QUICK CHECK 3 Explain how to simplify the integrand of $\int \frac{x+1}{x-1}\,dx$ before integrating. ◄

EXAMPLE 6 **Complete the square** Evaluate $\int \frac{dx}{\sqrt{-x^2 - 8x - 7}}$.

SOLUTION We don't see an integral in Table 8.1 that looks like the given integral, so some preliminary work is needed. In this case, the key is to complete the square on the polynomial in the denominator. We find that

$$\begin{aligned} -x^2 - 8x - 7 &= -(x^2 + 8x + 7) \\ &= -(x^2 + 8x \underbrace{+ 16 - 16}_{\text{add and subtract 16}} + 7) && \text{Complete the square.} \\ &= -((x+4)^2 - 9) && \text{Factor and combine terms.} \\ &= 9 - (x+4)^2. && \text{Rearrange terms.} \end{aligned}$$

After a change of variables, the integral is recognizable:

$$\begin{aligned} \int \frac{dx}{\sqrt{-7 - 8x - x^2}} &= \int \frac{dx}{\sqrt{9 - (x+4)^2}} && \text{Complete the square.} \\ &= \int \frac{du}{\sqrt{9 - u^2}} && u = x + 4,\ du = dx \\ &= \sin^{-1}\frac{u}{3} + C && \text{Table 8.1} \\ &= \sin^{-1}\left(\frac{x+4}{3}\right) + C. && \text{Replace } u \text{ with } x + 4. \end{aligned}$$

Related Exercises 31, 37 ◄

QUICK CHECK 4 Express $x^2 + 6x + 16$ in terms of a perfect square. ◄

The techniques illustrated in this section are designed to transform or simplify an integrand before you apply a specific method. In fact, these ideas may help you recognize the best method to use. Keep them in mind as you learn new integration methods and improve your integration skills.

SECTION 8.1 EXERCISES

Getting Started

1. What change of variables would you use for the integral $\int (4 - 7x)^{-6}\,dx$?
2. Evaluate $\int (\sec x + 1)^2\,dx$. (*Hint:* Expand $(\sec x + 1)^2$ first.)
3. What trigonometric identity is useful in evaluating $\int \sin^2 x\,dx$?
4. Let $f(x) = \frac{4x^3 + x^2 + 4x + 2}{x^2 + 1}$. Use long division to show that $f(x) = 4x + 1 + \frac{1}{x^2 + 1}$ and use this result to evaluate $\int f(x)\,dx$.
5. Describe a first step in integrating $\int \frac{10}{x^2 - 4x - 9}\,dx$.
6. Evaluate $\int \frac{2x + 1}{x^2 + 1}\,dx$ using the following steps.
   a. Fill in the blanks: By splitting the integrand into two fractions, we have $\int \frac{2x+1}{x^2+1}\,dx = \int \_\_\_\, dx + \int \_\_\_\, dx$.
   b. Evaluate the two integrals on the right side of the equation in part (a).

Practice Exercises

7–64. Integration review *Evaluate the following integrals.*

7. $\int \frac{dx}{(3 - 5x)^4}$
8. $\int (9x - 2)^{-3}\,dx$
9. $\int_0^{3\pi/8} \sin\left(2x - \frac{\pi}{4}\right)dx$
10. $\int e^{3-4x}\,dx$
11. $\int \frac{\ln 2x}{x}\,dx$
12. $\int_{-5}^{0} \frac{dx}{\sqrt{4 - x}}$
13. $\int \frac{e^x}{e^x + 1}\,dx$
14. $\int_0^1 x\,3^{x^2+1}\,dx$
15. $\int_1^{e^2} \frac{\ln^2(x^2)}{x}\,dx$
16. $\int_0^1 \frac{t^2}{1 + t^6}\,dt$
17. $\int_1^2 s(s - 1)^9\,ds$
18. $\int_3^7 (t - 6)\sqrt{t - 3}\,dt$
19. $\int \frac{(\ln w - 1)^7 \ln w}{w}\,dw$
20. $\int e^x(1 + e^x)^9(1 - e^x)\,dx$
21. $\int \frac{x + 2}{x^2 + 4}\,dx$
22. $\int \frac{\sin x + 1}{\cos x}\,dx$

23. $\int e^x \csc(3e^x + 4)\,dx$

24. $\int_0^9 \frac{x^{5/2} - x^{1/2}}{x^{3/2}}\,dx$

25. $\int_0^{\pi/4} \frac{\sec\theta + \csc\theta}{\sec\theta\csc\theta}\,d\theta$

