CS5487 Problem Set 1

Probability Theory and Linear Algebra Review

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Probability Theory

Problem 1.1 Linear Transformation of a Random Variable

Let x be a random variable on $\mathbb{R}$, and $a, b \in \mathbb{R}$. Let $y = ax + b$ be the linear transformation of x. Show the following properties:

$E(y) = \frac{\sum y_i}{n} = \frac{\sum a x_i + b}{n} = a\cdot\bar{x} + \frac{bn}{n} = a\bar{x} + b = E(x) + b$

$$\mathbb{E}[y] = a\mathbb{E}[x] + b, \tag{1.1}$$

$$\text{var}(y) = a^2 \text{var}(x). \tag{1.2}$$

$Var(ax+b) = \frac{\sum(ax_i + b - \overline{ax+b})^2}{n-1} = \frac{\sum(a x_i - \bar{x})^2}{n-1} = a^2 \cdot Var(x)$

Now, let x be a vector r.v. on $\mathbb{R}^d$, and $A \in \mathbb{R}^{m\times d}$, $b \in \mathbb{R}^m$. Let $y = Ax + b$ be the linear transformation of x. Show the following properties:

$$\mathbb{E}[y] = A\mathbb{E}[x] + b, \tag{1.3}$$

$$\text{cov}(y) = A\text{cov}(x)A^T. \tag{1.4}$$

$Cov(y) = E[(y-m)(y-m)^T]$
$= E[(Ax-Am)(Ax-Am)^T]$
$= E[A(x-m)(x-m)^T A^T]$
$= A\,E[(x-m)(x-m)^T]A^T$
$= A \ cov\,x \ A^T$

$E(y) = E(Ax+b)$
$= E(Ax) + b$
$= AE(x) + b$

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Problem 1.2 Properties of Independence

Let x and y be statistically independent random variables ($x \perp y$). Show the following properties:

$$\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y], \tag{1.5}$$

$$\text{cov}(x, y) = 0. \tag{1.6}$$

$E(xy) = \iint p(x,y)\cdot x\cdot y \, dx\, dy$
$= \int\int p(x)\cdot p(y)\cdot x\cdot y \, dx \cdot dy$
$= \int E(x)\cdot p(y)\cdot y \, dy$
$= E(x)\cdot E(y)$

$cov(x,y)$
$= E[(x-\bar{x})(y-\bar{y})]$
$= E[xy - \bar{x}y - x\bar{y} + \bar{x}\bar{y}]$
$= E[xy] - E(x)\cdot E(y) - E(x)E(y) + E(x)\cdot E(y)$
$= E[xy] - E(x)E(y)$

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Problem 1.3 Uncorrelated vs Independence

Two random variables x and y are said to be *uncorrelated* if their covariance is 0, i.e., $\text{cov}(x, y) = 0$. For statistically independent random variables, their covariance is always 0 (see Problem 1.2), and hence independent random variables are always uncorrelated. However, the converse is generally not true; uncorrelated random variables are not necessarily independent.

Consider the following example. Let the pair of random variables (x, y) take values $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$, each with equal probability (1/4).

$\bar{x} = 0 \quad \bar{y} = 0$

(a) Show that $\text{cov}(x, y) = 0$, and hence x and y are uncorrelated.

$cov(x,y) = E[(x-\bar{x})(y-\bar{y})]$
$= E(xy) - E(x)E(y) = 0 - 0 = 0$

(b) Calculate the marginal distributions, $p(x)$ and $p(y)$. Show that the $p(x, y) \neq p(x)p(y)$ and thus x and y are not independent.

$p(x=0) = \frac{1}{2}$ $\quad p(y=0) = \frac{1}{2}$
$p(x=1) = \frac{1}{4}$ $\quad p(y=1) = \frac{1}{4}$
$p(x=-1) = \frac{1}{4}$ $\quad p(y=-1) = \frac{1}{4}$

$p(x=1, y=0) = \frac{1}{4} \neq p(x=1)\cdot p(y=0) = \frac{1}{8}$

(c) Now consider a more general example. Assume that x and y satisfy

$$\mathbb{E}[x|y] = \mathbb{E}[x], \tag{1.7}$$

i.e., the mean of x is the same regardless of whether y is known or not (the above example satisfies this property). Show that x and y are uncorrelated.

