Quant Interview

August 2024

1. Two people are tossing a coin. The first person wins if they get "HHT" (two heads followed by a tail), and the second person wins if they get "HTH" (head, tail, head). What are the probabilities of each person winning, and why? (6 times)

Answer: Using Markov Chain:

Define the states:

S: initial states

H: Heads

HH: heads followed by heads

HT: heads followed by tail

Now let $P(S), P(H), P(HH), P(HT)$ define the probability of player winning from the different states, we have the following:

$$P(S) = 0.5 * P(H) + 0.5 * P(S)$$

$$P(H) = 0.5 * P(HH) + 0.5 * P(HT)$$

$$P(HH) = 0.5 * P(HH) + 1$$

$$P(HT) = 0.5 * 0 + 0.5 * P(S)$$

solve the equation, we get $P(S) = 2/3$. Therefore, the probability of the first player win is 2/3, and the probability of second win is 1/3.

For expected number need to sucess, using law of total expectation. Set partition of the event space, for example, for player one, it needs to reach HHT, the sample space will be:

$\{T, HT, HHT, HHH\}$

2. Probability of Both Travels Being International Two trips occurred last year, and at least one of them was international in December. What is the probability that both trips were international? (4 times)

Already known that one travel is an international travel happened in December. Derive all combinations:

All possible cases:

One international December travel, the other one:

Any travel in any cases:

11*2*2=44 (11 months, 2 travel type, two orders).

All in December:

One international/One domestic: 2 (different orders)

Both international: 1.(The only cases).

Counting both are international:

11*2(The given December travel can be the first time or the second time)+1(both in December).

In total:$\frac{23}{47}$.

3. X **is uniformly** $[0, 100]$**. Now take one from two envelopes with** X **and** $2X$**. You have the chance to change the envelop, please give the strategy to maximize your number.** (4 times) Answer:

Define the events:

Let Y be the numbers that were observed from the envelope

$A1$: observed Y is X.

$A2$: observed Y is $2X$.

Using Bayesian theorem to calculate the two probabilities:

$$P(A1|Y) = \frac{P(Y|A1) * P(A1)}{P(Y)}$$

$$P(A2|Y) = \frac{P(Y|A2) * P(A2)}{P(Y)}$$

$P(A2) = P(A1) = 0.5$

Now calculate the PDF of X. When $Y \in [0, 100]$, it can come from both envelopes.

$Y = X$, the density function is $\frac{1}{2} \times \frac{1}{100} = \frac{1}{200}$.

$Y = 2X, X = \frac{Y}{2}$. Now the density is $\frac{1}{2} \times \frac{1}{100} \times \frac{1}{2} = \frac{1}{400}$(times the Jacobian $\frac{dX}{dY} = \frac{1}{2}$)

Therefore,

$$f_Y(y) = \frac{1}{200} + \frac{1}{400} = \frac{3}{400}$$

When $Y \in (100, 200]$It can only come from the second envelop, $Y = 2X$, the density is:

$$f_Y(y) = \frac{1}{2} \times \frac{1}{100} \times \frac{1}{2} = \frac{1}{400}$$

Therefore, we have:

$$f_Y(y) = \begin{cases} \frac{3}{400}, & \text{when } 0 \leq y \leq 100 \\ \frac{1}{400}, & \text{when } 100 < y \leq 200 \\ 0, & \text{other cases} \end{cases}$$

For $y \in [0, 100]$:

$$P(Y = X \mid Y = y) = \frac{P(Y = y \mid Y = X) \cdot P(Y = X)}{f_Y(y)} = \frac{\frac{1}{100} \cdot \frac{1}{2}}{\frac{3}{400}} = \frac{\frac{1}{200}}{\frac{3}{400}} = \frac{2}{3}$$

$$P(Y = 2X \mid Y = y) = \frac{P(Y = y \mid Y = 2X) \cdot P(Y = 2X)}{f_Y(y)} = \frac{\frac{1}{100} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{400}} = \frac{\frac{1}{400}}{\frac{3}{400}} = \frac{1}{3}$$

