

Measuring Chaos from Spatial Information

Journal club

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Solé, R. V., & Bascompte, J. (1995). Measuring chaos from spatial information. *Journal of Theoretical Biology*, 175(2), 139–147. <https://doi.org/10.1006/jtbi.1995.0126>

Introduction

- In most of the nonlinear system, **detecting chaos** is always one of the most important topics.
- Conventional method is to use **Lyapunov exponent (LE) in time scale**.
- Basically, run the simulation/experiment for **quite a long time**, and record the LE
- However, some system is not able to have data with that many time steps (E.g., some ecology system can be affected by **transitions between attractors** when running time is long)
- So here, it introduces new method to get LE using **spatiotemporal information** [1]

Introduction – what is Lyapunov exponent

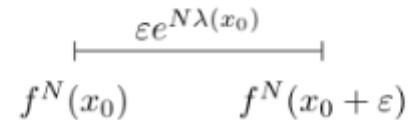
- Lyapunov exponent of a dynamical system is to characterizes the rate of separation of infinitesimally close trajectories. [2]
- Formula for LE [3] (Eq 2a in paper):

$$\lambda = \lim_{t \rightarrow \infty} \lim_{|\delta \mathbf{Z}_0| \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

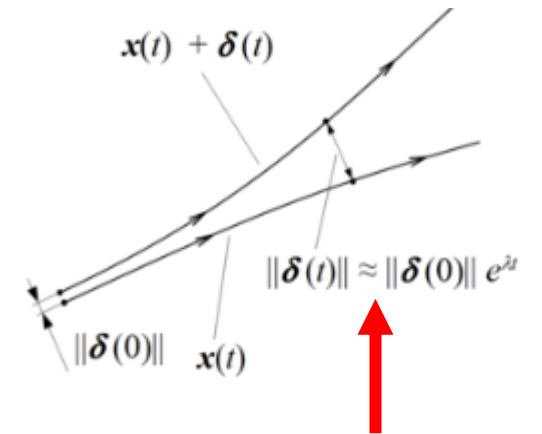
Distance between two start values:


$$\begin{array}{c} \varepsilon \\ \hline x_0 \qquad x_0 + \varepsilon \end{array}$$

After N iterations:


$$\begin{array}{c} \varepsilon e^{N\lambda(x_0)} \\ \hline f^N(x_0) \qquad f^N(x_0 + \varepsilon) \end{array}$$

- $\lambda > 0$: chaotic system
 - $\delta Z(t) \gg \delta Z(0)$ (difference has exp growth)
- $\lambda = 0$: periodic
 - $\delta Z(t) = \delta Z(0)$ (difference has exp growth)
- $\lambda < 0$: stable (convergent to fix point)



This means chaos

Introduction – what is Lyapunov exponent

- There is another formula for LE, which cannot apply here
- Another formula for LE (Eq 1 in paper):

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- This one needs to know the exact motional expression (since we need to get derivative) of dynamical system
- But for most of the cases, we have **only a temporal series** recorded, usually without information about mechanism behind

Introduction – what is Lyapunov exponent

- So, it applies Eq below for conventional method

$$\lambda = \lim_{t \rightarrow \infty} \lim_{|\delta \mathbf{Z}_0| \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

- Now, as we can see, it requires a long time series (usually $O(1000)$ data points) and so it cannot be applied to current data without a long temporal scale.
- And this paper introduces a new one.

