6.1 Basics of Counting

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The basics of counting

- Let *m* be the number of ways to do task 1 and *n* the number of ways to do task 2,
  - with each number independent of how the other task is done
  - and also assume that no way to do task 1 simultaneously accomplishes task 2
- Then, we have the following rules:
  - The ***sum rule***: The task "do either task 1 or task 2, but not both" can be done in ***m+n*** ways.
  - The ***product rule***: The task "do both task 1 and task 2" can be done in ***mn*** ways.

Set Theoretic Version

- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then:
  - The ways to do either task 1 or 2 are $A \cup B$, and

$$|A \cup B| = |A| + |B|$$

  - The ways to do both task 1 and 2 can be represented as $A \times B$, and

$$|A \times B| = |A| \cdot |B|$$

Extended multiple rules

- **Definition:**
  - Suppose that tasks $T_1, T_2, \ldots, T_k$ are to be performed in sequence. If T_1 can be performed in n_1 ways, and for each of these ways T_2 can be performed in n_2 ways, and for each of these $n_1 n_2$ ways of performing $T_1 T_2$ in sequence, T_3 can be performed in n_3 ways, and so on, then the sequence $T_1 T_2 \ldots T_k$ can be performed in exactly $n_1 n_2 \ldots n_k$ ways.
- **Example:**
  - A label identifier, for a computer system, consists of one letter followed by three digits. If repetitions are allowed, how many distinct label identifiers are possible?
- **Solution:**
  - There are 26 possibilities for the beginning letter and there are 10 possibilities for each of the three digits.
  - Thus, by the extended multiplication principle, there are $26 \times 10 \times 10 \times 10$ or 26,000 possible label identifiers.

Extended multiple rules

- **Example:**
  - Let A be a set with n elements. How many subsets does A have?
- **Solution:**
  - When the elements of A are listed in an arbitrary order, there is a one-to-one correspondence between subsets of A and bit strings of length $|A|$. When the ith element is in the subset, the bit string has a 1 in the ith position and a 0 otherwise.
  - By the product rule, there are $2^{|A|}$ such bit strings, and therefore $2^{|A|}$ subsets.
  - The solution is 2^n

a_1	a_2	a_3	……	a_n
1/0	**1/0**	**1/0**	**……**	**1/0**

Extended multiple rules

- **Example:**
  - Let A be any set with n elements, and suppose that $1 \le r \le n$. How many different sequences, each of length r, can be formed using elements from A if
  - (a) elements in the sequence may be repeated?
  - (b) all elements in the sequence must be distinct?
- **Solution:**
  - First we note that any sequence of length r can be formed by filling r boxes in order from left to right with elements of A.

  - Let T_1 be the task "fill box 1" , let T_2 be the task "fill box 2", and so on.

Extended multiple rules

- **Case (a):**
  - T_1 can be accomplished in n ways, since we may copy any element of A for the first position of the sequence.
  - The same is true for each of the tasks $T_2, T_3, \ldots, T_r$.
  - Then by the extended multiplication principle, the number of sequences that can be formed is $n \bullet n \bullet \ldots \bullet n = n^r$.

- **Theorem:**
  - Let A be a set with n elements and $l \leq r \leq n$. Then the number of sequences of length r that can be formed from elements of A, allowing repetitions, is $\boldsymbol{n^r}$.

Extended multiple rules

- **Case (b):**
  - T_1 can be performed in n ways, since any element of A can be chosen for the first position.
  - Whichever element is chosen, only $(n - 1)$ elements remain, so that T_2 can be performed in $(n - 1)$ ways, and so on, until finally T_r can be performed in $n - (r - 1))$ or $(n - r + 1)$ ways.
- **Theorem:**
  - By the extended principle of multiplication, a sequence of r distinct element from A can be formed in

$$n(n - 1)(n - 2) \ldots (n - r + 1) \text{ ways.}$$

Counting Functions

- **Counting Functions**: How many functions are there from a set with m elements to a set with n elements?
  - **Solution**: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

- **Counting One-to-One Functions**: How many one-to-one functions are there from a set with m elements to one with n elements?
  - **Solution**: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

IP Address Example

- **Question:**
  - Some facts about the Internet Protocol, version 4: Valid computer addresses are in one of 3 types:
    - A *class A* IP address contains a 7-bit "netid" $\neq \mathbf{1}^7$, and a 24-bit "hostid"
    - A *class B* address has a 14-bit netid and a 16-bit hostid.
    - A *class C* address has 21-bit netid and an 8-bit hostid.
  - The 3 classes have distinct headers (0, 10, 110)
  - Hostids that are all 0s or all 1s are not allowed.
  - How many valid computer addresses are there?