For $y \in (100, 200]$: It can only come from the second envelope, $Y = 2X$,therefore:

$$P(Y = 2X \mid Y = y) = 1$$

$$P(Y = X \mid Y = y) = 0$$

- **If choosing to switch:**
  - **When** $y \in [0, 100]$**:**

    $$E[\text{switch} \mid Y = y] = P(Y = X \mid Y = y) \times 2y + P(Y = 2X \mid Y = y) \times \frac{y}{2} = \frac{4y}{3} + \frac{y}{6} = \frac{9y}{6} = 1.5y$$

  - **When** $y \in (100, 200]$**:**

    $$E[\text{switch} \mid Y = y] = P(Y = 2X \mid Y = y) \times \frac{y}{2} = 1 \times \frac{y}{2} = \frac{y}{2}$$

- **If choosing not to switch:**
  - Regardless of y, the expected value is y.

- **When** $y \in [0, 100]$**:**
  - **Switch:** $E = 1.5y$
  - **Don't switch:** $E = y$

  Since $1.5y > y$, in this case, **switching** increases the expected value.

- **When** $y \in (100, 200]$**:**
  - **Switch:** $E = \frac{y}{2}$
  - **Don't switch:** $E = y$

  Since $\frac{y}{2} < y$, in this case, **not switching** maintains a higher expected value.

Based on the above analysis, the **optimal strategy** is:

- **If the amount in the chosen envelope** $Y \leq 100$**:**
  - **Switch** to the other envelope, as the expected value after switching $1.5Y > Y$.

- **If the amount in the chosen envelope** $Y > 100$**:**
  - **Don't switch**, as the expected value after switching $\frac{Y}{2} < Y$.

4. Suppose you and I are playing a game, and it could be any game. We have bet 10 dollars which would go to the winner. At some point, I offer to double the bet to 20 dollars. If you accept, the game continues with the new bet. If you refuse. you lose the game, along with the original ten dollars. What is the minimum probability of winning the game that you would need to accept the increased bet?

Refuse: Lose 10 dollar: -10.

Accept: probability p of win 20 or loss 20, expected value:

$$E(X) = 20p - 20(1-p) = 40p - 20$$

To accept, $40p - 20 > -10$, $p > 0.25$.

5. Pick three numbers from 1 to 20, what is the probability that one of them is the average of the other two?

To solve this problem, we need to compute all combinations of the numbers:

$$C(20,3) = \frac{20!}{3!17!} = \frac{20 * 19 * 18}{3 * 2 * 1} = 1140$$

Then, we count how many combinations fulfill those conditions:

Assume a, b, c

$$b = \frac{a+c}{2} \quad \Rightarrow \quad a + c = 2b$$

In this case, a and c must be both even or odd numbers, to make sure that b is an integer, also $a < b < c$.

In first case, $1 < b < 11$:

If $b = 2$, we have $(1, 2, 3)$

If $b = 3$, we have $(1/2, 3, 5/4)$

If $b = 4$, we have $(1/2/3, 4, 7/6/5)$

So for every value of b, there will be in total $(b-1)$ different combinations, the total values are

$$\sum_{i=2}^{10} (b_i - 1) = 45$$

In the second case: $10 < b < 20$, we need to make sure that $c < 20$. For example:

If $b = 19$, we have $(18, 19, 20)$ 1 combinations If $b = 18$, we have $(16/17, 18, 20/19)$ 2 combinations: There will be $20 - b$ combinations for different b values. In total we have:

$$\sum_{i=11}^{20} (20 - b_i) = 200 - (11 + \cdots + 20) = 200 - (30 * 4 + 20 + 15) = 45$$

In total: 90 combinations.

Another simple idea:

$$\frac{C(10,2)+C(10,2)}{C(20,3)}$$

6. There are 99 fair coins and one biased coin with two heads. You pick one randomly and flip it 7 times. You get 7 heads. What is the probability that you pick the biased coin given you got 7 heads?

