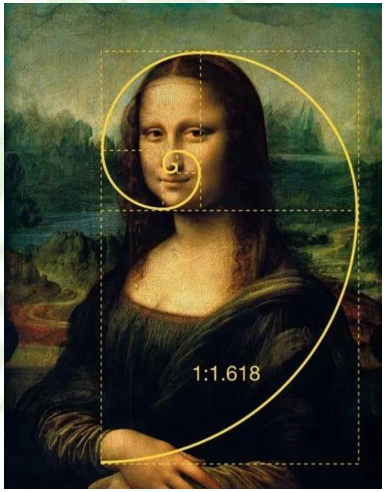
Sequences: Fibonacci sequence

- The ***Fibonacci sequence***, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$,and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

- **Example 7**
  - Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.







Sequences: factorial function

- **Example 8**
  - Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the ***factorial function*** at the integer n, where $n = 1, 2, 3, \ldots$.
  - Because $n! = n((n - 1)(n - 2) \ldots 2 \cdot 1) = n(n - 1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$.

Sequences: closed Formula

- **Definition:**
  - We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a ***closed formula***, for the terms of the sequence.

- **Example 9**
  - Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

    $$a_n = 2\times3(n-1)-3(n-2)=6n-6-3n+6=3n$$

  - Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

    $$a_n = 2\times2^{n-1} - 2^{n-2} = 3\times2^{n-2}$$

- **Example 10**
  - Solve the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, \ldots$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_2 &= 2 + 3\\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2\\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3\\
&\ \vdots\\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1).
\end{aligned}
$$

$$
\begin{aligned}
a_n &= a_{n-1} + 3\\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2\\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3\\
&\ \vdots\\
&= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1).
\end{aligned}
$$

Sequences: closed Formula

- **Iteration:**
  - The technique used in Example 10 is called **iteration**.
  - The first approach is called **forward substitution** – we found successive terms beginning with the initial condition and ending with a_n.
  - The second approach is called **backward substitution**, because we began with a_n and iterated to express it in terms of falling terms of the sequence until we found it in terms of a_1.

Sequences: Special sequences

- **Special integer sequences**
- **Example 12**
  - Find formulae for the sequences with the following first five terms:
    - (a) 1, $1/2$, $1/4$, $1/8$, $1/16$
    - (b) 1, 3, 5, 7, 9
    - (c) 1, −1, 1, −1, 1.
- **Example 13**
  - How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Sequences: Special sequences

- **Example 14**
  - How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41,47, 53, 59?

- **Example 15**
  - How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123?

$$a_n = a_{n-1} + a_{n-2} \quad \text{with } a_0 = 2,\ a_1 = 1$$

Summation

- **Notation:**
  - Given a series $\{a_n\}$, an integer ***lower bound*** (or *limit*) $j \geq 0$, and an integer ***upper bound*** $k \geq j$, then the *summation of* $\{a_n\}$ *from j to k* is written and defined as follows:

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

  - Here, i is called the *index of summation*.

Generalized Summations

- For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i \equiv a_j + a_{j+1} + \ldots$$

- To sum a function over all members of a set $X=\{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) \equiv f(x_1) + f(x_2) + \ldots$$

- Or, if $X=\{x|P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) \equiv f(x_1) + f(x_2) + \ldots$$

Summation

- **Simple Summation Example**

$$\sum_{i=2}^{4} i^2 + 1 = (2^2+1)+(3^2+1)+(4^2+1)$$
$$= (4+1)+(9+1)+(16+1)$$
$$= 5+10+17$$
$$= 32$$

- **Examples:**
  - Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 = 2^2+3^2+5^2+7^2 = 4+9+25+49 = 87$$

Summation: Manipulations

- **Manipulations:**
  - Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$

Distributive law.

$$\sum_x (f(x)+g(x)) = \left(\sum_x f(x)\right) + \sum_x g(x)$$

An application of commutativity.

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$

Index shifting.

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

Series splitting.

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

Order reversal.

$$\sum_{i=0}^{2k} f(i) = \sum_{i=0}^{k} \left(f(2i) + f(2i+1)\right)$$

Grouping.

Summation: Euler's Trick

- **Example: Impress Your Friends**
  - Boast, "I'm so smart; give me any 2-digit number *n*, and I'll add all the numbers from 1 to *n* in my head in just a few seconds."
  - *I.e.*, Evaluate the summation: $\sum_{i=1}^{n} i$
  - There is a simple closed-form formula for the result, discovered by Euler at age 12!

