➤ In Example 4a, the two methods produce results that look different but are equivalent. This is common when evaluating trigonometric integrals. For instance, evaluate $\int \sin^4 x\,dx$ using reduction formula 1, and compare your answer to

$$\frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C,$$

the solution found in Example 1b.

Because the integrand also has an odd power of tan x, an alternative solution is to split off a factor of sec x tan x and prepare the integral for the substitution $u = \sec x$:

$$\begin{aligned}
\int \tan^3 x \sec^4 x\,dx &= \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec^3 x \cdot \sec x \tan x\,dx \\
&= \int (\sec^2 x - 1)\sec^3 x \cdot \sec x \tan x\,dx \\
&= \int (u^2 - 1)\,u^3\,du && u = \sec x;\ du = \sec x \tan x\,dx \\
&= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C. && \text{Evaluate; } u = \sec x.
\end{aligned}$$

The apparent difference in the two solutions given here is reconciled by using the identity $1 + \tan^2 x = \sec^2 x$ to transform the second result into the first, the only difference being an additive constant, which is part of C.

b. In this case, we write the even power of tan x in terms of sec x:

$$\begin{aligned}
\int \tan^2 x \sec x\,dx &= \int (\sec^2 x - 1)\sec x\,dx && \tan^2 x = \sec^2 x - 1 \\
&= \underbrace{\int \sec^3 x\,dx}_{\text{reduction formula 4}} - \int \sec x\,dx && \text{Expand.} \\
&= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx - \int \sec x\,dx && \text{Use reduction formula.} \\
&= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C. && \text{Add secant integrals; use Table 8.1 in Section 8.1.}
\end{aligned}$$

Related Exercises 33–35 ◀

Table 8.3 summarizes the methods used to integrate $\int \tan^m x \sec^n x\,dx$. Analogous techniques are used for $\int \cot^m x \csc^n x\,dx$.

Table 8.3

$\int \tan^m x \sec^n x\,dx$	**Strategy**
n even and positive, m real	Split off $\sec^2 x$, rewrite the remaining even power of sec x in terms of tan x, and use $u = \tan x$.
m odd and positive, n real	Split off sec x tan x, rewrite the remaining even power of tan x in terms of sec x, and use $u = \sec x$.
m even and positive, n odd and positive	Rewrite the even power of tan x in terms of sec x to produce a polynomial in sec x; apply reduction formula 4 to each term.

SECTION 8.3 EXERCISES

Getting Started

1. State the half-angle identities used to integrate $\sin^2 x$ and $\cos^2 x$.
2. State the three Pythagorean identities.
3. Describe the method used to integrate $\sin^3 x$.
4. Describe the method used to integrate $\sin^m x \cos^n x$, for m even and n odd.
5. What is a reduction formula?
6. How would you evaluate $\int \cos^2 x \sin^3 x\,dx$?
7. How would you evaluate $\int \tan^{10} x \sec^2 x\,dx$?
8. How would you evaluate $\int \sec^{12} x \tan x\,dx$?

Practice Exercises

9–61. Trigonometric integrals *Evaluate the following integrals.*

9. $\int \cos^3 x\,dx$
10. $\int \sin^3 x\,dx$
11. $\int \sin^2 3x\,dx$
12. $\int \cos^4 2\theta\,d\theta$
13. $\int \sin^5 x\,dx$
14. $\int \cos^3 20x\,dx$
15. $\int \sin^3 x \cos^2 x\,dx$
16. $\int \sin^2\theta \cos^5\theta\,d\theta$
17. $\int \cos^3 x \sqrt{\sin x}\,dx$
18. $\int \sin^3\theta \cos^{-2}\theta\,d\theta$
19. $\int_0^{\pi/3} \sin^5 x \cos^{-2} x\,dx$
20. $\int \sin^{-3/2} x \cos^3 x\,dx$

