A 2 EM & Optics

① Electromagnetism
② Optics
③ Others

① Electromagnetism (excluding miscellaneous)

☆ Maxwell's eqns (in materials)

$$\begin{cases} \nabla \cdot \underline{D} = \rho_f & \text{(Gauss)} \\ \nabla \cdot \underline{B} = 0 & \\ \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} & \text{(Faraday)} \\ \nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} + \underline{J} & \text{(Maxwell-Ampere)} \end{cases}$$

where $\begin{cases} \underline{D} = \varepsilon_0 \underline{E} + \underline{P} \\ \underline{B} = \mu_0 (\underline{H} + \underline{M}) \end{cases}$

Bound charges: $\rho_b = -\nabla \cdot \underline{P}$ $\quad \sigma_b = \underline{P} \cdot \hat{n}$
Bound currents: $\underline{J_b} = \nabla \times \underline{M}$ $\quad \underline{K_b} = \underline{M} \times \hat{n}$

Linear dielectrics

$\varepsilon_r = 1 + \chi_e$ $\qquad \mu_r = 1 + \chi_m$
$P = \varepsilon_0 \chi_e E$ $\qquad M = \chi_m H$
$D = \varepsilon_0 \varepsilon_r E = \varepsilon E$ $\qquad B = \mu_0 \mu_r H = \mu H$
(electric) $\qquad$ (magnetic)

Dipole moments

Electric: $\underline{p}$ where $V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\underline{p} \cdot \hat{r}}{r^2}$

Polarization: $\underline{P} = n\underline{p} = n\alpha \underline{E}$

Magnetic: $\underline{m}$ where $\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{r}}{r^2}$
$(\underline{B} = \nabla \times \underline{A})$ → magnetic vector potential

Magnetization: $\underline{M} = n\underline{m}$ ~~$= n\beta \underline{B}$~~

$\begin{cases} \hat{\underline{n}} \cdot \Delta \underline{E} = \frac{\sigma}{\varepsilon_0} \\ \hat{\underline{n}} \times \Delta \underline{B} = \mu_0 \underline{S} \end{cases}$ &

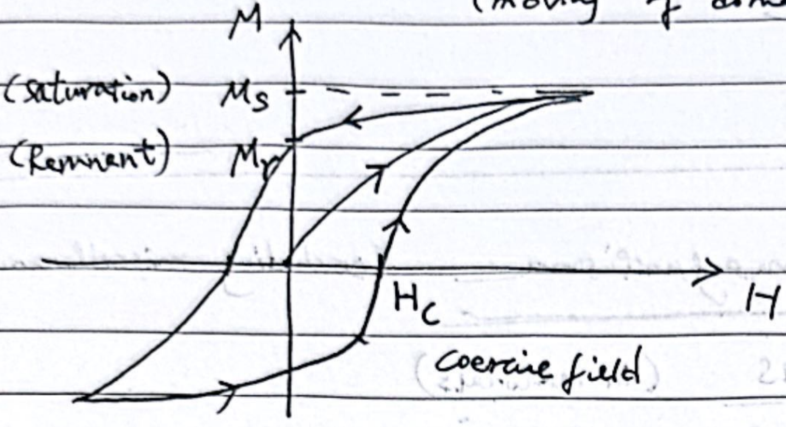
Boundary conditions → $\boxed{\underset{\hat{n}\times\underline{E}}{E_{\parallel}},\ \underset{\hat{n}\times\underline{H}}{H_{\parallel}},\ \underset{\hat{n}\cdot\underline{P}}{D_{\perp}},\ \underset{\hat{n}\cdot\underline{B}}{B_{\perp}} \text{ are continuous.}}$

Useful particularly for solving Laplace!

Magnetism in Materials

① Diamagnetism $\quad \chi_m < 0 \quad$ weak, all materials show
② Paramagnetism $\quad \chi_m > 0 \quad$ unpaired electrons lined up by $\underline{H}$

③ Ferromagnetism   nonlinear, retain magnetization over time. (moving of domain walls, i.e.)



Soft: H_c, M_r small (transformers)

Hard: H_c, M_r large (permanent magnets)

EM Waves in materials

(I) Assume no free currents/charges, $\rho_f = \underline{J}_f = 0$

(free plane waves) $\hookrightarrow$ $\begin{cases} \nabla^2 \underline{E} = \mu\varepsilon \dfrac{\partial^2 \underline{E}}{\partial t} \\ \nabla^2 \underline{B} = \mu\varepsilon \dfrac{\partial^2 \underline{B}}{\partial t} \end{cases}$ $v = \dfrac{1}{\sqrt{\varepsilon\mu}} = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_r\mu_r}}$

$n = \sqrt{\varepsilon_r\mu_r}$ (refractive index)

Impedance: $Z = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_r}{\varepsilon_0}}\, Z_0 \longrightarrow \left(Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\Omega\right)$

It can be shown from Maxwell's:

$$\frac{\partial}{\partial t}\Big(\underbrace{\frac{1}{2}\underline{E}\cdot\underline{D}}_{u_E} + \underbrace{\frac{1}{2}\underline{B}\cdot\underline{H}}_{u_B}\Big) + \nabla\cdot(\underbrace{\underline{E}\times\underline{H}}_{\underline{S}}) = -\underline{E}\cdot\underline{J}$$

(Continuity equation of energy (density)) $\underline{S}$ → Poynting Vector (transport of energy)

$$\begin{cases} u_E = \frac{1}{2}\underline{E}\cdot\underline{D} & \text{Electrostatic energy density} \\ u_B = \frac{1}{2}\underline{B}\cdot\underline{H} & \text{Magnetostatic energy density} \end{cases}$$

Radiation pressure: (transport of momentum) $P_{rad} = \begin{cases} \dfrac{\langle S\rangle}{c} & \text{for perfect absorber} \\ \dfrac{2\langle S\rangle}{c} & \text{for perfect reflector} \end{cases}$

others:

(II) Conductors

only difference (to free waves) is putting $\underline{J} = \sigma\underline{E} \neq 0$

Maxwell's wave eqn for conductor: $\nabla^2 \underline{E} = \mu\sigma \frac{\partial \underline{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}$

Ansatz: $\underline{E}(z,t) = E_0 \hat{x} e^{i(\tilde{k}z - \omega t)}$ $\quad (\tilde{k} = k + iK)$

$\Rightarrow \tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$

solve for k & K independently, ...