e.g., www.bupt.edu.cn is 211.68.69.240

IP Address Example

- **Solution:**
  - (# addrs)
    = (# class A) + (# class B) + (# class C)   (by sum rule)
  - # class A = (# valid netids)·(# valid hostids) (by product rule)
  - (# valid class A netids) $2^7 - 1 = 127$.
  - (# valid class A hostids) $2^{24} - 2 = 16{,}777{,}214$.
  - (# valid class A addrs) 127*16,777,214 = 2,130,706,178
  - So as class B and Class C addrs.

Subtraction Rule (Inclusion-Exclusion)

- Suppose that $k \le m$ of the ways of doing task 1 <u>also</u> simultaneously accomplish task 2.
- Then, the number of ways to accomplish "Do either task 1 or task 2" is $m+n-k$.
- **Set theory:** If A and B are not disjoint, then
  $$|A \cup B| = |A| + |B| - |A \cap B|.$$ (容斥原理)
  - If they are disjoint, this simplifies to $|A|+|B|$.

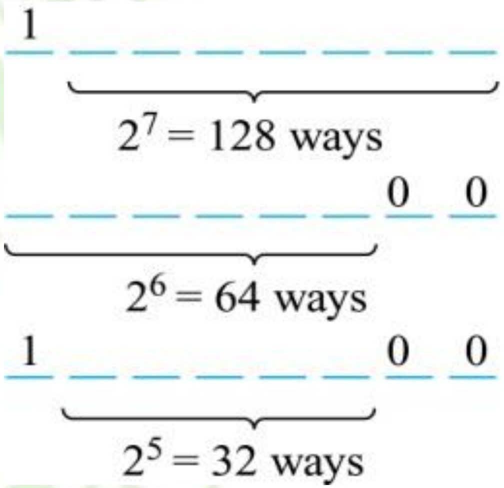
Subtraction Rule (Inclusion-Exclusion)

- **Example: Some hypothetical rules for passwords.**
  - Passwords must be 2 characters long. Each character must be a letter a-z, a digit 0-9, or one of the 10 punctuation characters !@#$%^&*().
  - Each password must contain <u>at least 1</u> digit or punctuation character.

- **Solution:**
  - A legal password has a digit or puctuation character in position 1 **or** position 2. These cases overlap, so the principle applies.
  - (required symbol in position #1): (10+10)·(10+10+26)=920
  - (required sym. in pos. #2): 46·20=920
  - (required sym. in both places): 20·20=400
  - Answer: 920+920−400 = 1440

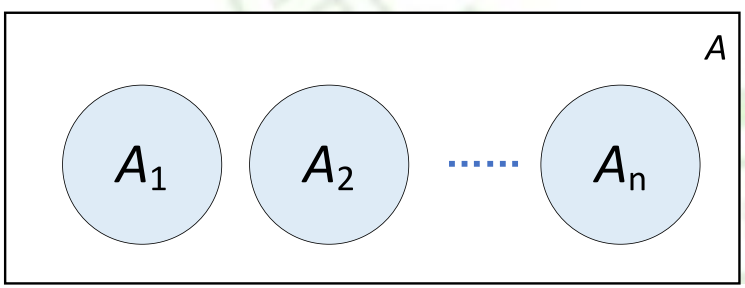
Counting Bit Strings

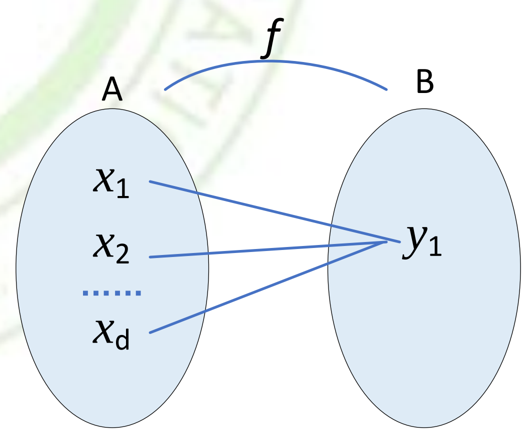
- **Example:**
  - How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
- **Solution**:
  - Use the subtraction rule.
  - Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
  - Number of bit strings of length eight that start with bits 00: $2^6 = 64$
  - Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
  - Hence, the number is $128 + 64 - 32 = 160$.



