Beta (a, b), $a>0$, $b>0$

(generalization of uniform, x is bounded, unlike normal $(-\infty, +\infty)$]

PDF $f(x) = c x^{a-1}(1-x)^{b-1}$, $0<x<1$. (c is scaling factor)

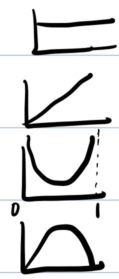
- flexible family of continuous distribution $(0,1)$ $\Rightarrow$ useful modeling tool

  $a=b=1$

  $a=2, b=1$

  $a=\frac{1}{2}=b$

  $a=b=2$



- often used as prior for a parameter in $(0,1)$
- "conjugate prior" to Binomial
- connections to other distributions

Conjugate prior for Binomial $\quad X|p \sim \mathrm{Bin}(n,p)$, $p \sim \mathrm{Beta}(a,b)$ [prior]

Find posterior dist $p|X$. $f(p|X=k) = \dfrac{P(X=k|p)\,f(p)}{P(X=k)}$ ← does not depend on p integrated out.

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} c \cdot p^{a-1} (1-p)^{b-1}}{P(X=k)}$$

(ignore constant that does not depend on p) $\propto p^{a+k-1}(1-p)^{b+n-k-1}$

$\Rightarrow p|X \sim \mathrm{Beta}(a+k, b+n-k)$

intuitive $\quad k$ success, $n-k$ failure

prior $\quad a$ success $\quad b$ failure

reformulate $\quad$ Find $\int_0^1 \binom{n}{k} x^k (1-x)^{n-k} dx$ without using calculus

using a story (Bayes Billiards)

① $n+1$ billiard balls, all white

paint one pink, throw them on $(0,1)$ independently

alternatively ② first throw, than paint one pink.

①, ② ⇒ equivalent

Let X = # balls on the left of pink one

① throw pink first (does not matter)

$$P(X=k) = \int_0^1 P(X=k|p) \underbrace{f(p)}_{1,\ \text{uniform}}\, dp$$

$$= \int_0^1 \binom{n}{k} p^k (1-p)^{n-k}\, dp$$

$$= \frac{1}{n+1}$$ ⇒ select a pink ball ②

Gamma distribution

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Gamma Function

$$\Gamma(a) = \int_0^\infty x^a e^{-x} \frac{dx}{x} = \int_0^\infty x^{a-1} e^{-x}\, dx \text{, for real } a>0$$

($a \neq 0$, since it is not well-defined at $x=0$)

$(\Gamma(n) = (n-1)!)$

$(\Gamma(x+1) = x\cdot\Gamma(x))$

$(\Gamma(\frac{1}{2}) = \sqrt{\pi})$

Distribution

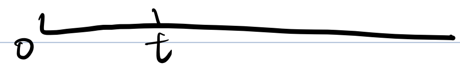
$$1 = \frac{1}{\Gamma(a)} \int_0^\infty x^a \cdot e^{-x} \frac{dx}{x}$$ gamma$(a,1)$ PDF

more generally $Y \sim \text{Gamma}(a, \lambda)$

$Y = \frac{X}{\lambda}$, $X \sim$ gamma$(a, 1)$

$$f_Y(y) = f_X(x)\cdot\frac{dx}{dy}$$
$$= \frac{1}{\Gamma(a)}(\lambda y)^a e^{-\lambda y}\cdot\frac{1}{\lambda y}\cdot\lambda$$
$$= \frac{1}{\Gamma(a)}(\lambda y)^a e^{-\lambda y}\frac{1}{y} \quad (y>0)$$

Gamma, Exponential connection



emails

N_t = # emails up to time t $\sim Pois(\lambda t)$

arrivals in disjoint intervals are independent.

$$P(T_1 > t) = P(N_t = 0)$$
$$= e^{-\lambda t}$$

$\Rightarrow$ inter-arrival time i.i.d. $\text{expo}(\lambda)$

T_n = (time of nth arrival) continuous

$= \sum_{j=1}^{n} X_j$, X_j i.i.d.

