5.3.118

a. If m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[a,x]$ of width h_i, we have $m^* h_i \le f(x_i^*)h_i \le M^* h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[a,x]$ and taking the limit as $n \to \infty$ gives $m^*(x-a) \le \int_a^x f(t)\,dt \le M^*(x-a)$, so $m^*(x-a) \le A(x) \le M^*(x-a)$. Now consider $\lim_{x\to a^+} m^*(x-a) = \lim_{x\to a^+} M^*(x-a) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x\to a^+} A(x) = 0 = A(a)$, so A is continuous from the right at $x = a$.

b. First note that $A(b) - A(x) = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt = \int_x^b f(t)\,dt$. Now if m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[b,x]$ of width h_i, we have $m^* h_i \le f(x_i^*)h_i \le M^* h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[b,x]$ and taking the limit as $n \to \infty$ gives $m^*(b-x) \le \int_x^b f(t)\,dt \le M^*(b-x)$, so $m^*(b-x) \le A(b) - A(x) \le M^*(b-x)$. Now consider $\lim_{x\to b^-} m^*(b-x) = \lim_{x\to b^-} M^*(b-x) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x\to b^-} (A(b) - A(x)) = 0$, so $\lim_{x\to b^-} A(x) = A(b)$, and A is continuous from the left at $x = b$.

5.3.119

a. By definition of Reimann sums, $\int_a^b f'(x)\,dx$ is approximated by $\sum_{k=1}^{n} f'(x_{k-1})\Delta x$. But $f'(x_{k-1}) = \lim_{h\to 0} \frac{f(x_{k-1}+h) - f(x_{k-1})}{h}$. If $h = \Delta x$, then we have

$$f'(x_{k-1}) \approx \frac{f(x_{k-1}+\Delta x) - f(x_{k-1})}{\Delta x} = \frac{f(x_k) - f(x_{k-1})}{\Delta x},$$

so that

$$\int_a^b f'(x)\,dx \approx \sum_{k=1}^{n} \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x.$$

b. Canceling the Δx factors we obtain

$$\begin{aligned}
\int_a^b f'(x)\,dx &\approx \sum_{k=1}^{n} \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x \\
&= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \\
&= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1})) \\
&= f(x_n) - f(x_0) = f(b) - f(a).
\end{aligned}$$

c. The analogy between the two situations is that both (a) the sum of difference quotients and (b) integral of a derivative are equal to the difference in function values at the endpoints.

5.4 Working with Integrals

5.4.1 If f is odd, it is symmetric about the origin, which guarantees that between $-a$ and a, there is as much area above the axis and under f as there is below the axis and above f, so the net area must be 0.

5.4.2 If f is even, it is symmetric about the the $y-$axis, which guarantees that the region between $-a$ and 0 has the same net area as the region between 0 and a, so $\int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

5.4.3

a. Because $\int_{-8}^{8} f(x)\,dx = 18 = 2\int_0^8 f(x)\,dx$, we have $\int_0^8 f(x)\,dx = \frac{18}{2} = 9$.

b. Because $xf(x)$ is an odd function when $f(x)$ is even, we have $\int_{-8}^{8} xf(x)\,dx = 0$.

5.4.4

a. Because $f(x)$ is odd, $\int_{-4}^{0} f(x)\,dx = -\int_0^4 f(x)\,dx$. We have

$$\int_{-4}^{8} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_0^8 f(x)\,dx = -\int_0^4 f(x)\,dx + \int_0^8 f(x)\,dx = -3 + 9 = 6.$$

b. $\int_{-8}^{4} f(x)\,dx = \int_{-8}^{0} f(x)\,dx + \int_0^4 f(x)\,dx = -9 + 3 = -6.$

5.4.5 The integrand can be written as $(5x^4 + 2x^2 + 1) + (3x^3 + x)$. The function $f(x) = 5x^4 + 2x^2 + 1$ is an even function and the function $g(x) = 3x^3 + x$ is an odd function. Thus,

$$\int_{-4}^{4} (f(x) + g(x))\,dx = \int_{-4}^{4} f(x)\,dx + \int_{-4}^{4} g(x)\,dx = 2\int_0^4 f(x)\,dx + 0 = 2\int_0^4 (5x^4 + 2x^2 + 1)\,dx.$$