Division Rule

- **Definition:**
  - There are *n/d* ways to do a task if it can be done using a procedure that can be carried out in *n* ways, and for every way *w*, exactly *d* of the *n* ways correspond to way *w*.
  - If the finite set *A* is the union of *n* pairwise disjoint subsets each with d elements, then $n = |A|/d$.
  - If *f* is a function from *A* to *B* where *A* and *B* are finite sets, and that for every value $y \in B$ there are exactly *d* values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$



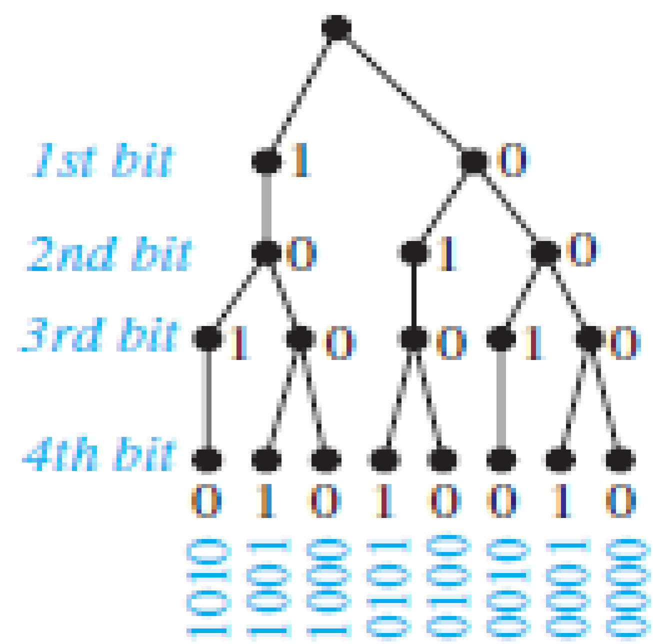


Division Rule

- **Example:**
  - How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
- **Solution:**
  - Number the seats around the table from 1 to 4 proceeding clockwise.
  - There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.
  - Therefore, by the division rule, there are 24/4=6 different seating arrangements.

Tree Diagrams

- **Example**
  - How many bit strings of length four do not have two consecutive 1s? (长度为4的比特串不包含两个连续的1的个数)



Tree Diagrams

- **Example:**
  - Suppose that T-shirts come in five different sizes: S, M, L, XL, and XXL.
  - Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black.
  - How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

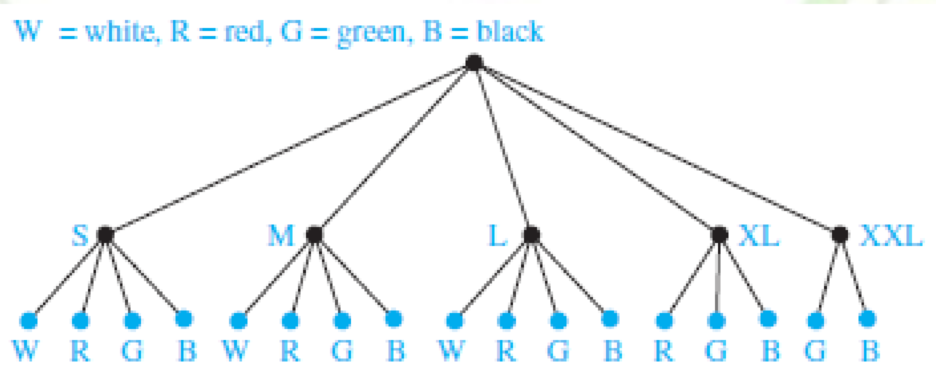


FIGURE 4 Counting Varieties of T-Shirts.

6.2 Pigeonhole Principle

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Pigeonhole Principle

- A.k.a. the "**Dirichlet drawer principle (狄利克雷抽屜原理)**"
- **Definition:**
  - If $\geq k+1$ objects are assigned to k places, then at least 1 place must be assigned ≥ 2 objects.
  - In terms of the assignment function: If $f{:}A \rightarrow B$ and $|A| \geq |B|+1$, then some element of B has ≥ 2 preimages under f.
  - I.e., f is not one-to-one.

(1805–1859)
Germany

Pigeonhole Principle

- **Example:**
  - There are 101 possible numeric grades (0-100) rounded to the nearest integer.
  - Also, there are >101 students in this class.
  - Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - I.e., the function from students to rounded grades is *not* a one-to-one function.

