② Classical Mechanics (S7)

Lagrangian Mechanics

· Calculus of variations: minimise functional $F[\underline{x}]$ of the path $\underline{x}(t)$.

$$F[\underline{x}] = \int_{t_1}^{t_2} f(\underline{x}, \dot{\underline{x}}, t)\,dt$$

$$\delta F[\underline{x}] = 0 \rightarrow \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial \underline{x}} \cdot \delta \underline{x} + \frac{\partial f}{\partial \dot{\underline{x}}} \cdot \delta \dot{\underline{x}}\right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial \underline{x}} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\underline{x}}}\right) \cdot \delta \underline{x}\, dt = 0$$

→ Euler-Lagrange eqn. $\frac{d}{dt}\frac{\partial f}{\partial \dot{\underline{x}}} - \frac{\partial f}{\partial \underline{x}} = 0$

· In the context of mechanics: Action: $S[\underline{q}(t)]$ (is the functional to be minimised)

$$S = \int_{t_1}^{t_2} L(\underline{q}, \dot{\underline{q}}, t)\,dt$$

$L = T - V$ (to retrieve Newton's 2nd)

Principle of Least Action: Actual path taken by the system is an extremum of S.

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}} - \frac{\partial L}{\partial \underline{q}} = 0 \quad \text{(E-L eqn.)}$$

For generalised-coordinates $\underline{q} = (q_1, q_2, \ldots, q_{3N})$

$i = 1, 2, \ldots, 3N$ (N particles in 3D)

(Configuration space is 3N-dim)

Then for each q_i: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Generalised momenta: $\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$

Generalised forces: $F_j = \frac{\partial L}{\partial q_j}$

Cyclic coordinates: (ignorable) $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$

thus $\frac{\partial L}{\partial \dot{q}_i}$ is conserved.

Normally $L = T(\underline{q}, \dot{\underline{q}}) - V(\underline{q})$, L is motion-invariant for $\underline{L' = \alpha L}$ or $\underline{L' = L + \frac{df}{dt}}$

(leaves E-L same)

General Coordinate system (3D)

$$ds^2 = \sum_{i,j} g_{ij}(q_1, q_2, q_3)\, dq_i dq_j$$

$$\Rightarrow L = \sum_{i,j} \frac{m}{2} \underbrace{g_{ij}(q_1, q_2, q_3)\, \dot{q}_i \dot{q}_j}_{\dot{\rho}_{ij}^2} - V(q_1, q_2, q_3)$$

① spherical polars: (r, θ, ϕ) $\quad L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - V(r, \theta, \phi)$

② cylindrical: (r, ϕ, z) $\quad L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - V(r, \phi, z)$

Rotating coordinates

$$\dot{\underline{r}}_{inertial} = \dot{\underline{r}} + \underline{\omega} \times \underline{r}$$

(in rotating frame) $L = \frac{m}{2} \dot{\underline{r}}_{inertial} \cdot \dot{\underline{r}}_{inertial} - V(\underline{r})$

(use) $|\underline{\omega} \times \underline{r}|^2 = (\underline{\omega}\cdot\underline{\omega})(\underline{r}\cdot\underline{r}) - (\underline{\omega}\cdot\underline{r})^2$

$$\text{E-L:} \quad m\ddot{\underline{r}} = \underbrace{-2m\underline{\omega}\times\dot{\underline{r}}}_{\text{Coriolis force}} \underbrace{- m\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{Centrifugal force}} - \frac{\partial V}{\partial \underline{r}}$$

Rigid Bodies ★

$$m = \sum_i m_i \quad (= \int dm)$$

$\rightarrow \dot{\underline{r}}_i = \dot{\underline{r}}_{CM} + \underline{\Omega} \times (\underline{r}_i - \underline{r}_{CM})$ where $\underline{r}_{CM} = \frac{1}{m}\sum_i m_i \underline{r}_i$

let $\underline{r}_{CM} = 0 \rightarrow \dot{\underline{r}}_i = \dot{\underline{r}}_{CM} + \underline{\Omega} \times \underline{r}_i$

$$\hookrightarrow \boxed{T = \frac{m}{2} \dot{\underline{r}}_{CM} \cdot \dot{\underline{r}}_{CM} + \frac{1}{2} I_{\alpha\beta} \Omega_\alpha \Omega_\beta} \quad ★$$

where $I_{\alpha\beta} = \sum_i m_i (\delta_{\alpha\beta}\, \underline{r}_i \cdot \underline{r}_i - (r_i)_\alpha (r_i)_\beta)$

