Topics in Covariant Electromagnetism

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Introduction

This note is predominantly based on topics covered in the third-year undergraduate Physics course *(B2) Symmetry and Relativity* at the University of Oxford. The author does not claim originality for any of the following content, nor does he seek commercial gains from it, so please respect this and only distribute this document for academic discussion purposes. The purpose of the document is for sharing intellectual knowledge in classical electromagnetism, a fundamental cornerstone of classical field theory. (In fact the sole purpose to write this document is because the crazy author is bored by a lot of tedious things and wants to have some mathematical-theoretical fun!)

The author could not cover the whole covariant formulation of electromagnetism in such a short text, so the focus here is mainly regarding the covariant electromagnetic tensor and the related stress-energy tensor. We proceed by discussing a few interesting properties of these quantities with concise but non-trivial derivations, guided by a few short questions which are adapted from Lecturer's problem sets and past papers examined in the *Oxford Finals*.

The author assumes that the reader has had undergraduate-level introductory courses in electromagnetism, special relativity and linear algebra, to sufficiently understand the topics covered here. Quantum mechanics is not important at all because the whole theory is completely classical (no QFT yet!). However, most of the derivations are set up step-by-step without assumption of too much prior knowledge.

One further note on some notations in this text. *Italic fonts* are used where the author thinks the concept is important and worthwhile checking from other sources, whilst **bold fonts** are key concepts which are central to our discussion.

1 Covariant Formulation

1.1 Lorentz Invariance

This text is not meant to discuss the history of modern physics, but a brief walk is nonetheless meaningful. Particularly, to proceed with the covariant relativistic theory of electromagnetism, we first need to look at **Galilean invariance**, the basis of Newtonian mechanics. Galilean invariance states that *acceleration* is invariant across all (inertial) frames, and time is also the same across all frames, while the space is 3-dimensional *Euclidean*, where all the spatial translations and rotations are independent of time.

One of key characteristics of electromagnetism, particularly Maxwell's equations, is its violation of Galilean invariance, unlike Newton's laws of motion (which are Galilean-invariant). Therefore, to the end of the 19th century there was a hot debate as to whether EM or Galilean invariance is fundamentally wrong.

It turns out that EM and invariance are fine concepts, but Galilean invariance is not universal. Rather, our universe behaves with the concept of **Lorentz invariance**. One needs to introduce the notion of the 4-dimensional *Minkowski spacetime*. All the coordinate transformations are elements of the *Poincaré group* $\mathcal{P}(1,3)$, composed as

$$\mathcal{P}(1,3) = \mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3) \tag{1}$$

where $\mathbb{R}^{1,3}$ describes the set of translations in space (3-components) or/and time (1-component), whilst $\mathrm{O}(1,3)$ is the *Lorentz group* which contains spatial rotations and Lorentz boosts.

For our purposes we mainly concern with the $\mathrm{SO}(1,3)$ and refer to that as the 'Lorentz group', for which all the elements, represented by matrices Λ, will be both orthogonal ($\Lambda^T g \Lambda = g$, where the metric tensor g will be discussed later) and 'special' ($\det(\Lambda) = +1$). The latter condition may seem arbitrary, and it rules out some transformations such as reflections, but it actually allows the application of a powerful tool known as *Lie Algebra*, in deriving the Lorentz transformations explicitly, which is a suitable topic for another day.

These matrices Λ are known as the **Lorentz transformations** which connect space-time coordinates between different inertial frames moving relative to each other at some constant velocities. Assume one has a frame (S') which moves at constant velocity $+v$ in the $\hat{\boldsymbol{x}}$ direction relative to another frame (S), one may explicitly write down the (4×4) Lorentz transformation matrix in the space-time basis $\{ct, x, y, z\}$

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where $\beta = v/c$ and the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$.

If one denotes $\boldsymbol{X} = (ct, x, y, z)^T$ as the space-time coordinates in (S) while $\boldsymbol{X'} = (ct', x', y', z')^T$ as the counterpart in (S'), then one has the relationship

$$\boldsymbol{X'} = \Lambda\boldsymbol{X}$$

where Λ is basically a transformation matrix connecting two sets of coordinates.

