1.5 Nested Quantifiers

Wenjing Li

wjli@bupt.edu.cn

School of Computer Science

Beijing University of Posts & Telecommunications

Outline

- Translating Predicate Logic involving nested quantifiers
- Translating Statements into Predicate Logic
- Negating nested quantifiers
- Order of nested quantifiers

Translating statements involving nested quantifiers

- Assume that the universe of discourse for the variables x and y consists of all real numbers.
- **Example 1**
  - $\forall xy(x+y=y+x)$ $x + y = y + x$ for all real numbers x and y
    commutative law for addition of real numbers
  - $\forall x \exists y(x+y=0)$ for every real number x there is a real number y such that $x + y = 0$
    every real number has an **additive inverse**
  - $\forall xyz(x+(y+z)=(x+y)+z)$
    associative law for addition of real numbers

Translating statements involving nested quantifiers

- **Example 2**
  - $\forall xy((x>0)\wedge(y<0)\rightarrow(xy<0))$

- For every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.
- For real numbers x and y, if x is positive and y is negative, then xy is negative.
- "The product of a positive real number and a negative real number is always a negative real number."

Translating statements involving nested quantifiers

- Assume the universe of discourse for *x*, *y* and *z* consists of all students in your school.
  - $C(x)$ is "*x* has a computer"
  - $F(x,y)$ is "*x* and *y* are friends"

- **Example 9**
  - $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$

  Every student in your school has a computer or has a friend who has a computer.

- **Example 10**
  - $\exists x \forall y \forall z((F(x,y) \wedge F(x,z) \wedge (y \neq z) \rightarrow \sim F(y,z))$

  There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into logical expressions

- **Example 6**
  - The sum of two positive integers is positive.
- **Solution:**
  - Rewrite the statement to find quantifiers and domains

    "For every two integers, if these integers are both positive, then the sum of these integers is positive."
  - Introduce the variables *x* and *y*, and specify the domain:

    Where the domain of x and y is all positive integers

    $\forall x \ \forall y\ (x + y > 0)$
  - Another result:

    where the domain of both variables consists of all integers

    $\forall x \ \forall y\ ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$

- **Example 7**
  - Every real number except zero has a multiplicative inverse.

Multiplicative inverse of a real number x is a real number y such that $\boldsymbol{xy = 1}$.

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1)) \qquad (U=R)$$

- **Example 11**
  - If a person is female and is a parent, then this person is someone' s mother.
    - ***U.D: all people***
    - ***F(x): "x is female,"***
    - ***P (x): "x is a parent,"***
    - ***M(x, y): "x is the mother of y."***

    - $\forall x((F(x) \land P(x)) \rightarrow \exists y M(x, y))$ ***OR***
    - $\forall x \exists y((F(x) \land P(x)) \rightarrow M(x, y))$

- **Example 12**
  - Everyone has exactly one best friend.
    - ***U.D: all people***
    - ***B(x, y) :"y is the best friend of x,"***
    - $\forall x \exists ! y (B(x, y))$
    - $\forall x \exists y (B(x, y) \wedge \forall z ((z \neq y) \rightarrow \neg B(x, z)))$

- **Example 13**
  - There is a woman who has taken a flight on every airline in the world.

- ***Solution***

• ***P(w,f):"w has taken f " ; Q(f,a):"f is a flight on a ."***

• ***The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.***

• $\exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$

Negating quantifiers

- Expanding quantifiers: If u.d.=a,b,c,…

  $$\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$$

  $$\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$$

- From those, we can prove the laws:

  $$\neg\forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$$

  $$\neg\exists x\, P(x) \Leftrightarrow \forall x\, \neg P(x)$$

- Which *propositional* equivalence laws can be used to prove this?