Very easy, conditional Bayesian probability:

$$P(biased|7) = \frac{P(7|biased)P(Biased)}{P(7)}$$

$$= \frac{P(7|biased)P(Biased)}{P(7|biased)P(Biased) + P(7|unbiased)P(unbiased)}$$

$$= \frac{1 * 0.01}{1 * 0.01 + 0.5^7 * 0.99}$$

$$= 0.5638$$

2. A white cube is painted green on the outside. Cut it into three equal parts along each dimension (like a Rubik's cube). Dissemble the 27 cubes and roll them as dice. Now we found a cube sitting on the ground of which the 5 visible faces are all white, but we don't know the color of the bottom face. What's the probability that this cube is the one that was at the center before dissembling.

Answer: Using Bayesian theorem.

$$P(Cencter|FiveWhite) = \frac{P(FiveWhite|Cencter) * P(Center)}{P(FiveWhite)}$$

Here we need to notice that in order to make sure the conditional probability holds, the two events should be non-exclusive. Therefore, the probability of center should not be $\frac{1}{7}$ because the sample space is not the 27 dices, but only the dices that are in the center and the center dices in every front.

Therefore, the probability can be calculated as follows:

$P(Center) = \frac{1}{7}$

$P(FiveWhite|Center) = 1$

$P(FiveWhite) = P(FiveWhite|Cencter) * P(Center) + P(FiveWhite|NotCencter) * P(NotCenter) = 1 * \frac{1}{7} + \frac{1}{6}\frac{6}{7} = \frac{2}{7}$

The total probability is then $\frac{1}{2}$.

3. Given a standard deck of cards, you draw one card from the top of a shuffled deck. How much would you bet that the next card drawn will be higher than the card you just drew? Follow-up questions include whether the size of the bet and the order matter.

Answer:

Total combinations: 52*51 = 2652

Define A the largest, then assume the first card is r, then the cases that the second card is larger can be counted by: $4 * (13 - r + 1) = (14 - r) * 4$.

So in total we have:

$$\sum_{i=2}^{14} = (14 - r_i) * 4 = 318$$

Because we have in total 4 different colors, all possible combinations will be 318*4 = 1248.

4. Five friends walk into a restaurant. Each has a different birthday. There is a round table with five seats. What is the probability that they will be seated in the correct age order, either clockwise or counterclockwise?

Answer:

Since this is a round table, the total different combinations of all possible cases would be $(n-1)!$. Here we have $n = 5$, the total combinations will be 24.

If they seat as the correct order, then the total probability will be $\frac{2}{24} = \frac{1}{12}$. Clockwise and counterclockwise two cases.

5. You flip four fair coins. If all four coins show heads in a row, I give you 10. Otherwise, you pay me 1. Should you play this game?

Answer: simple expected value. Binomial distribution.

$P(X = 4|n = 4, p = 0.5) = 0.0625$.

Expected value: 0.0625*10+(-1)*(1-0.0625) = 0.5625.

Since the expected value is smaller than 1, I will not play.

8. In a population: 50% are non-smokers, 20% are heavy smokers, and 30% are light smokers. Heavy smokers are twice as likely to die as light smokers, and light smokers are twice as likely to die as non-smokers. What is the probability that a person who has died was a heavy smoker?

Answer: Very simple Bayesian theorem, answer = 8/19

10. You own a piece of land. There is a 10% chance it contains oil worth 1 million, a 30% chance it contains cows worth 500,000, and if it contains nothing, it is worth 200,000. What is the fair value of the land?

Expected value: 370,000

11. Two points are uniformly distributed around the circumference of a unit circle. What is the expected value of the distance (chord length) between them?

Let the angle be θ and the length be L. The θ is uniformly distributed on $[0, 2\pi]$. Ina unit circle, $L = 2sin(\theta/2)$, because $L = 2Rsin(\theta/2)$ and here $R = 1$.