$$\begin{cases} k = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)^{1/2} \\ K = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)^{1/2} \end{cases} \Rightarrow \underline{E}(z,t) = E_0\hat{x}\underbrace{e^{-Kz}}_{\text{decaying part}} e^{i(kz-\omega t)}$$

defined skin depth: $\boxed{\delta \equiv \frac{1}{K}}$

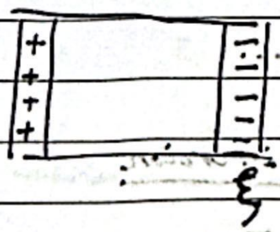
(characteristic length of exp. decay).

for $\sigma \gg \varepsilon\omega$ (good conductors):

$$k = K = \sqrt{\frac{\mu\omega\sigma}{2}} \Rightarrow \delta = \sqrt{\frac{2}{\mu\omega\sigma}} \quad \text{(very fast decay)}$$

(III.) Plasma

plasma: (neutral) gas of charged particles (ions + free e^-)



E-field generated

$$E = \frac{\sigma}{\varepsilon_0} = \frac{ne\xi}{\varepsilon_0}$$

$$m\ddot{\xi} = -eE = -\frac{ne^2}{\varepsilon_0}\xi$$

(SHM)

$\Rightarrow \boxed{\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}}}$ (plasma frequency)

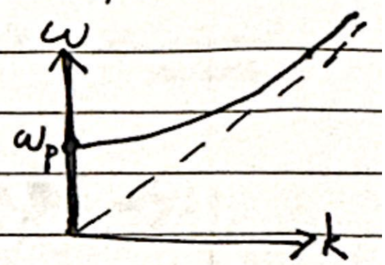
· Assume $\omega \gg \gamma$ } damping negligible

For a plane EM-wave $E = E_0 e^{-i\omega t}$ (ignore spatial parts)

$$m\ddot{\xi} = -eE_0 e^{-i\omega t}$$

Let $\xi = \xi_0 e^{-i\omega t}$

$\to m\xi_0(-\omega^2) = -eE_0 \to \xi_0 = \frac{eE_0}{m\omega^2}$



★ Polarization $P = np = -ne\xi_0 = -\frac{ne^2E_0}{m\omega^2}$

(key quantity)

Let $P = \varepsilon_0(\varepsilon_r - 1)E_0 = -\frac{ne^2E_0}{m\omega^2}$

dispersion relation

$$\varepsilon_r = 1 - \frac{ne^2}{m\omega^2\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow \boxed{\omega_p^2 + c^2k^2 = \omega^2}$$

$\downarrow$

$(= n^2 = \frac{c^2}{v^2})$ $\quad (= \frac{v^2}{k^2})$

· Plasma waves are dispersive.

· No propagation for $\omega < \omega_p$.

Dispersion & Absorption

(treating electron response to EM-wave driven as a damped SHO.)

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = qE_0 e^{-i\omega t}$$

$\Rightarrow$ Use $x = x_0 e^{-i\omega t}$,

$$(-\omega^2 - i\omega\gamma + \omega_0^2)\, x_0 = \frac{qE_0}{m}$$

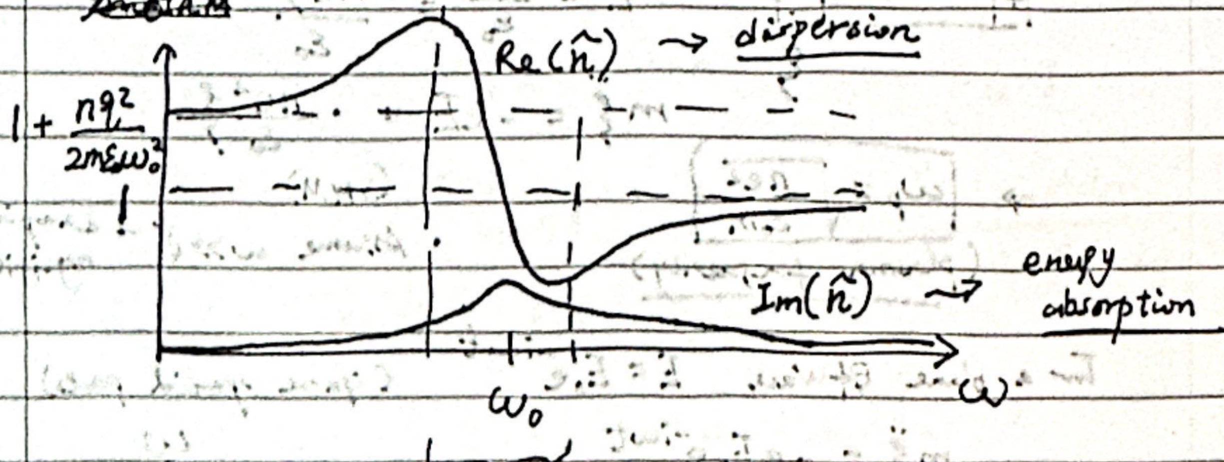
polarization: $P = n\underbrace{qx_0}_{p} = \frac{nq^2E_0}{m}\left(\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}\right) = \varepsilon_0(\tilde{\varepsilon}_r - 1)E_0$

Complex: $\hat{\varepsilon}_r = 1 + \frac{nq^2}{m\varepsilon_0}\left(\frac{\omega_0^2-\omega^2+i\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\right)$

$$\begin{cases} Re(\hat{\varepsilon}_r) = 1 + \frac{nq^2}{m\varepsilon_0}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \\ Im(\hat{\varepsilon}_r) = \frac{nq^2}{m\varepsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \end{cases}$$

$\hat{n} = \sqrt{\hat{\varepsilon}_r} = 1 + \frac{1}{2}(\text{small term}) + \cdots$

~~Anomalous~~



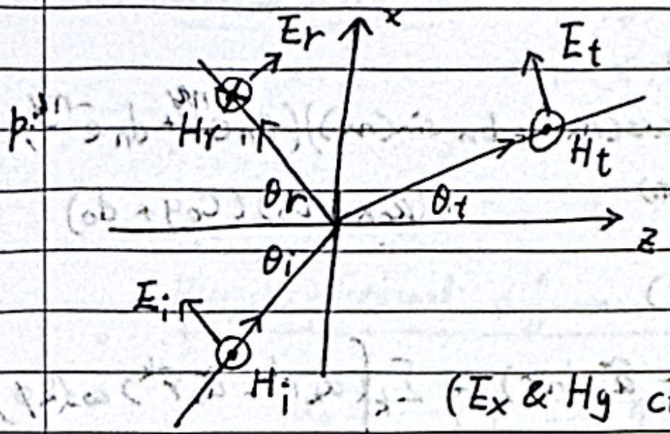
Anomalous dispersion ($\hat{n}$ drops with frq. in this interval)