Method – a new spatiotemporal way

- Coupled map lattices (CML) is used to prove this method
- Every point in CML is just like logistic map, but also affected by its neighboring (Laplacian):

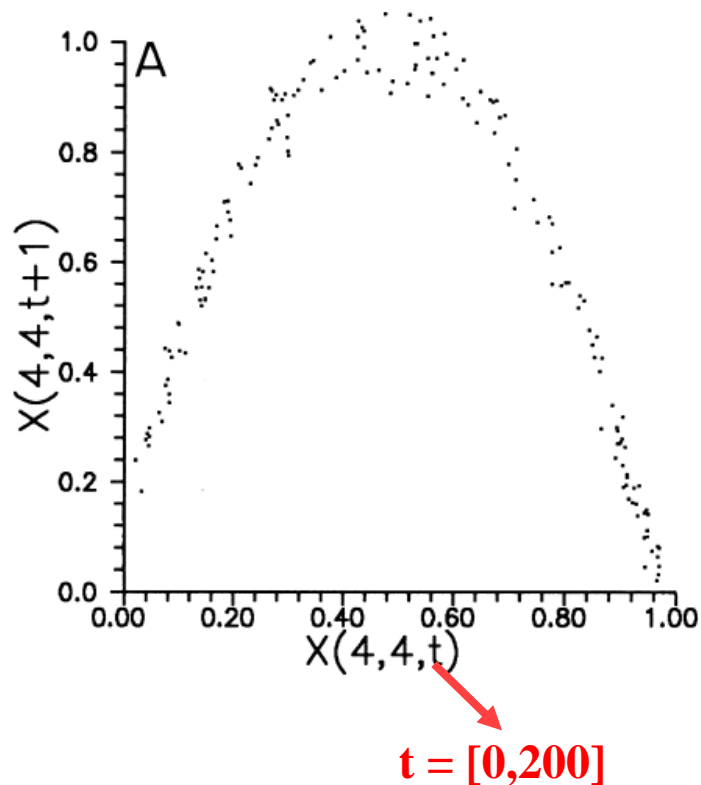
$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D \nabla^2 x_n(r),$$

- Now, in this 2D lattice map, it runs twice, first time recording one point with long time (200 steps)
- Second time recording all points in lattice, with a short time

