Discrete Mathematics and Its Applications

2.1 Sets

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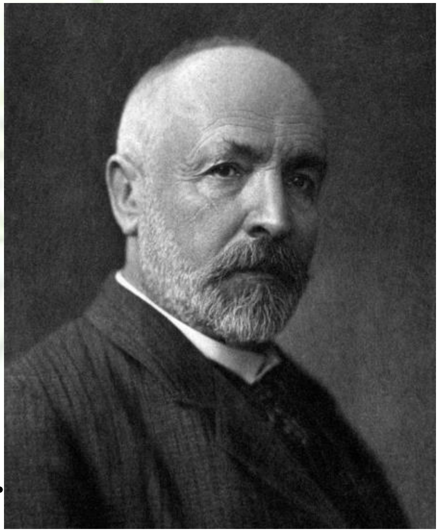
Introduction to Set Theory

- **Introduction:**
  - A ***set*** is a new type of structure, representing an ***unordered*** collection (group, plurality) of zero or more ***distinct*** (different) objects.
  - ***Set theory*** deals with operations between, relations among, and statements about sets.
  - Sets are ubiquitous in computer software systems.
  - All of mathematics can be defined in terms of some form of set theory (using predicate logic).

SETS（集合）

- **Definition:**
  - A ***set*** is a collection or group of *objects（对象）* or *elements（元素）* or *members（成员）*. (Cantor 1895, Germany)
  - A set is said to *contain（包含）* its elements.
  - There must be an underlying universal set（全集） U, either specifically stated or understood.
- **Notation (Roster method):**
  - list the elements between braces:

    S = {a, b, c, d}={b, c, a, d, d}
- **Note:**
  - listing object more than once does't change the set.
  - Ordering means nothing.



Cantor 1845-1918

Sets（集合）

- **Notation：specification by predicates（谓词）**

  $$S = \{x \mid P(x)\}$$

  - $P(x)$ denote a sentence or statement P concerning the variable object x.
  - S contains all the elements from U which make the predicate P true.
  - x is a member of S or x is an element of S:

    $x \in S$.
  - x is not an element of S:

    $x \notin S$.

Sets（集合）

- **Examples:**
  - $\{x \mid x$ is a positive integer less than $4\} = \{1,2,3\}$
  - $\{x \mid x$ is a letter in the word "byte"$\} = \{b, y, t, e\}$
  - If A={BASIC,PASCAL,ADA} and B={ADA,PASCAL,BASIC}, then A=B.

- Two sets A and B are ***equal***（***相等***） if and only if they have the same elements, we write $A=B$.

SETS（集合）

- **Common Universal Sets:**
  - $\mathbb{R}$ = Reals
  - $\mathbb{N}$ = Nature numbers = {0, 1, 2, 3…..}, the counting numbers
  - $\mathbb{Z}$ = All numbers = {…, -3, -2, -1, 0, 1, 2, 3…}
  - $\mathbb{Z}^+$ = The set of positive numbers
  - $\mathbb{Q}$ = The Rational numbers

Empty set（空集）

- **Definition:**
  - The *void* set, the *null* set, the *empty* set, denoted { } or $\varnothing$, is the set with no members.
- **Example:**
  - $\{x \mid x$ is a real number and $x^2=-1\}= \varnothing$.

Subsets（子集）

- **Definition:**
  - The set A is a *subset* of the set B, or A contained in（包含于）B, denoted $A \subseteq B$, iff（当且仅当）

$$\forall x[x \in A \rightarrow x \in B]$$

  - If A is not a subset of B ,denote $A \not\subseteq B$.

- **Note:**
  - The assertion $x \in \varnothing$ is always false. Hence $\forall x[x \in \varnothing \rightarrow x \in B]$ is always true(vacuously). Therefore, $\varnothing$ is a subset of every set, $\varnothing \subseteq B$.
  - A set B is always a subset of itself, $B \subseteq B$.