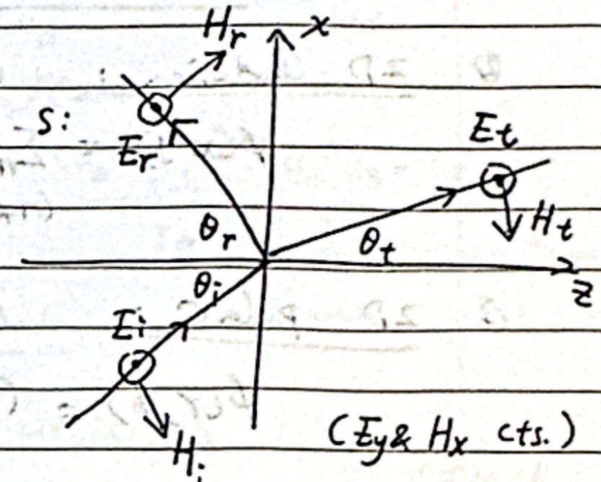
Reflection & Transmission at Interface of Media $(\theta_r = \theta_i)$

Snell's law: $\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2} = \frac{n_1}{n_2}$ ★ $E_{\parallel}$ & $H_{\parallel}$ are continuous.

p-polarization: $E \parallel$ plane of incidence $\left(E_j/H_j = Z_j = \sqrt{\frac{\mu}{\varepsilon_j}}\right)$

s-polarization: $E \perp$ plane of incidence





(E_x & H_y cts.) (E_y & H_x cts.)

$\left(Z \propto \frac{1}{n}\right)$ (Z_1) (Z_2) (Z_1) (Z_2)

$$p: \begin{cases} E_i\cos\theta_i + E_r\cos\theta_i = E_t\cos\theta_t \\ \frac{E_i}{Z_1} - \frac{E_r}{Z_1} = \frac{E_t}{Z_2} \end{cases} \qquad s: \begin{cases} E_i + E_r = E_t \\ -\frac{E_i}{Z_1}\cos\theta_i + \frac{E_r}{Z_1}\cos\theta_i = -\frac{E_t\cos\theta_t}{Z_2} \end{cases}$$

(Note consistency in signs)

i.e. solve for $r_p = \frac{E_r}{E_i} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}$

$r_p = 0 \leftrightarrow \theta_i = \arctan\left(\frac{n_2}{n_1}\right) \equiv \theta_B$ Brewster's angle (perfect transmission)

Always $\theta_B < \theta_c$ for $(n_2 < n_1)$

↳ $\theta_c = \arcsin\left(\frac{n_2}{n_1}\right)$ critical angle (total internal reflection)

Poisson's equation

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (generally)

For $\rho = 0 \rightarrow \boxed{\nabla^2 V = 0}$ Laplace's Eqn.

- For given boundary conditions, the solutions of V are unique (up to a constant addition).

(Relat for ∇^2 → see data sheet)

① In <u>spherical polars</u>: Solutions for (θ, ϕ) are combinations of Associated-Legendre polynomials $P_l^m(\cos\theta)$

For azimuthal symmetry (no m-dependence, $m=0$):

$$\boxed{V(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)}$$ (by separation of vars.)

② <u>2D Cartesian</u>: (x, y)

$$\phi(x,y) = \sum_n \left(a_n \cos(nx) + b_n \sin(nx)\right)\left(c_n e^{ny} + d_n e^{-ny}\right) + (a_0 x + b_0)(c_0 y + d_0)$$

$(n \neq 0,\ \text{int.})$

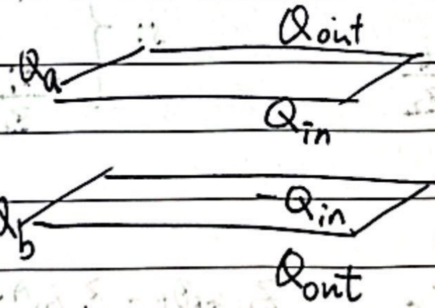
③ <u>2D polar</u>: (r, ϕ)

$$\phi(r,\phi) = (a_0 + \tilde{a}_0 \ln r) + \sum_k \left[(a_k r^k + \tilde{a}_k r^{-k})\cos(k\phi) + (b_k r^k + \tilde{b}_k r^{-k})\sin(k\phi)\right]$$

$(k = 1, 2, 3, \ldots)$

(Solve angular part as 'SHM' and come back to radial)

<u>On Capacitors</u>:

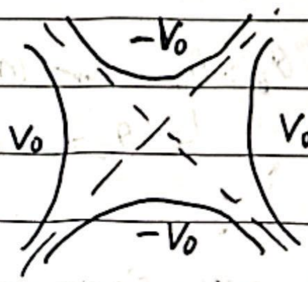


$$\begin{cases} Q_{out} + Q_{in} = Q_a \\ -Q_{in} + Q_{out} = Q_b \end{cases}$$

(E-field inside conductor $= 0$)

<u>Special solutions</u> i.e. $\boxed{V = \frac{V_0}{d^2}(x^2 - y^2)}$ 2D-Quadrupole field potential

non-separable



(2) Optics

Generalised amplitude : $u = u_0 e^{i\phi} = u_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)}$

Irradiance/intensity $I \propto \langle u^* u \rangle_t$ (time avg)

Incoherent light : $I = I_A + I_B$

Coherent light : $k_A = k_B$, $\omega_A = \omega_B$

$$I \propto \langle (u_A + u_B)^* (u_A + u_B) \rangle$$

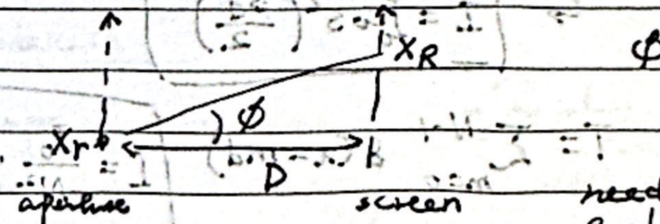
$$\propto \underbrace{\langle u_A^* u_A \rangle}_{I_A} + \underbrace{\langle u_B^* u_B \rangle}_{I_B} + \underbrace{\langle u_A^* u_B + u_B^* u_A \rangle}_{\text{Interference term}}$$

(cause fringes)

Huygens-Fresnel diffraction integral

$$u_P = -\frac{i}{\lambda}\int_S \frac{u_P}{r}\eta(\underline{n},\underline{r})\exp[ikr]\,dS$$

Fraunhofer condition : relative phases would be linear in x_r (plane of aperture)



$$\phi = k\sqrt{D^2 + (x_R - x_r)^2}$$

$$\approx kD + \frac{k(x_R - x_r)^2}{2D}$$

need quadratic term $\frac{k x_r^2}{2D} \ll \pi$

$\frac{\pi x_r^2}{\lambda D} \ll \pi \rightarrow \frac{x_r^2}{\lambda D} \ll 1$ Fraunhofer regime (far-field)

(i.e. aperture size $a \sim x_r$: $\frac{a^2}{\lambda D} \ll 1$ needed)

If $\frac{a^2}{\lambda D} \sim 1 \rightarrow$ Fresnel diffraction $k\sin\theta \equiv$ spatial frq.