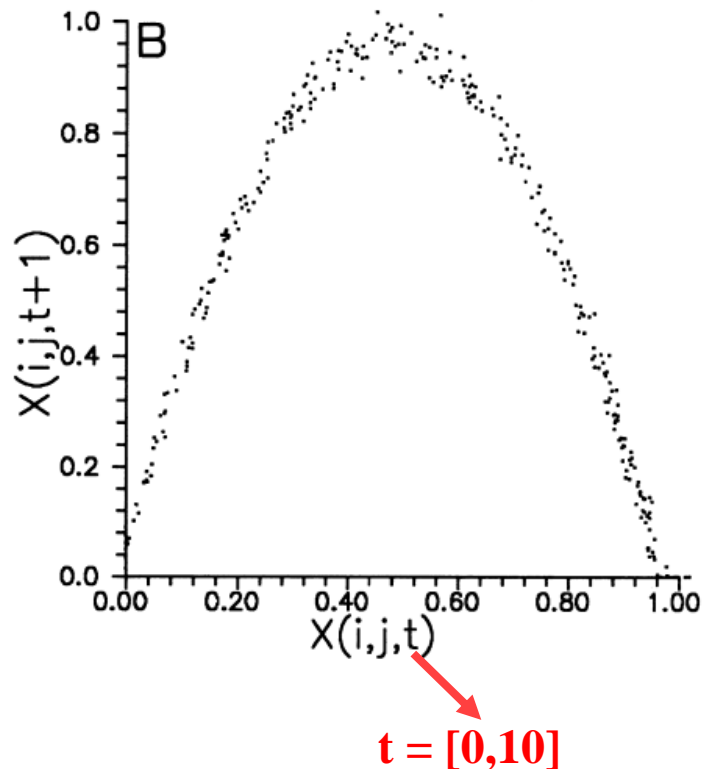
Method – a new spatiotemporal way

- First time one point with long time; Second time all points with a short time

First, **one point only**

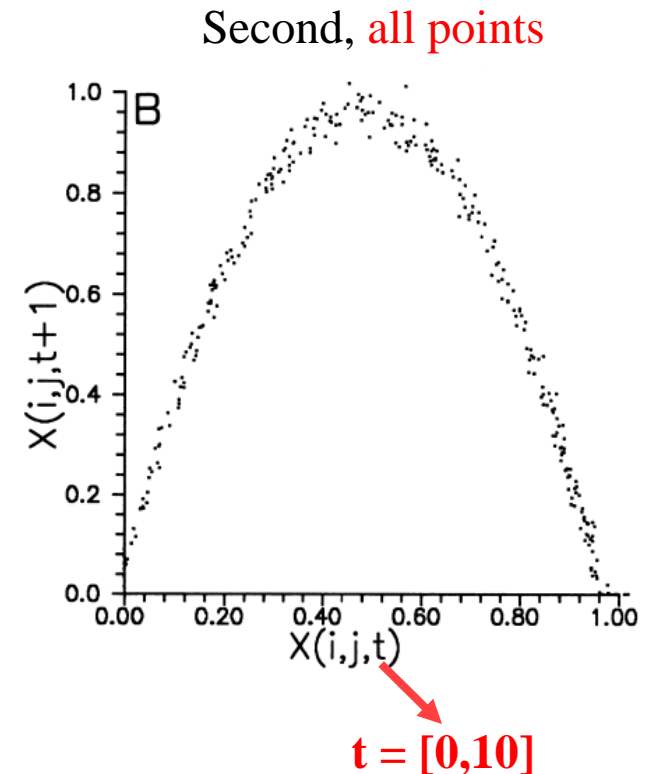
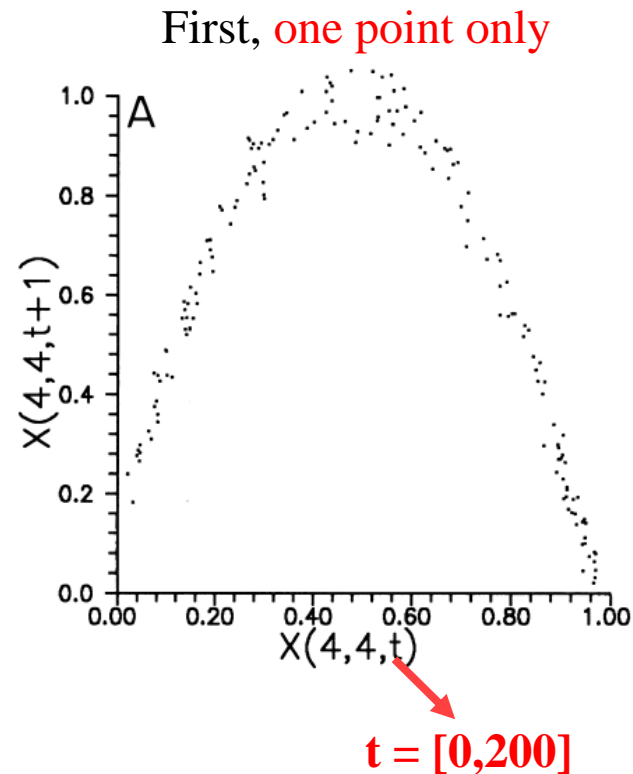


Second, **all points**



Method – a new spatiotemporal way

- Now we see that they are very similar.
- The lack of **temporal** information is **compensated** by the **spatial** information



Method – a new spatiotemporal way

- The lack of **temporal** information is **compensated** by the **spatial** information
- Then we can also apply it onto LE
- A new spatiotemporal LE is defined:

$$\lambda_s(d) = \frac{1}{N_p} \sum_{i=1}^{m-d} \sum_{\langle \mathbf{k}, \mathbf{h} \rangle} Ln \left[\frac{\| \mathbf{X}_{i+1}^j(\mathbf{k}) - \mathbf{X}_{i+1}^j(\mathbf{h}) \|}{\| \mathbf{X}_i^j(\mathbf{k}) - \mathbf{X}_i^j(\mathbf{h}) \|} \right]. \quad \| \mathbf{X}_i^j(\mathbf{k}) - \mathbf{X}_i^j(\mathbf{h}) \| = \left[\sum_{u=i}^{i+d-1} (x_u^j(\mathbf{k}) - x_u^j(\mathbf{h}))^2 \right]^{1/2} < \epsilon$$