Pigeonhole Principle

- **Theorem (Fun Pigeonhole proof):**
  - $\forall n \in \mathbf{N}$, $\exists$ a multiple $m>0$ of n, m has only 0's and 1's in its decimal expansion!
- **Proof:**
  - Consider the n+1 decimal integers 1, 11, 111, …, $\underbrace{1\cdots1}_{n+1}$. They have only n possible remainders mod n.
  - So, take the difference of two that have the same remainder. The result is the answer!

$a \equiv b \pmod{n}$ iff $n \mid a-b$

Pigeonhole Principle

- **A specific Case (Fun Pigeonhole proof):**
  - Let n=3. Consider 1, 11, 111, 1111.
    - 1 mod 3 = 1
    - 11 mod 3 = 2
    - 111 mod 3 = 0 ← Lucky extra solution.
    - 1,111 mod 3 = 1
  - 1,111 − 1 = 1,110 = 3*370.
    - Its difference mod 3 = 0, so it's a multiple of 3.
    - It has only 0's and 1's in its expansion.

Note same remainder.

Generalized Pigeonhole Principle

- **Theorem:**
  - If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.

- **Example:**
  - There are N=52 students in this class. There are k=12 months in the year.
  - Therefore, there must be at least 1 month during which at least $\lceil 52/12 \rceil = \lceil 4.3 \rceil = 5$ students in the class have a birthday.

广义鸽巢原理：G.P.P

G.P.P.

- **Proof:**
  - By contradiction. Suppose every place has $< \lceil N/k \rceil$ objects, thus $\leq \lceil N/k \rceil - 1$.
  - Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\left(\frac{N}{k}\right) = N$$

  - So, there are less than N objects, which contradicts our assumption of N objects!

- **Example 7:**
  - How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit (相同花色) are chosen?

Solution: At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

  - How many must be selected to guarantee that at least three hearts are selected?

Solution： A deck contains 13 hearts and 39 cards which are not hearts. when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)

Appl. of Pigeonhole Principle

- **Example 10 (Baseball Problem):**
  - Suppose in June, the Marlins baseball team plays at least 1 game a day, but ≤45 games total. Show there must be some sequence of consecutive days in June during which they play *exactly* 14 games.
- **Proof:**
  - Let a_j be the number of games played on or before day j.
  - Then, $a_1,\ldots,a_{30} \in \mathbf{Z}^+$ is a sequence of 30 distinct (increasing) integers with $1 \le a_j \le 45$. Therefore $a_1+14,\ldots,a_{30}+14$ is a sequence of 30 distinct integers with $15 \le a_j+14 \le 59$.
  - Thus, $(a_1,\ldots,a_{30},a_1+14,\ldots,a_{30}+14)$ is a sequence of 60 integers from the set $\{1,..,59\}$.
  - By the Pigeonhole Principle, two of them must be equal, but $a_i \neq a_j$ for $i \neq j$. So, $\exists ij$: $a_i = a_j+14$. Thus, 14 games were played on days $a_j+1, \ldots, a_i$.

Appl. Of Pigeonhole Principle

- **Baseball Example** of $\{a_i\}$: Note all elements are distinct.
  - 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 21, 22, 23, 25, 27, 29, 30,31, 33, 34, 36, 37, 39, 40, 41, 43, 45
  - Then $\{a_i+14\}$ is the following sequence: 15, 16, 18, 19, 21, 22, 24, 25, 27, 28,30, 32, 33, 35, 36, 37, 39, 41, 43, 44,45, 47, 48, 50, 51, 53, 54, 55, 57, 59
- In any 60 integers from 1-59 there must be some duplicates, indeed we find the following ones:
  - 16, 19, 21, 22, 25, 27, 30, 33, 36, 37, 39, 41, 43, 45

Thus, for example, exactly 14 games were played during days 3 to 11: 2+1+2+1+2+1+2+1+2

Appl. of Pigeonhole Principle

- **Example 11:**
  - Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
- **Proof:**
  - Each of the $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ can be written as a power of 2 times an odd integer. **(why?)**
  - In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n+1$, where k_j is a nonnegative integer and all q_j are odd positive integers less than $2n$.
  - Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal.
  - Therefore, there are distinct integers i and j such that $q_i = q_j = q$. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i.