DeMorgan's

Negating nested quantifiers

- **Example 14**
  - Express the negation of the statement $\forall x \exists y(xy=1)$.
  - $\sim\forall x \exists y(xy=1)$

$\Leftrightarrow \exists x \sim\exists y(xy=1)$

$\Leftrightarrow \exists x \forall y \sim(xy=1)$

$\Leftrightarrow \exists x \forall y\ (xy\neq 1)$

Negating nested quantifiers

- **Example 15**
  - There dose not exist a woman who has taken a flight on every airline in the world.
- **Solution**
  - $\neg\exists w\ \forall a\ \exists f\ (P(w,f) \wedge Q(f,a))$

$\Leftrightarrow \forall w\ \neg\ \forall a\ \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$

$\Leftrightarrow \forall w\ \exists a\ \neg\exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$

$\Leftrightarrow \forall w\ \exists a\ \forall f\ \neg(P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$

$\Leftrightarrow \forall w\ \exists a\ \forall f\ (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Negating nested quantifiers

- **Example 16**
  - Express the fact that $\lim_{x \to a} f(x) = L$ does not exist.

$$\lim_{x \to a} f(x) \neq L$$

$$\neg \forall \epsilon > 0\ \exists \delta > 0\ \forall x (0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$

$$\equiv \exists \epsilon > 0\ \neg \exists \delta > 0\ \forall x (0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$

$$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \neg \forall x (0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$

$$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \exists x\ \neg (0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$

$$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

The last step uses the equivalence $\neg(p \to q) \equiv p \wedge \neg q$

Order of Nested quantifiers

- **Example**
  - U=R： the real number
  - Let $P(x,y)$ be the statement "$x+y=y+x$".
  - what is the truth value of the quantification $\forall x \forall y\, P(x,y)$?

- **Example**
  - Let $Q(x,y)$ denote "$x+y=0$".
  - what are the truth values of the quantifications $\exists y \forall x\, Q(x,y)$ and $\forall x \exists y\, Q(x,y)$?

Order of Nested quantifiers

- Let $U = R$, the real numbers,
- $P(x,y)$: $xy= 0$
  - $\forall x \forall y P(x, y)$
  - $\forall x \exists y P(x, y)$
  - $\exists x \forall y P(x, y)$
  - $\exists x \exists y P(x, y)$
- Suppose $P(x, y)$ is the predicate $x/y=1$?

Order of Nested quantifiers

- **Example**
  - U=R
  - Let $Q(x,y,z)$ be the statement "$x+y=z$".
  - what are the truth values of the statements $\forall x \forall y \exists z\, Q(x,y,z)$ and $\exists z \forall x \forall y\, Q(x,y,z)$?

Order of Nested quantifiers

- 例： “存在最小的自然数”。
- 解1: 采用全体自然数作为个体域.

  设: G(x,y): $x \le y$;

  原命题符号化成: **$\exists x \forall y G(x,y)$**

  注意量词顺序: $\forall y \exists x G(x,y)$ “没有最小的自然数”

- 解2: 采用全总个体域

  设: F(x): x是自然数; G(x,y): $x \le y$;

  原命题符号化成: **$\exists x(F(x) \land \forall y(F(y) \rightarrow G(x,y)))$**

Other examples

- **例：“不存在最大的自然数”。**

  解：采用全总个体域

  设: F(x): x是自然数；G(x,y): x≤y;

  原命题符号化成:

