CS5487 Lecture Notes (2022B)
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Tutorial 1

PS 1.6 m.v. Gaussian

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\underbrace{\|x-\mu\|^2_\Sigma}_{(x-\mu)^T\Sigma^{-1}(x-\mu)}}$$

a) assume Σ is diagonal: $\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}$

$$|\Sigma| = \prod_{i=1}^d \sigma_i^2$$ (det. of a diagonal matrix is the product of the diagonal)

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_d^2 \end{bmatrix}$$ (inverse of a diag mtx just inverts the diagonal elements)

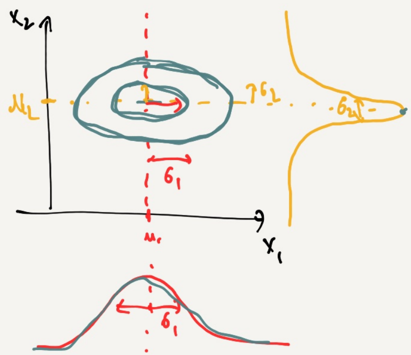
$$\|x-\mu\|^2_\Sigma = (x-\mu)^T\Sigma^{-1}(x-\mu)$$
$$= \underbrace{(x-\mu)^T \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_d^2 \end{bmatrix} \begin{pmatrix} x_1-\mu_1 \\ x-\mu \end{pmatrix}}_{} = \sum_{i=1}^d \frac{1}{\sigma_i^2}(x_i-\mu_i)^2$$

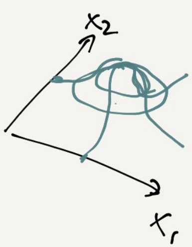
$$p(x) = \underbrace{\frac{1}{(2\pi)^{d/2}\prod_{i=1}^d \sigma_i}}_{} e^{\underbrace{-\frac{1}{2}\sum_{i=1}^d \frac{1}{\sigma_i^2}(x_i-\mu_i)^2}_{\Pi}}$$

$$= \left[\prod_{i=1}^d \frac{1}{(2\pi)^{1/2}\sigma_i}\right]\left[\prod_{i=1}^d e^{-\frac{1}{2}\frac{1}{\sigma_i^2}(x_i-\mu_i)^2}\right]$$

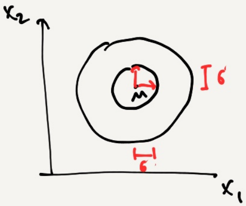
$$= \prod_{i=1}^d \underbrace{\frac{1}{(2\pi)^{1/2}\sigma_i} e^{-\frac{1}{2}\frac{1}{\sigma_i^2}(x_i-\mu_i)^2}}_{\text{univariate Gaussian}} = \boxed{\prod_{i=1}^d N(x_i|\mu_i,\sigma_i^2)}$$

product of marginal distributions
$\Rightarrow$ x_i's are independent





if $\sigma_i^2 = \sigma^2 \Rightarrow \Sigma = \sigma^2 I$ (scaled identity, isotropic Covariance / Gaussian, iid covariance. "circular" Gaussian)



d) General Case Σ $\qquad (\Sigma - \lambda I)v = 0$

$\Sigma v_i = \lambda_i v_i$

eigenvector $\rightarrow$ d eigenpairs (λ_i, v_i)

eigenvalue

"same"

Looking at all eigen pairs:

$$\Sigma \underbrace{[v_1 \cdots v_d]}_{V = \text{matrix of eigenvectors}} = [\lambda_1 v_1 \cdots \lambda_d v_d] = [v_1 \cdots v_d] \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}}_{\Lambda = \text{diag mtx of eigenvalues}}$$

$\Sigma V = V \Lambda$

$\downarrow$ multiply by V^{-1} on right

$\Sigma \cancel{V V^{-1}} = V \Lambda V^{-1}$

$\downarrow$ $V^T V = I$ for symmetric mtx.

eigendecomposition: $\Sigma = V \Lambda V^T$

general Σ:

$$\Sigma^{-1} = (V \Lambda V^T)^{-1}$$
$$= V^{T^{-1}} \Lambda^{-1} V^{-1}$$
$$= V \Lambda^{-1} V^T$$

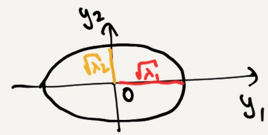
$\downarrow$ $(AB)^{-1} = B^{-1} A^{-1}$

$\downarrow$ $V^{-1} = V^T$

Mahal distance: $\|x - \mu\|^2_\Sigma = \underbrace{(x-\mu)^T V}_{y^T} \Lambda^{-1} \underbrace{V^T (x - \mu)}_{y}$

$= y^T \Lambda^{-1} y$, $\quad y = V^T(x - \mu)$

$\uparrow$ diagonal



$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_d} \end{bmatrix}$$

A Gaussian in y-space w/ $\mu = 0$ & diagonal cov Λ.

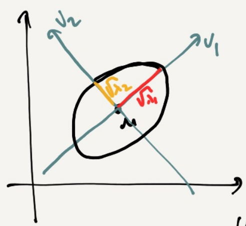
Rewrite in terms of x:

$y = V^T(x - \mu)$

$Vy = x - \mu$

$\Rightarrow x = Vy + \mu$

rotation $\qquad$ translation



rotated ellipse as the iso contours (according to the eigenvectors of Σ)