- s means spatial
- k = every point of 2D map from t_i to t_(i+d)
- h = neighboring points of k from t_i to t_(i+d) (neighbor distance < a small value)
- d = embedded dimension
- N_p = number of <k, h> pairs (take average)
- j = type of X variable
- i = time step

Method – a new spatiotemporal way

- A new spatiotemporal LE is defined
- Compared to the old one:

$$\lambda_s(d) = \frac{1}{N_p} \sum_{i=1}^{m-d} \sum_{\langle \mathbf{k}, \mathbf{h} \rangle} \text{Ln} \left[\frac{\| \mathbf{X}_{i+1}^j(\mathbf{k}) - \mathbf{X}_{i+1}^j(\mathbf{h}) \|}{\| \mathbf{X}_i^j(\mathbf{k}) - \mathbf{X}_i^j(\mathbf{h}) \|} \right].$$

$$\lambda = \lim_{t \rightarrow \infty} \lim_{|\delta \mathbf{Z}_0| \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

- $t = [0, m-d]$
- Points = all

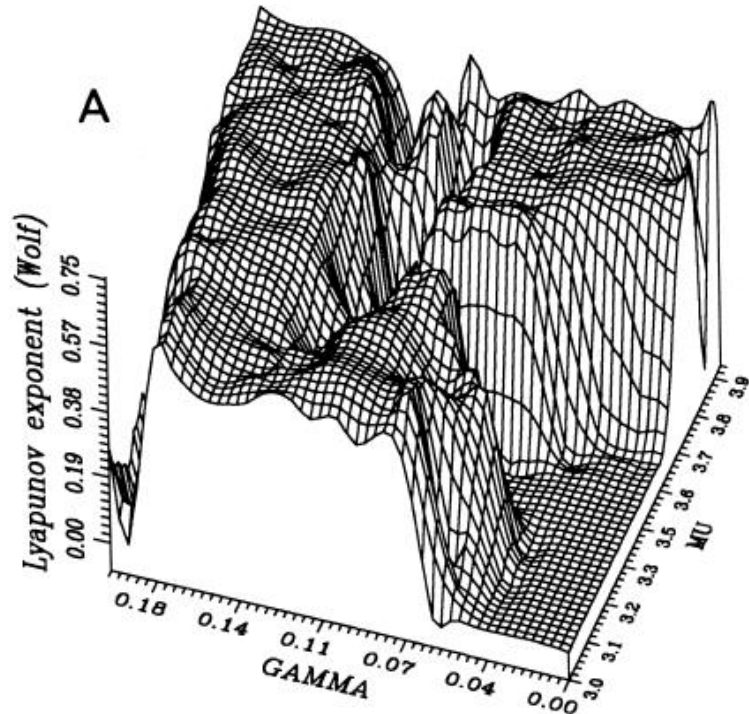
require less time
but more spatial info

$t = [0, \text{inf}]$
points = one

Results – LE obtained from new method

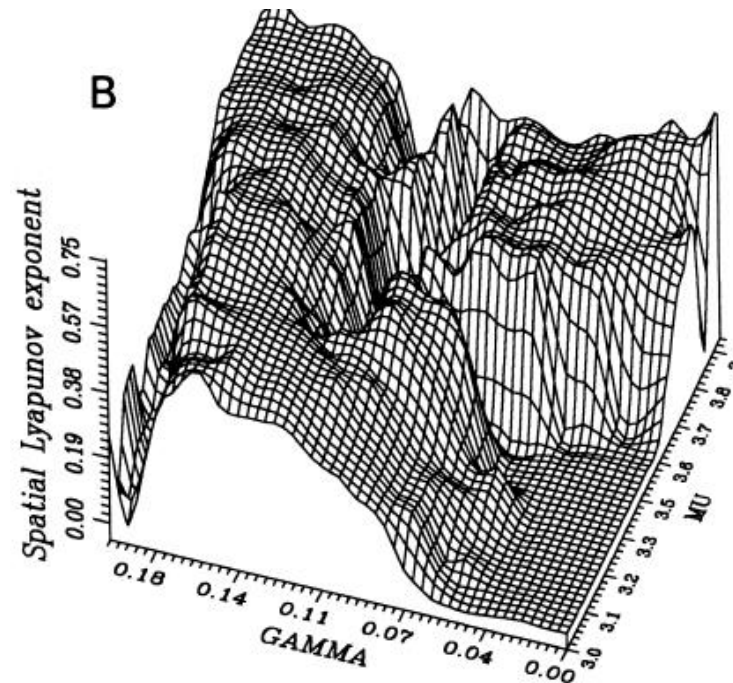
- 1. Logistic CML

1500 time steps after 1500 transients



Conventional LE, Gamma = D

12 time steps after 1500 transients



New method LE

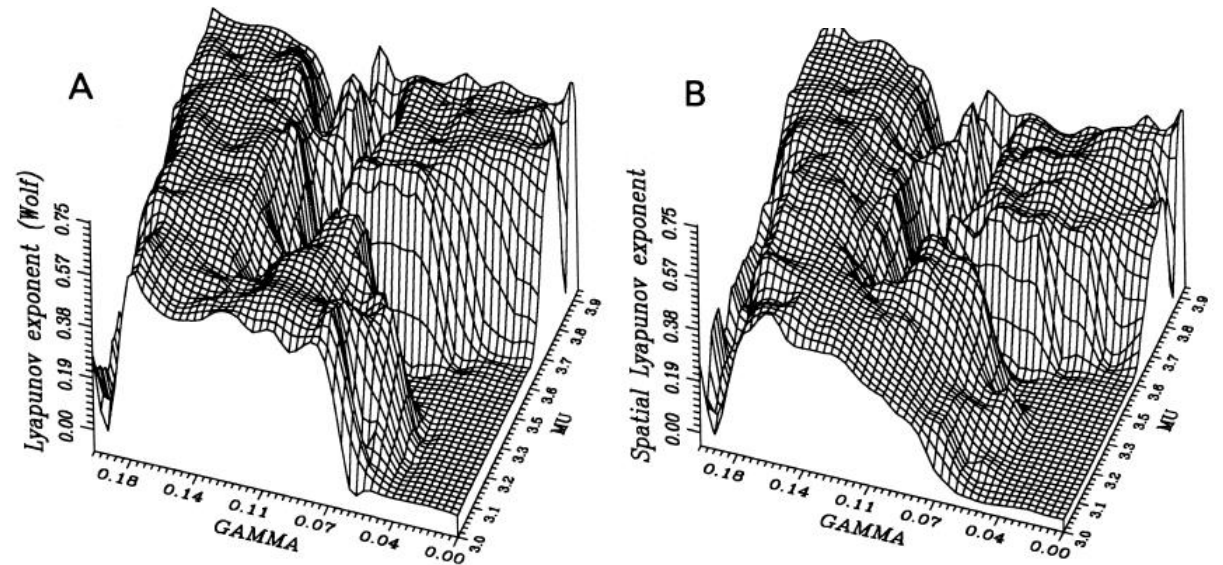
$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D \nabla^2 x_n(r),$$

with D the diffusion rate and

$$\begin{aligned} \nabla^2 x_n(r) = & x_n(i-1, j) + x_n(i+1, j) \\ & + x_n(i, j-1) + x_n(i, j+1) - 4x_n(i, j) \end{aligned}$$

Results – LE obtained from new method

- The two exponents show the **same domains** of stable, periodic and chaotic attractors
- For further evidence of the validity, host–parasitoid CML is applied.