  $$\sim\exists x(F(x)\land\forall y(F(y)\rightarrow G(y,x)))\ \text{或}$$

  $$\forall x(F(x)\rightarrow\exists y(F(y)\land G(x,y)))$$

- **例：“存在唯一的对象满足性质P”。**

  解： 设： P(x): x满足性质P

  原命题符号化成（**全总个体域**）：

  $$\exists!xP(x)\qquad \text{或}$$

  $$\exists x(\ P(x)\land\forall y(\ P(y)\rightarrow x=y\ )\ )$$

Other examples

- **例：“火车比汽车快”。**

  解: 设：F(x): x是火车；G(x): x是汽车；H(x,y)：x比y快

  原命题符号化成（全总个体域）：

  $$\forall x(F(x)\rightarrow\forall y(G(y)\rightarrow H(x,y)))$$ **或**

  $$\forall x\forall y((F(x)\wedge G(y))\rightarrow H(x,y))$$

- **例：“有的汽车比火车快”。**

  解: 设: F(x): x是火车；G(x): x是汽车；H(x,y)：x比y快

  原命题符号化成（全总个体域）：

  $$\exists y(F(y)\wedge\exists x(G(x)\wedge H(x,y)))$$ **或**

  $$\exists y\exists x(F(y)\wedge G(x)\wedge H(x,y))$$

Summary: Predicate Logic

- **论域/个体域(U.D./scope/domain):** 个体词的取值范围, 缺省(default)采用全总个体域.
- **全总个体域:** 世界上的万事万物
- **特性谓词：P( )** 表示所关注的对象的性质
- **两类量词：**∀， ∃
- **量词的否定、嵌套和顺序**
- **谓词逻辑的基础应用：**
  - 谓词逻辑到自然语言的翻译
  - 自然语言到谓词逻辑的翻译

Homework

- § 1.5
  - 10, 28, 34, 46

Discrete Mathematics and Its Applications

Suppl: 一阶逻辑合式公式

Wenjing Li

wjli@bupt.edu.cn

School of Computer Science

Beijing University of Posts & Telecommunications

一阶逻辑的字母表

- 个体常项： $a, b, c, \ldots, a_1, b_1, c_1, \ldots$
- 个体变项： $x, y, z, \ldots, x_1, y_1, z_1, \ldots$
- 函数符号： $f, g, h, \ldots, f_1, g_1, h_1, \ldots$
- 谓词符号： $F, G, H, \ldots, F_1, G_1, H_1, \ldots$
- 量词符号： $\exists, \forall$
- 联结词符号： $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$
- 括号与逗号： (,), ,

个体词

- **个体常项（Constant）**
  - 表示具体的特定对象
  - 用小写英文字母a,b,c,…来表示
  - 例如： a:王大明，b:王小明，G(x,y): x与y是兄弟,
    “王大明与王小明是兄弟”： G(a,b)
- **个体变项（Variable）**
  - 表示不确定的泛指对象
  - 用小写英文字母x,y,z,...来表示
  - 例如： F(x): x是人。G(x): x是数。
    “存在着人”： $\exists xF(x)$
    “仅有一人”： $\exists !xF(x)$
    “万物皆数”：$\forall xG(x)$

一阶逻辑的合式公式

- **项(term)**
  - 个体常项和个体变项是项
  - 若$f(x_1,x_2,\ldots,x_n)$是n元函数, $t_1,t_2,\ldots,t_n$是项, 则$f(t_1,t_2,\ldots,t_n)$是项
  - 所有的项都是有限次地应用上述规则形成的
  - 例如: a, x, f(a), g(a, x), g(x, f(a))
- **原子公式(atomic formula)**
  - 若$R(x_1,x_2,\ldots,x_n)$是n元谓词, $t_1,t_2,\ldots,t_n$是项, 则$R(t_1,t_2,\ldots,t_n)$是原子公式
  - 例如: F(a), G(a, y), F(f(a)), G(x, g(a, y))

一阶逻辑的合式公式

- 合式公式(**well-formed formula**)
  - 原子公式是合式公式
  - 若A是合式公式，则(~A)是合式公式
  - 若A,B是合式公式，则(A∧B), (A∨B), (A→B), (A↔B)也是合式公式
  - 若A是合式公式，则∃xA, ∀xA也是合式公式
  - 有限次地应用上述规则形成的符号串是合式公式
    - ***F(f(a,a),b)***
    - ***∃x(F(x)∧∀y(G(y)→H(x,y)))***
- 约定：省略多余括号
  - 最外层
  - 优先级递减：∃, ∀; ~; ∧, ∨; →, ↔

合式公式中的变项

- **量词辖域:** 在$\exists xA$, $\forall xA$中, A是量词的辖域. 例如:

  $\exists x(\underline{F(x)\wedge\forall y(G(y)\rightarrow H(x,y))})$

  $\exists xF(x)\wedge\forall y(G(y)\rightarrow H(x,y))$

- **指导变项:** 紧跟在量词后面的个体变项.例如:

  $\exists x(F(x)\wedge\forall y(G(y)\rightarrow H(x,y)))$

- **约束出现:** 在辖域中与指导变项同名的变项. 例如:

  $\exists x(F(x)\wedge\forall y(G(y)\rightarrow H(x,y)))$

- **自由出现:** 既非指导变项又非约束出现. 例如:

  $\forall y(G(y)\rightarrow H(x,y))$

合式公式中的变项

- $H(x,y) \vee \exists \underline{x} F(x) \vee \forall \underline{y}(G(y) \to H(x,y))$
  - $\underline{x}$与$\underline{y}$是指导变项
  - x与y是约束出现
  - x与y是自由出现

闭式(closed form)

- **闭式：无自由出现的变项**
- 一般来说，闭式表示的是命题，例如
  - $F(a)$
  - $\exists x F(x)$
- 不是闭式
  - $F(x)$
  - $\forall y(G(y) \rightarrow H(x,y))$

合式公式的解释

- F(f(a,a),b)
- 例1：个体域是全体自然数;

  a: 2; b: 4;

  f(x,y)=x+y;   F(x,y): x=y

  原公式解释为：“2+2=4”

- 例2：个体域是全体实数;

  a: 3; b: 5;

  f(x,y)=x-y;   F(x,y): x>y

  原公式解释为：“3-3>5”

对合式公式的解释包括给出：
- 个体域（论域）
- 个体常项，个体变项
- 函数
- 谓词

的具体含义

一阶逻辑类型

- **永真式Tautology:**
  - 在各种**解释**下取值均为真(**逻辑有效式**)
  - 命题逻辑永真式: 在各种*赋值*下取值均为真(**重言式**)

- **永假式Contradiction:**
  - 在各种**解释**下取值均为假(**矛盾式**)
  - 命题逻辑永假式: 在各种赋值下取值均为假(**矛盾式**)

- **可满足式:** 非永假式

SUPPL: 一阶逻辑等值式

WENJING LI

wjli@bupt.edu.cn

SCHOOL OF COMPUTER SCIENCE

BEIJING UNIVERSITY OF POSTS & TELECOMMUNICATIONS

一阶逻辑等值式(定义)

- 等值： $A \Leftrightarrow B$
  - 读作：A等值于B
  - 含义：A与B在各种解释下取值均相等

- $A \Leftrightarrow B$ 当且仅当 $A \leftrightarrow B$是永真式
  - 例如： $\sim\forall xF(x) \Leftrightarrow \exists x\sim F(x)$

一阶逻辑等值式(来源)

- **1.命题逻辑等值式的代换实例**
- **2.与量词有关的**
  - 2.1 有限个体域量词消去
  - 2.2 量词否定
  - **2.3 量词辖域收缩与扩张**
  - **2.4 量词分配**
- **3.与变项有关的**
  - 3.1 换名规则
  - 3.2 代替规则

1.代换实例

- 在命题逻辑等值式中, 代入一阶逻辑公式所得到的式子, 称为原来公式的代换实例.

- 例1：A⇔~~A，令A=∀xF(x)，得到

  $$\forall xF(x) \Leftrightarrow \sim\sim \forall xF(x)$$

- 例2：$A\rightarrow B\Leftrightarrow \sim A\lor B$，令A=F(x)，B=G(y)，得到

  $$F(x)\rightarrow G(y)\Leftrightarrow \sim F(x)\lor G(y)$$

2.1 有限个体域量词消去

- 设个体域为有限集$D=\{a_1, a_2,\ldots, a_n\}$, 则

$$\forall xA(x)\Leftrightarrow A(a_1)\wedge A(a_2)\wedge\ldots\wedge A(a_n)$$

$$\exists xA(x)\Leftrightarrow A(a_1)\vee A(a_2)\vee\ldots\vee A(a_n)$$

- 例：个体域$D=\{a,b,c\}$, 则 $\exists x\forall yF(x,y)$

$$\Leftrightarrow\exists x\,(F(x,a)\wedge F(x,b)\wedge F(x,c))$$

$$\Leftrightarrow\quad (F(a,a)\wedge F(a,b)\wedge F(a,c))\vee$$
$$(F(b,a)\wedge F(b,b)\wedge F(b,c))\vee$$
$$(F(c,a)\wedge F(c,b)\wedge F(c,c))$$