(Inertia tensor)

Angular momentum $\boxed{M_\alpha = \frac{\partial L}{\partial \Omega_\alpha} = I_{\alpha\beta}\Omega_\beta}$ $(= \sum_i m_i (\underline{r}_i \times \dot{\underline{r}}_i)_\alpha)$

Principle axes: when $I_{\alpha\beta}\Omega_\beta = \lambda \Omega_\alpha$ $\quad (\lambda = I_1, I_2, I_3)$

(can construct diagonal $\hat{I}$) $\quad \downarrow (I_\alpha)$ $\quad$ Principle M.o.I.

Then simply $L = \frac{M}{2}\dot{r}_{CM}\cdot\dot{r}_{CM} + \frac{1}{2}\sum_i I_i \Omega_i^2 - V$

$\downarrow$

i.e. $I_x = \sum_i m_i(y_i^2 + z_i^2)$

(from the tensor formula)

· Consider conservation of $\underline{M}$: $\dot{\underline{M}} = 0$

$M_\alpha = I_\alpha \Omega_\alpha \hat{e}_\alpha$ $\quad$ ~~$\underline{M} = \sum_\alpha I_\alpha \Omega_\alpha \hat{e}_\alpha$ (principal)~~

$\sum_\alpha \left(I_\alpha \frac{d\Omega_\alpha}{dt}\hat{e}_\alpha + I_\alpha\Omega_\alpha \frac{d\hat{e}_\alpha}{dt}\right) = 0$ $\quad$ $\alpha = 1,2,3.$

Use $\left(\frac{d\hat{e}_\alpha}{dt} = \underline{\Omega}\times\hat{e}_\alpha\right)$ ☆

Euler's equations

$$\hookrightarrow \begin{cases} I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2) = 0 \\ I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3) = 0 \\ I_3\dot{\Omega}_3 + \Omega_1\Omega_2(I_2 - I_1) = 0 \end{cases}$$

i.e. Free Tops : ① $I_1 = I_2 = I_3$ (sphere) $\Rightarrow$ $\underline{\Omega_\alpha = \text{const.}}$

② $I_1 = I_2 \neq I_3$ (symmetric top)

$$I_1\dot{\Omega}_1 = \Omega_2\Omega_3(I_1 - I_3)$$
$$I_2\dot{\Omega}_2 = -\Omega_1\Omega_3(I_1 - I_3)$$
$$I_3\dot{\Omega}_3 = 0$$

$$\underline{(\Omega_1, \Omega_2) = \Omega_0(\sin\Omega t, \cos\Omega t)}$$

where $\Omega = \frac{\Omega_3}{I_1}(I_1 - I_3)$

· Direction of spin not constant,
but they precess about $\hat{e}_3$ axis with frq. Ω.

Tennis racket theorem (Asymmetric Top stability)

Consider initially $\Omega_1 = \Omega_0$; $\Omega_2 = \Omega_3 = 0$
with small perturbation $\delta\Omega_1, \delta\Omega_2, \delta\Omega_3$

$$\begin{cases} \delta\dot{\Omega}_1 = 0 \\ \delta\dot{\Omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\Omega_0\,\delta\Omega_3 \\ \delta\dot{\Omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\Omega_0\,\delta\Omega_2 \end{cases}$$

Giving $\delta\ddot{\Omega}_{2,3} = \underbrace{\left[\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\Omega_0\right]}_{A}\delta\Omega_{2,3}$

$$\begin{cases} A>0 \rightarrow \text{unstable (exp. growth)} \\ A<0 \rightarrow \text{oscillatory, stable} \end{cases} \rightarrow \text{If } I_2 < I_1 < I_3 \text{ or } I_3 < I_1 < I_2$$

- Can rotate about axes of smallest or largest but not of intermediate M.O.I.