Subsets（子集）

- **Examples:**
  - $\mathbb{Z}^+ \subseteq \mathbb{Z}$
  - $\mathbb{Z}^+ \subseteq \mathbb{R}$
  - Let A={1, 2, 3, 4, 5, 6}, B={2, 4, 5}, C={1, 2, 3, 4, 5}
  - Then
    - $B \subseteq A$, $B \subseteq C$, $C \subseteq A$
    - $A \not\subseteq B$, $A \not\subseteq C$, $C \not\subseteq B$

Proper subset（真子集）

- **Definition:**
  - If $A \subseteq B$ but $A \neq B$ then we say A is a *proper subset*（真子集） of B, denoted $A \subset B$ (in some texts).

- **Examples:**
  - $\mathbb{Z}^+ \subset \mathbb{Z}$
  - $\mathbb{Z}^+ \subset \mathbb{R}$

CARDINALITY（基数）

- **Definition:**
  - The number of distinct elements in set A, denoted $|A|$, is called the *cardinality* of A.
  - If the cardinality is a natural number (in N), then the set is called *finite*, else *infinite*.
- **Example:**
  - A={a, b}, |A|=?
  - B={a, b, a, c}, |B|=?
  - $C=\varnothing$, |C|=?
  - D=N, |D|=|N|=?

- N is ***infinite*** since |N| is not a natural number.
- It is called a ***transfinite cardinal number***.

THE POWER SET （幂集）

- **Definition:**
  - The set of all subset of a set *A*, denoted *P(A)*, is called the *power set*（*幂集*） of *A*.
  - A is finite and so is P(A).

- **Example:**
  - If A = {a, b} then

    P(A) = {∅, {a}, {b}, {a,b}}

- 幂集中的元素都是集合

THE POWER SET （幂集）

- **Note:**
  - Sets can be both members and subsets of other sets.
- **Example:**
  - $A = \{\emptyset,\{\emptyset\}\}$.
  - A has two elements and hence four subsets:
    - $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$. $\{\emptyset,\{\emptyset\}\}$
  - Note that $\emptyset$ is both a member of A and a subset of A!
- **Example:**
  - Let A be a set and let $B=\{A,\{A\}\}$,Then
  - $A\in B$ and $\{A\} \in B$， $\{A\}\subseteq B$ and $\{\{A\}\} \subseteq B$. $A \not\subseteq B$
  - $P(B)=?$

Cardinality of Power Set

- $A=\{a, b\}$, $|A|=2$
- $P(A) =\{\varnothing, \{a\}, \{b\},\{a,b\}\}$, $|P(A)|=4$
- **Useful fact**: $|A|=n$ implies $|P(A)|=2^n$.

- **Examples:**
  - $A=\{1, 2, 3\}$, $|A|=3$
  - $P(A) = ?$, $|P(A)|=?$

Ordered n-tuples

- **Definition:**
  - These are like sets, except that duplications matter, and the order makes a difference.
  - For $n \in N$, an ordered n-tuple or a sequence or list of length n is written $(a_1, a_2, \ldots, a_n)$, and it's first element is a_1, etc.
  - Empty sequence, singlet, pairs, triples, quadruples, quintuples, …, n-tuples
- **Examples:**
  - $(1, 2) \neq (2, 1) \neq (2, 1, 1)$
  - $\{1, 2\} = \{2, 1\} = \{2, 1, 1\}$

Cartesian product（笛卡尔乘积）

- **Definition:**
  - The *Cartesian product* of A with B, denoted $A \times B$, is the set of ordered pairs $\{<a, b> \mid a \in A \wedge b \in B\}$ **$\{(a, b) \mid a \in A \wedge b \in B\}$**
  - Notation: $$\overset{n}{\underset{i=1}{\times}} A_i = \{<a_1, a_2, \ldots, a_n> \mid a_i \in A_i\}$$
  - Note: The Cartesian product of anything with $\varnothing$ is $\varnothing$. (why?)

- **Example**
  - $A = \{a, b\}$, $B = \{1, 2, 3\}$
  - $A \times B$

  $= \{<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>\}$
  - What is $B \times A$? $A \times B \times A$?
  - When $A \times B = B \times A$?
  - If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

1596-1650，France

Cartesian product（笛卡尔乘积）

- **Properties:**
  - $A \times B \neq B \times A \quad (A \neq \varnothing \wedge B \neq \varnothing \wedge A \neq B)$
  - $(A \times B) \times C \neq A \times (B \times C) \quad (A \neq \varnothing \wedge B \neq \varnothing \wedge A \neq B)$
  - When $A_1 = A_2 = \ldots = A_n$, then $A_1 \times A_2 \ldots \times A_n = A^n$

- 笛卡尔乘积不满足交换律
- 笛卡尔乘积不满足结合律

- **Example 22**
  - What do the statements $\forall x \in R\ (x^2 \geq 0)$ and $\exists x \in Z(x^2=1)$ mean?
  - For every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative."