2.1 有限个体域量词消去

- **U={1, 2, 3}**
- Find an expression equivalent to $\forall x \exists y P(x, y)$, where the variables are bound.
- Expand from inside out or outside in.
- Outside in:

$$\forall x \exists y P(x, y)$$

$$\Leftrightarrow \exists y P(1, y) \wedge \exists y P(2, y) \wedge \exists y P(3, y)$$

$$\Leftrightarrow [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$

$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$

$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

2.2 量词否定等值式

- $\sim\forall xA(x)\Leftrightarrow\exists x\sim A(x)$
- $\sim\exists xA(x)\Leftrightarrow\forall x\sim A(x)$

- P(x): x是必然的， $\forall xP(x)$
- 并非一切都是必然的
  - $\sim\forall xP(x)$
- 有的事物是偶然的
  - $\exists x$(*x不是必然的*)
  - $\Leftrightarrow \exists x\sim P(x)$
- 所以
  - $\sim\forall x\, P(x) \Leftrightarrow \exists x\, \sim P(x)$

- *P*(*x*): x是孤立的， $\exists xP(x)$
- 不存在孤立事物
  - $\sim\exists xP(x)$
- 一切事物都不是孤立的
  - $\forall x\sim P(x)$
- 所以
  - $\sim\exists x\, P(x) \Leftrightarrow \forall x\, \sim P(x)$

2.3 量词辖域收缩与扩张(∀)

- **假设B中不含x的出现**
  - 1）$\forall x(A(x) \vee B) \Leftrightarrow \forall x A(x) \vee B$
  - 2）$\forall x(A(x) \wedge B) \Leftrightarrow \forall x A(x) \wedge B$
  - 3）$\forall x(A(x) \to B) \Leftrightarrow \exists x A(x) \to B$
  - 4）$\forall x(B \to A(x)) \Leftrightarrow B \to \forall x A(x)$

∀的辖域收缩与扩张

- 例1: $\forall x(F(x) \vee G(y))$

  $\Leftrightarrow \forall x F(x) \vee G(y)$ **辖域收缩:$\forall x(A(x) \vee B) \Leftrightarrow \forall x A(x) \vee B$**

- 例2: $\forall x \forall y(F(x) \wedge G(y))$

  $\Leftrightarrow \forall x(F(x) \wedge \forall y G(y))$ **辖域收缩:$\forall x(A(x) \wedge B) \Leftrightarrow \forall x A(x) \wedge B$**

  $\Leftrightarrow \forall x F(x) \wedge \forall y G(y)$ **辖域收缩:$\forall x(A(x) \wedge B) \Leftrightarrow \forall x A(x) \wedge B$**

2.3 量词辖域收缩与扩张(∀)

- **证3)：** $\forall x(A(x)\to B) \Leftrightarrow \exists x A(x)\to B$

证明： $\forall x(A(x)\to B)$

$\Leftrightarrow \forall x(\sim A(x)\vee B)$

$\Leftrightarrow \forall x\sim A(x)\vee B$

$\Leftrightarrow \sim\exists x A(x)\vee B$

$\Leftrightarrow \exists x A(x)\to B$

- **证4)：** $\forall x(B\to A(x)) \Leftrightarrow B\to\forall x A(x)$

证明： $\forall x(B\to A(x))$

$\Leftrightarrow \forall x(\sim B\vee A(x))$

$\Leftrightarrow \sim B\vee\forall x A(x)$

$\Leftrightarrow B\to\forall x A(x)$

2.3 量词辖域收缩与扩张(∃)