Appl. of Pigeonhole Principle

- **Strictly increasing and strictly decreasing**
- **Definition:**
  - Suppose that $a_1, a_2, \ldots a_n$ is a sequence of real numbers. A subsequence of this sequence is a sequence of the form $a_{i1}, a_{i2}, \ldots a_{im}$,where $1 \le i_1 < i_2 < \ldots < i_m \le N$.
  - A sequence is called strictly increasing if each term is larger than the one that precedes it.
  - A sequence is called strictly decreasing if each term is smaller than the one that precedes it.

Appl. Of Pigeonhole Principle

- **Theorem 3:**
  - Every sequence of n^2+1 distinct real number contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate (i_k, d_k) to the term a_k, where i_k is the length of the longest increasing subsequence starting at a_k, and d_k is the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, 2, \ldots, n^2 + 1$. Hence, by the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal. In other words, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible. Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$. If $a_s < a_t$, then, because $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking a_s followed by an increasing subsequence of length i_t beginning at a_t. This is a contradiction. Similarly, if $a_s > a_t$, the same reasoning shows that d_s must be greater than d_t, which is a contradiction. ◁

Ramsey theory (拉姆齐理论)

- **Example 13**
  - Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.
- **Proof**
  - Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements.
  - In the former case, suppose that B, C, and D are friends of A. ①If any two of these three individuals are friends, then these two and A form a group of three mutual friends. ②Otherwise, B, C, and D form a set of three mutual enemies.
  - The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner.

应用：地图着色问题

Homework

- **§ 6.1**
  - 48, 52
- **§ 6.2**
  - 12, 26（选作）, 46

6.3 Permutations and Combinations

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Permutations

- **Definition:**
  - A ***permutation*** of a set S of objects is a sequence that contains each object in S exactly once.
  - An ordered arrangement of r **distinct** elements of S is called an ***r-permutation of S***.
- **Theorem 1:**
  - The number of r-permutations of a set with $|S|=n$ elements is

$$P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$$

Solving Counting Problems

- **Example:**
  - In the movie, a terrorist has planted an armed nuclear bomb in the city, and it is actor's job to disable it by cutting wires to the trigger device.
  - There are 10 wires to the device. If the actor cuts exactly the right three wires, in exactly the right order, actor will disable the bomb, otherwise it will explode! If the wires all look the same, what are the actor's chances of survival?

$$P(10,3) = 10 \cdot 9 \cdot 8 = 720$$

so there is a 1 in 720 chance that the actor will survive!

Solving Counting Problems

- **Example 4:**
  - How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

- **Solution:**

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems

- **Example 6:**
  - Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
- **Solution:**
  - The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

  - If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems

- **Example 7:**
  - How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
- **Solution:**
  - We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

- **Definition:**
  - An r-combination of elements of a set S is simply a subset $T \subseteq S$ with r members, $|T|=r$.
- **Theorem 2:**
  - The number of r-combinations of a set with $|S|=n$ elements is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

  - **Corollary:** $C(n,r) = C(n, n-r)$
    - Because choosing the r members of T is the same thing as choosing the $n-r$ non-members of T.

Combinations

- **Example:**
  - How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.
  - Answer $C(52,7) = P(52,7)/P(7,7)$

    $= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \,/\, 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

    $= 133{,}784{,}560$

6.5 Generalized Permutations and Combinations

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Permutation with repetition

- **Theorem 1:**
  - The number of r-permutations of a set of n objects with repetition allowed is n^r.
- **Example 1:**
  - How many strings of length r can be formed from the uppercase letters of the English alphabet?

Combinations with Repetition

- **Example 2:**
  - How many ways are there to select four pieces of fruit from a bowl containing apples ,oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

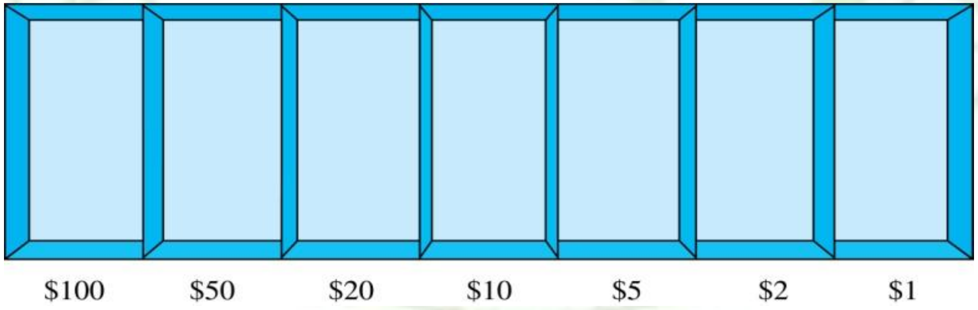
Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