Euler angles
(i) ϕ about $\hat{z}$ axis
(ii) θ about $\hat{n}$ axis
(iii) ψ about $\hat{e}_3$ axis

$$\underline{\Omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{n} + \dot{\psi}\hat{e}_3$$

$$\Downarrow$$

$$\underline{\Omega} = \underbrace{(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)}_{\Omega_1}\hat{e}_1 + \overbrace{(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)}^{\Omega_2}\hat{e}_2 + \underbrace{(\dot{\phi}\cos\theta + \dot{\psi})}_{\Omega_3}\hat{e}_3$$

Small oscillations

- Around equilibrium where $f_i(q_0^1, \dots, q_0^N) = 0$

$$q^i(t) = q_0^i + \underbrace{\eta^i(t)}_{\text{small perturbations}} \quad (\text{for } i = 1, \dots, N)$$

$$\ddot{\eta}^i(t) \approx \left.\frac{\partial f_i}{\partial q^j}\right|_{q=q_0} \eta^j \quad (\text{by Taylor expansion})$$

More useful
Exponential $\rightarrow$ unstable (imaginary ω)
oscillatory $\rightarrow$ stable (real ω)
(for $e^{i\omega t}$ form) in normal modes.

Noether's Theorem (Lagrangian way)

The flow : $\underline{q} \rightarrow \underline{q} + \frac{d\underline{q}}{d\lambda}\delta\lambda$ (any $\underline{q}$ in config space)
(in configuration space)

If this leaves Hamiltonian Least Action invariant, then generally changes Lagrangian by a total time derivative.

$$\delta L = \frac{\partial L}{\partial \underline{q}}\cdot\delta\underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\delta\dot{\underline{q}} = \frac{dG}{dt}\delta\lambda$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\delta\underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{d}{dt}\delta\underline{q} = \frac{dG}{dt}\delta\lambda$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{d\underline{q}}{d\lambda} - G\right) = 0$$

(where $G = G(\underline{q}, \dot{\underline{q}}, t)$ arbitrary func.)

Conservation of Noether's charge.

i.e. $d\lambda = dt$, $G = L$

$\Rightarrow \quad \underline{\underline{\frac{\partial L}{\partial \dot{\underline{q}}} \cdot \dot{\underline{q}} - L = \text{constant}}}$ (Hamiltonian included in temporal symmetry)

Constraints

- Holonomic constraints: $f_\alpha(\underline{q}, \dot{\underline{q}}, t) = 0$ (can be eliminated from the Lagrangian) $(\alpha = 1, 2, \ldots, n)$

① Modify Lagrangian by: $\boxed{L^* = L + \sum_\alpha \lambda_\alpha f_\alpha}$

② Then apply E-L for L^* (one constraint $\to L + \lambda f$)

- Non-holonomic constraint: $f_a(\underline{q}, \dot{\underline{q}}, t) = 0$ ×

Cannot be written as ⅅ.

Hamiltonian Mechanics

Hamiltonian

$$H = \underline{p}\cdot\dot{\underline{q}} - L \qquad \left(\underline{p} \equiv \frac{\partial L}{\partial \dot{\underline{q}}}\right)$$

$$\rightarrow dH = \dot{\underline{q}}\cdot d\underline{p} - \dot{\underline{p}}\cdot d\underline{q} - \frac{\partial L}{\partial t}dt$$

$$H = H(\underline{p}, \underline{q}, t)$$

Hamilton's equations $\left\{ \begin{array}{l} \frac{\partial H}{\partial \underline{p}} = \dot{\underline{q}} \\ \frac{\partial H}{\partial \underline{q}} = -\dot{\underline{p}} \end{array} \right.$

Also, $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$

$\therefore$ H conserved if L is not explicit on time.

Particle in Potential field: $\quad \underbrace{H = \frac{\underline{p}\cdot\underline{p}}{2m} + V(\underline{q})}$

Rotating frame: $\quad L = \frac{m}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} + m\dot{\underline{r}}\cdot(\underline{\omega}\times\underline{r}) + \frac{m}{2}|\underline{\omega}\times\underline{r}|^2 - V(\underline{r})$

$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = \boxed{m(\dot{\underline{r}} + \underline{\omega}\times\underline{r})} \quad (= \underline{p}_{inertial})$$

$$H = \underline{p}\cdot\dot{\underline{r}} - L = \underline{p}\cdot\left(\frac{\underline{p}}{m} - \underline{\omega}\times\underline{r}\right) - \left(\frac{\underline{p}\cdot\underline{p}}{2m} - V\right)$$

$$H(\underline{p}, \underline{q}) = \frac{\underline{p}\cdot\underline{p}}{2m} + V - \underline{\omega}\cdot\underbrace{(\underline{r}\times\underline{p})}_{\underline{M}} \quad \leftarrow \text{additional term}$$

also written $\hookrightarrow$ $H = \frac{m}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} + V - \underbrace{\frac{m}{2}|\underline{\omega}\times\underline{r}|^2}_{\text{centrifugal potential}}$