- **假设: B中不含x的出现**
  - 1）$\exists x(A(x)\vee B) \Leftrightarrow \exists xA(x)\vee B$
  - 2）$\exists x(A(x)\wedge B) \Leftrightarrow \exists xA(x)\wedge B$
  - 3）$\exists x(A(x)\rightarrow B) \Leftrightarrow \forall xA(x)\rightarrow B$
  - 4）$\exists x(B\rightarrow A(x)) \Leftrightarrow B\rightarrow\exists xA(x)$

∃的辖域收缩与扩张

- 例1: $\exists x(F(x)\vee G(y))$

  $\Leftrightarrow \exists xF(x)\vee G(y)$ 辖域收缩: $\exists x(A(x)\vee B)\Leftrightarrow \exists xA(x)\vee B$

- 例2: $\forall x\exists y(F(x)\wedge G(y))$

  $\Leftrightarrow\forall x(F(x)\wedge\exists yG(y))$ 辖域收缩: $\exists x(A(x)\wedge B)\Leftrightarrow \exists xA(x)\wedge B$

  $\Leftrightarrow\forall xF(x)\wedge\exists yG(y)$ 辖域收缩: $\forall x(A(x)\wedge B)\Leftrightarrow \forall xA(x)\wedge B$

2.3 量词辖域收缩与扩张(∃)

- **证3)：** $\exists x(A(x)\rightarrow B) \Leftrightarrow \forall x A(x)\rightarrow B$

证明：$\exists x(A(x)\rightarrow B)$

$\Leftrightarrow \exists x(\sim A(x)\vee B)$

$\Leftrightarrow \exists x\sim A(x)\vee B$

$\Leftrightarrow \sim\forall x A(x)\vee B$

$\Leftrightarrow \forall x A(x)\rightarrow B$

- **证4)：** $\exists x(B\rightarrow A(x)) \Leftrightarrow B\rightarrow\exists x A(x)$

证明：$\exists x(B\rightarrow A(x))$

$\Leftrightarrow \exists x(\sim B\vee A(x))$

$\Leftrightarrow \sim B\vee\exists x A(x)$

$\Leftrightarrow B\rightarrow\exists x A(x)$

2.4 量词分配

- $\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall x A(x) \wedge \forall x B(x)$
  - 班里每位同学都上了微积分课和离散数学课
    - $\forall x\,(P(x) \wedge Q(x))$
  - 班里每位同学都上了微积分课，同时每位同学也都上了离散数学课
    - $\forall x\, P(x) \wedge \forall x\, Q(x)$
  - 显然: $\forall x\,(P(x) \wedge Q(x)) \Leftrightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$

- $\exists x(A(x) \vee B(x)) \Leftrightarrow \exists x A(x) \vee \exists x B(x)$
  - 有一数，它或是偶数或是素数
    - $\exists x\,(P(x) \vee Q(x))$
  - 或者有一个偶数，或者有一个素数
    - $\exists x\, P(x) \vee \exists x\, Q(x)$
  - 显然: $\exists x\,(P(x) \vee Q(x)) \Leftrightarrow \exists x\, P(x) \vee \exists x\, Q(x)$

2.4 量词分配（反例）

- $\forall x(A(x) \lor B(x)) \not\Rightarrow \forall x A(x) \lor \forall x B(x)$

  反例：个体域为全体自然数; A(x): x是偶数，B(x): x是奇数

  左$\Leftrightarrow$1, 右$\Leftrightarrow$0

- $\forall x(A(x) \lor B(x)) \Leftarrow \forall x A(x) \lor \forall x B(x)$ （逻辑蕴含式）

- $\exists x(A(x) \land B(x)) \Rightarrow \exists x A(x) \land \exists x B(x)$ （逻辑蕴含式）
- $\exists x(A(x) \land B(x)) \not\Leftarrow \exists x A(x) \land \exists x B(x)$

  反例：个体域为全体自然数; A(x): x是偶数，B(x): x是奇数

  左$\Leftrightarrow$0, 右$\Leftrightarrow$1

- 全称量词对合取满足分配律
- 存在量词对析取满足分配律
- 相反不成立

2.4 量词分配（蕴含）

- $\exists x\,(A(x) \rightarrow B(x)) \Leftrightarrow \forall x\, A(x) \rightarrow \exists x\, B(x)$