1.2 Notations

However, this matrix notation is often not the clearest notation to use. One often needs to introduce the **index notation**, and, alongside with it, the **four vectors**, such as the space-time one X^μ. Then the Lorentz transformations would become $\Lambda^\mu{}_\nu$. The previous relationship thus becomes

$$X^\mu = \Lambda^\mu{}_\nu X^\nu \tag{3}$$

where μ or ν are simply arbitrary indices running through $\{0, 1, 2, 3\}$ completely corresponded with the basis $\{ct, x, y, z\}$.

The *Einstein summation convention* is obeyed where one never sees a repeated upper index (or lower index), but one takes a summation where one upper index meets the same exact lower index. For instance, in (3), ν is the 'dummy index' which is summed over since one lower ν meets an upper ν, but μ is kept on both sides without being contracted. This procedure of contraction and summation convention with well-labeled indices will prove extremely handy in problem-solving, especially for high-dimensional tensor objects, but the essential idea is not much different than applying a matrix operator to a vector.

The special theory of relativity will not be physically important if the only possible case is the spacetime 4-vector X^μ. In fact, all elements of the Lorentz group can be covariantly transformed across frames by the same Lorentz transformation $\Lambda^\mu{}_\nu$. One only has to make 4-vector quantities which will satisfy the rule to be compatible to special relativity. This is the case to 'fix up' some Newtonian quantities, leading to objects such as the 4-velocity $U^\mu = \gamma(c, \boldsymbol{v})^T$. When one tries to find out the 4-force F^μ, things get very nasty. However, electromagnetism (we will see) proves quite elegant because it is relativistically consistent already.

Before we move on, we need to briefly discuss **tensors**. Tensors are mathematical objects which are (sort of) generalisations of matrices. They contain descriptive 'coordinates' in them which vary depending on arbitrary choices of frames, but their physical essence is invariant across frames (one can think of a pen on the table; it doesn't matter in which direction one looks at it, as it is always there). We can thus say (in the context of special relativity) that tensors are objects which essentially stay invariant under Lorentz transformations even though their components transform. General rules for computing high-dimensional tensors can be found in many mathematical texts, but the 2-indices case is most relevant here:

$$(K')^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\rho}K^{\sigma\rho} \tag{4}$$

We also need to introduce the **Minkowski metric** $g_{\mu\nu}$ (diag$(-,+,+,+)$) convention in this text)

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{5}$$

which can transform a *contravariant* (upper-index) 4-vector to a *covariant* (lower-index) one, vice versa. (Note the fact that the Minkowski metric is its own inverse is not universally true for all metrics, so more care is needed for the general case)

$$A_{\mu} = g_{\mu\nu}A^{\nu} \tag{6}$$

One also defines the 4-vector *inner product* by:

$$A_{\mu}B^{\mu} = g_{\mu\nu}A^{\mu}B^{\nu} \tag{7}$$

For example, $X_{\mu}X^{\mu} = -(ct)^2 + \boldsymbol{x}^2$ is the famous space-time invariant interval. This notion of invariance applies to any such inner products which lead to complete contraction of indices, and scalars are certainly frame-invariant as expected.

2 Fields and Energy

2.1 Electromagnetic Tensor

We now follow a series of short questions.

(I) Write down the relationship between electric and magnetic fields and the scalar and vector potentials. Also find the **electromagnetic tensor** $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ in terms of electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$.

In terms of the scalar electric potential ϕ and the magnetic vector potential $\mathbf{A}$, the electric and magnetic fields can be expressed as

$$\begin{cases} \mathbf{E} = -\nabla\phi - \dfrac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases} \tag{8}$$

We also note that

$$\partial^\mu = g^{\mu\nu}\,\partial_\nu = \left(-\frac{1}{c}\partial_t\,, \nabla\right)^T$$

$$A^\mu = \left(\frac{\phi}{c}\,, \mathbf{A}\right)^T$$

Since $F^{\mu\nu} = -F^{\nu\mu}$, the electromagnetic tensor is essentially anti-symmetric, hence all diagonal terms must be 0. One then only cares about off-diagonal terms. Consider F^{0i} and F^{ij} respectively, where i or/and j runs $\{1, 2, 3\}$

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -\frac{1}{c}\partial_t A_i - \partial_i \frac{\phi}{c} = \frac{E_i}{c}$$

$$F^{i0} = -F^{0i} = -\frac{E_i}{c}$$

At first attempt you may think it is wise to plug in some values for i, j to see what happens to F^{ij}, but I will show you some magic:

$$\begin{aligned} F^{ij} &= \partial^i A^j - \partial^j A^i \\ &= \partial_i A_j - \partial_j A_i \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_l A_m \\ &= \epsilon_{kij}\epsilon_{klm}\partial_l A_m \\ &= \epsilon_{ijk} B_k \end{aligned}$$

which mostly follow from rules of computing *Levi-Civita tensor* and *Kronecker delta*, which should be familiar from any introductory linear algebra course. Thus, we assemble $F^{\mu\nu}$ as

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{9}$$

which composes all the relevant information of electromagnetic fields for our purposes. A theoretically-minded reader may also construct a Lagrangian from $F^{\mu\nu}$ in the field theory approach and try to solve the *Euler-Lagrange Equation* to retrieve key physical laws, but it is unnecessary here.