Think of $E(X|Y)$ as a random variable w.r.t. Y, so $\boxed{E(E[X|Y]) = E[X]}$ $\Rightarrow$ You know $E(X|Y)$ if you know $Y = y_i$ for some y_i

We need to show $E[xy] - E[x]E[y] = 0$

$$E(xy) = \sum\sum p(x = x_i, y = y_j) \cdot x_i y_j$$
$$= \sum\sum p(x = x_i | y = y_j) \cdot p(y = y_j)\, x_i y_j$$
$$= \sum E(x|y_j) \cdot p(y = y_j)\, y_j$$
$$= E(x) \cdot E(y)$$

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Problem 1.4 Sum of Random Variables

Let x and y be random variables (possibly dependent), show the following property:

$$\mathbb{E}[x + y] = \mathbb{E}[x] + \mathbb{E}[y]. \tag{1.8}$$

Furthermore, if x and y are statistically independent ($x \perp y$), show that

$$\mathrm{var}(x + y) = \mathrm{var}(x) + \mathrm{var}(y). \tag{1.9}$$

$$Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$$
$$= E[X^2 + Y^2 + 2XY] - E^2(X) - E^2(Y) - 2E(X)E(Y)$$
$$= E(X^2) + E(Y^2) - E^2(X) - E^2(Y) + 2E(XY) - 2E(X)E(Y)$$
$$= Var(X) + Var(Y) + 2cov(X, Y)$$

However, in general this is not the case when x and y are dependent.

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Problem 1.5 Expectation of an Indicator Variable

Let x be an indicator variable on $\{0, 1\}$. Show that

$$\mathbb{E}[x] = p(x = 1), \tag{1.10}$$
$$\mathrm{var}(x) = p(x = 0)p(x = 1). \tag{1.11}$$

$$Var(X) = p(X=0) \times p(X=1)^2 + p(X=1)(1 - p(X=1))^2$$
$$= p(X=0) \cdot p(X=1)$$

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Problem 1.6 Multivariate Gaussian

The multivariate Gaussian is a probability density over real vectors, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \in \mathbb{R}^d$, which is parameterized by a mean vector $\mu \in \mathbb{R}^d$ and a covariance matrix $\Sigma \in \mathbb{S}^d_+$ (i.e., a d-dimensional positive-definite symmetric matrix). The density function is

$$p(x) = \mathcal{N}(x|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}\|x-\mu\|^2_\Sigma}, \tag{1.12}$$

where $|\Sigma|$ is the determinant of Σ, and

$$\|x - \mu\|^2_\Sigma = (x - \mu)^T \Sigma^{-1} (x - \mu) \tag{1.13}$$

is the *Mahalanobis distance*. In this problem, we will look at how different covariance matrices affect the shape of the density.

First, consider the case where Σ is a *diagonal matrix*, i.e., the off-diagonal entries are 0,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}. \tag{1.14}$$

$$p(x) = \frac{1}{(2\pi)^{d/2}\left(\prod_{i=1}^{d}\alpha_i^2\right)^{\frac{1}{2}}} \cdot e^{-\frac{1}{2}\sum_{i=1}^{d}\frac{1}{\alpha_i^2}(x_i-\mu_i)^2}$$

$$= \prod_{i=1}^{d} \frac{1}{(2\pi)^{\frac{1}{2}}\alpha_i} \cdot e^{-\frac{1}{2\alpha_i^2}(x_i-\mu_i)^2}$$

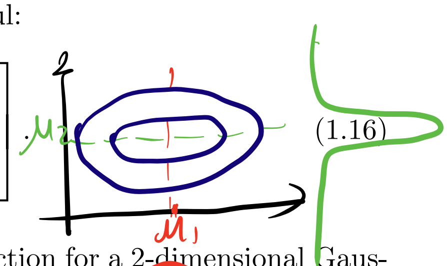
(a) Show that with a diagonal covariance matrix, the multivariate Gaussian is equivalent to assuming that the elements of the vector are independent, and each is distributed as a univariate Gaussian, i.e.,

diagonal assumes each individual variables are independent

$$\mathcal{N}(x|\mu,\Sigma) = \prod_{i=1}^{d} \mathcal{N}(x_i|\mu_i,\sigma_i^2). \tag{1.15}$$

Hint: the following properties of diagonal matrices will be useful:

$$|\Sigma| = \prod_{i=1}^{d}\sigma_i^2, \quad \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_d^2} \end{bmatrix}. \tag{1.16}$$



(b) Plot the Mahalanobis distance term and probability density function for a 2-dimensional Gaussian with $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$. How is the shape of the density affected by the diagonal terms?