  左边$\Leftrightarrow \exists x(\sim A(x) \vee B(x))$

  $\Leftrightarrow \exists x \sim A(x) \vee \exists x B(x)$　　　∃量词分配

  $\Leftrightarrow \sim \forall x A(x) \vee \exists x B(x)$

  $\Leftrightarrow \forall x A(x) \rightarrow \exists x B(x)$

- $\exists x A(x) \rightarrow \forall x B(x) \Rightarrow \forall x\,(A(x) \rightarrow B(x))$

  左边$\Leftrightarrow \sim \exists x A(x) \vee \forall x B(x))$

  $\Leftrightarrow \forall x \sim A(x) \vee \forall x B(x)$

  $\Rightarrow \forall x(\sim A(x) \vee B(x))$　　　逻辑蕴含

  $\Leftrightarrow \forall x(A(x) \rightarrow B(x))$

2.4 量词分配（蕴含）

- $\forall x\,(A(x) \to B(x)) \Rightarrow \forall x A(x) \to \forall x B(x)$

$\forall x\,(A(x) \to B(x)) \wedge \forall x A(x)$ 附加前提法

$\Leftrightarrow \forall x(\sim A(x) \vee B(x)) \wedge \forall x A(x)$

$\Leftrightarrow \forall x((\sim A(x) \vee B(x)) \wedge A(x))$ ∀量词分配

$\Leftrightarrow \forall x((\sim A(x) \wedge A(x)) \vee (B(x)) \wedge A(x))$ 分配律

$\Leftrightarrow \forall x(B(x) \wedge A(x))$

$\Leftrightarrow \forall x B(x) \wedge \forall x A(x)$

$\Rightarrow \forall x B(x)$ 合取消去

- $\forall x\,(A(x) \leftrightarrow B(x)) \Rightarrow \forall x\, A(x) \leftrightarrow \forall x B(x))$

3.1 换名(RENAME)规则

- **Rename:**
  - 把某个指导变项和其量词辖域中所有同名的约束变项, 都换成新的个体变项符号.

- **Example:**
  - $\forall x(A(x)\wedge B(x)) \Leftrightarrow \forall y(A(y)\wedge B(y))$
  - $\forall xA(x)\wedge \forall xB(x) \Leftrightarrow \forall yA(y)\wedge \forall zB(z)$
  - $H(x,y)\vee \exists xF(x)\vee \forall y(G(y)\rightarrow H(x,y))$

$\Leftrightarrow H(x,y)\vee \exists zF(z)\vee \forall u(G(u)\rightarrow H(x,u))$

3.2 代替(SUBSTITUTE)规则

- **Substitute:**
  - 把某个自由变项的所有出现, 都换成新的个体变项符号.

- **Example:**
  - $A(x) \wedge B(x) \Leftrightarrow A(y) \wedge B(y)$
  - $\forall x A(x) \wedge B(x) \Leftrightarrow \forall x A(x) \wedge B(y)$
  - $H(x,y) \vee \exists x F(x) \vee \forall y(G(y) \to H(x,y))$

$\Leftrightarrow H(s,t) \vee \exists x F(x) \vee \forall y(G(y) \to H(s,y))$

谓词公式的范式(NF)

- **前束范式（Prenex Normal Form ）**
  - 如果量词均在合式公式的开头，它们的辖域延伸到整个公式末尾，则该公式称为前束范式。
    - $\forall x \forall y \exists z(Q(x,y) \rightarrow R(z))$
    - $\forall y \forall x(\sim P(x,y) \rightarrow Q(y))$
- **存在性定理：**
  - 谓词逻辑合式公式均存在与之等值的前束范式
- **构造方法：**
  - 利用换名、代替、量词的收缩和扩张等进行等值变换

前束范式PNF

- 示例
  - Put the statements in prenex normal form.