5.4.6 The number 2 should go in the first blank and the function $\cos x$ in the second.

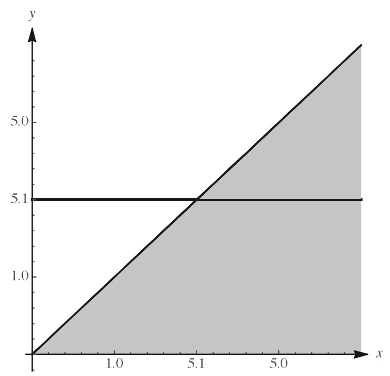
$$\int_{-\pi}^{\pi} (\sin x + \cos x)\,dx = \int_{-\pi}^{\pi} \sin x\,dx + \int_{-\pi}^{\pi} \cos x\,dx = 0 + 2\int_0^{\pi} \cos x\,dx = 2\int_0^{\pi} \cos x\,dx.$$

5.4.7 $f(x) = x^{12}$ is an even function, because $f(-x) = (-x)^{12} = x^{12} = f(x)$. $g(x) = \sin x^2$ is also even, because $g(-x) = \sin((-x)^2) = \sin x^2 = g(x)$.

5.4.8 The average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. This is analogous to "adding up all the value of f and dividing by how many there are" – in the sense that computing the interval is like adding up all the values of the function, and dividing by $b - a$ is like dividing by how many x values there are.

5.4.9 The average value of a continuous function on a closed interval $[a, b]$ will always be between the maximum and the minimum value of f on that interval. Because the function is continuous, the Intermediate Value Theorem assures us that the function will take on each value between the maximum and the minimum somewhere on the interval.

5.4.10 Note that the area of the triangle is $\frac{1}{2}\cdot 2\cdot 2 = 2$, so the rectangle needs to have a height of 1 and a base of 2 so that its area is 2.



5.4.11 Because x^9 is an odd function, $\int_{-2}^{2} x^9\,dx = 0$.

5.4.12 Because $2x^5$ is an odd function, $\int_{-200}^{200} 2x^5\,dx = 0$.

5.4.13 $\int_{-2}^{2}(3x^8-2)\,dx = 2\int_0^2(3x^8-2)\,dx = 2\left(\frac{x^9}{3}-2x\right)\Big|_0^2 = \left(\frac{1024}{3}\right)-8 = \frac{1000}{3}$.

5.4.14 $\int_{-\pi/4}^{\pi/4}\cos x\,dx = 2\int_0^{\pi/4}\cos x\,dx = 2\,(\sin x)\Big|_0^{\pi/4} = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$.

5.4.15 $\int_{-2}^{2}(x^2+x^3)\,dx = 2\int_0^2 x^2\,dx = 2\left(\frac{x^3}{3}\right)\Big|_0^2 = 2\left(\frac{8}{3}-0\right) = \frac{16}{3}$.

5.4.16 Because $x^2\sin x$ is an odd function, $\int_{-\pi}^{\pi} x^2\sin x\,dx = 0$

5.4.17 Note that the first two terms of the integrand form an odd function, and the last two terms form an even function. $\int_{-2}^{2}(x^9-3x^5+2x^2-10)\,dx = 2\int_0^2(2x^2-10)\,dx = 2\left(\frac{2x^3}{3}-10x\right)\Big|_0^2 = \frac{32}{3}-40 = -\frac{88}{3}$.

5.4.18 $\int_{-\pi/2}^{\pi/2} 5\sin x\,dx = 0$ because the integrand is an odd function.

5.4.19 Because the integrand is an odd function and the interval is symmetric about 0, this integral's value is 0.

5.4.20 $\int_{-1}^{1}(1-|x|)\,dx = 2\int_0^1(1-x)\,dx = 2\left(x-\frac{x^2}{2}\right)\Big|_0^1 = 2\left(1-\frac{1}{2}\right) = 1$.

5.4.21 $\sec^2 x$ is even, so the value of this integral is $2\int_0^{\pi/4}\sec^2 x\,dx = 2\,(\tan x)\Big|_0^{\pi/4} = 2\cdot(1-0) = 2$.

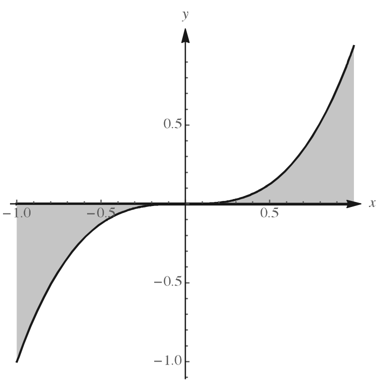
5.4.22 Recall that the tangent function is an odd function, so the value of this integral is 0.