(c) Plot the Mahalanobis distance term and pdf when the variances of each dimension are the same, e.g., $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is sometimes called an i.i.d. (independently and identically distributed) covariance matrix, isotropic covariance matrix, or circular covariance matrix.

$\Sigma = \alpha^2 I$ (scaled identity)

Next, we will consider the general case for the covariance matrix.

(d) Let $\{\lambda_i, v_i\}$ be the eigenvalue/eigenvector pairs of Σ, i.e.,

Σ is positive definite symmetric since it is covariance matrix

$$\Sigma v_i = \lambda_i v_i, \quad i \in \{1, \cdots, d\}. \tag{1.17}$$

Show that Σ can be written as

$\Sigma \cdot V = V \cdot \Lambda$

$\Sigma V \cdot V^{-1} = V \Lambda V^{-1}$ (For semmetric mtx V is orthogonal)

$\Sigma = V \Lambda V^T$

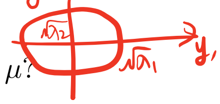
$$\Sigma = V\Lambda V^T, \tag{1.18}$$

where $V = [v_1, \cdots, v_d]$ is the matrix of eigenvectors, and $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_d)$ is a diagonal matrix of the eigenvalues.

$\Sigma^{-1} = V^{T^{-1}} \Lambda^{-1} V^{-1} = V \Lambda^{-1} V^T$

(e) Let $y = V^T(x - \mu)$. Show that the Mahalanobis distance $\|x - \mu\|^2_\Sigma$ can be rewritten as $\|y\|^2_\Lambda$, i.e., a Mahalanobis distance with a diagonal covariance matrix. Hence, in the space of y, the multivariate Gaussian has a diagonal covariance matrix. (Hint: use Problem 1.12)

$\|x-\mu\|^2_\Sigma = (x-\mu)^T \cdot V \Lambda^{-1} V^T (x-\mu) = y^T \Lambda^{-1} y \rightarrow \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_d} \end{bmatrix}$



(f) Consider the transformation from y to x: $x = Vy + \mu$. What is the effect of V and μ?

$y = V^T(x-\mu) \quad Vy = x - \mu \quad x = Vy + \mu$

orthogonal ⇓ rotation

translation

(g) Plot the Mahalanobis distance term and probability density function for a 2-dimensional Gaussian with $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix}$. How is the shape of the density affected by the eigenvectors and eigenvalues of Σ?

$\det(\Sigma - \lambda I) = 0$

$\det\begin{bmatrix} \frac{5}{8}-\lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8}-\lambda \end{bmatrix} = 0$

$\frac{25}{64} + \lambda^2 - \frac{5}{4}\lambda - \frac{9}{64} = 0$

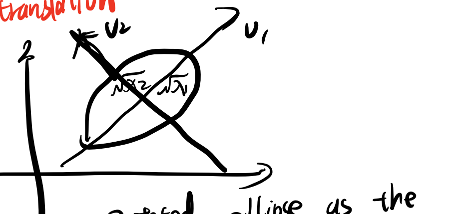
$\lambda^2 - \frac{5}{4}\lambda + \frac{1}{4} = 0$

$(\lambda - \frac{5}{8})^2 = \frac{9}{64}$

$\lambda - \frac{5}{8} = \pm\frac{3}{8}$

. . . .