Fraunhofer diffraction $\rightarrow$ $e^{ikr} = e^{ikr_0} e^{ikx\sin\theta} \propto e^{ikx\sin\theta}$

1D: $u_P \propto u_0 \int dx\, T(x)\, e^{ikx\sin\theta}$ Fourier transform of the transmission function.

2D: $u_P \propto u_0 \int dx dy\, T(x,y)\, e^{i(kx\sin\theta + ky\sin\phi)}$

Convolution Theorem $\int_{-\infty}^{\infty} f(x') g(x-x') dx' = f(x) * g(x)$

(convolution def.)

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)] \quad ☆$$

Convolution $\qquad\qquad$ product

- Convolution of simple aperture transmission functions can give intensity patterns which are products of independent cases.

Examples $\quad (\beta_i = k \sin\theta_i) \quad i = x, y, z$

Single-slit (a): $T = \begin{cases} 1 & |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \rightarrow \boxed{I = I_0 \operatorname{sinc}^2\left(\frac{a\beta}{2}\right)}$

Rectangular aperture (a, b): $T = \begin{cases} 1 & |x| < \frac{a}{2} \ \& \ |y| < \frac{b}{2} \\ 0 & \text{otherwise} \end{cases} \rightarrow \boxed{I = I_0 \operatorname{sinc}^2\left(\frac{a\beta_x}{2}\right) \operatorname{sinc}^2\left(\frac{b\beta_y}{2}\right)}$

triangular ⇓ sinc^4 (conv. of 2 hats)

Double-slit (d): $T = \delta(x - \frac{d}{2}) + \delta(x + \frac{d}{2})$

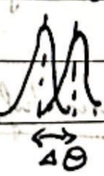
$\hookrightarrow \boxed{I = I_0 \cos^2\left(\frac{\beta d}{2}\right)}$

Multiple-slits (grating): $T = \sum_{m=0}^{N-1} \delta(x - md) \quad \boxed{I = \frac{I_0}{N^2} \frac{\sin^2\left(\frac{N\beta d}{2}\right)}{\sin^2\left(\frac{\beta d}{2}\right)}}$

Practical importance:
The in-focus image of an optical system is convolved by the Fraunhofer diffraction intensity pattern of the entrance aperture.

Rayleigh criterion: just resolved for $\Delta\theta \sim \theta_{\text{1st min}}$

Telescope: $\theta_{\text{1st min}} \simeq 1.22 \frac{\lambda}{D}$ (circular)



resolving power (R) $\equiv \frac{1}{\Delta\theta} \simeq \boxed{\frac{D}{1.22\lambda}}$

(wish R to be large → increase D)

Reflection gratings: now $\beta = k(\sin\theta - \sin\alpha)$ ($\alpha \rightarrow \theta_i$) $\quad$ still set ($\beta d = 2n\pi$)

Blaze angle: $\boxed{\theta_B = \frac{\theta_p - \theta_i}{2}}$ ⟸ optimal for p-th order max

Diffraction grating spectrograph

$\frac{m}{N} \neq$ integer.

$(d\sin\theta = \frac{m\lambda}{N}$ for min$)$

n-th order max $\rightarrow \frac{kd\sin\theta}{2} = n\pi \rightarrow d\sin\theta = p\lambda$

Angular dispersion $\frac{d\theta}{d\lambda} = \frac{n}{d\cos\theta}$ (deviation from n-th order)

(~~Linear dispersion: xf~~)

· minima separated by $\boxed{\Delta\sin\theta = \frac{\lambda}{Nd}}$

Useful approx: $\Delta\cos\theta \simeq \sin\theta\Delta\theta$ $(\Delta\theta << 1)$

(by disp. relation)

$\rightarrow d\Delta\sin\theta = p\Delta\lambda \rightarrow \boxed{\Delta\lambda_{INST} = \frac{\lambda}{NP}}$ instumental width in wavelength

$\boxed{R \equiv \frac{\lambda}{\Delta\lambda_{INST}} = Np}$ (resolving power)

N → num. of slits; p → order

$\boxed{\Delta\lambda_{FSR} = \frac{\lambda}{p}}$

We need $\underbrace{\Delta\lambda_{INST}}_{\text{min width for separation of wavelengths}} < \Delta\lambda < \underbrace{\Delta\lambda_{FSR}}_{\text{uniquely determine order of diffraction}}$ for good spectroscopy.

Coherence

Temporal coherence: $\tau_c \simeq \frac{1}{\Delta\nu}$

τ_c → temporal coherence time

$\Delta\nu$ → width of intensity spectral

→ characterises the timescale on which the wave has a consistent phase.

$\Delta x = c\tau_c$ → coherence length (for light)

Visibility $V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$

Recall that intensity (total) is:

$$I = I_A + I_B + \underbrace{\langle u_A^* u_B + u_B^* u_A \rangle}_{= 2\sqrt{I_A I_B}\,\mathrm{Re}[\gamma]}$$

where γ characterises the degree of coherence

$$\begin{cases} \gamma = 0 & \text{fully incoherent} \\ |\gamma| = 1 & \text{fully coherent} \\ 0 < |\gamma| < 1 & \text{partially coherent} \end{cases}$$

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_A I_B}}{I_A + I_B}|\gamma| \overset{\text{if } I_A = I_B}{=} |\gamma|$$

Transverse incoherence : extended light source's different parts produce uncorrelated wavefronts.

Interference by division of wavefront: different components of a wavefront need to maintain a consistent phase with each other (require transverse coherence).
(i.e. Young's slits)

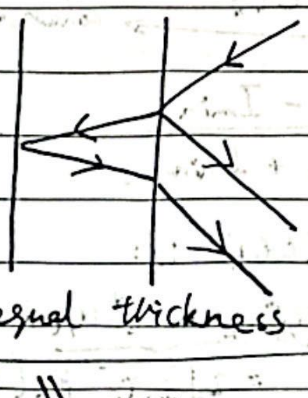
Interference by division of amplitude: a same wavefront is split into separate beams of different amplitudes and recombines to form interference pattern (require temporal coherence) (i.e. Michelson interferometer)

Localisation

Non-localised : not restricted to a plane, real
(i.e. Young's slits

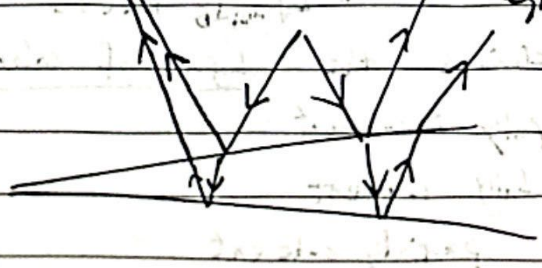
Localised: in a particular plane, virtual or real, produced by extended light sources
(point sources can also produce this).