2.2 Maxwell's Equations

(II) The electromagnetic tensor satisfies the equation $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$ where J^μ is the 4-current. Use this equation to obtain 2 of the **Maxwell equations**. Also obtain the other two from $F^{\mu\nu}$.

Given

$$J^\mu = (\rho c, \mathbf{J})^T \tag{10}$$

Consider first $\mu = 0$ case,

$$\begin{aligned}
\partial_\nu F^{0\nu} &= \mu_0 J^0 \\
\partial_0 F^{00} + \partial_i F^{0i} &= \mu_0 J^0 \\
0 + \frac{1}{c}\nabla \cdot \mathbf{E} &= \mu_0 \rho c
\end{aligned}$$

hence obtaining Gauss's Law in electrostatics

$$\nabla \cdot \mathbf{E} = \mu_0 \rho c^2 = \frac{\rho}{\epsilon_0} \tag{11}$$

Now consider the case $\mu = i$ where $i = \{1, 2, 3\}$,

$$\begin{aligned}
\partial_\nu F^{i\nu} &= \mu_0 J^i \\
\partial_0 F^{i0} + \partial_j F^{ij} &= \mu_0 J^i \\
-\frac{1}{c^2}\partial_t E_i + \epsilon_{ijk}\partial_j B_k &= \mu_0 J_i \\
\epsilon_{ijk}\partial_j B_k &= \mu_0 \left(J_i + \epsilon_0 \partial_t E_i\right)
\end{aligned}$$

In vector notation this becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \partial_t \mathbf{E}\right) \tag{12}$$

which is the Ampère-Maxwell Law.

Now it is left as an exercise for the reader to derive the other two Maxwell equations by simply operating ∇ on (8) in clever ways. This is arguably the easiest route of proof. However, we can also apply a particular case of the so-called *Bianchi identity*, taking the mathematical form

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{13}$$

which is an elegant alternative. For example, taking $\{\mu, \nu, \lambda\} = \{0, i, j\}$, we get

$$\begin{aligned}
\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} &= 0 \\
-\partial_t \epsilon_{ijk} B_k - \partial_i E_j + \partial_j E_i &= 0 \\
\epsilon_{mij}(\partial_i E_j - \partial_j E_i) &= -\epsilon_{ijm}\epsilon_{ijk}\partial_t B_k \\
2\epsilon_{mij}\partial_i E_j &= -2\delta_{mk}\partial_t B_k
\end{aligned}$$

$$\epsilon_{mij}\partial_i E_j = -\partial_t B_m$$

which is Faraday's Law of electromagnetic induction

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{14}$$

Or we can just plug in values iteratively like $\{\mu, \nu, \lambda\} = \{1, 2, 3\}$, and we can just read off the entries of (9), to get

$$\begin{aligned} \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} &= 0 \\ \partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \\ \partial_i B_i &= 0 \end{aligned}$$

which is Gauss's Law (again) but in magnetostatics

$$\nabla \cdot \mathbf{B} = 0 \tag{15}$$

This shows that by writing consistent equations related to the electromagnetic tensor $F^{\mu\nu}$, one can retrieve all the Maxwell equations effectively, which not only demonstrates the completeness and usefulness of the electromagnetic tensor, but more over the 'Lorentz covariant' nature of the Maxwell equations which are inherently consistent and equivalent to our relativistically covariant formulation of electromagnetism, one of the most successful classical field theories.

2.3 Stress-Energy Tensor

Before diving into anything energy-related, we also need to compute the covariant tensor $F_{\mu\nu}$. We know that

$$F_{\mu\nu} = g_{\mu\sigma} g_{\nu\rho} F^{\sigma\rho} \tag{16}$$

with the following rules

$$\begin{cases} F_{0i} = -F^{0i} \\ F_{i0} = -F^{i0} \\ F_{ij} = F^{ij} \end{cases} \tag{17}$$

therefore, we can write explicitly

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{18}$$

which may be useful for direct evaluations.