(why? consider $\langle Ax, Ay \rangle = y^T A^T A x = y^T x = \langle x, y \rangle$)



rotated ellipse as the iso contours (according to the eigenvectors of Σ)

$y = V^T(x-\mu)$

$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \cdot x = \begin{bmatrix} \cos 45 & \sin 45 \\ \sin 45 & -\cos 45 \end{bmatrix}$

$\lambda_1 = 1, \lambda_2 = \frac{1}{4}$

$\Rightarrow \begin{bmatrix} -\frac{3}{8}, & \frac{3}{8} \\ \frac{3}{8}, & -\frac{3}{8} \end{bmatrix} x = 0$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \begin{bmatrix} 1 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix} \times \frac{1}{2} = \Sigma$

Problem 1.7 Product of Gaussian Distributions

Show that the product of two Gaussian distributions, $\mathcal{N}(x|\mu_1, \sigma_1^2)$ and $\mathcal{N}(x|\mu_2, \sigma_2^2,)$, is a scaled Gaussian,

$$\mathcal{N}(x|\mu_1, \sigma_1^2)\mathcal{N}(x|\mu_2, \sigma_2^2) = Z\mathcal{N}(x|\mu_3, \sigma_3^2), \tag{1.19}$$

where

$$\mu_3 = \sigma_3^2(\sigma_1^{-2}\mu_1 + \sigma_2^{-2}\mu_2), \tag{1.20}$$

$$\sigma_3^2 = \frac{1}{\sigma_1^{-2} + \sigma_2^{-2}}, \tag{1.21}$$

$$Z = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{\frac{-1}{2(\sigma_1^2+\sigma_2^2)}(\mu_1-\mu_2)^2} = \mathcal{N}(\mu_1|\mu_2, \sigma_1^2 + \sigma_2^2). \tag{1.22}$$

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Problem 1.8 Product of Multivariate Gaussian Distributions

Show that the product of two d-dimensional multivariate Gaussians distributions, $\mathcal{N}(x|a, A)$ and $\mathcal{N}(x|b, B)$, is a scaled multivariate Gaussian,

$\Sigma'\left(\sum_{i=1}^{n-1} \frac{m_i}{\Sigma_i}\right)$

$$\mathcal{N}(x|a, A)\mathcal{N}(x|b, B) = Z\mathcal{N}(x|c, C), \tag{1.23}$$

where $\frac{1}{(2\pi)^{\frac{d}{2}}|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}} e^{-\frac{1}{2}\|x-a\|_A - \frac{1}{2}\|x-b\|_B}$

induction:

cov

$\checkmark \ \frac{1}{\Sigma} = \frac{1}{\Sigma'} + \frac{1}{\Sigma_n} = \sum_{i=1}^{n} \frac{1}{\Sigma_i}$

mean.

$m = \Sigma\left(\frac{m'}{\Sigma'} + \frac{m_n}{\Sigma_n}\right) = \Sigma\left(\frac{1}{\Sigma'} \cdot \Sigma'\left(\sum_{i=1}^{n-1} \frac{m_i}{\Sigma_i}\right) + \frac{m_n}{\Sigma_n}\right) = \Sigma\left(\sum_{i=1}^{n} \frac{m_i}{\Sigma_i}\right)$

$$c = C(A^{-1}a + B^{-1}b), \tag{1.24}$$

$$C = (A^{-1} + B^{-1})^{-1}, \tag{1.25}$$

$$Z = \frac{1}{(2\pi)^{\frac{d}{2}}|A+B|^{\frac{1}{2}}} e^{-\frac{1}{2}(a-b)^T(A+B)^{-1}(a-b)} = \mathcal{N}(a|b, A+B). \tag{1.26}$$

Hint: after expanding the exponent term, apply the result from Problem 1.10 and (1.35).

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Problem 1.9 Correlation between Gaussian Distributions

Using the result from Problem 1.8, show that the correlation between two multivariate Gaussian distributions is

$$\int \mathcal{N}(x|a, A)\mathcal{N}(x|b, B)dx = \mathcal{N}(a|b, A+B). \tag{1.27}$$

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Linear Algebra

Problem 1.10 Completing the square

Let $x, b \in \mathbb{R}^n$, $A \in \mathbb{S}^n$, $c \in \mathbb{R}$, and let $f(x)$ be a quadratic function of x,

$$f(x) = x^T A x - 2x^T b + c. \tag{1.28}$$

Show that $f(x)$ can be rewritten in the form

$$f(x) = (x-d)^T A (x-d) + e, \tag{1.29}$$

where

$$d = A^{-1} b, \tag{1.30}$$

$$e = c - d^T A d = c - b^T A^{-1} b. \tag{1.31}$$

Rewriting the quadratic function in (1.28) as (1.29) is a procedure known as "completing the square", which is very useful when dealing with products of Gaussian distributions.