Fringes of equal inclination: any fringe traces a locus of equal inclination angle.



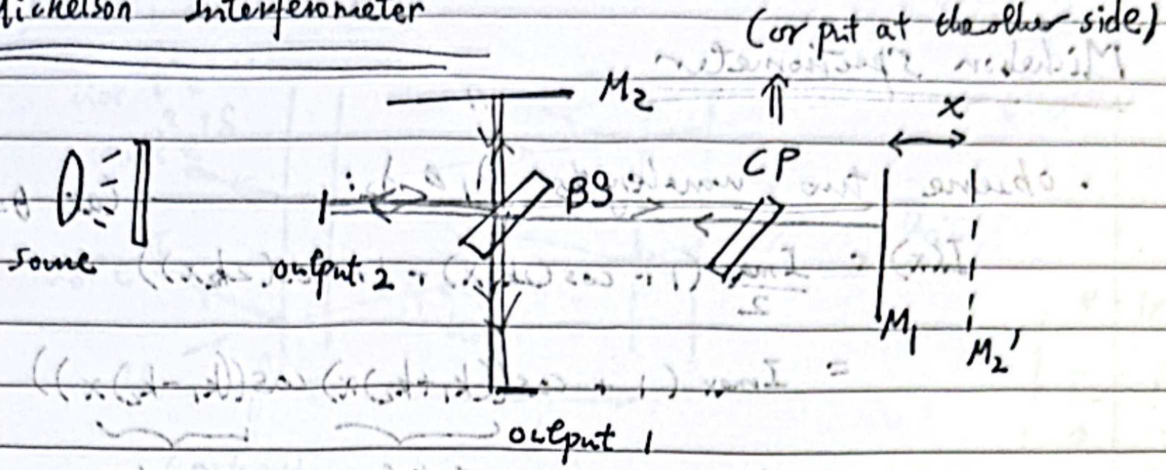
(i.e. localised at infty)

Fringes of equal thickness : any fringe traces a locus of equal surface separation.



(i.e. real or virtual localised fringes)

Michelson Interferometer



- CP ensures beams traverse equal thickness of glass along two pathways.
- phase difference $\boxed{\delta = 2kx\cos\theta = 4\pi\bar{\nu}x\cos\theta}$ $\left(\bar{\nu} = \frac{1}{\lambda} = \frac{k}{2\pi}\right)$
- Parallel alignment of M_1 & M_2' → fringes of equal inclination, localised at infty, circular.
- Not aligned → fringes of equal thickness, localised fringes.
- phase shift for external reflections is π, so output 1 & 2 must be dark-bright, vice versa.

$$U = u_0 + u_0 e^{i\delta} = u_0(1+e^{i\delta})$$

$$I = u^* u = |u_0|^2(1+\cos\delta + i\sin\delta)(1+\cos\delta - i\sin\delta)$$

$$= |u_0|^2(2+2\cos\delta) = \boxed{\frac{I_0}{2}(1+\cos\delta)}$$

$(I_0 = 4u_0^2)$

$\Rightarrow$ max, when $2x\cos\theta_0 = 2x = p_0\lambda$
$(\theta = 0)$

$p_0 = \frac{2x}{\lambda}$ → actually highest order

(linear displacement from center is too large for large angles, so phase difference is small in comparison)

other orders: $2x\cos\theta_p = (p_0 - p)\lambda$

$$2x\left(1 - \frac{\theta_p^2}{2} + \dots\right) = p_0\lambda - p\lambda$$

$$x\theta_p^2 \approx p\lambda \rightarrow \theta_p^2 \approx \frac{p\lambda}{x}$$

separation of neighboring orders: $\theta_{p+1}^2 - \theta_p^2 \approx \frac{\lambda}{x}$

$\hookrightarrow r_{p+1}^2 - r_p^2 \approx \frac{f^2\lambda}{x}$

(detected with a focal length f)

Michelson spectrometer

- observe two wavelengths λ_1 & λ_2: (at $\theta=0$)

$$I(x) = \frac{I_{max}}{2}(1+\cos(2k_1x) + 1 + \cos(2k_2x))$$

$$= I_{max}(1 + \underbrace{\cos((k_1+k_2)x)}_{\text{Interference}}\underbrace{\cos((k_1-k_2)x)}_{\text{Envelope}})$$

To detect presence of two wavelengths from envelope of fringe:

set $(k_1-k_2)x = \Delta k\, x = \pi$

$$\Delta k = \frac{\pi}{x} \rightarrow \boxed{\Delta\bar{\nu}_{INST} = \frac{1}{2x}}$$

Thus $RP = \frac{\lambda}{\Delta\lambda_{INST}} = \frac{\bar{\nu}}{\Delta\bar{\nu}_{INST}} = 2x.\bar{\nu} = \frac{2x}{\lambda}$.

Fourier Transform Spectrometer

$$I(x) = \frac{1}{2}\int_0^\infty \underbrace{S(\bar{\nu})}_{\text{spectral intensity of source}}(1+\cos(4\pi\bar{\nu}x))\,d\bar{\nu}$$