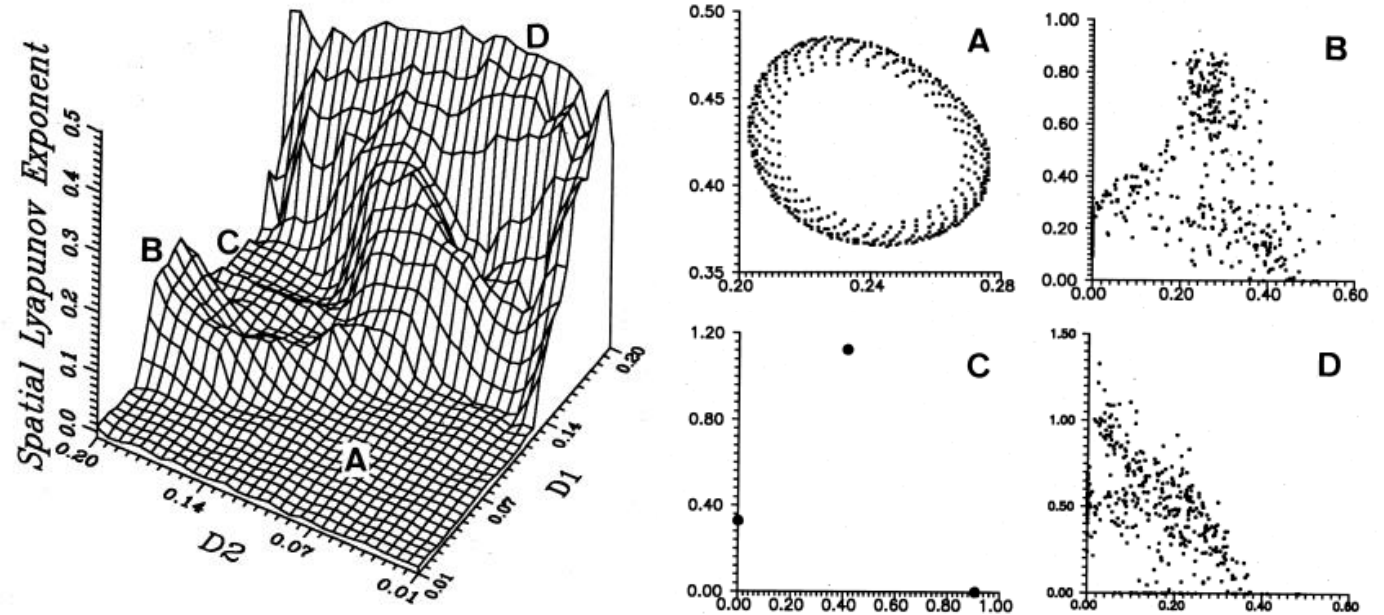


Results – LE obtained from new method

• 2. Host–parasitoid CML.

$$\begin{aligned}x_{n+1}(\mathbf{k}) &= \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) \\ &\quad \times \exp(-\beta y_n(\mathbf{k})) + D_1 \nabla^2 x_n(r) \\ y_{n+1}(\mathbf{k}) &= x_n(\mathbf{k})(1 - \exp(-\beta y_n(\mathbf{k}))) \\ &\quad + D_2 \nabla^2 y_n(r).\end{aligned}$$