(III) The (symmetric) **stress-energy tensor** of the electromagnetic field may be written as

$$T^{\mu\nu} = -\frac{1}{\mu_0}\left(g^{\alpha\mu}F_{\alpha\rho}F^{\rho\nu} + \frac{1}{4}g^{\nu\mu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{19}$$

Find the top row of this tensor (i.e. $T^{0\nu}$) in terms of $\mathbf{E}$ and $\mathbf{B}$ for a general field, and identify the physical quantities obtained.

The calculations get quite complicated, but it is nonetheless worth going through the full derivation. We first attempt to calculate T^{00}:

$$\begin{aligned} T^{00} &= -\frac{1}{\mu_0}\left(g^{\alpha 0}F_{\alpha\rho}F^{\rho 0} + \frac{1}{4}g^{00}F^{\alpha\beta}F_{\alpha\beta}\right) \\ &= \frac{1}{\mu_0}\left(\delta^{\alpha 0}F_{\alpha\rho}F^{\rho 0} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\right) \end{aligned}$$

One can separate T^{00} into $\alpha = 0$ and $\alpha = \{1,2,3\}$ cases separately for clarity.

$$\begin{aligned} T^{00}|_{\alpha=0} &= \frac{1}{\mu_0}\left(F_{0\rho}F^{\rho 0} + \frac{1}{4}F^{0\beta}F_{0\beta}\right) \\ &= \frac{1}{\mu_0}\left(F_{0\rho}F^{\rho 0} - \frac{1}{4}F^{\beta 0}F_{0\beta}\right) \\ &= \frac{3}{4\mu_0}F_{0\rho}F^{\rho 0} \\ &= \frac{3}{4\mu_0}\left(\frac{\mathbf{E}}{c}\right)^2 \\ &= \frac{3}{4}\epsilon_0\mathbf{E}^2 \end{aligned}$$

and

$$\begin{aligned} T^{00}|_{\alpha=1,2,3} &= \frac{1}{\mu_0}\left(0 + \frac{1}{4}F^{i\beta}F_{i\beta}\right) \\ &= \frac{1}{4\mu_0}F^{i\beta}F_{i\beta} \\ &= \frac{1}{4\mu_0}\left(F^{i0}F_{i0} + F^{ij}F_{ij}\right) \\ &= \frac{1}{4\mu_0}\left(-\frac{\mathbf{E}^2}{c^2} + 2\mathbf{B}^2\right) \\ &= -\frac{1}{4}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \end{aligned}$$

thus

$$T^{00} = \frac{3}{4}\epsilon_0\mathbf{E}^2 + \left(-\frac{1}{4}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) = \frac{1}{2}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$$

or simply

$$T^{00} = \mathcal{E} \tag{20}$$

where $\mathcal{E} \equiv \epsilon_0 \mathbf{E}^2/2 + \mathbf{B}^2/(2\mu_0)$ is the electromagnetic energy density in vacuum.

Now evaluate T^{0j}, where $j = \{1, 2, 3\}$

$$\begin{aligned}
T^{0j} &= -\frac{1}{\mu_0}\left(g^{\alpha 0} F_{\alpha\rho} F^{\rho j} + \frac{1}{4} g^{j0} F^{\alpha\beta} F_{\alpha\beta}\right) \\
&= \frac{1}{\mu_0} \delta^{\alpha 0} F_{\alpha\rho} F^{\rho j} \\
&= \frac{1}{\mu_0} F_{0\rho} F^{\rho j} \\
&= \frac{1}{\mu_0} \left(0 + F_{0i} F^{ij}\right) \\
&= \frac{1}{\mu_0} \left(-\frac{E_i}{c}\right) \epsilon_{ijk} B_k \\
&= \frac{1}{\mu_0 c} \epsilon_{jik} E_i B_k
\end{aligned}$$

which is simply

$$T^{0j} = \frac{S_j}{c} \tag{21}$$

where S_j is the $j-$component of the *Poynting vector* $\mathbf{S}$ defined by $\mathbf{S} \equiv (\mathbf{E} \times \mathbf{B})/\mu_0$ which describes the energy/power flux of the electromagnetic field. By the symmetric nature of the stress-energy tensor defined here, $T^{j0} = T^{0j} = S_j/c$ too. The other leftover elements are given as

$$T^{ij} = -\sigma_{ij} \tag{22}$$

which are the stress components exerted by the electromagnetic field, which is rather nasty and not to be systematically discussed further here.