since even function: → turns to general Fourier Transform

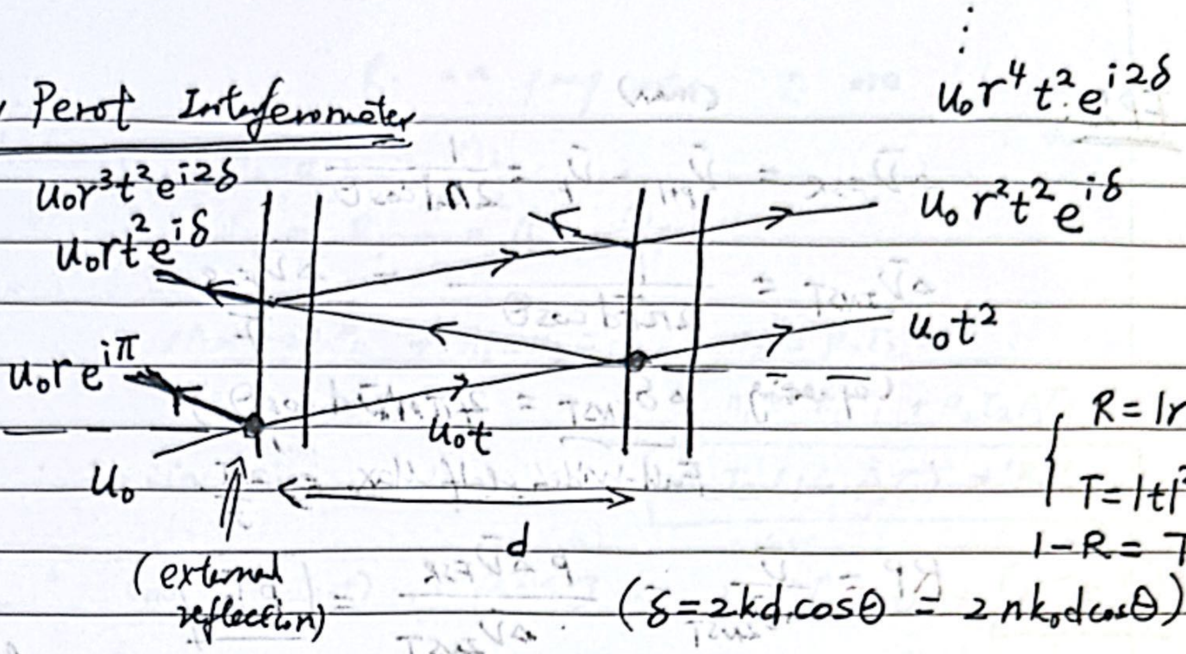
$$I(x) = \int_{-\infty}^{\infty} S(\bar{\nu})\exp(-4\pi i\bar{\nu}x)\,d\bar{\nu} = \int_{-\infty}^{\infty} S(\bar{\nu})\exp(-2\pi i\bar{\nu}L)\,d\bar{\nu}$$

$(L = 2x)$

$$S(\bar{\nu}) = \int_{-\infty}^{\infty} I(x)\exp(2\pi i\bar{\nu}L)\,dL$$

(up to const. prefactors of 2π, $\sqrt{2\pi}$)

Fabry Perot Interferometer



$$\begin{cases} R = |r|^2 \\ T = |t|^2 \\ 1-R = T \end{cases}$$

$(\delta = 2kd\cos\theta = 2nk_0 d\cos\theta)$

Transmission

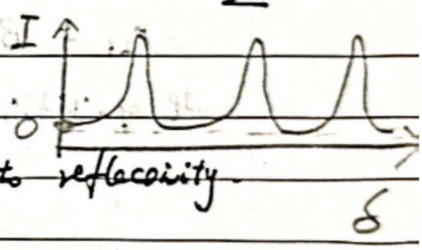
$$U_T(\delta) = u_0 t^2 + u_0 r^2 t^2 e^{i\delta} + u_0 r^4 t^2 e^{i2\delta} + \dots$$

$$= \frac{u_0 t^2}{1 - r^2 e^{i\delta}}$$

$$I_T(\delta) = \frac{|u_0|^2 |t|^4}{(1 - r^2\cos\delta - ir^2\sin\delta)(1 - r^2\cos\delta + ir^2\sin\delta)}$$

$$= \frac{I_0(1-R)^2}{(1-R\cos\delta)^2 + R^2\sin^2\delta} = \frac{I_0(1-R)^2}{(1-R)^2 + 4R\sin^2\frac{\delta}{2}}$$

$$= I_0\left[1 + \frac{4F^2}{\pi^2}\sin^2\left(\frac{\delta}{2}\right)\right]^{-1}$$



Finesse: $\boxed{F = \frac{\pi\sqrt{R}}{1-R}}$ F related to reflectivity.

Reflection

$$U_R(\delta) = -u_0 r + u_0 r t^2 e^{i\delta} + u_0 r^3 t^2 e^{i2\delta}$$

$$= u_0 r\left[-1 + t^2 e^{i\delta}(1 + r^2 e^{i\delta} + \cdots)\right]$$

$$= u_0 r\left[-1 + \frac{t^2 e^{i\delta}}{1 - r^2 e^{i\delta}}\right] = u_0 r\left[\frac{e^{i\delta} - 1}{1 - Re^{i\delta}}\right]$$

$$\Rightarrow I_R(\delta) = |u_0|^2 R\left[\frac{1 - e^{-i\delta}}{1 - Re^{-i\delta}}\right]\left[\frac{1 - e^{i\delta}}{1 - Re^{i\delta}}\right]$$

$$= \frac{I_0 4F^2}{\pi^2}\sin^2\frac{\delta}{2}\left[1 + \frac{4F^2}{\pi^2}\sin^2\left(\frac{\delta}{2}\right)\right]^{-1}$$

Hence: $I_T + I_R = I_0 \Rightarrow$ Conservation of energy.

FPI:

$$\Delta\bar{\nu}_{FSR} = \bar{\nu}_{p+1} - \bar{\nu}_p = \frac{1}{2nd\cos\theta} \quad (\max)$$

$$\Delta\bar{\nu}_{INST} = \frac{1}{2n\mathcal{F}d\cos\theta} = \frac{\Delta\bar{\nu}_{FSR}}{\mathcal{F}}$$

(equating $\underbrace{\Delta\delta_{INST}}_{\text{Full-Width Half Max} \approx \frac{2\pi}{\mathcal{F}}} = 4\pi n\Delta\bar{\nu} d\cos\theta$)

$$RP = \frac{\bar{\nu}}{\Delta\bar{\nu}_{INST}} = \frac{p\Delta\bar{\nu}_{FSR}}{\Delta\bar{\nu}_{INST}} = \boxed{p\mathcal{F}}$$

independent of refractive index, or angles.

Lasers (Basics)

$A \rightarrow$ spontaneous emission

$B \rightarrow$ stimulated emission or absorption

$u(\omega) \rightarrow$ spectral energy density (based on SHO model)

$r_{12} = B_{12} n_1 u(\omega)$

$r_{21} = B_{21} n_2 u(\omega) + A_{21} n_2$

At equilibrium: $\dot{n}_1 = \dot{n}_2 = 0 \rightarrow B_{12} n_1 u(\omega) = B_{21} n_2 u(\omega) + A_{21} n_2$

$$n_2/n_1 = \frac{B_{12} u(\omega)}{A_{21} + B_{21} u(\omega)} < 1$$

(cannot have population inversion in only 2-level atom).