Here we need to calculate the expected value, which is:

$$E[L] = \frac{1}{2\pi}\int_0^{2\pi} 2\sin(\frac{\theta}{2})d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin(\frac{\theta}{2})d\theta$$

$$= \frac{1}{\pi}[-2\cos(\frac{\theta}{2})]_0^{2\pi} \; = \frac{1}{\pi}[-2\cos(\pi) - (-2\cos(0))] \; = \frac{1}{\pi}[2 - (-2)] \; = \frac{4}{\pi}$$

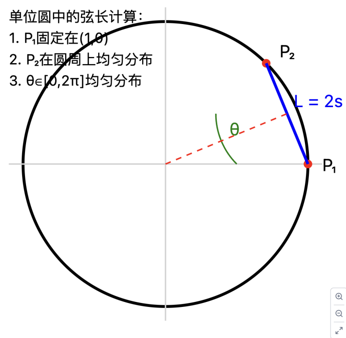

Figure 1: Expected chord length in a unit circle. P_1 is fixed at $(1, 0)$, while P_2 is uniformly distributed on the circumference. The chord length $L = 2\sin(\theta/2)$ where θ is the central angle between P_1 and P_2. The expected value $E[L] = \frac{4}{\pi}$ is obtained by integrating over $\theta \in [0, 2\pi]$.

12. A company's current stock price is 10. There is a 50% chance it will be acquired. If acquired, there is an 80% chance the stock price will go up to 15 and a 20% chance the price will become xxx. If the acquisition does not happen, there is a 60% chance the company goes bankrupt and a 40% chance the stock price will drop to 5. What is the fair price of a call option with a strike price of 21?

13. Three people sit in eight available seats. What is the probability that all three people sit next to each other?

Answer: (1.) all possible cases: A(8,3): combination $8!/(n-3)! = 336$. (2).put the three people as a whole, calculate the internal and external position possibility: Internal: factorial of $3! = 6$.

External: also 6 different case.

All combinations : 6*6 = 36

Probability: 36/ 336

13. What is the probability that three randomly chosen points on the circumference of a circle will form an acute-angled triangle?

1. Problem Analysis:

- Choose three points randomly on the circumference of a circle
- Calculate the probability that they form an acute triangle
- Key is to understand the conditions for an acute triangle

2. Key Observation:

- On a circle, an angle is obtuse $\iff$ its corresponding arc is greater than a semicircle
- Triangle is acute $\iff$ all three arcs are less than a semicircle

- This is because inscribed angle equals half the central angle

3. Solution Approach:

- Fix one point (due to symmetry, this doesn't affect the result)
- Require arcs between other two points and first point to be less than π
- Also require arc between these two points to be less than π

4. Calculation:

- Points on circumference can be represented by angles
- Assume first point is at 0
- Let θ_1 be angle of second point, θ_2 angle of third point
- θ_1 ranges in $[0, 2\pi]$
- θ_2 ranges in $[0, 2\pi]$

5. Conditions:

- All three arcs must be less than π:

$$\begin{aligned} |\theta_1| &< \pi \\ |\theta_2| &< \pi \\ |\theta_2 - \theta_1| &< \pi \end{aligned}$$

6. Solution:

- By symmetry, consider only θ_1 in $[0, \pi]$ and multiply by 2
- For given θ_1, θ_2 must satisfy:

$$\begin{aligned} \theta_2 &> \theta_1 - \pi \\ \theta_2 &< \theta_1 + \pi \end{aligned}$$

7. Area Calculation:

- Total area $= 4\pi^2$
- Area satisfying conditions $= \pi^2$
- Probability $= \frac{\pi^2}{4\pi^2} = \frac{1}{4} = 0.25$

Therefore, the probability that three randomly chosen points on the circumference of a circle will form an acute-angled triangle is $\frac{1}{4}$ or 25%.

15. Dice Game Winning Probability Player A and Player B take turns rolling a die, stopping when a 6 is rolled. What is the probability that Player A wins, and what is the expected value?

$$P = \begin{pmatrix} P(A \to A) & P(A \to B) & P(A \to END_A) & P(A \to END_B) \\ P(B \to A) & P(B \to B) & P(B \to END_A) & P(B \to END_B) \\ P(END_A \to A) & P(END_A \to B) & P(END_A \to END_A) & P(END_A \to END_B) \\ P(END_B \to A) & P(END_B \to B) & P(END_B \to END_A) & P(END_B \to END_B) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ \frac{5}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_A = P(A \to END_A) \times 1 + P(A \to B) \times p_B = \frac{1}{6} \times 1 + \frac{5}{6} \times p_B$$