26. $\int \frac{4 + e^{-2x}}{e^{3x}}\,dx$

27. $\int \frac{2 - 3x}{\sqrt{1 - x^2}}\,dx$

28. $\int \frac{3x + 1}{\sqrt{4 - x^2}}\,dx$

29. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x}\,dx$

30. $\int_{e^{\pi/4}}^{e^{\pi/3}} \frac{\cot(\ln x)}{x}\,dx$

31. $\int \frac{dx}{x^2 - 2x + 10}$

32. $\int_0^2 \frac{x}{x^2 + 4x + 8}\,dx$

33. $\int \frac{x^3 + 2x^2 + 5x + 3}{x^2 + x + 2}\,dx$

34. $\int_2^4 \frac{x^2 + 2}{x - 1}\,dx$

35. $\int_0^1 \frac{t^4 + t^3 + t^2 + t + 1}{t^2 + 1}\,dt$

36. $\int \frac{t^3 - 2}{t + 1}\,dt$

37. $\int \frac{d\theta}{\sqrt{27 - 6\theta - \theta^2}}$

38. $\int \frac{x}{x^4 + 2x^2 + 1}\,dx$

39. $\int \frac{d\theta}{1 + \sin\theta}$

40. $\int \frac{1 - x}{1 - \sqrt{x}}\,dx$

41. $\int \frac{dx}{\sec x - 1}$

42. $\int \frac{d\theta}{1 - \csc\theta}$

43. $\int \frac{\cosh 3x}{1 + \sinh 3x}\,dx$

44. $\int_0^{\sqrt{3}} \frac{6x^3}{\sqrt{x^2 + 1}}\,dx$

45. $\int \frac{e^x}{e^x - 2e^{-x}}\,dx$

46. $\int \frac{e^{2z}}{e^{2z} - 4e^{-z}}\,dz$

47. $\int \frac{dx}{x^{-1} + 1}$

48. $\int \frac{dy}{y^{-1} + y^{-3}}$

49. $\int \sqrt{9 + \sqrt{t + 1}}\,dt$

50. $\int_4^9 \frac{dx}{1 - \sqrt{x}}$

51. $\int_{-1}^0 \frac{x}{x^2 + 2x + 2}\,dx$

52. $\int_{\pi/6}^{\pi/2} \frac{dy}{\sin y}$

53. $\int e^x \sec(e^x + 1)\,dx$

54. $\int_0^1 \sqrt{1 + \sqrt{x}}\,dx$

55. $\int \sin x \sin 2x\,dx$

56. $\int_0^{\pi/2} \sqrt{1 + \cos 2x}\,dx$

57. $\int \frac{dx}{x^{1/2} + x^{3/2}}$

58. $\int_0^1 \frac{dp}{4 - \sqrt{p}}$

59. $\int \frac{x - 2}{x^2 + 6x + 13}\,dx$

60. $\int_0^{\pi/4} 3\sqrt{1 + \sin 2x}\,dx$

61. $\int \frac{e^x}{e^{2x} + 2e^x + 1}\,dx$

62. $\int \frac{-x^5 - x^4 - 2x^3 + 4x + 3}{x^2 + x + 1}\,dx$

63. $\int_1^3 \frac{2}{x^2 + 2x + 1}\,dx$

64. $\int_0^2 \frac{2}{s^3 + 3s^2 + 3s + 1}\,ds$

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\int \frac{3}{x^2 + 4}\,dx = \int \frac{3}{x^2}\,dx + \int \frac{3}{4}\,dx.$

b. Long division simplifies the evaluation of the integral $\int \frac{x^3 + 2}{3x^4 + x}\,dx.$

c. $\int \frac{1}{\sin x + 1}\,dx = \ln|\sin x + 1| + C.$

d. $\int \frac{dx}{e^x} = \ln e^x + C.$

66–67. Integrals of cot *x* and csc *x*

66. Use a change of variables to prove that $\int \cot x\,dx = \ln|\sin x| + C$. (*Hint:* See Example 1.)

67. Prove that $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$. (*Hint:* See Example 2.)

68. Different methods

a. Evaluate $\int \cot x \csc^2 x\,dx$ using the substitution $u = \cot x$.

b. Evaluate $\int \cot x \csc^2 x\,dx$ using the substitution $u = \csc x$.

c. Reconcile the results in parts (a) and (b).

69. Different substitutions

a. Evaluate $\int \tan x \sec^2 x\,dx$ using the substitution $u = \tan x$.

b. Evaluate $\int \tan x \sec^2 x\,dx$ using the substitution $u = \sec x$.

c. Reconcile the results in parts (a) and (b).

70. Different methods Let $I = \int \frac{x + 2}{x + 4}\,dx$.

a. Evaluate I after first performing long division on the integrand.

b. Evaluate I without performing long division on the integrand.

c. Reconcile the results in parts (a) and (b).

71. Different methods Let $I = \int \frac{x^2}{x + 1}\,dx$.

a. Evaluate I using the substitution $u = x + 1$.

b. Evaluate I after first performing long division on the integrand.

c. Reconcile the results in parts (a) and (b).

72. Area of a region between curves Find the area of the entire region bounded by the curves $y = \frac{x^3}{x^2 + 1}$ and $y = \frac{8x}{x^2 + 1}$.

73. Area of a region between curves Find the area of the region bounded by the curves $y = \frac{x^2}{x^3 - 3x}$ and $y = \frac{1}{x^3 - 3x}$ on the interval $[2, 4]$.

74. Volume of a solid Consider the region R bounded by the graph of $f(x) = \frac{1}{x + 2}$ and the x-axis on the interval $[0, 3]$. Find the volume of the solid formed when R is revolved about the y-axis.

75. Volume of a solid Consider the region R bounded by the graph of $f(x) = \sqrt{x^2 + 1}$ and the x-axis on the interval $[0, 2]$. Find the volume of the solid formed when R is revolved about the y-axis.

Explorations and Challenges

76. Different substitutions

a. Show that $\int \frac{dx}{\sqrt{x - x^2}} = \sin^{-1}(2x - 1) + C$ using either $u = 2x - 1$ or $u = x - \frac{1}{2}$.