2.4 Energy Conservation

> **(IV)** Find the $\mu = 0$ component of $\partial_\nu T^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, and state how it is related to the current density $\mathbf{J}$.

Consider

$$\partial_\nu T^{0\nu} = \partial_0 T^{00} + \partial_i T^{0i} = \frac{1}{c}\left(\partial_t \mathcal{E} + \nabla \cdot \mathbf{S}\right) \tag{23}$$

We need to revise Maxwell equations to find out the relation to $\mathbf{J}$ specifically. Consider the Ampère-Maxwell Law(12), dotting it with $\mathbf{E}$ to get

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \left(\mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mathbf{E} \cdot \partial_t \mathbf{E}\right)$$

$$\frac{1}{\mu_0}\left[-\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{E})\right] = \left(\mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mathbf{E} \cdot \partial_t \mathbf{E}\right)$$

$$
\begin{aligned}
-\nabla \cdot \mathbf{S} - \frac{1}{\mu_0}\mathbf{B}\cdot\partial_t\mathbf{B} - \epsilon_0\mathbf{E}\cdot\partial_t\mathbf{E} &= \mathbf{E}\cdot\mathbf{J} \\
-\nabla \cdot \mathbf{S} - \partial_t\left(\frac{1}{2}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) &= \mathbf{E}\cdot\mathbf{J} \\
-(\nabla\cdot\mathbf{S} + \partial_t\mathcal{E}) &= \mathbf{E}\cdot\mathbf{J}
\end{aligned}
$$

Thus one can finally relate (23) with $\mathbf{J}$ as

$$
\partial_\nu T^{0\nu} = \frac{1}{c}\left(\partial_t\mathcal{E} + \nabla\cdot\mathbf{S}\right) = -\frac{1}{c}\mathbf{E}\cdot\mathbf{J} \tag{24}
$$

which is in fact the **energy conservation** equation for the energy transport of electromagnetic fields in vacuum.

2.5 Example: A Moving Capacitor

(V) A parallel-plate capacitor has its plates parallel to the xz plane and moves relative to the laboratory in the x direction at speed v. Let E be the electric field between the plates in the rest frame of the capacitor. Write down the electromagnetic tensor for this field in the rest frame, and hence obtain the stress–energy tensor, first in the rest frame and then in the laboratory. Hence find the Poynting vector of the field observed in the laboratory.

Firstly consider the rest frame of the capacitor where it is not moving, since the capacitor (ideal, edge-fields neglected) is placed parallel to the xz plane, the electric field must be $\mathbf{E} = E\hat{\mathbf{y}}$, without any presence of $\mathbf{B}$ at all. So in the rest frame,

$$
F^{\mu\nu} = \begin{pmatrix} 0 & 0 & E/c & 0 \\ 0 & 0 & 0 & 0 \\ -E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{25}
$$

First consider the non-stress elements, $T^{\nu 0}$ and $T^{0\nu}$. We note

$$
\begin{aligned}
T^{00} &= \mathcal{E} = \frac{1}{2}\epsilon_0 E^2 \\
T^{0j} &= T^{j0} = 0
\end{aligned}
$$

where the last line is due to the fact that $\mathbf{B} = 0$ everywhere.

Now we must worry about the nasty stress components σ_{ij} which are the T^{ij} terms essentially. To greatly simplify problems, we first will show that off-diagonal terms all vanish. Let $i \neq j$, then

$$
\begin{aligned}
T^{ij} &= -\frac{1}{\mu_0}\left(g^{\alpha i}F_{\alpha\rho}F^{\rho j} + \frac{1}{4}g^{ji}F^{\alpha\beta}F_{\alpha\beta}\right) \\
&= -\frac{1}{\mu_0}\left(\delta^{\alpha i}F_{\alpha\rho}F^{\rho j} + 0\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\mu_0} F_{i\rho} F^{\rho j} \\
&= -\frac{1}{\mu_0} F_{i0} F^{0j}
\end{aligned}
$$