Taking $u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left(\frac{1}{e^{\beta\hbar\omega} - 1}\right)$

· In thermal equilibrium: $\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\frac{E_2 - E_1}{k_B T}\right] = \frac{g_2}{g_1}\exp\left(-\beta(E_2 - E_1)\right)$

Thus $\rightarrow$ if treating $g_1 B_{12} = g_2 B_{21}$ & $g_1 = g_2$. $B_{12} = B_{21} = B$

$$\boxed{\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}}$$

Lasing condition: Stim. Emission > Stim. Absorption

(pop. inversion) $N_2 B_{21} u(\omega) > N_1 B_{12} u(\omega)$

$\hookrightarrow N_2 > N_1$ (for B equal cases)

Pumping

R_i are pump rates, τ_i are lifetimes at i-th level

$$\dot{n}_2 = R_2 - \frac{n_2}{\tau_2}$$
$$\dot{n}_1 = R_1 + n_2 A - \frac{n_1}{\tau_1}$$

· steady state → $\dot{n}_1 = \dot{n}_2 = 0$ $\quad n_2 = R_2\tau_2$

$\quad n_1 = R_1\tau_1 + R_2\tau_2 A\tau_1$

For inversion: $n_2 > n_1$ $\quad \boxed{R_2\tau_2(1 - A\tau_1) > R_1\tau_1}$

(not sufficient) necessary : $1 - A\tau_1 > 0 \rightarrow \boxed{A\tau_1 < 1}$

Coherence length/time measurement $(nk_0\Delta x)$

$\Delta x = c\tau_c$ — Any path difference greater than such will lead to disappearance of fringes (measured via Michelson interferometer).

Δx: coh. length; τ_c: coh. time

Polarisation :

Unpolarised : 'random', no consistent polarisation pattern.

Completely polarised: electric field vector can be described by:

$$\underline{E}: \begin{cases} \underline{E_x} = E_{x0}\cos(kz - \omega t)\,\hat{x} \\ \underline{E_y} = E_{y0}\cos(kz - \omega t + \phi)\,\hat{y} \end{cases}$$

ϕ → a constant phase.

(where $\underline{S}$ propagates along $\hat{z}$)

Partially polarised : mixture of unpolarised and a fraction of polarised light (degree of polarisation)

(ie. Reflections from surfaces & Rayleigh scattering)

Linear polarisation: $\phi = 0, \pi$; $\tan\alpha = \frac{E_{y0}}{E_{x0}}$

Circular polarisation: $\phi = \pm\frac{\pi}{2}$; $E_{0x} = E_{0y} = E_0$; $E_x^2 + E_y^2 = E_0^2$

Ellipsoidal polarisation : ϕ any arbitrary value

in an arbitrary rotated system

$$1 = \left(\frac{E_{x'}}{E_{x'0}}\right)^2 + \left(\frac{E_{y'}}{E_{y'0}}\right)^2 \quad \text{rotated by } \alpha : \quad \tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\phi}{(E_{0x}^2 - E_{0y}^2)}$$

<u>Jones vector</u> $\underline{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_x \begin{pmatrix} 1 \\ |E_y|/|E_x| \, e^{i\phi} \end{pmatrix}$

(off-syllabus)

relative phase

<u>Polariser</u>

- can be realised by crystals or grid of conducting wires.
- Achieved through <u>Dichroism</u> (selective absorption of one polarisation state than the other).

<u>Waveplates</u>

- Change the phase of the two components of E-field.
- Represented by Jones matrices.
- Can be implemented through <u>birefringent waveplates</u> (uniaxial)

<u>Birefringence</u> (i.e. Wollaston prism (two birefringent crystals aligned $\perp$ in axes))

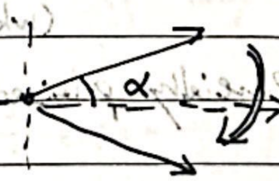
- Electrons oscillate more/less easily in one direction.
- Rays of different $\underline{E}$ directions are refracted differently $\begin{cases} n_o \text{ (ordinary) } & \text{irrespective, } \perp \text{ optic axis} \\ n_e \text{ (extraordinary) } & \text{dependent on orientation, } // \text{ optic axis} \end{cases}$
- <u>relative retardation</u>: $\underline{\delta = kd(n_o - n_e)}$

phase difference.

$\delta = 2\pi \Rightarrow$ full wave plate.

$\delta = \pi \Rightarrow$ half-wave plate

$\delta = \frac{\pi}{2} \Rightarrow$ quarter-wave plate



change

generally: Linear $\Longleftrightarrow$ Elliptical

special cases: $\alpha = 0, \pm\pi \Rightarrow$ still linear

$\alpha = \pm\frac{\pi}{4} \Rightarrow$ circular polarisation.

(linear)

<u>Malus's law</u>: $I = I_0 \cos^2\theta$

$\hookrightarrow$ passed through a linear polariser with relative angle θ.

Analysis of polarised light (elliptic)

① use linear polariser (rotate to find max/min transmission)
② orient the axes to be along/normal those. Set quarter-wave plate before linear polariser (check for max transmission and complete extinction)
③ $\tan\theta = \frac{|E_{y0}|}{|E_{x0}|}$ θ is the angle between vector of the now-linear polarised light ($\perp$ to extinction) and the axes of waveplate. (optic axis used)

Table of Jones Vec. Formalism

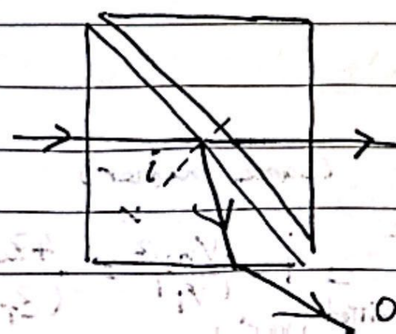
State (orthonormal bases)
$$\begin{cases}|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\ |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\end{cases} \quad \begin{cases}|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\\ |R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\end{cases} \quad \begin{cases}|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}\\ |V\rangle = \begin{pmatrix}0\\1\end{pmatrix}\end{cases}$$

Polariser (projector)
$$\Pi_H = |H\rangle\langle H| = \begin{pmatrix}1&0\\0&0\end{pmatrix} \quad \Pi_V = |V\rangle\langle V| = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$
$$\Pi_L = |L\rangle\langle L| = \frac{1}{2}\begin{pmatrix}1&-i\\i&1\end{pmatrix} \quad \Pi_R = |R\rangle\langle R| = \frac{1}{2}\begin{pmatrix}1&i\\-i&1\end{pmatrix}$$

waveplate (unitary)
$$S_{1/2} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}; \quad S_{1/4} = \begin{pmatrix}1&0\\0&\pm i\end{pmatrix}$$

Phase-shift at boundary (air-to-glass, $n_1 < n_2$)
$r_s < 0$ always $\rightarrow$ always π-phase shift
$r_p < 0$ for $\theta > \theta_B$ undergo phase shift.
(If $\theta < \theta_B$, no phase shift for r_p)
(very close to normal)

Polarising Beam Splitter



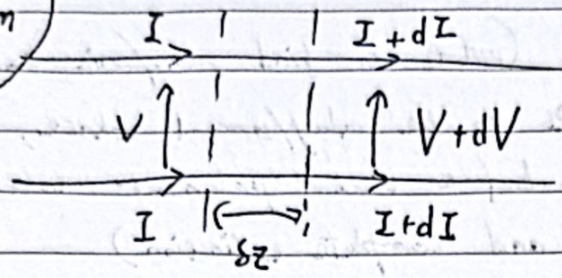
e-ray $\left(\frac{1}{n_o} < \sin i < \frac{1}{n_e}\right)$

o-ray (total internal reflection at 1st boundary)

③ Others (Transmission lines & Fourier, etc.)