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Problem 1.11 Eigenvalues

Let $\{\lambda_i\}$ be the eigenvalues of $A \in \mathbb{R}^{n\times n}$. Derive the following properties:

$\det|A - \lambda I| = 0$

$$\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i, \qquad |A| = \prod_{i=1}^n \lambda_i. \tag{1.32}$$

$\sum_{k=0}^{n} a_k \lambda^k = \lambda^n - \sum_{i=1}^{n} a_{ii} \lambda^{n-1} \cdots (-1)^n |A| = 0$

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Problem 1.12 Eigenvalues of an inverse matrix

Let $\{\lambda_i, x_i\}$ be the eigenvalues/eigenvectors of $A \in \mathbb{R}^{n\times n}$. Show that $\{\frac{1}{\lambda_i}, x_i\}$ are the eigenvalues/eigenvectors of A^{-1}.

$Ax = \lambda x$

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$A^{-1}Ax = \lambda A^{-1} x$

$x = \lambda A^{-1} x$

$A^{-1} x = \frac{1}{\lambda} x$

Problem 1.13 Positive definiteness

Derive the following properties:

1. A symmetric matrix $A \in \mathbb{S}^n$ is positive definite if all its eigenvalues are greater than zero.
2. For any matrix $A \in \mathbb{R}^{m\times n}$, $G = A^T A$ is positive semidefinite.

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1. $x^T A x > 0$ for all $x \neq 0$

$x^T S \Lambda S^T x$

$= (S^T x)^T \Lambda S^T x > 0$

(S real symmetric $\Downarrow$ $S^T x \neq 0$ for $x \neq 0$)

2. $x^T G x$

$= x^T A^T A x$

$= (Ax)^T A x \geq 0$

Problem 1.14 Positive definiteness of inner product and outer product matrices

Let $X = [x_1, \cdots, x_n] \in \mathbb{R}^{d \times n}$ be a matrix of n column vectors $x_i \in \mathbb{R}^d$. We can think of each column vector x_i as a sample in our dataset X.

features $\Rightarrow$ columns of $X^T \Rightarrow$ linearly independent

(a) outer-product: Prove that $\Sigma = XX^T$ is always *positive semi-definite* ($\Sigma \succeq 0$). When will Σ be strictly *positive definite*?

$x^T X X^T x = (X^T x)^T X^T x \geq 0$ $\quad$ $X^T x$ has no 0 entry for $x \neq 0$

(b) inner-product: Prove that $G = X^T X$ is always *positive semi-definite*. When will G be strictly *positive definite*?

$(Xx)^T (Xx) \geq 0$ $\quad$ columns of X

Note: If $\{x_1, \cdots, x_n\}$ are zero mean samples, then Σ is n times the sample covariance matrix. G is sometimes called a *Gram matrix* or *kernel matrix*.

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Problem 1.15 Useful Matrix Inverse Identities

$\Rightarrow B^T R^{-1} B P B^T + B^T = B^T + B^T R^{-1} B P B^T$

Show that the following identities are true:

$B^T R^{-1}(BPB^T + R) = (P^{-1} + B^T R^{-1} B) P B^T$

$$(P^{-1} + B^T R^{-1} B)^{-1} B^T R^{-1} = P B^T (B P B^T + R)^{-1}, \tag{1.33}$$

$$(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1} B = B(A + B)^{-1} A, \tag{1.34}$$

$(A+B)^{-1} = A^{-1}(A^{-1}+B^{-1})^{-1}B^{-1} = (B\cdot(A^{-1}+B^{-1})\cdot A)^{-1} = ((BA^{-1}+I)\cdot A)^{-1} = (A+B)^{-1}$

$$(A^{-1} + B^{-1})^{-1} = A - A(A + B)^{-1} A = B - B(A + B)^{-1} B, \tag{1.35}$$

$$(A^{-1} + U C^{-1} V^T)^{-1} = A - AU(C + V^T A U)^{-1} V^T A \tag{1.36}$$

The last one is called the Matrix Inversion Lemma (or Sherman-Morrison-Woodbury formula) Hint: these can be verified by multiplying each side by an appropriate matrix term.