We note that i, j cannot both be 2, thus it is impossible to have non-zero terms surviving, hence,

$$T^{ij} = 0 \ (i \neq j) \tag{26}$$

Now we are left with diagonal terms, consider

$$
\begin{aligned}
T^{ii} &= -\frac{1}{\mu_0} \left(g^{\alpha i} F_{\alpha\rho} F^{\rho i} + \frac{1}{4} g^{ii} F^{\alpha\beta} F_{\alpha\beta} \right) \\
&= -\frac{1}{\mu_0} \left(\delta^{\alpha i} F_{\alpha\rho} F^{\rho i} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \right) \\
&= -\frac{1}{\mu_0} \left(F_{i\rho} F^{\rho i} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \right) \\
&= -\frac{1}{\mu_0} \left(F_{i0} F^{0i} + \frac{1}{4} \left(F^{02} F_{02} + F^{20} F_{20} \right) \right) \\
&= -\frac{1}{\mu_0} \left(\delta_{2i} E^2/c - \frac{1}{2} E^2/c \right) \\
&= \mathcal{E}(1 - 2\delta_{2i})
\end{aligned}
$$

For $i = 1, 3$, we have

$$T^{11} = T^{33} = \mathcal{E} \tag{27}$$

The special case is where $i = 2$, where the sign is flipped:

$$T^{22} = -\mathcal{E} \tag{28}$$

Thus we assemble the stress-energy tensor for our xz capacitor as

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & & & \\ & \mathcal{E} & & \\ & & -\mathcal{E} & \\ & & & \mathcal{E} \end{pmatrix} \tag{29}$$

All these are in the rest frame. Transforming back to the lab frame, we apply the tensor transformation rule (4), where

$$(T')^{\mu\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\rho} T^{\sigma\rho} \tag{30}$$

In matrix form, considering the unitary nature of Lorentz transformation (2), this becomes

$$T' = \Lambda T \Lambda \tag{31}$$

Hence, explicitly

$$(T')^{\mu\nu} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E} & & & \\ & \mathcal{E} & & \\ & & -\mathcal{E} & \\ & & & \mathcal{E} \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{32}$$

where the '+' sign of $\beta\gamma$ is because of transforming back from the rest frame to the lab frame.

Evaluating all the matrix entries of $(T')^{\mu\nu}$ is tedious and unnecessary. Since we mostly care about energy transport, we try evaluate $(T')^{0i}$

$$\begin{aligned} (T')^{0i} &= \Lambda^0{}_\rho \Lambda^i{}_\sigma T^{\rho\sigma} \\ &= \Lambda^0{}_0 \Lambda^i{}_0 T^{00} + \Lambda^0{}_1 \Lambda^i{}_1 T^{11} \\ &= \beta\gamma^2 \delta_{i1} \mathcal{E} + \beta\gamma^2 \delta_{i1} \mathcal{E} \\ &= 2\delta_{i1} \beta\gamma^2 \mathcal{E} \end{aligned}$$

Hence, from the condition containing Kronecker delta, only $i = 1$ contributes

$$(T')^{01} = S_1/c = 2\beta\gamma^2\mathcal{E} \tag{33}$$

Hence the Poynting vector

$$\mathbf{S} = \gamma^2 v \epsilon_0 E^2 \hat{\mathbf{x}} \tag{34}$$

So the propagation of electrostatic energy stored in the capacitor is in the $\hat{\mathbf{x}}$ direction as it moves.

(VI) Repeat the calculation for a capacitor moving in the same way whose plates are parallel to the yz plane.

Much of the procedure is unchanged, essentially now $\mathbf{E} = E\hat{\mathbf{x}}$, so one can eventually show that the '$-$' sign in $T^{\mu\nu}$ (rest frame) is simply swapped $2 \to 1$, explicitly

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & & & \\ & -\mathcal{E} & & \\ & & \mathcal{E} & \\ & & & \mathcal{E} \end{pmatrix} \tag{35}$$

with the same Λ transforming between frames, this leads to

$$(T')^{0i} = \beta\gamma^2\delta_{i1}\mathcal{E} - \beta\gamma^2\delta_{i1}\mathcal{E} = 0 \tag{36}$$

which implies $\mathbf{S} = 0$, no spatial energy transport occurs at all. This means the motion of capacitors, parallel to $\mathbf{E}$, does not affect the electrostatic energy density in space.

3 Angular Momentum in EM Field

Previously we have only worked with tensors up to rank-2, now we will look at a rank-3 tensor, the **angular momentum tensor** denoted by $M^{\alpha\beta\gamma}$. Similar to the electromagnetic tensor $F^{\mu\nu}$ or the stress-energy tensor $T^{\mu\nu}$, $M^{\alpha\beta\gamma}$ is associated with key physical laws retained within it to be retrieved properly.