Leonhard Euler (1707-1783)

Summation: Euler's Trick

- **Euler's Trick, Illustrated**
  - Consider the sum:

$$1+2+\ldots+(n/2)+((n/2)+1)+\ldots+(n-1)+n$$

$$n+1$$

$$\vdots$$

$$n+1$$

$$n+1$$

  - We have $n/2$ pairs of elements, each pair summing to $n+1$, for a total of $(n/2)(n+1)=n(n+1)/2$.

Summation: Euler's Trick

- **Symbolic Derivation of Trick**

For case where n is even…

$$\sum_{i=1}^{n} i = \sum_{i=1}^{2k} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=k+1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (i+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} ((n-(k+1))-i)+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (n-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n-(i-1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n+1-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k} (n+1-i) = \dots$$

Summation: Euler's Trick

- **Concluding Euler's Derivation**

$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k}(n+1-i) = \sum_{i=1}^{k}(i+n+1-i)$$

$$= \sum_{i=1}^{k}(n+1) = k(n+1) = \tfrac{n}{2}(n+1)$$

$$= n(n+1)/2$$

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd *n* (prove this at home).

Summation: Geometric Progression

- **Example: Geometric Progression**
  - A *geometric progression* is a series of the form $a, ar, ar^2, ar^3, \ldots, ar^k$, where $a,r \in \mathbf{R}$.
  - The sum of such a series is given by:

$$S = \sum_{i=0}^{k} ar^i$$

  - We can reduce this to *closed form* via clever manipulation of summations...

- **Derivation Example**

$$S = \sum_{i=0}^{n} ar^{i}$$

$$rS = r\sum_{i=0}^{n} ar^{i} = \sum_{i=0}^{n} rar^{i} = \sum_{i=0}^{n} arr^{i} = \sum_{i=0}^{n} ar^{1}r^{i}$$

$$= \sum_{i=0}^{n} ar^{1+i} = \sum_{i=1}^{n+1} ar^{1+(i-1)} = \sum_{i=1}^{n+1} ar^{i}$$

$$= \left(\sum_{i=1}^{n} ar^{i}\right) + \sum_{i=n+1}^{n+1} ar^{i} = \left(\sum_{i=1}^{n} ar^{i}\right) + ar^{n+1} = \ldots$$

Summation: Geometric Progression

- **Derivation Example (Cont)**

$$S = \sum_{i=0}^{k} ar^{i}$$

$$rS = \left(\sum_{i=1}^{n} ar^{i}\right) + ar^{n+1} = (ar^{0} - ar^{0}) + \left(\sum_{i=1}^{n} ar^{i}\right) + ar^{n+1}$$

$$= ar^{0} + \left(\sum_{i=1}^{n} ar^{i}\right) + ar^{n+1} - ar^{0}$$

$$= \left(\sum_{i=0}^{0} ar^{i}\right) + \left(\sum_{i=1}^{n} ar^{i}\right) + ar^{n+1} - a$$

$$= \left(\sum_{i=0}^{n} ar^{i}\right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)$$

Summation: Geometric Progression

- **Concluding Derivation Example**

$$S = \sum_{i=0}^{k} ar^i$$

$$rS = S + a(r^{n+1} - 1)$$

$$rS - S = a(r^{n+1} - 1)$$

$$S(r-1) = a(r^{n+1} - 1)$$

$$S = a\left(\frac{r^{n+1}-1}{r-1}\right) \qquad \text{when } r \neq 1$$

$$\text{When } r = 1,\ S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$$

Some Shortcut Expressions

$$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$$

Geometric series.

$$\sum_{k=1}^{n} k = n(n+1)/2$$

Euler's trick.

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$

Quadratic series.

$$\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$$

Cubic series.

Some Shortcut Expressions

- **Example:**
  - Evaluate $\sum_{k=50}^{100} k^2$
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925.$$

Nested Summations

- These have the meaning you'd expect.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6\cdot 10 = 60$$

- Note issues of free vs. bound variables, just like in quantified expressions, integrals, etc.

Review

- **Sequence:**
  - Definition and Notation
  - Recurrence relation and closed formula
  - Special sequences: Fibonacci series, factorial series…

- **Summation:**
  - How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulation laws
  - Euler's Trick
  - Geometric Progression
  - Shortcut closed-form formulas and how to use them.

Homework

- **§ 2.4**
  - 6(a,c,e,g), 12(a,d), 18
  - 26, 30, 34(b)