$$p_B = P(B \to END_A) \times 0 + P(B \to A) \times p_A + P(B \to END_B) \times 0 = \frac{5}{6} \times p_A$$

every time

$$p_A - \frac{25}{36} p_A = \frac{1}{6} \frac{11}{36} p_A = \frac{1}{6} p_A = \frac{1/6}{11/36} = \frac{6}{11}$$

$$E_A = 1 + P(A \to END_A) \times 0 + P(A \to B) \times E_B = 1 + \frac{1}{6} \times 0 + \frac{5}{6} \times E_B$$

$$E_B = 1 + P(B \to END_B) \times 0 + P(B \to A) \times E_A = 1 + \frac{1}{6} \times 0 + \frac{5}{6} \times E_A$$

$$E_A - \frac{25}{36} E_A = 1 + \frac{5}{6} \frac{11}{36} E_A = \frac{11}{6} E_A = \frac{11/6}{11/36} = 6$$

16. People sit in a circle and exchange cards. What is the expected number of people who receive cards?

17. Ten people sit around a round table. Three people know each other. What is the probability that at least two of them sit next to each other?

Using total probability:

All combinations: $(n-1)!$ because of round table. Here $9!$.

Pick 5 peole who does not know each other to not sit with each other: $A(7,5) = \frac{7!}{(5-2)!}$.

For the rest seats not neighbored, we arrange the three people, which is: $A(5,3) = \frac{5}{(5-3)!}$

At least two people know each other sit next to each other:

$1 - P(AllThreeNotNeighbour) = 1 - \frac{\frac{7!}{2!}\frac{5!}{2!}}{9!} = 1 - 5/12$

18. What is the probability of getting "HTH" (head, tail, head) before getting "HHT" (head, head, tail)?

19. A bag contains 2 red balls and 1 blue ball, every time take out a ball and replace it by a blue ball, what is the expected number that needed, in order to have all blue balls in the bag?

Answer: define Markov State and Markov probability:

State: S_2, S_1, S_0, denotes the number of red balls still in the bag.

Transition probability:

$S_2 to S_2$: 2/3

$S_2 to S_1$: 1/3

$S_2 to S_0$: 0

$S_1 to S_2$: 0

$S_1 to S_1$: 1/3

$S_1 to S_0$: 2/3

Expectation:

$$E_2 = 1 + \frac{1}{3}E_2 + \frac{2}{3}E_1$$

$$E_1 = 1 + \frac{2}{3}E_1 + \frac{1}{3}E_0$$

$$E_0 = 0$$

Solve it we get $E_2 = 4.5, E_1 = 3$.

20. Take a yellow cube, paint the surface green, and cut it twice horizontally and twice vertically (that is, connect 27 small cubes in series). Ask about the probability of randomly picking up a small cube and throwing it out, with the green side facing up.

Answer: law of total probability:

$$\begin{aligned} &\frac{6}{27} * \frac{1}{6} + \frac{8}{27} * \frac{1}{2} + \frac{12}{27} * \frac{2}{6} + \frac{1}{27} * 0 \\ &= \frac{1}{27} + \frac{4}{27} + \frac{12 * 4}{27 * 3} \\ &= \frac{9}{27} = \frac{1}{3} \end{aligned}$$

21. Randomly select three numbers from 0, 1, 2, and 3 and ask the probability that the median is between 1 and 2

Without replacement:

All possible cases: $C(4,3) = C(4,1) = 4!/(1! * 3!) = 4$

List all cases: $(0,1,2), (0,1,3), (0,2,3), (1,2,3)$.

Median: 1, 1,2,2

Probability: 1.

With replacement:

All possible cases: $4^3 = 64$

Median 1 or 2:

Median 1: one of the values is equal to 1, for the rest two cases, one of them smaller or equal, the other one of them larger or equal to 1.

Situation 1:one 1, one 0, one larger than 1

values: $\{0,1,1\},\{0,1,2\},\{0,1,3\}$

$\{0,1,1\}$:three:$(011,101,110)$

$0,1,2$:6 types$(012,021,102,120,201,210)$

$0,1,3$:6 types$(013,031,103,130,301,310)$

Situation 2: three times 1 $\{1,1,1\}$ In total:$3+6+6+1=16$

Same analysis for median equals to 2, in total we have 35 cases.