5.4.23 The integrand is an odd function, so the value of this integral is zero.

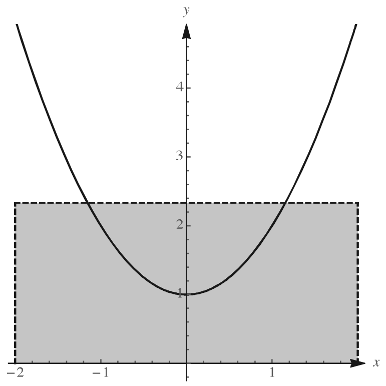
5.4.24 The function $1-|x|^3$ is even, so the value of this integral is $2\int_0^2(1-x^3)\,dx = 2\left(x-\frac{x^4}{4}\right)\Big|_0^2 = 2(2-4) = -4$.

5.4.25

The average value is $\frac{1}{1-(-1)}\int_{-1}^{1}x^3\,dx = \frac{1}{2}\left(x^4/4\right)\Big|_{-1}^{1} = 0$.



5.4.26

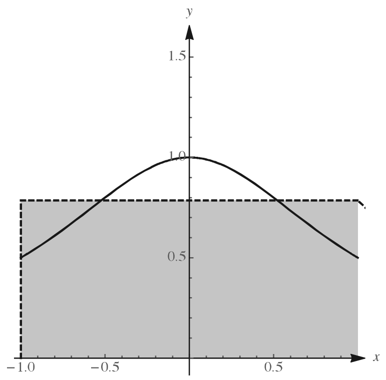


The average value is

$$\frac{1}{2-(-2)}\int_{-2}^{2}(x^2+1)\,dx$$

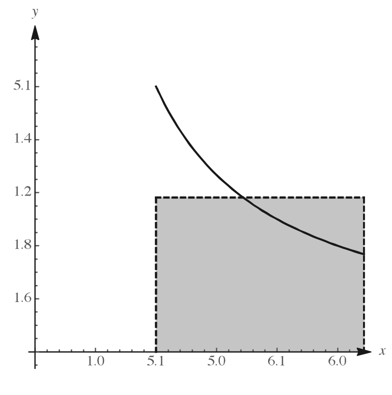
$$= \frac{1}{4}\left(x^3/3+x\right)\Big|_{-2}^{2} = \frac{8/3+2-(-8/3-2)}{4} = \frac{7}{3}.$$

5.4.27



The average value is $\frac{1}{1-(-1)}\int_{-1}^{1}\frac{1}{x^2+1}\,dx = \frac{1}{2}\tan^{-1}x\Big|_{-1}^{1} = \frac{\pi/4-(-\pi/4)}{2} = \frac{\pi}{4}.$

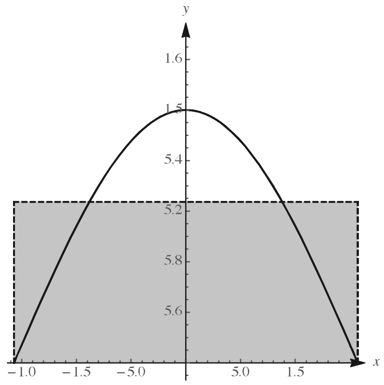
5.4.28



The average value is $\frac{1}{e-1}\int_{1}^{e}\frac{1}{x}\,dx = \frac{1}{e-1}\left(\ln|x|\right)\Big|_{1}^{e} = \frac{1}{e-1} \approx 0.582.$

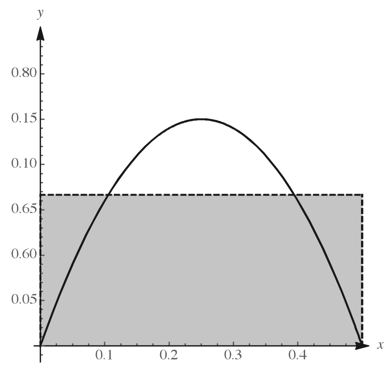
5.4.29

The average value is
$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{1}{\pi}\left(\sin x\right)\Big|_{-\pi/2}^{\pi/2}$
$= \frac{1}{\pi}\cdot(1 - -1) = \frac{2}{\pi} \approx 0.6366.$