    $A=\{x \in R \mid x^2 \geq 0\}$
  - There exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1).

    $A=\{x \in Z \mid x^2 = 1\}$

Truth Sets of Quantifiers

- **Definition:**
  - Given a predicate P, and a domain D, The ***truth set*** of $P(x)$ is denoted by $\{x \in \mathrm{D} \mid P(x)\}$.

- **Example 23:**
  - What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers?
    - $P(x) : |x| = 1$
    - $Q(x) : x^2 = 2$
    - $R(x) : |x| = x$

Review of 2.1 Set

- Definition of Sets
- Special sets: $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.....
- Set notations: $\{a, b, \ldots\}$, $\{x \mid P(x)\}$ ...
- Relations: $x \in S$, $S \subseteq T$, $S \subset T$, $S = T$...
- Cardinality
- Power set
- Cartesian Product

Homework

- **§ 2.1**
  - 8, 12, 18, 24, 48

2.2 Set Operations

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Introduction

- **Logical calculus** （逻辑演算）and **set theory**（集合论） are both instances of an algebraic system （代数系统）called a

  Boolean Algebra（布尔代数）

- The **operators** in set theory are defined in terms of the corresponding operator in propositional calculus

- As always there must be a **universe U**, all sets are assumed to be **subsets of U**.

Equal（相等）

- **Definition:**
  - Two sets A and B are *equal*, denoted $A = B$, iff

    $\forall x[x \in A \leftrightarrow x \in B]$.
- **Note:**
  - By a previous logical equivalence we have

    $A = B$ iff $\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$

  or

  A = B* iff *A* $\subseteq$ *B* and *B* $\subseteq$ *A

Set Operations

- The ***union*（并集）** of A and B, denoted $A \cup B$, is the set

  $\{x \mid x \in A \vee x \in B\}$

- The ***intersection*（交集）** of A and B, denoted $A \cap B$, is the set

  $\{x \mid x \in A \wedge x \in B\}$

  - Note: If the intersection is void, A and B are said to be *disjoint(不相交)*

- The ***complement*（补集）** of A, denoted $\overline{A}$, is the set

  $\{x \mid \neg(x \in A)\}$

  - Note: Alternative notation is A^c, and $\{x|x \notin A\}$.

The Addition principle（加法原理）

- **Theorem**
  - If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$
  - It also called the ***inclusion-exclusion principle***（容斥原理）
  - The Addition principle for Disjoint Sets: $|A \cup B| = |A| + |B|$
- **Example:**
  - Let $A=\{a, b, c, d, e\}$ and $B=\{c, e, f, h, k, m\}$.
  - Verify inclusion-exclusion principle.
- **Solution:**
  - $A \cup B=\{a, b, c, d, e, f, h, k, m\}$ and $A \cap B=\{c, e\}$
  - $|A|= 5$, $|B|= 6$, $|A \cup B|= 9$ and $|A \cap B|= 2$
  - $|A|+|B|-|A \cap B| = 9 = |A \cup B|$

Q.E.D

The difference of Sets

- **Definitions :**
  - The ***difference（差）*** of A and B, or the *complement* of B *relative to* A, denoted A - B, is the set

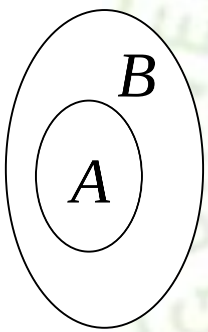
$$A \cap \overline{B}$$

  - Note: The (absolute) complement of B is U – B (i.e. $\overline{\text{B}}$)
  - The ***symmetric difference（对称差）*** of A and B, denoted $A \oplus B$, is the set