22. In a round table with 8 seats, 3 people randomly select seats. Ask the probability that no two of these three people sit next to each other.

Answer:

Total combinations:

$$C(8,3)=\frac{8!}{3!5!}=56$$

Combination of three people are all not sit next to each other:

$$c(n-k,k)+c(n-k-1,k-1)=c(8-3,3)c(8-3-1,3-1)=c(5,3)+c(4,2)=10+6=16$$

Idea: fix someone in one seat. Then the rest seat that can be choose will be 5.

Among the 5 seats, choose two seats and not neighbored, the combination will be $c(5-2+1,2)=c(4,2)=6$.

The total combinations will be $6\frac{8}{3}=16$, because we have 8 seats and three people.

Therefore, the probability of at least two people sitting together is $\frac{5}{7}$

23. There are two red balls and two blue balls in a bag. I take one ball from the bag every time, and then you guess the color of the ball. If you guess correctly, you win a dollar, otherwise you win no money. How much are you willing to spend to play this game.

Condition not accurately described: Depends on the condition. For example, how many times? Is the sampling procedure with or without replacement?

24. You live near two bus stops. Bus line A comes every 12 minutes on average, and bus line B comes every 15 minutes on average. Both follow Poisson distributions. You need to get to work and can take either bus. You randomly pick one bus stop to wait at, and you'll take the first bus that comes. What's your expected waiting time? If after waiting for 10 minutes at your chosen stop and no bus has arrived yet, what's your expected additional waiting time?

Answer: 1. This is the connection between the Poisson and exponential distribution. The expectation of the two Poisson distribution is $\lambda_1=1/12$ and $\lambda_2=1/15$.

Now sum them, the summation of two poisson distribution is also a poisson distribution, with the parameters simply added together. Therefore, we have $\lambda=\lambda_1+\lambda_2=\frac{3}{20}$. The expected waiting time is $\frac{20}{3}=6.67$. Then, because for exponential distribution, the memoryless property, the expected time after waiting 10 mins is still the same.

Follow up:

If you could check your phone and see that the last A bus came 3 minutes ago, and the last B bus came 8 minutes ago, which stop would you go to? What would be your expected waiting time in this case? If instead of randomly picking a stop, you can see both stops from your home and can walk to either one as soon as you see a bus coming (assume the time to walk to either stop is the same and negligible), what's your expected waiting time now?

26. There are two children in a family. Now it is known that one is a girl. What is the probability that the other one is also a girl?

Conditional probability:

$$P(FF|F) = \frac{P(F|FF) * P(FF)}{P(F)} = \frac{1 * \frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

27. A 2-meter string, one on the wall and one on your hand, randomly cut the string until the string on the wall is less than 1 meter, what is the expected number of cuts?

$$E(L) = 1 + \int_0^1 E(UL)dU = 2 + \log(L)$$

1 is the current cutting, $\int_0^1 E(UL)dU$ is the expected number of cuts after cutting, the length becoming UL.

Set $E(L) = a\log(L) + b$

Solve

$$a\log(L) + b = 1 + \int_0^1 (a\log(UL) + b)dU$$

solve we have $a = 1$, and $b = 1$

28. Draw a integer x uniformly on $[0, 10]$, if x is a primer, gain x, otherwise, lose $2x$, what is the expected value of the game?

$$E(X) = \frac{1}{11}(2 + 3 + 5 + 7) - \frac{1}{11} * \frac{1}{2} * (1 + 4 + 6 + 8 + 9 + 10)$$

29. A rectangle table with 10 seats, (5 seats on each longer side), 7 people randomly pick a seat, what is the probability that A and B sit face to face?

Total possibility of choice: $P(10, 7) = \frac{10!}{3!}$.

Select one from 5 and times two: 10

Arrange the rest 5 people in 8 different seats: $P(8, 5) = \frac{8!}{3!}$

Probability: $\frac{1}{9}$

30. You and I both toss 3 times of a fair coin, if we have a same number of heads, I win 2, otherwise, you win 1, should you play?