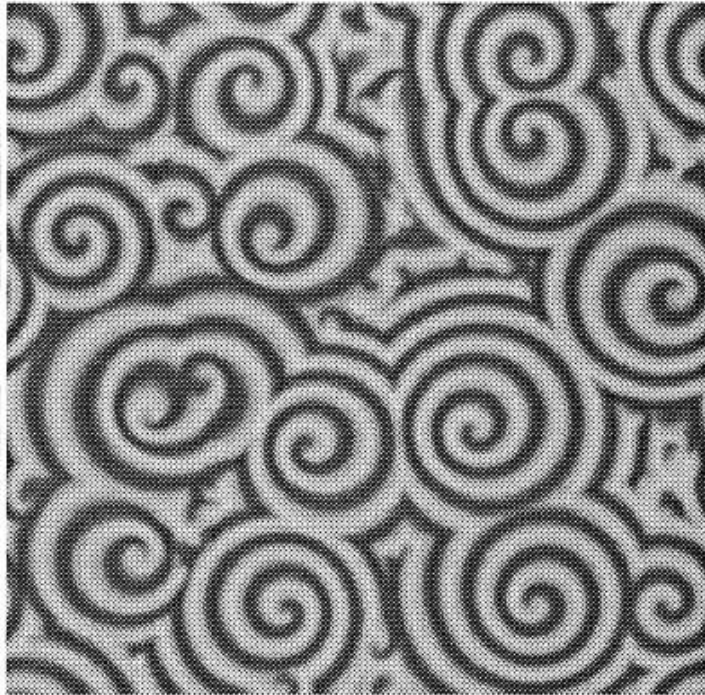
- A: periodic, $\lambda = 0$
- B: chaotic, $\lambda > 0$
- C: stable, $\lambda < 0$
- D: chaotic, $\lambda > 0$



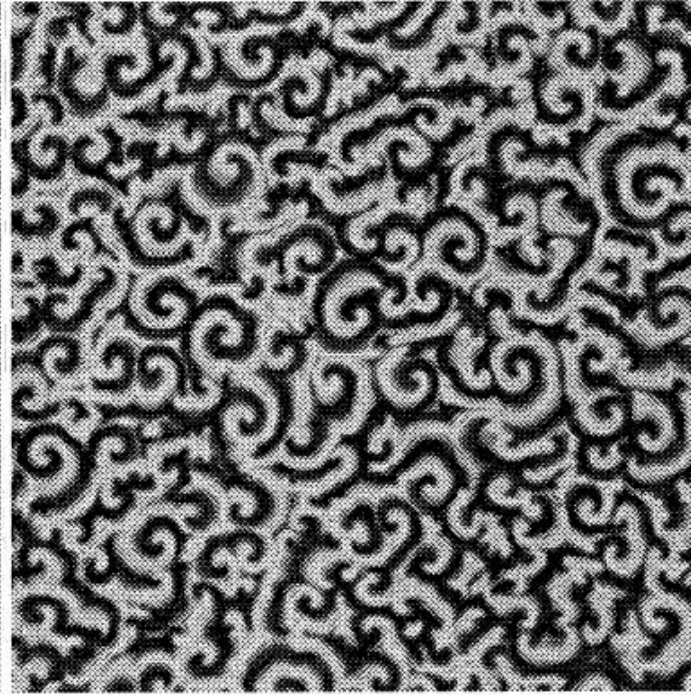
Results – LE obtained from new method

- λ get from **spatiotemporal** method (Spiral waves showed by the host–parasitoid CML)

- $\lambda = 0.013$ (**quasiperiodic**)



- $\lambda = 0.125$ (**chaotic**)



$$\begin{aligned}x_{n+1}(\mathbf{k}) &= \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) \\ &\quad \times \exp(-\beta y_n(\mathbf{k})) + D_1 \nabla^2 x_n(r) \\ y_{n+1}(\mathbf{k}) &= x_n(\mathbf{k})(1 - \exp(-\beta y_n(\mathbf{k}))) \\ &\quad + D_2 \nabla^2 y_n(r).\end{aligned}$$

Further discussion on dimensionality

- Recall that d is the value it used to find neighbor points

$$\| \mathbf{X}_i^j(\mathbf{k}) - \mathbf{X}_i^j(\mathbf{h}) \| = \left[\sum_{u=i}^{i+d-1} (x_u^j(\mathbf{k}) - x_u^j(\mathbf{h}))^2 \right]^{1/2} < \epsilon$$