$$(\exists x P(x) \vee \forall y Q(y)) \to \exists x R(x)$$

原式$\Leftrightarrow (\exists x P(x) \vee \forall y Q(y)) \to \exists z R(z)$	换名规则
$\Leftrightarrow \exists x (P(x) \vee \forall y Q(y)) \to \exists z R(z)$	辖域扩张
$\Leftrightarrow \exists x \forall y (P(x) \vee Q(y)) \to \exists z R(z)$	辖域扩张
$\Leftrightarrow \forall x (\forall y ((P(x) \vee Q(y)) \to \exists z R(z)))$	辖域扩张:$\exists x A(x) \to B \Leftrightarrow \forall x (A(x) \to B)$
$\Leftrightarrow \forall x \exists y ((P(x) \vee Q(y)) \to \exists z R(z))$	辖域扩张:$\forall x A(x) \to B \Leftrightarrow \exists x (A(x) \to B)$
$\Leftrightarrow \forall x \exists y \exists z ((P(x) \vee Q(y)) \to R(z))$	辖域扩张:$B \to \exists x A(x) \Leftrightarrow \exists x (B \to A(x))$

课堂练习

- Put the statements in prenex normal form.

$$\forall x P(x,y) \rightarrow \exists y Q(x,y)$$

原式 $\Leftrightarrow \forall x P(x,y) \rightarrow \exists t Q(s,t)$　　换名规则，代替规则

$\Leftrightarrow \exists t(\forall x P(x,y) \rightarrow Q(s,t))$　　辖域扩张: $B \rightarrow \exists x A(x) \Leftrightarrow \exists x(B \rightarrow A(x))$

$\Leftrightarrow \exists t(\exists x(P(x,y) \rightarrow Q(s,t)))$　　辖域扩张: $\forall x A(x) \rightarrow B \Leftrightarrow \exists x(A(x) \rightarrow B)$

$\Leftrightarrow \exists t \exists x(P(x,y) \rightarrow Q(s,t))$

原式 $\Leftrightarrow \forall x P(x,y) \rightarrow \exists t Q(s,t)$　　换名规则，代替规则

$\Leftrightarrow \sim\forall x P(x,y) \vee \exists t Q(s,t)$　　蕴含等值式

$\Leftrightarrow \exists x \sim P(x,y) \vee \exists t Q(s,t)$　　否定等值式

$\Leftrightarrow \exists x(\sim P(x,y) \vee \exists t Q(s,t))$　　辖域扩张

$\Leftrightarrow \exists x \exists t\ (\sim P(x,y) \vee Q(s,t))$　　辖域扩张

课堂练习

- Put the statements in prenex normal form.

$$\forall x P(x,y) \rightarrow \exists y Q(x,y)$$

原式$\Leftrightarrow \forall x P(x,s) \rightarrow \exists y Q(t,y)$　　代替规则

$\Leftrightarrow \exists x (P(x,s) \rightarrow \exists y Q(t,y))$　　辖域扩张:$\forall x A(x) \rightarrow B \Leftrightarrow \exists x (A(x) \rightarrow B)$

$\Leftrightarrow \exists x \exists y (P(x,s) \rightarrow Q(t,y)))$　　辖域扩张: $B \rightarrow \exists x A(x) \Leftrightarrow \exists x (B \rightarrow A(x))$

合式公式的前束范式是不唯一的

HOMEWORK

- 1. 设个体域D={a,b,c}, 消去下列各式的量词：

  (1) $\forall x\,(F(x) \wedge G(x))$

  (2) $\forall x\,(F(x) \rightarrow \exists y G(y))$

- 2. 证明等值式:

  $\forall x F(x) \vee \sim\exists x G(x) \Leftrightarrow \forall x \forall y (F(x) \vee \sim G(y))$

- 3. 求下列各式的前束范式:

  (1) $\forall x F(x) \vee \exists x G(x)$

  (2) $\forall x \forall y (\exists z (P(x,z) \wedge P(y,z)) \rightarrow \exists u Q(x,y,u))$