Principle of Least Action:

$$\delta S = 0 \quad \text{with} \quad S \equiv \int_{t_1, \underline{q}_1}^{t_2, \underline{q}_2} (\underline{p}\cdot\dot{\underline{q}} - H(\underline{p}, \underline{q}, t))\, dt$$

$$= \int_{t_1, \underline{q}_1}^{t_2, \underline{q}_2} \left(\left\{ \dot{\underline{q}} - \frac{\partial H}{\partial \underline{p}} \right\}\cdot\delta\underline{p} - \left\{ \dot{\underline{p}} + \frac{\partial H}{\partial \underline{q}} \right\}\cdot\delta\underline{q} \right) dt$$

$= 0$, retrieve Hamilton's eqns.

Liouville's theorem

phase space: $\{\underline{p}, \underline{q}\}$

$$\sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = \sum_i \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0$$

$\Rightarrow$ 'Phase velocities' are non-divergent

$$\Delta V = \sum_i \left(\frac{\Delta V}{\delta q_i} \delta q_i + \frac{\Delta V}{\delta p_i} \delta p_i \right)$$

$$\frac{d}{dt}\Delta V = \sum_i \left(\frac{\Delta V}{\delta q_i} \delta \dot{q}_i + \frac{\Delta V}{\delta p_i} \delta \dot{p}_i \right) = \Delta V \sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = 0$$

Corollary: 2n-dim, 'volume' of elements in place space is conserved over time. (for Hamiltonian system)

In a Hamiltonian system: cannot have attractors/repulsors.

(otherwise $\frac{\partial \dot{\theta}}{\partial \theta} + \frac{\partial \dot{p}_\theta}{\partial p_\theta} \neq 0$, i.e.)

Motion in E-M field ☆

write (Maxwell)
$$\begin{cases} \underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t} \\ \underline{B} = \nabla \times \underline{A} \end{cases}$$

(fields as potentials)

$$\begin{cases} \phi = \phi(\underline{r}) & \text{Electric pot.} \\ \underline{A} = \underline{A}(\underline{r}) & \text{Magnetic vec. pot.} \end{cases}$$

Trial Lagrangian: $\mathcal{L} = \frac{m}{2} \dot{\underline{r}} \cdot \dot{\underline{r}} + q(\dot{\underline{r}} \cdot \underline{A} - \phi)$

E-L eqn: $\frac{d}{dt}(m\dot{\underline{r}} + q\underline{A}) - q\nabla(\dot{\underline{r}} \cdot \underline{A} - \phi) = 0$

Note that $\frac{d}{dt}\underline{A} = \frac{\partial \underline{A}}{\partial t} + (\dot{\underline{r}} \cdot \nabla)\underline{A}$

Hence $m\ddot{\underline{r}} = q\left(-\frac{\partial \underline{A}}{\partial t} - (\dot{\underline{r}} \cdot \nabla)\underline{A} - \nabla\phi + \nabla(\dot{\underline{r}} \cdot \underline{A})\right)$

$$= q(\underline{E} + \dot{\underline{r}} \times \underbrace{(\nabla \times \underline{A})}_{\underline{B}}) \rightarrow \text{Lorentz force } \checkmark$$

Gauge invariance

$$\phi \rightarrow \phi - \frac{\partial f}{\partial t}$$
$$\underline{A} \rightarrow \underline{A} + \nabla f$$

Effect on $\mathcal{L}$: $\mathcal{L} + q\left(\frac{\partial f}{\partial t} + \dot{\underline{r}} \cdot \nabla f\right) = \mathcal{L} + \frac{d}{dt}(qf)$

(invariant for adding total time-deriv.)