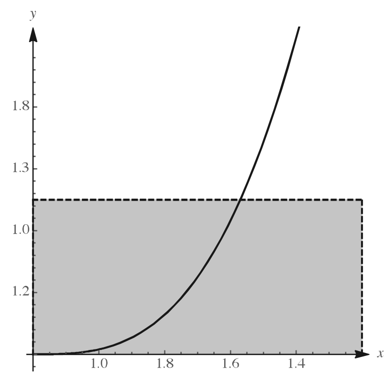
5.4.30

The average value is $\frac{1}{1}\int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx 0.1667.$



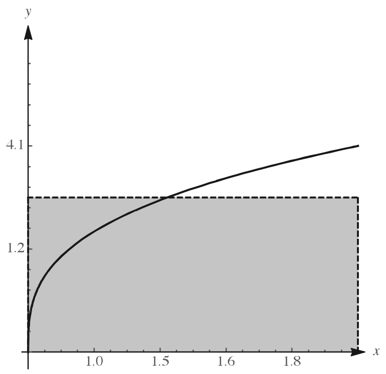
5.4.31

The average value is $\frac{1}{1}\int_0^1 x^n\,dx = \left(\frac{x^{n+1}}{n+1}\right)\Big|_0^1 = \frac{1}{n+1}$. The picture shown is for the case $n = 3$.



5.4.32

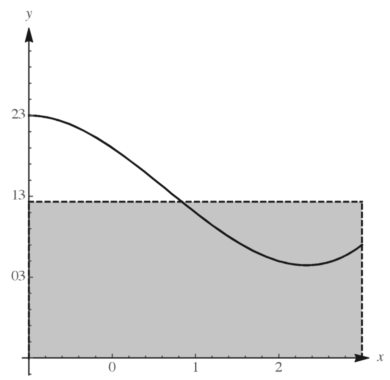
The average value is $\frac{1}{1}\int_0^1 x^{1/n}\,dx = \left(\frac{x^{(n+1)/n}}{(n+1)/n}\right)\Big|_0^1 = \frac{n}{n+1}$. The picture shown is for the case $n = 3$.



5.4.33 The average distance to the axis is given by $\frac{1}{20}\int_0^{20} 30x(20-x)\,dx$. This is equal to $\frac{1}{20}\int_0^{20}(600x - 30x^2)\,dx = \frac{1}{20}\left(300x^2 - 10x^3\right)\Big|_0^{20} = 2000$.

5.4.34

The average value is $\frac{1}{4-0}\int_0^4 (x^3 - 5x^2 + 30)\,dx = \frac{1}{4}\left(\frac{x^4}{4} - \frac{5x^3}{3} + 30x\right)\Big|_0^4 = \frac{1}{4}(64 - \frac{320}{3} + 120) - 0 = \frac{58}{3}$.



5.4.35 The average velocity is

$$\frac{1}{6-0}\int_0^6 (t^2+3t)\,dt = \frac{1}{6}\left(\frac{t^3}{3} + \frac{3t^2}{2}\right)\Big|_0^6 = \frac{1}{6}(72+54-0) = \frac{1}{6}(126) = 21\,\text{m/s}.$$

5.4.36 The average velocity is

$$\frac{1}{4-0}\int_0^4 (-32t+64)\,dt = \frac{1}{4}\left(-16t^2+64t\right)\Big|_0^4 = \frac{1}{4}(-256+256) = 0\,\text{ft/s}.$$

5.4.37 The average height is $\frac{1}{\pi}\int_0^{\pi} 10\sin x\,dx = \frac{1}{\pi}\left(-10\cos x\right)\Big|_0^{\pi} = \frac{1}{\pi}(10 - -10) = \frac{20}{\pi}$.

5.4.38 The average height is $\frac{1}{2\pi}\int_{-\pi}^{\pi}(5+5\cos x)\,dx = \frac{1}{2\pi}\left(5x + 5\sin x\right)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}(5\pi - -5\pi) = 5$.

5.4.39 The average value is $\frac{1}{4}\int_0^4 (8-2x)\,dx = \frac{1}{4}\left(8x - x^2\right)\Big|_0^4 = 4$. The function has a value of 4 when $8 - 2x = 4$, which occurs when $x = 2$.