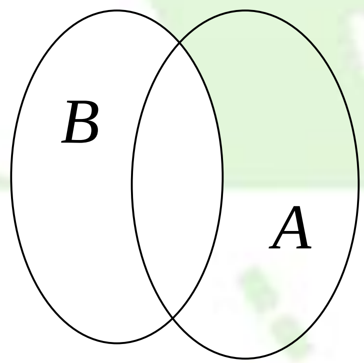
$$(A - B) \cup (B - A)$$

VENN DIAGRAMS（文氏图）

- Diagrams used to show relationships between sets after the British logician John Venn
- Example:



$A \subseteq B$

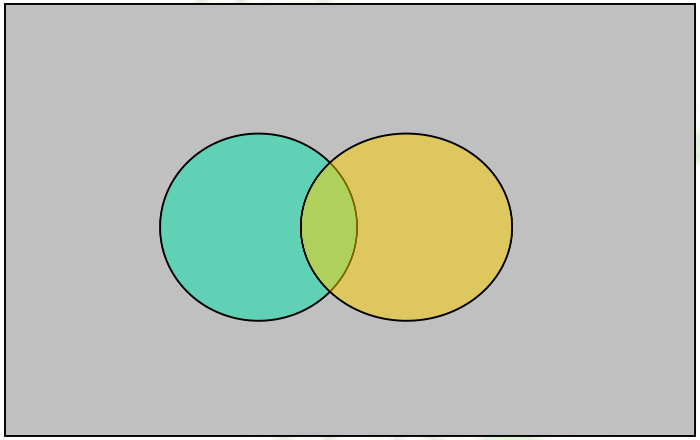


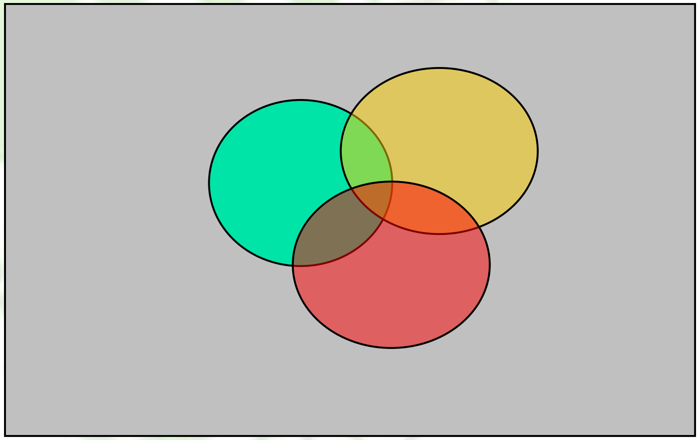
$A \not\subseteq B$

John Venn
1834-1923

Venn Diagrams

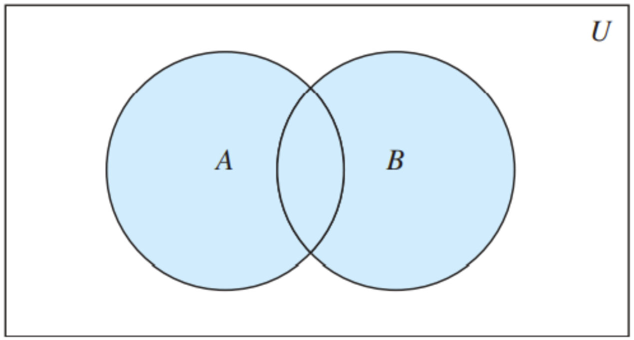
- A useful geometric visualization tool (for 3 or less sets)
  - The Universe U is the rectangular box
  - Each set is represented by a circle and its interior
  - All possible combinations of the sets must be represented



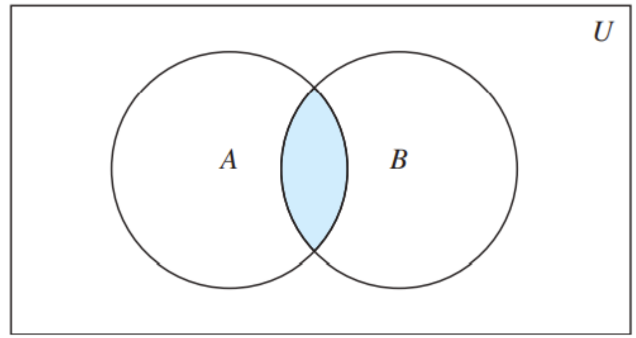


Shade the appropriate region to represent the given set operation.