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Problem 1.16 Useful Matrix Determinant Identities

Verify that the following identities are true:

$\det\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -B \\ A & I \end{pmatrix} = \det\begin{pmatrix} BA+I & \\ A & I \end{pmatrix} = \det(I+BA)$

$$\left|I + AB^T\right| = \left|I + B^T A\right| \tag{1.37}$$

$\det\begin{pmatrix} I & -B \\ A & I \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} I & \\ A & AB+I \end{pmatrix} = \det(AB+I)$

$$\left|I + ab^T\right| = 1 + b^T a \tag{1.38}$$

$$\left|A^{-1} + UV^T\right| = \left|I + V^T A U\right| \left|A^{-1}\right| \tag{1.39}$$

$= |I + UV^T A||A^{-1}|$

$= |A^{-1} + UV^T|$

eg: $\begin{pmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 4 & \\ & & & & 4 \end{pmatrix}\left(\cdots\right)$

$= A + [\vdots][1\,1\,1\,1\,1]$

$= \det(A)\det[I + [1\,1\,1\,1\,1] A^{-1} [\vdots]]$

$= 4^5 \times (1 + \frac{5}{4}) = 9 \times 4^4$

The last one is called the Matrix Determinant Lemma.

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Problem 1.17 Singular Value Decomposition (SVD)

The singular value decomposition (SVD) of a real $n \times m$ matrix A is a set of three matrices $\{U, S, V\}$, such that

$$A = USV^T, \tag{1.40}$$

where

- $U \in \mathbb{R}^{n\times m}$ is an orthonormal matrix of left-singular vectors (columns of U), i.e., $U^TU = I$.
- $S \in \mathbb{R}^{m\times m}$ is a diagonal matrix of *singular values*, i.e. $S = \mathrm{diag}(s_1, \cdots, s_m)$. The singular values are usually ordered $s_1 \geq s_2 \geq \cdots \geq s_m \geq 0$
- $V \in \mathbb{R}^{m\times m}$ is an orthonormal matrix of right-singular vectors (columns of V), i.e., $V^TV = I$.

The SVD has an intuitive interpretation, which shows how the matrix A acts on a vector $x \in \mathbb{R}^m$. Consider the matrix-vector product

$$z = Ax = USV^T x. \tag{1.41}$$

This shows that the matrix A performs 3 operations on x, namely rotation by V^T, scaling along the axis via S and another rotation by U. The SVD is also closely related to the eigen-decomposition, matrix inverse, and pseudoinverses.

$AA^T = USV^TVS^TU^T = USS^TU^T$

(a) Show that the singular values of A are the square roots of the eigenvalues of the matrix $B = AA^T$, and that the left-singular vectors (columns of U) are the associated eigenvectors.

(b) Show that the singular values of A are the square roots of the eigenvalues of the matrix $C = A^TA$, and that the right-singular vectors (columns of V) are the associated eigenvectors.

$A^TA = VS^TU^TUSV^T = VS^TSV^T$

(c) Suppose $A \in \mathbb{R}^{n\times n}$ is a square matrix of rank n. Show that the inverse of A can be calculated from the SVD,

$$A^{-1} = VS^{-1}U^T. \tag{1.42}$$

(d) Suppose $A \in \mathbb{R}^{n\times m}$ is a "fat" matrix ($n < m$) of rank n. The Moore-Penrose pseudoinverse of A is given by

$$A^\dagger = A^T(AA^T)^{-1}. \tag{1.43}$$

Likewise, for a "tall" matrix ($n > m$) of rank m, the pseudoinverse is

$$A^\dagger = (A^TA)^{-1}A^T. \tag{1.44}$$

Show that in both cases the pseudoinverse can also be calculated using the SVD,

$$A^\dagger = VS^{-1}U^T. \tag{1.45}$$

Note: Properties in (c) and (d) also apply for lower rank matrices. In these cases, some of the singular values will be 0, and S^{-1} is hence replaced with $S^\dagger$, where $S^\dagger$ replaces all non-zero diagonal elements by its reciprocal.

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