5.4.40 The average value is $\frac{1}{2}\int_0^2 e^x\,dx = \frac{1}{2}(e^x)\Big|_0^2 = \frac{e^2-1}{2}$.The function attains this value when $\frac{e^2-1}{2} = e^x$, which is when $x = \ln\left(\frac{e^2-1}{2}\right) \approx 1.1614$.

5.4.41 The average value is $\frac{1}{a}\int_0^a \left(1-\frac{x^2}{a^2}\right)dx = \frac{1}{a}\left(x-\frac{x^3}{3a^2}\right)\Big|_0^a = \frac{2}{3}$. The function attains this value when $\frac{2}{3} = 1-\frac{x^2}{a^2}$, which is when $x^2 = \frac{a^2}{3}$, which on the given interval occurs for $x = \sqrt{3}a/3$.

5.4.42 The average value is $\frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x\,dx = \frac{1}{4}(-\cos x)\Big|_0^\pi = \frac{1}{4}(1--1) = \frac{1}{2}$. The function attains this value when $\frac{1}{2}\cdot\frac{4}{\pi} = \sin x$, which is when $x = \sin^{-1}\frac{2}{\pi} \approx 0.690107$ and for $x \approx 2.45149$.

5.4.43 The average value is $\frac{1}{2}\int_{-1}^1 (1-|x|)\,dx = \frac{1}{2}\int_{-1}^0 (1+x)\,dx + \frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\left(x+\frac{x^2}{2}\right)\Big|_{-1}^0 + \frac{1}{2}\left(x-\frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$. The function attains this value twice, once on $[-1,0]$ when $1+x = \frac{1}{2}$ which occurs when $x = -\frac{1}{2}$, and once on $[0,1]$ when $1-x = \frac{1}{2}$ which occurs when $x = \frac{1}{2}$.

5.4.44 The average value is given by $\frac{1}{3}\int_1^4 1/x\,dx = \frac{1}{3}(\ln x)\Big|_1^4 = \frac{1}{3}(\ln 4)$. The function attains this value when $x = \frac{3}{\ln 4} \approx 2.164$.

5.4.45

a. True. Because of the symmetry, the net area between 0 and 4 will be twice the net area between 0 and 2.

b. True. This follows because the symmetry implies that the net area from a to $a+2$ is the opposite of the net area from $a-2$ to a.

c. True. If $f(x) = cx+d$ on $[a,b]$ the value at the midpoint is $c\cdot\frac{a+b}{2}+d$, and the average value is $\frac{1}{b-a}\int_a^b (cx+d)\,dx = \frac{1}{b-a}\left(\frac{cx^2}{2}+dx\right)\Big|_a^b = \frac{1}{b-a}\left(\frac{cb^2}{2}+db-\left(\frac{ca^2}{2}+da\right)\right) = \frac{c}{2}\cdot(a+b)+d$.

d. False, for example, when $a=1$, we have that the maximum value of $x-x^2$ on $[0,1]$ occurs at $\frac{1}{2}$ and is equal to $\frac{1}{4}$, but the average value is $\int_0^1 (x-x^2)\,dx = \left(\frac{x^2}{2}-\frac{x^2}{3}\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$.

5.4.46

a. $d^2 = x^2+y^2 = x^2+b^2(1-(x^2/a^2))$. The average value of d^2 is $\frac{1}{2a}\int_{-a}^a \left(b^2+\left(1-\frac{b^2}{a^2}\right)x^2\right)dx = \frac{1}{2a}\left(b^2x+\frac{(1-(b^2/a^2))\,x^3}{3}\right)\Big|_{-a}^a = \frac{1}{2a}\left(b^2a+\frac{a^3}{3}-\frac{b^2a}{3}-\left(-b^2a-\frac{a^3}{3}+\frac{b^2a}{3}\right)\right) = \frac{2b^2}{3}+\frac{a^2}{3}$.

b. If $a=b=R$, the above becomes $\frac{2R^2}{3}+\frac{R^2}{3} = R^2$.