Conjugate momentum: $\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} + q\underline{A}$

Hamiltonian: $H = (m\dot{\underline{r}} + q\underline{A}) \cdot \dot{\underline{r}} - \left(\frac{m}{2}\dot{\underline{r}} \cdot \dot{\underline{r}} + q(\dot{\underline{r}} \cdot \underline{A} - \phi)\right)$

$$= \frac{m}{2}\dot{\underline{r}} \cdot \dot{\underline{r}} + q\phi$$

$$= \frac{m}{2}\frac{1}{m^2}|\underline{p} - q\underline{A}|^2 + q\phi$$

Thus $\boxed{H = \frac{|\underline{p} - q\underline{A}|^2}{2m} + q\phi}$ defines correctly H under both E & M fields

Poisson brackets

$$\{A, B\} \equiv \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right)$$

Note: $\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right) = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \right)$

$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}}$$

if $\frac{\partial A}{\partial t} = \{A, H\} = 0 \Rightarrow \frac{dA}{dt} = 0$

A is conserved if A is not explicit on time & $\{A, H\} = 0$.

Hamilton's equations: $\begin{cases} \dot{q}_i = \{q_i, H\} \\ \dot{p}_i = \{p_i, H\} \end{cases}$

Property of Poisson brackets: (analogous of commutators)

① Anti-symmetry $\{A, B\} = -\{B, A\}$

② Fundamentals $\{q_i, q_j\} = \{p_i, p_j\} = 0$

$\{q_i, p_j\} = \delta_{ij}$

③ Algebra $\{A, B+C\} = \{A, B\} + \{A, C\}$

$\{AB, C\} = A\{B, C\} + B\{A, C\}$ (Leibniz)

$\frac{\partial}{\partial t}\{A, B\} = \{\frac{\partial A}{\partial t}, B\} + \{A, \frac{\partial B}{\partial t}\}$

④ Jacobi identity

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0$$

⑤ Poisson's theorem $\dot{A} = 0,\ \dot{B} = 0 \Rightarrow \frac{d}{dt}\{A, B\} = 0$

(can be helpful for finding conserved quantities)

Angular momentum $M_i = \varepsilon_{ijk} x_j p_k$

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k$$

(quantum, just multiplies by $i\hbar$)

$(M^2 \equiv \sum_i M_i^2)$

$$\{M^2, M_i\} = 0$$

Revisit Noether's Theorem:

Generally, consider generator of motion $G = G(p, q)$

where $\begin{cases} \frac{\partial G}{\partial p} = \frac{\delta q}{\delta \lambda} \\ \frac{\partial G}{\partial q} = -\frac{\delta p}{\delta \lambda} \end{cases} \Leftrightarrow \delta G = \frac{\delta q}{\delta \lambda} \delta p + \left(-\frac{\delta p}{\delta \lambda}\right) \delta q$

(no explicit time-dependence)

$$\frac{\delta H}{\delta \lambda} = \frac{\partial H}{\partial q} \cdot \frac{\delta q}{\delta \lambda} + \frac{\partial H}{\partial p} \cdot \frac{\delta p}{\delta \lambda}$$

$$= \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = \{H, G\}$$

Thus: $\boxed{\frac{\delta H}{\delta \lambda} = \{H, G\} = -\{G, H\} = -\frac{dG}{dt}}$

$\begin{cases} \text{(I) Transformation generated by } G \text{ leaves } H \text{ invariant } \left(\frac{\delta H}{\delta \lambda} = 0\right) \\ \text{(II) } G \text{ is conserved } \left(\frac{dG}{dt} = 0\right) \end{cases}$

The two statements above are equivalent.

Symmetry $\{H, G\} = 0 \Leftrightarrow$ conservation law $\{G, H\} = 0$

i.e.

① Time-translation: $G = H$

$\begin{cases} \frac{\partial H}{\partial p} = \dot{q} \\ \frac{\partial H}{\partial q} = -\dot{p} \end{cases}$ $\{H, H\} = 0$ (trivial)

H invariant under time-translation, & H is conserved.

② Spatial-translation: $G = \left(\sum_i p_i\right) \cdot \hat{n}$ $\delta \lambda = \delta x$

$$\hat{n} = \frac{\delta q}{\delta x}$$

$$0 = \frac{\delta p}{\delta x}$$

If $\frac{\delta H}{\delta x} = 0 \rightarrow \{H, G\} = 0 \rightarrow \frac{d}{dt}\left(\sum_i p_i \cdot \hat{n}\right) = 0$

(angular mom. conservation in direction of $\hat{n}$)

③ Rotational-translation: $G = \left(\sum_i r_i \times p_i\right) \cdot \hat{n}$ $\delta \lambda = \delta \phi$

(similar reasoning to follow)