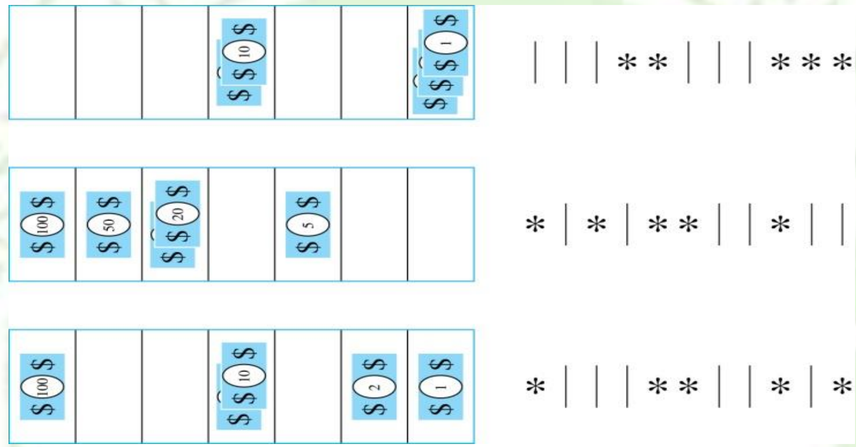
The solution is the number of 4-combinations with repetition allowed from a three-element set, $\{apple, orange, pear\}$. ◀

Combinations with Repetition

- **Example 3:**
  - How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100? Assume the order in which the bills are chosen does not matter.
- **Solution**:
  - Place the selected bills in the appropriate position of a cash box illustrated below:



- **Solution(cont):**
  - Some possible ways of placing the five bills:



  - The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
  - This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

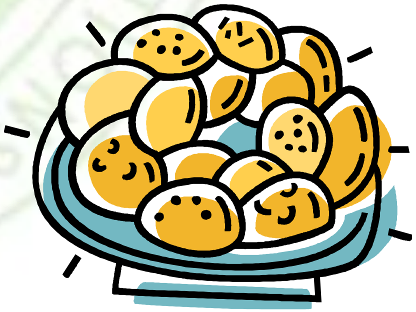
- **Theorem 2:**
  - The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

- **Proof:**
  - Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star for each time the ith element of the set occurs in the combination.
  - The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars.
  - This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars.

Combinations with Repetition

- **Example 4:**
  - Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?



Combinations with Repetition

- **Example 5:**
  - How many solutions does the equation
    $x_1 + x_2 + x_3 = 11$
    have, where x_1 , x_2 and x_3 are nonnegative integers?
- **Solution:**
  - Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.
  - By Theorem 2 it follows that there are
    $$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 12}{1\cdot 2} = 78$$
- **Add conditions:**
  - ***Let $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.***

Combinations with Repetition

- **Excercise:**
  - How many solutions are there to the equation

    $x_1 + x_2 + x_3 + x_4 + x_5 = 20$,

    where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

    ① $x_i \geq 2$ for i=1, 2, 3, 4, 5

    ② $2 \leq x_2 < 4$

Summarizing the Formulas

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\dfrac{n!}{(n-r)!}$
r-combinations	No	$\dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

- **Example 7:**
  - How many different strings can be made by reordering the letters of the word *SUCCESS*.
- **Solution:**
  - There are seven possible positions for the 3 Ss, 2 Cs, 1 U, and 1 E.
  - The 3 Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
  - The 2 Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
  - The U can be placed in $C(2,1)$ different ways, leaving one position free.
  - The E can be placed in $C(1,1)$ way.

  By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

- **Theorem 3**
  - The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,…and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- **Proof:**
  - By the product rule the total number of permutations is:

    $C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$
  - since:
    - The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
    - Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
    - Continue in this fashion, until n_k objects of type k are placed in $C(n-n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.
  - The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!}=\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- **Distinguishable objects and distinguishable boxes**
- **Theorem 4**
  - The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i=1,2,\ldots k$, equals
$$\frac{n!}{n_1!n_2!\cdots n_k!}$$
- **Example 8**
  - How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
  - There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.

Distributing Objects into Boxes

- **Indistinguishable objects and distinguishable boxes**
  - There are $C(n + r-1, r)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.

- **Example 9**
  - How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

Homework

- § 6.3
  - 30, 46
- § 6.5
  - 14, 20, 34, 48