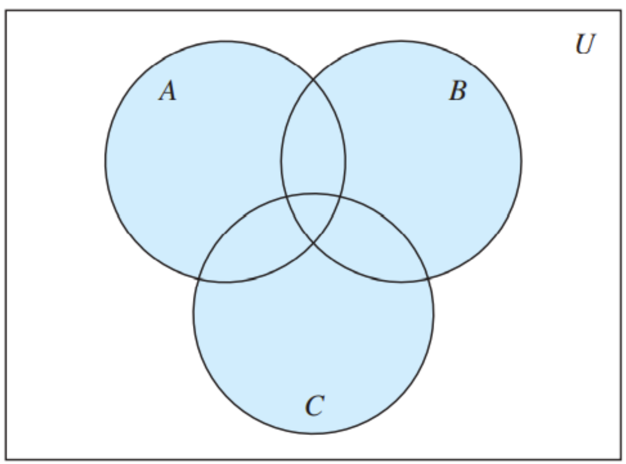
Venn Diagrams



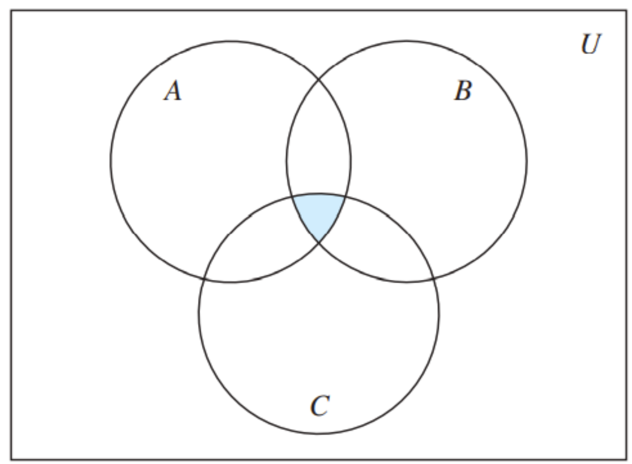
$A \cup B$ is shaded.



$A \cap B$ is shaded.

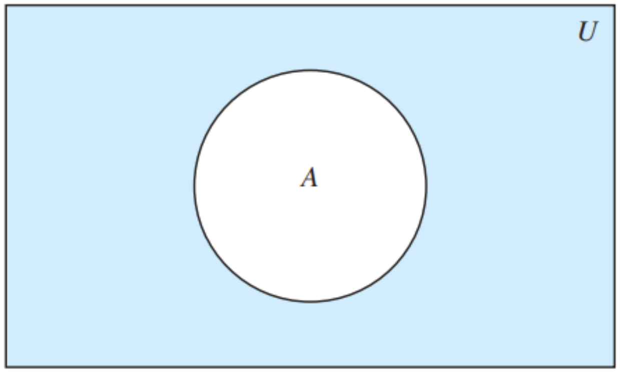


(a) $A \cup B \cup C$ is shaded.

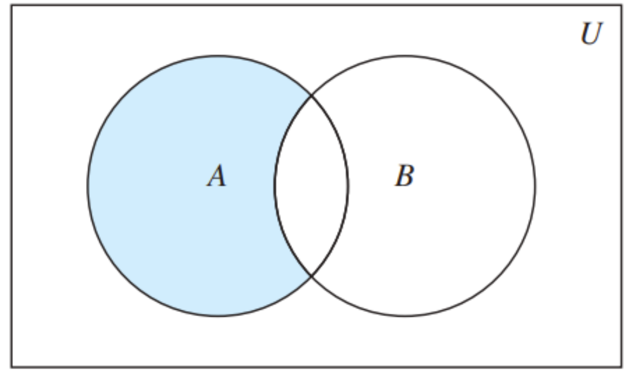


(b) $A \cap B \cap C$ is shaded.