21. $\int_0^{\pi/2} \cos^3 x \sqrt{\sin^3 x}\, dx$

22. $\int_{\pi/4}^{\pi/2} \sin^2 2x \cos^3 2x\, dx$

23. $\int \sin^2 x \cos^2 x\, dx$

24. $\int \sin^3 x \cos^5 x\, dx$

25. $\int \sin^2 x \cos^4 x\, dx$

26. $\int \sin^3 x \cos^{3/2} x\, dx$

27. $\int \tan^2 x\, dx$

28. $\int 6 \sec^4 x\, dx$

29. $\int \cot^4 x\, dx$

30. $\int \tan^3 \theta\, d\theta$

31. $\int 20 \tan^6 x\, dx$

32. $\int \cot^5 3x\, dx$

33. $\int 10 \tan^9 x \sec^2 x\, dx$

34. $\int \tan^9 x \sec^4 x\, dx$

35. $\int \tan x \sec^3 x\, dx$

36. $\int \tan 4x \sec^{3/2} 4x\, dx$

37. $\int \frac{\sec^4(\ln \theta)}{\theta}\, d\theta$

38. $\int \tan^5 \theta \sec^4 \theta\, d\theta$

39. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 \theta - 1}\, d\theta$

40. $\int_0^{\pi/6} \tan^5 2x \sec 2x\, dx$

41. $\int_0^{\pi/4} \sec^7 x \sin x\, dx$

42. $\int \sqrt{\tan x} \sec^4 x\, dx$

43. $\int \tan^3 4x\, dx$

44. $\int \frac{\sec^2 x}{\tan^5 x}\, dx$

45. $\int \sec^2 x \tan^{1/2} x\, dx$

46. $\int \sec^{-2} x \tan^3 x\, dx$

47. $\int \frac{\csc^4 x}{\cot^2 x}\, dx$

48. $\int \csc^{10} x \cot x\, dx$

49. $\int_{\pi/20}^{\pi/10} \csc^2 5w \cot^4 5w\, dw$

50. $\int \csc^{10} x \cot^3 x\, dx$

51. $\int (\csc^2 x + \csc^4 x)\, dx$

52. $\int_0^{\pi/8} (\tan 2x + \tan^3 2x)\, dx$

53. $\int_0^{\pi/4} \sec^4 \theta\, d\theta$

54. $\int_0^{\sqrt{\pi/2}} x \sin^3(x^2)\, dx$

55. $\int_{\pi/6}^{\pi/3} \cot^3 \theta\, d\theta$

56. $\int_0^{\pi/4} \tan^3 \theta \sec^2 \theta\, d\theta$

57. $\int_0^{\pi} (1 - \cos 2x)^{3/2}\, dx$

58. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \cos 4x}\, dx$

59. $\int_0^{\pi/2} \sqrt{1 - \cos 2x}\, dx$

60. $\int_0^{\pi/8} \sqrt{1 - \cos 8x}\, dx$

61. $\int_0^{\pi/4} (1 + \cos 4x)^{3/2}\, dx$

62. **Arc length** Find the length of the curve $y = \ln(\sec x)$, for $0 \le x \le \pi/4$.

63. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. If m is a positive integer, then $\int_0^{\pi} \cos^{2m+1} x\, dx = 0$.
   b. If m is a positive integer, then $\int_0^{\pi} \sin^m x\, dx = 0$.

64. **Sine football** Find the volume of the solid generated when the region bounded by $y = \sin x$ and the x-axis on the interval $[0, \pi]$ is revolved about the x-axis.

65. **Volume** Find the volume of the solid generated when the region bounded by $y = \sin^2 x \cos^{3/2} x$ and the x-axis on the interval $[0, \pi/2]$ is revolved about the x-axis.

66. **Particle position** A particle moves along a line with a velocity (in m/s) given by $v(t) = \sec^4 \frac{\pi t}{12}$, for $0 \le t \le 5$, where t is measured in seconds. Determine the position function $s(t)$, for $0 \le t \le 5$. Assume $s(0) = 0$.

67–70. Integrals of the form $\int \sin mx \cos nx\, dx$ *Use the following three identities to evaluate the given integrals.*

$$\sin mx \sin nx = \frac{1}{2}(\cos((m - n)x) - \cos((m + n)x))$$

$$\sin mx \cos nx = \frac{1}{2}(\sin((m - n)x) + \sin((m + n)x))$$

$$\cos mx \cos nx = \frac{1}{2}(\cos((m - n)x) + \cos((m + n)x))$$

67. $\int \sin 3x \cos 7x\, dx$

68. $\int \sin 5x \sin 7x\, dx$

69. $\int \sin 3x \sin 2x\, dx$

70. $\int \cos x \cos 2x\, dx$

Explorations and Challenges

71. Prove the following **orthogonality relations** (which are used to generate *Fourier series*). Assume m and n are integers with $m \ne n$.
   a. $\int_0^{\pi} \sin mx \sin nx\, dx = 0$
   b. $\int_0^{\pi} \cos mx \cos nx\, dx = 0$
   c. $\int_0^{\pi} \sin mx \cos nx\, dx = 0$, for $|m + n|$ even

72. **A sine reduction formula** Use integration by parts to obtain the reduction formula for positive integers n:

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n - 1) \int \sin^{n-2} x \cos^2 x\, dx.$$

Then use an identity to obtain the reduction formula

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n - 1}{n} \int \sin^{n-2} x\, dx.$$

Use this reduction formula to evaluate $\int \sin^6 x\, dx$.

73. **A tangent reduction formula** Prove that for positive integers $n \ne 1$,

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n - 1} - \int \tan^{n-2} x\, dx.$$

Use the formula to evaluate $\int_0^{\pi/4} \tan^3 x\, dx$.

74. **A secant reduction formula** Prove that for positive integers $n \ne 1$,

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n - 1} + \frac{n - 2}{n - 1} \int \sec^{n-2} x\, dx.$$

(*Hint:* Integrate by parts with $u = \sec^{n-2} x$ and $dv = \sec^2 x\, dx$.)