Using binomial distribution calculated

$$P(X=0,Y=0)=P(X=0)P(Y=0)$$

$$P(X=1,Y=1)=P(X=1)P(Y=1)$$

The probability of winning is $\frac{20}{64}$. Should play.

Consider a circle (a) What is the expected number of regions into which the circle is divided after making these two random cuts? (b) What is the expected number of regions into which the cirvle is divided after n cuts?

A circle on which four points are randomly and uniformly distributed along its circumference. These four points are connected in pairs by two straight chords (cuts). Depending on the configuration of these points, the chords may or may not intersect within the circle, thereby dividing the circle into distinct regions.

In total: $C(4,2)=6$ different cuts.

Among the 6 cases, two of the are not intersected, therefore, the circle will be divided into 4 different cases.

For the rest 4 cases, there will be two 3 pieces. Therefore, the expected value is

$$E(X)=3*\frac{4}{6}+4*\frac{2}{6}=\frac{10}{3}$$

In case of n cuts:
Total $C2n=\frac{n(n-1)}{2}$ cuts.
Expected intersections:

$$E(int)=\frac{1}{3}C(n,2)=\frac{1}{3}\frac{n(n-1)}{2}$$

Number of regions = cuts + intersections + 1

$$E(X)=n+E(int)+1=\frac{n^2+5n+6}{6}$$

Let X, Y, and Z be three independent and identically distributed (i.i.d.) random variables, each following a uniform distribution on the interval $[0,3]$, denoted as $U[0,3]$. Determine the probability that the median (the middle value) of these three variables lies between 1 and 2.

Think the three different regions as different boxes. Fix the middle number in the middle box, and calculate the probability.

Total number of cases:

$$3^3=27$$

3 scenarios: 1st: One in the middle, one in the left and one in the right: 3*2=6 (choose one of three numbers in the middle, then the other two will be either in A or in C).

2nd: Two of them in the middle, one in the left or in the right. $2*C(3,2)=6$
3rd: All of them in the middle: 1
In total: $P=\frac{6+6+1}{27}=\frac{13}{27}$

Consider an unfair coin that lands on heads with probability p and tails with probability $1-p$. The coin is tossed consecutively 16 times. Define a "streak of heads" as a sequence of one or more consecutive heads. What is the expected number of streaks of heads in these 16 tosses? Can this problem be solved using Markov chains or the linearity of expectation?

2. Solution Using Linearity of Expectation Let X_i be an indicator variable such that $X_i=1$ if position i is the start of a streak, and $X_i=0$ otherwise.

For $i=1$:

$$X_1 = 1 \text{ if first toss is H}$$

$$X_1 = 0 \text{ if first toss is T}$$

For $2 \leq i \leq 16$:

$$X_i = 1 \text{ if position } i \text{ is H and position } i-1 \text{ is T}$$

$$X_i = 0 \text{ otherwise}$$

The total number of streaks N is:

$$N = X_1 + X_2 + X_3 + \cdots + X_{16}$$

By linearity of expectation:

$$\mathbb{E}[N] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \cdots + \mathbb{E}[X_{16}]$$

For $i=1$:

$$\mathbb{E}[X_1] = P(X_1 = 1) = P(\text{first toss is H}) = p$$

For $2 \leq i \leq 16$:

$$\mathbb{E}[X_i] = P(X_i = 1) = P(\text{H at } i \text{ and T at } i-1) = p(1-p)$$

Therefore:

$$\mathbb{E}[N] = p + 15p(1-p) = 16p - 15p^2$$

From a group of 30 people, 5 individuals are randomly selected. It is observed that these 5 people are arranged in either strictly increasing or strictly decreasing order of their heights. What is the probability that the first person in this arrangement is the shortest among all 30 people?

All possible choices:$C(30,5)*2$.

Answer fulfill the potential choice: $1*C(29,4)$: Explanation: 1 means the shortest person, $C(29,4)$ means the rest four people, and 2 times decreasing or increasing orders.

Probability:

$$\frac{C(29,4)}{2*C(30,5)} = \frac{1}{12}$$