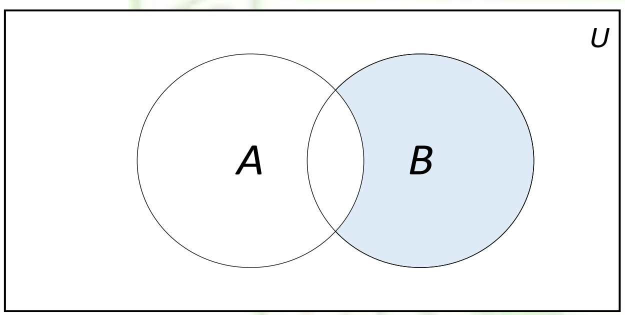
Venn Diagrams



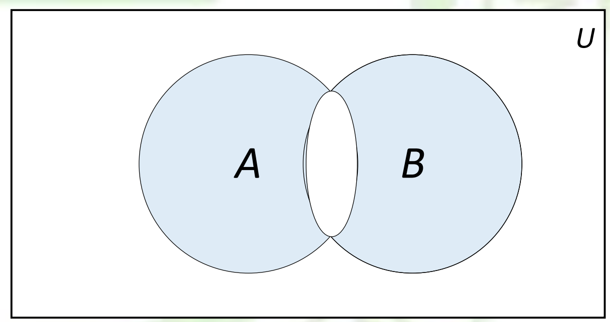
$\overline{A}$ is shaded.



$A - B$ is shaded.



B-A is shaded



$A \oplus B$ is shaded

Examples of Set Operations

- U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- A= {1, 2, 3, 4, 5}, B = {4, 5, 6, 7, 8}. Then
  - $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
  - $A \cap B = \{4, 5\}$
  - $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
  - $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
  - $A - B =$
  - $B - A =$
  - $A \oplus B =$

Algebraic Properties of Set Operations

- ***Commutative properties (law)（交换律）***

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

- ***Associative properties（结合律）***

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- ***Distributive properties（分配律）***

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Algebraic Properties of Set Operations

- ***Idempotent properties*（幂等律）**

$$A \cup A = A$$
$$A \cap A = A$$

- ***Properties of the Complement*（补集性质）**

$$\overline{\overline{A}} = A$$
$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \varnothing$$
$$\overline{\varnothing} = U$$
$$\overline{U} = \varnothing$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Algebraic Properties of Set Operations

- ***Properties of a Universal Set***

$$A \cup U = U$$

$$A \cap U = A$$

- ***Properties of the Empty Set***

$$A \cup \varnothing = A$$

$$A \cap \varnothing = \varnothing$$

- ***Absorption law***

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Algebraic Properties of Set Operations

- ***Properties of Cartesian product***
  - $A\times(B\cup C)=(A\times B)\cup(A\times C)$
  - $(B\cup C)\times A=(B\times A)\cup(C\times A)$
  - $A\times(B\cap C)=(A\times B)\cap(A\times C)$
  - $(B\cap C)\times A=(B\times A)\cap(C\times A)$

- 笛卡尔积对并和交运算满足分配律

- ***Example：*** $A=\{1, 2\}$, $B=\{a, b\}$, $C=\{a, c\}$. Then
  - $A\times(B\cup C)=$
  - $(A\times B)\cup(A\times C)=$
  - $(B\cap C)\times A=$
  - $(B\times A)\cap(C\times A)=$

Proving Set Identities

- **Method:**

  To prove statements about sets, of the form $S_1 = S_2$ (where the Si are set expressions), here are three useful techniques:

  - **1) Prove $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$ separately.**
  - **2) Use set builder notation & logical equivalences.**
  - **3) Use a *membership table*.**

Method 1: Mutual subsets

- **Example:**

  Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$. (***Distributive properties***)

  - **Part 1: Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.**
    - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
    - We know that $x\in A$, and $x\in B$ or $x\in C$.
      - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
      - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
    - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
    - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - **Part 2: Show $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$.**
    - …

Method 2: Use set builder notation

- **Example 11: Proof that** $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	by definition of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	by definition of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by the first De Morgan law for logical equivalences
$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by definition of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by definition of union
$= \overline{A} \cup \overline{B}$	by meaning of set builder notation

Method 3: Membership Tables

- **Just like truth tables for propositional logic**
  - Columns for different set expressions.
  - Rows for all combinations of memberships in constituent sets.
  - Use "1" to indicate membership in the derived set, "0" for non-membership.
  - Prove equivalence with identical columns.