Transmission lines

(L, C are per unit length)



① $Q = C\,\delta z\,V$

$$\frac{dQ}{dt} = \left(-\frac{\partial I}{\partial z}\right)\delta z = C\frac{\partial V}{\partial t}\delta z$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

② $\Phi = I\,\delta z\,L$

$$\frac{d\Phi}{dt} = \left(-\frac{\partial V}{\partial z}\right)\delta z = L\frac{\partial I}{\partial t}\delta z$$

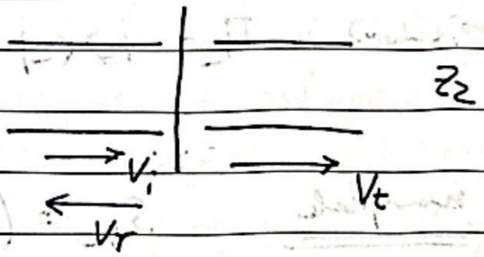
$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

Thus: $\boxed{\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial z^2}}$

$v = \frac{1}{\sqrt{LC}}$ Propagation speed $\qquad Z = \left|\frac{\partial V}{\partial I}\right| = \sqrt{\frac{L}{C}}$ Impedance

EE convention: $\boxed{\propto e^{i(\omega t - kz)}}$

Transmission line joined



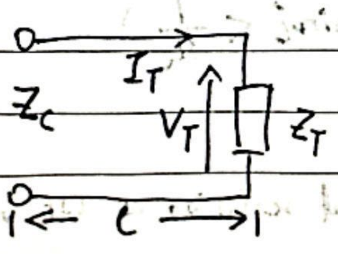
$$V = \begin{cases} V_i e^{-ik_1 x} + V_r e^{ik_1 x} \\ V_t e^{-i(k_2 x)} \end{cases} \qquad (Z \propto k)$$

· Continuity of boundary for $V(x=0)$ & $I(x=0)$ ~~$\frac{\partial}{\partial x}$~~

$$\begin{cases} V_i + V_r = V_t \\ \frac{V_i}{Z_1} - \frac{V_r}{Z_1} = \frac{V_t}{Z_2} \end{cases} \Rightarrow \begin{cases} r = \frac{V_r}{V_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \\ t = \frac{V_t}{V_i} = \frac{2Z_2}{Z_2 + Z_1} \end{cases}$$

Flow of Power $= \frac{V_i^2}{Z_1} = \frac{V_r^2}{Z_1} + \frac{V_t^2}{Z_2} \rightarrow 1 = r^2 + \frac{Z_1}{Z_2}t^2$

Termination by a load



$$\frac{V_r}{V_i} = \frac{Z_T - Z_C}{Z_T + Z_C} \quad \text{(same manner)}$$

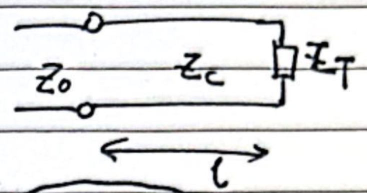
$$P_{\text{transmitted (ratio)}} = 1 - \left|\frac{V_r}{V_i}\right|^2 = \frac{4Z_C Z_T}{(Z_T + Z_C)^2}$$

Define input impedance at $z=-l$

$$Z_{in} \equiv \frac{V(-l)}{I(-l)} = \frac{V_i e^{ikl} + V_r e^{-ikl}}{\frac{V_i}{Z_c} e^{ikl} - \frac{V_r}{Z_c} e^{-ikl}}$$

$$\boxed{Z_{in} = Z_c \left(\frac{Z_T \cos kl + i Z_c \sin kl}{Z_c \cos kl + i Z_T \sin kl} \right)}$$

* Impedance Matching condition: (for a line of Z_0) $\underline{Z_0 = Z_{in}}$



$l = \frac{\lambda}{2}$ $(kl=\pi)$ $\Rightarrow$ $Z_{in} = Z_T$ $\quad$ $l = \frac{\lambda}{4}$ $(kl = \frac{\pi}{2})$ $\boxed{Z_{in} = \frac{Z_c^2}{Z_T}}$

short circuit: $Z_T = 0 \Rightarrow Z_{in} = i Z_c \tan kl$

open circuit: $Z_T \to \infty \Rightarrow Z_{in} = -i Z_c \cot kl$

$$\underline{(Z_{in})_0 (Z_{in})_\infty = Z_c^2}$$

Signals

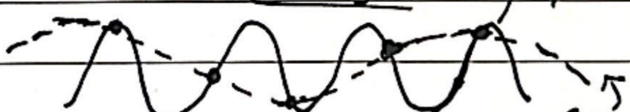
Nyquist Theorem

- If $V(t)$ is band-limited to ω_{max}, then can only be perfectly sampled by $\underline{f_s \geq 2 f_{max}}$ (i.e. $T \leq \frac{\pi}{\omega_{max}}$)

  → Nyquist frequency f_{max}

  Nyquist rate $= 2 f_{max}$

- If $\omega_s < 2\omega_{max}$: aliasing (many possibilities, high-low frq. misrepresented)



Auto-correlation function

ACF: $\gamma(t) = \int f(\tau) f^*(\tau - t)\, d\tau$ (for signal $f(t)$)

F.T. $\rightarrow$ $\tilde{\gamma}(\omega) = \iint f(\tau) f^*(\tau - t) e^{-i\omega t}\, dt\, d\tau = |\tilde{f}(\omega)|^2$

Wiener-Khinchin theorem: FT. of the autocorrelation function is the power spectrum (in frq.).

Johnson noise $\langle V^2 \rangle = 4 k_B T R \Delta f$ (thermal effect)

Shot noise $\langle I^2 \rangle = 2e \langle I \rangle \Delta f$ (non-zero current flow)