3.1 Angular Momentum Tensor

The *angular momentum density* of the field $\mathbf{L}$ can be defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{g}. \tag{37}$$

where the *momentum density* of the electromagnetic field is given by $\mathbf{g} = \mathbf{S}/c^2$. This is related to the third-rank angular momentum tensor:

$$M^{\alpha\beta\gamma} \equiv X^\gamma T^{\alpha\beta} - X^\beta T^{\alpha\gamma}, \tag{38}$$

where $T^{\alpha\beta}$ is the symmetric stress-energy tensor.

Up to this stage a sensible reader will realise that it becomes increasingly infeasible to treat tensors as 'matrices' or 'vectors' since this is essentially 'cubic' in scale, making it impossible to explicitly write down a planar representation of entries (although analogues or attempts of visualisation are very much possible). We need to fully trust the mathematical framework of tensor algebra and only retrieve the 'vector-like' form at the very end of proof.

(VII) Show that $\partial_\alpha M^{\alpha\beta\gamma} = 0$. *Hint:* use the fact that $T^{\alpha\beta}$ is symmetric and that $\partial_\mu T^{\mu\nu} = 0$.

We start by operating on the angular momentum tensor (38), so

$$\begin{aligned}
\partial_\alpha M^{\alpha\beta\gamma} &= \partial_\alpha (X^\gamma T^{\alpha\beta} - X^\beta T^{\alpha\gamma}) \\
&= (\partial_\alpha X^\gamma) T^{\alpha\beta} + X^\gamma \partial_\alpha T^{\alpha\beta} - (\partial_\alpha X^\beta) T^{\alpha\gamma} - X^\beta \partial_\alpha T^{\alpha\gamma} \\
&= \delta^\gamma_\alpha T^{\alpha\beta} + 0 - \delta^\beta_\alpha T^{\alpha\gamma} - 0 \\
&= T^{\gamma\beta} - T^{\beta\gamma} \\
&= 0
\end{aligned}$$

which means structural similarity between $M^{\alpha\beta\gamma}$ and $T^{\mu\nu}$, which should not come as a surprise since the former is defined by the latter.

3.2 Conservation Laws

A few conservation laws relating angular momentum, momentum and energy can be obtained from $M^{\alpha\beta\gamma}$ by suitable choices of contraction. Recall that the stress on a surface with a normal in the

k-direction is given by

$$\mathbf{T}^{(k)} = -\sigma_{ik}\hat{\mathbf{x}}_i - \sigma_{jk}\hat{\mathbf{x}}_j - \sigma_{kk}\hat{\mathbf{x}}_k, \tag{39}$$

where the stress-tensor components σ_{ij} are symmetric.

(VIII) Rewrite the equation

$$\partial_\alpha M^{\alpha ij} = 0, \qquad \text{where } i, j = \{1, 2, 3\}$$

in terms of $\mathbf{L}$ and the stress exerted by the electromagnetic field.

There is some rather nasty calculation, but we will work through it to good ends. Consider

$$\partial_0 M^{0ij} + \partial_k M^{kij} = 0 \tag{40}$$

Recall the elements $T^{0j} = S^j/c = cg^j$ (upper indices for consistency even though the components are identical either way) as given in (21), then we have

$$\begin{aligned}
\partial_0 M^{0ij} &= \frac{1}{c}\partial_t(X^j T^{0i} - X^i T^{0j}) \\
&= \partial_t(x^j g^i - x^i g^j) \\
&= \partial_t(\delta^j_\alpha \delta^i_\beta - \delta^i_\alpha \delta^j_\beta) x^\alpha g^\beta \\
&= \partial_t \epsilon^{kji}\epsilon_{k\alpha\beta} x^\alpha g^\beta
\end{aligned}$$

where we recognise (37), so

$$\partial_0 M^{0ij} = \epsilon^{kji}\partial_t L_k \tag{41}$$

And

$$\begin{aligned}
\partial_k M^{kij} &= \partial_k(X^j T^{ki} - X^i T^{kj}) \\
&= \partial_k(x^j T^{ki} - x^i T^{kj}) \\
&= \partial_k(\delta^j_\alpha \delta^i_\beta - \delta^i_\alpha \delta^j_\beta) x^\alpha T^{k\beta} \\
&= \partial_k \epsilon^{mji}\epsilon_{m\alpha\beta} x^\alpha T^{k\beta} \\
&= \epsilon^{kji}\partial_m \epsilon_{k\alpha\beta} x^\alpha T^{m\beta}
\end{aligned}$$

where in the last line we swapped indices m and k, leading to

$$\partial_k M^{kij} = \epsilon^{kij}(\nabla \cdot (\mathbf{r} \times \mathbb{T}))_k \tag{42}$$