- **Example:**
  - Prove $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Method 3: Membership Tables

- **Exercise:**

  Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A B C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0 0 0	0	0			0
0 0 1	0	0			0
0 1 0	1	1			1
0 1 1	1	0			0
1 0 0	1	1			1
1 0 1	1	0			0
1 1 0	1	1			1
1 1 1	1	0			0

Generalized Union

- Union & intersection are ***commutative and associative***.
- Binary union operator: $A \cup B$
- n-ary union:

  $$A \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$$

  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$ or for infinite sets: $\bigcup_{i=1}^{\infty} A_i$
- Or more generally: $\bigcup_{A \in X} A$

Generalized Intersection

- Union & intersection are ***commutative and associative***.
- Binary intersection operator: $A \cap B$
- n-ary intersection:

  $$A_1 \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$$

  (grouping & order is irrelevant)

- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$ or for infinite sets: $\bigcap_{i=1}^{\infty} A_i$
- Or more generally: $\bigcap_{A \in X} A$

Generalized Unions & Intersections

- **Example 16:** ***closed interval, open interval***
  - Let $A_i = [i,\infty),\ 1 \le i < \infty$ (*For i is integer*)
  - Then

$$\bigcup_{i=1}^{n} A_i = ? \qquad \boxed{[1,\infty)}$$

$$\bigcap_{i=1}^{n} A_i = ? \qquad \boxed{[n,\infty)}$$

- **Exampe 17:**
  - Let $A_i = [1,i],\ 1 \le i < \infty$ (*For i is integer*)
  - Then

$$\bigcup_{i=1}^{\infty} A_i = ? \qquad \boxed{[1, \infty)}$$

$$\bigcap_{i=1}^{\infty} A_i = ? \qquad \boxed{\{1\}}$$

Representations

- A frequent theme of this course will be methods of *representing* one discrete structure using another discrete structure of a different type.
- ***E.g.*****,** one can represent natural numbers as
  - Sets: $\mathbf{0}:\equiv\varnothing$, $\mathbf{1}:\equiv\{\mathbf{0}\}$, $\mathbf{2}:\equiv\{\mathbf{0},\mathbf{1}\}$, $\mathbf{3}:\equiv\{\mathbf{0},\mathbf{1},\mathbf{2}\}$, …
  - Bit strings: $\mathbf{0}:\equiv 0$, $\mathbf{1}:\equiv 1$, $\mathbf{2}:\equiv 10$, $\mathbf{3}:\equiv 11$, $\mathbf{4}:\equiv 100$, …

Representing Sets with Bit Strings

- **Method:**
  - For an enumerable u.d. U with ordering $x_1, x_2, \ldots$, represent a finite set $S \subseteq U$ as the finite bit string $B = b_1 b_2 \ldots b_n$ where $\forall i: x_i \in S \leftrightarrow (i < n \wedge b_i = 1)$.

- **Example:**

  $U=\mathbf{N}$, $S=\{2,3,5,7,11\}$, B=001101010001.

  - In this representation, the set operators "$\cup$", "$\cap$", "$\overline{}$" are implemented directly by bitwise OR, AND, NOT!

Computer representation of sets

- **Example 18**
  - Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$.
  - What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

1010101010，0101010101，1111100000

Computer representation of sets

- **Example 19**
  - We have seen that the bit string for the set {1, 3, 5, 7, 9} (with universal set {1, 2, 3, 4,5, 6, 7, 8, 9, 10}) is 10 1010 1010.What is the bit string for the complement of this set?

- **Example 20**
  - The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010,respectively. Use bit strings to find the union and intersection of these sets.

1111100000
<u>1010101010</u>
1111101010

1111100000
<u>1010101010</u>
1010100000

Review of 2.2

- Set Operations: $\cup$, $\cap$, $-$, $\oplus$
- Set equality proof techniques:
  - Mutual subsets.
  - Derivation using logical equivalences.
  - Member table.

HOMEWORK

- **§ 2.2**
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