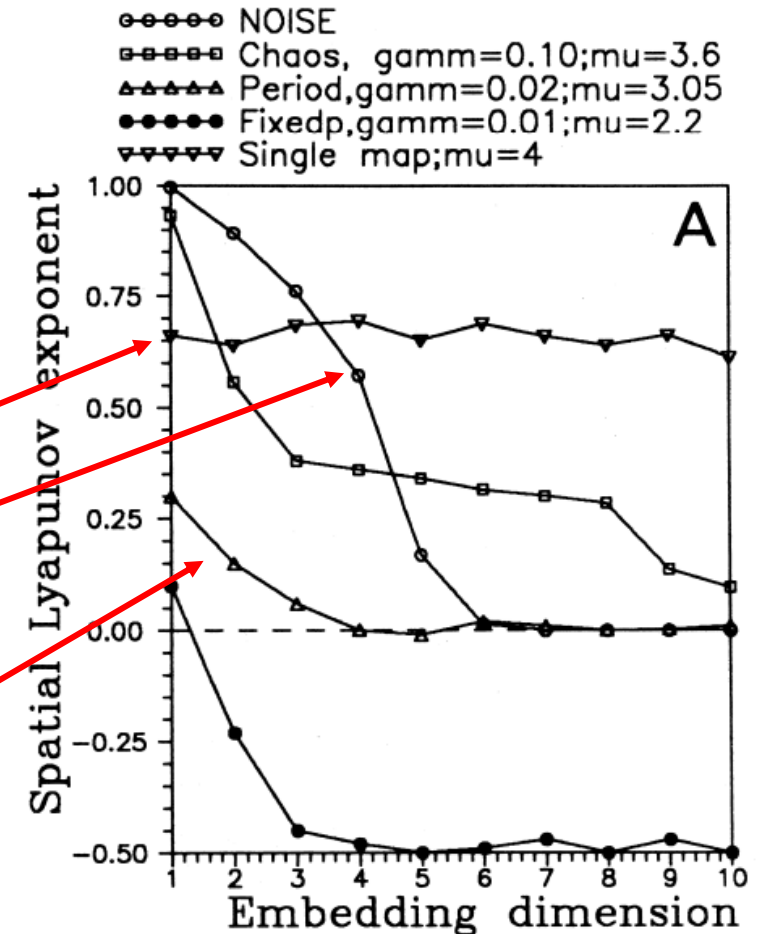
- $\lambda(d)$ shows a plateau after a certain $d=d_0$
- Where d_0 is an estimation of system's dimensionality
 - For single map (logistic map without coupled), its dimension = 1, so λ keeps almost constant when d increases ($d = d_0 = 1$)
 - For noise, $d_0 = \text{inf}$, so no plateau
 - For others, $d_0 = 3$, so logistic CML has dimension around 3

$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D \nabla^2 x_n(r),$$

with D the diffusion rate and

$$\nabla^2 x_n(r) = x_n(i-1, j) + x_n(i+1, j)$$

$$+ x_n(i, j-1) + x_n(i, j+1) - 4x_n(i, j)$$



For Logistic CML

Conclusion

- It detects the presence of **chaos** in very **short temporal series** with information in **different spatial points**.
- It used CML models to prove the validity of the method
- Furthermore, **dimension information** can be inferred from this method, and can be used for validity check

Reference

- [1]R. V. Solé and J. Bascompte, “Measuring chaos from spatial information,” *Journal of theoretical biology*, vol. 175, no. 2, pp. 139–147, 1995, doi: 10.1006/jtbi.1995.0126.
- [2]Lyapunov exponent [Online]. Available: https://en.wikipedia.org/wiki/Lyapunov_exponent
- [3]A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining Lyapunov exponents from a time series,” *Physica. D*, vol. 16, no. 3, pp. 285–317, 1985, doi: 10.1016/0167-2789(85)90011-9.

Thank you!