$\sim \text{Gamma}(n, \lambda)$

proof that $T = \sum_{j=1}^{n} X_j$, X_j i.i.d. $\text{Expo}(1)$ is $\text{Gamma}(n,1)$

MGF of X_j $\frac{1}{1-t}$, $t<1 \Rightarrow$ MGF of T_n is $\left(\frac{1}{1-t}\right)^n$, $t<1$

Let $Y \sim \text{Gamma}(n,1)$

$$E(e^{tY}) = \frac{1}{\Gamma(n)}\int_0^\infty e^{ty} y^n e^{-y}\frac{dy}{y}$$
$$= \frac{1}{\Gamma(n)}\int_0^\infty y^n e^{-(1-t)y}\frac{dy}{y}$$

handy $\Rightarrow$ does not change

$$\overset{\text{let } x=(1-t)y}{=} \frac{1}{\Gamma(n)}\int_0^\infty (1-t)^{-n} x^n e^{-x}\frac{dx}{x}$$
$$= (1-t)^{-n}$$

Let $X \sim \text{Gamma}(a,1)$, find $E(X^c)$

$$\frac{1}{\Gamma(a)}\int_0^\infty x^c\cdot x^a\cdot e^{-x}\frac{dx}{x} = \frac{1}{\Gamma(a)}\int_0^\infty x^{a+c}e^{-x}\frac{dx}{x}$$
$$= \frac{\Gamma(a+c)}{\Gamma(a)}, \text{ if } a+c>0$$

$$E(X) = \frac{\Gamma(a+1)}{\Gamma(a)} = a.$$

intuitively $\Rightarrow$ integer case linearity $\underset{(\lambda)}{a \times 1} = a.$

$$E(X^2) = \frac{\Gamma(a+2)}{\Gamma(a)} = (a+1) \cdot a$$

$$Var(X) = E(X^2) - (E(X))^2 = a$$

Gamma $(a\ \lambda) \Rightarrow$ mean $\frac{a}{\lambda}$ Var $\frac{a}{\lambda^2}$ (change of variable $Y = \frac{1}{\lambda} \cdot X$)

Connection Beta Gamma.

Bank – post office example

$X \sim Gamma(a, \lambda)$ wait at bank

$Y \sim Gamma(b, \lambda)$ wait at post-office.

distribution $T = X + Y$ (total time)

$W = \frac{X}{X+Y}$ (fraction of time)

(simplify let $\lambda = 1$)

joint PDF $f_{T,W}(t,w) = f_{X,Y}(x,y) \left| \left| \frac{\partial(x,y)}{\partial(t,w)} \right| \right|$

$$= \frac{1}{\Gamma(a)\Gamma(b)} x^a e^{-x} y^b e^{-y} \frac{1}{xy} \cdot t$$

$$\left(\begin{array}{l} x+y = t \\ \frac{x}{x+y} = w \\ x = tw \\ y = t(1-w) \end{array} \quad \left| \frac{\partial(x,y)}{\partial(t,w)} \right| = \begin{vmatrix} w, & t \\ 1-w, & -t \end{vmatrix} = -(-t - tw + tw) = t \right)$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \cdot t^a w^a e^{-tw} \cdot t^b (1-w)^b e^{-t(1-w)} \frac{1}{t^2(1-w) \cdot w} \cdot t$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \cdot w^{a-1}(1-w)^{b-1} \cdot t^{a+b} e^{-t} \cdot \frac{1}{t}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} ; \underline{t^{a+b} e^{-t} \cdot \frac{1}{t}}$$

$\Gamma(a+b)$

$$f_W(w) = \int_{-\infty}^{\infty} f_{T,W}(t,w)\cdot dt$$

$$= \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}} w^{a-1}(1-w)^{b-1}$$

$\Rightarrow$ normalized constant of Beta.

$T \sim \text{Gamma}(a+b,1)$
$W \sim \text{Beta}(a,b)$ } independent

Find $E(W)$, $W \sim \text{Beta}(a,b)$

$$E\left(\frac{X}{X+Y}\right) = \frac{E(X)}{E(X+Y)} = \frac{a}{a+b}$$

(be careful!) $\Rightarrow$ though true in this case

(why true here?

$E\left(\frac{X}{X+Y}\right) E(X+Y) = E(X)$ in this special case of Gamma-Beta

$\Rightarrow$ T, W INDEPENDENT