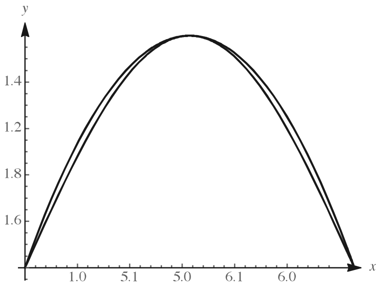
c. $D^2 = (x-\sqrt{a^2-b^2})^2+y^2 = x^2-2x\sqrt{a^2-b^2}+y^2+a^2-b^2 = \left(1-\frac{b^2}{a^2}\right)x^2-2\sqrt{a^2-b^2}x+a^2$. So the average value of D^2 is $\frac{1}{2a}\int_{-a}^a D^2\,dx = \frac{1}{2a}\int_{-a}^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]dx - \frac{1}{a}\int_{-a}^a x\sqrt{a^2-b^2}\,dx = \frac{1}{a}\int_0^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]dx + 0 = \frac{1}{3}(a^2-b^2)+a^2 = \frac{4a^2-b^2}{3}$.

5.4.47 The average height of the arch is given by

$$\frac{1}{630}\int_{-315}^{315}\left(630-\frac{630}{315^2}x^2\right)dx=\frac{630}{630}\left(x-\frac{x^3}{3\cdot 315^2}\right)\Bigg|_{-315}^{315}=(315-105-(-315+105))=420 \text{ ft.}$$

5.4.48

a. Note that $\frac{d}{dx}\sin x = \cos x$, which is zero when $x=\pi/2$, and because the derivative is positive on $(0,\pi/2)$ and negative on $(\pi/2,0)$, there is a maximum at $x=\pi/2$. Similarly, $\frac{d}{dx}\frac{4\pi x-4x^2}{\pi^2}=\frac{4\pi-8x}{\pi^2}$, which is zero when $x=\pi/2$, and this function is increasing on $(0,\pi/2)$ and decreasing on $(\pi/2,\pi)$, so it also has a maximum at $\pi/2$. Also, both functions have the value 1 at $x=\pi/2$.



b. On $(0,\pi)$, the sine function is always less than or equal to the other function.

c. The average values are

$$\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left(-\cos x\right)\Big|_0^{\pi}=\frac{2}{\pi}$$

and

$$\frac{1}{\pi}\cdot\frac{4}{\pi^2}\int_0^{\pi}\pi x-x^2\,dx=\frac{4}{\pi^3}\left(\frac{\pi x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^{\pi}=\frac{4}{\pi^3}\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)=\frac{2}{3}.$$

5.4.49 $f(g(-x))=f(g(x))$, so $f(g(x))$ is an even function, and $\int_{-a}^{a}f(g(x))\,dx=2\int_0^a f(g(x))\,dx$.

5.4.50 $f(p(-x))=f(-p(x))=f(p(x))$, and thus $f(p(x))$ is an even function. Therefore, $\int_{-a}^{a}f(p(x))\,dx=2\int_0^a f(p(x))\,dx$.

5.4.51 $p(g(-x))=p(g(x))$, so $p(g(x))$ is an even function, and $\int_{-a}^{a}p(g(x))\,dx=2\int_0^a p(g(x))\,dx$.

5.4.52 $p(q(-x))=p(-q(x))=-p(q(x))$, so $p(q(x))$ is an odd function, and $\int_{-a}^{a}p(q(x))\,dx=0$.

5.4.53

a. The average value is $\int_0^1(ax-ax^2)\,dx=\left(\frac{ax^2}{2}-\frac{ax^3}{3}\right)\Big|_0^1=\frac{a}{2}-\frac{a}{3}=\frac{a}{6}$.

b. The function is equal to its average value when $\frac{a}{6}=ax-ax^2$ which occurs when $6x-6x^2=1$, so when $6x^2-6x+1=0$. On the given interval, this occurs for $x=\frac{6\pm\sqrt{12}}{12}=\frac{3\pm\sqrt{3}}{6}$.

5.4.54

a. $f(0)=\dfrac{\int_a^b x\,dx}{\int_a^b 1\,dx}=\dfrac{\frac{b^2-a^2}{2}}{b-a}=\dfrac{a+b}{2}$.

b. $f\left(-\dfrac{3}{2}\right)=\dfrac{\int_a^b x^{-1/2}\,dx}{\int_a^b x^{-3/2}\,dx}=\dfrac{\frac{2}{1}(b^{1/2}-a^{1/2})}{-\frac{2}{1}\cdot(b^{-1/2}-a^{-1/2})}\cdot\dfrac{\sqrt{ab}}{\sqrt{ab}}=\dfrac{b^{1/2}-a^{1/2}}{b^{1/2}-a^{1/2}}\cdot\sqrt{ab}=\sqrt{ab}$.