Eventually we get

$$\partial_t \mathbf{L} + \nabla \cdot (\mathbf{r} \times \mathbb{T}) = 0 \tag{43}$$

which is a conservation law for the rate of change of angular momentum density. This looks weird indeed because the second term of (43) is not an ordinary divergence in 3d vector calculus. Rather, this implies a contraction of the rank-2 tensor formed by the 'vector-product' (also not in the ordinary sense) of $\mathbf{r}$ (the spatial 3-displacement vector) and $\mathbb{T}$ (the rank-2 stress-energy tensor).

If you think this does not help explain much more then you are probably right, and I find that coming back to the index notation might be more illuminating than struggling to understand the 'vector form'. The author is immensely grateful if any reader can find any alternative means of description.

(IX) Write the equation

$$\partial_\alpha M^{\alpha 0i} = 0, \qquad i = \{1, 2, 3\}$$

and show that it leads to

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{c^2\mathbf{G}}{E} \tag{44}$$

where $\mathbf{R}$ is the coordinate of the centre of mass of the electromagnetic field, defined by

$$\mathbf{R}\int_{\mathcal{V}} \mathcal{E}\,\mathrm{d}V = \int_{\mathcal{V}} \mathbf{r}\,\mathcal{E}\,\mathrm{d}V \tag{45}$$

where $\mathcal{E}$ is the electromagnetic energy density, and E and $\mathbf{G}$ are the total electromagnetic energy and momentum, respectively.

Again we note that

$$\partial_\alpha M^{\alpha 0i} = \partial_0 M^{00i} + \partial_j M^{j0i} = 0 \tag{46}$$

where

$$\begin{aligned}
\partial_0 M^{00i} &= \frac{1}{c}\partial_t(X^i T^{00} - X^0 T^{0i}) \\
&= \frac{1}{c}\partial_t(x^i\mathcal{E} - c^2 g^i t) \\
&= \frac{1}{c}\left(\partial_t(x^i\mathcal{E}) - c^2 g^i\right)
\end{aligned}$$

and

$$\partial_j M^{j0i} = \partial_j(X^i T^{j0} - X^0 T^{ji})$$

which is actually a special case of (42), a divergence of a finite quantity.

To proceed, we integrate both terms of (46) with a volume integral characterised by arbitrary volume $\mathcal{V}$,

$$\int_{\mathcal{V}} \frac{1}{c}\left(\partial_t(x^i\mathcal{E}) - c^2 g^i\right)\mathrm{d}V + \int_{\mathcal{V}} \left(\partial_j(X^i T^{j0} - X^0 T^{ji})\right)\mathrm{d}V = 0 \tag{47}$$

where we can eliminate the second term by applying the generalised *divergence theorem* which turns this volume integral with integrand $\nabla\cdot\mathbf{F}\,\mathrm{d}V$ into a closed surface integral $\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$. Since one can choose the surface to encompass an arbitrarily large volume $\to\infty$ (far away), whereby the T^{j0} and

T^{ji} contributions will vanish on that ∞ surface. The above discussion leads to only the first term surviving:

$$\int_{\mathcal{V}} \left(\partial_t(x^i\mathcal{E}) - c^2 g^i\right) \mathrm{d}V = 0 \tag{48}$$

which, in vector form, can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathbf{r}\,\mathcal{E}\,\mathrm{d}V = c^2\int_{\mathcal{V}} \mathbf{g}\,\mathrm{d}V \tag{49}$$

Now using (45),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}\int_{\mathcal{V}} \mathcal{E}\,\mathrm{d}V\right) = c^2\int_{\mathcal{V}} \mathbf{g}\,\mathrm{d}V$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}E\right) = c^2\mathbf{G}$$
$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{c^2\mathbf{G}}{E}$$

which is (44) shown finally. This tells us that the velocity of the centre of mass is proportional to the ratio of the total electromagnetic momentum to the total electromagnetic energy, actually retrieving an equation of motion.

4 References and Recommended Texts

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