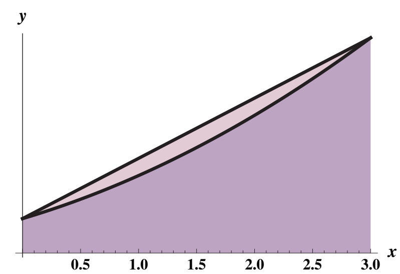
8.8.76 Let $x_j = a + j(b-a)/2n$ for $1 \le j \le 2n$ and let $x_i = a + i(b-a)/n$ and let $\overline{x_i}$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then

$$\begin{aligned} T(2n) &= \frac{b-a}{4n}(f(a) + 2\sum_{j=1}^{2n-1} f(x_j) + f(b)) \\ &= \frac{1}{2}\cdot\frac{b-a}{2n}(f(a) + 2\sum_{1\le j\le 2n-1, j \text{ even}} f(x_j) + f(b) + 2\sum_{1\le j\le 2n-1, j\text{ odd}} f(x_j)) \\ &= \frac{1}{2}\left(\frac{b-a}{2n}(f(a) + 2\sum_{i-1}^{n-1} f(x_i) + f(b)) + \frac{b-a}{n}\sum_{i=1}^{n} f(\overline{x_i})\right) = \frac{1}{2}(T(n) + M(n)). \end{aligned}$$

8.8.77 The trapezoidal rule will be an overestimate in this case. This is because of the fact that if the function is above the axis and concave up on the given interval, then each trapezoid on each subinterval lies over the area under the curve for that corresponding subinterval.



8.8.78 Let $x_i = a + i(b-a)/n$ for $1 \le i \le n$ and let $x_j = a + j(b-a)/2n$ for $1 \le j \le 2n$. We have $4T(2n) - T(n)$

$$\begin{aligned} &= \frac{b-a}{2n}(2(f(a) + 2\sum_{j=1}^{2n-1} f(x_j) + f(b)) - (f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)) \\ &= \frac{b-a}{2n}((f(a) + 4\sum_{1\le j\le 2n-1, j\text{ odd}} f(x_j) + 4\sum_{2\le j\le 2n-2, j\text{ even}} f(x_j) - 2\sum_{i=1}^{n-1} f(x_i) + f(b))) \\ &= \frac{b-a}{2n}(f(a) + 4\sum_{1\le j\le 2n-1, j\text{ odd}} f(x_j) + 4\sum_{2\le j\le 2n-2} f(x_j) - 2\sum_{2\le j\le 2n-1} f(x_j) + f(b)) \\ &= \frac{b-a}{2n}(f(a) + 4\sum_{1\le j\le 2n-1, j\text{ odd}} f(x_j) + 2\sum_{2\le j\le 2n-2, j\text{ even}} f(x_j) + f(b)) = 3S(2n). \end{aligned}$$

8.8.79 Using the given formula, we have

$$S(2n) = \frac{2M(n) + T(n)}{3}.$$

So $S(20) = \frac{1}{3}(2M(10) + T(10)) \approx 1.000001$.

8.9 Improper Integrals

8.9.1 The interval of integration is infinite or the integrand is unbounded on the interval of integration.

8.9.2 $\lim_{b\to\infty}\int_2^b \frac{dx}{x^3}$. We have

$$\lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{x^2}\right)\Bigg|_2^b = \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{4}\right) = \frac{1}{8}.$$

8.9.3 $\lim_{b\to\infty}\int_2^b \frac{dx}{x^{1/5}}$. We have

$$\lim_{b\to\infty}\frac{5}{4}\cdot x^{4/5}\Big|_2^b = \lim_{b\to\infty}\left(\frac{5}{4}\cdot b^{4/5} - \frac{5}{4}\cdot 2^{4/5}\right) = \infty.$$

The integral diverges.

8.9.4 $\int_0^1 \frac{1}{x^{1/5}}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^{1/5}}$. We have

$$\lim_{b\to 0^+}\frac{5}{4}\cdot x^{4/5}\Big|_b^1 = \lim_{b\to 0^+}\left(\frac{5}{4} - \frac{5}{4}\cdot b^{4/5}\right) = \frac{5}{4}.$$

8.9.5 This is $\int_{-\infty}^{\infty} f(x)\,dx$.

8.9.6 As shown in Example 2, this integral converges if and only if $p > 1$.

8.9.7 $\int_3^\infty x^{-2}\,dx = \lim_{b\to\infty}\int_3^b x^{-2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_3^b = \lim_{b\to\infty}\left(\frac{1}{3} - \frac{1}{b}\right) = \frac{1}{3}.$

8.9.8 $\lim_{b\to\infty}\frac{dx}{x} = \lim_{b\to\infty}\ln x\Big|_2^b = \lim_{b\to\infty}(\ln b - \ln 2) = \infty$. This integral diverges.

8.9.9 $\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_2^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\Big|_2^b = \lim_{b\to\infty} 2(\sqrt{b} - \sqrt{2}) = \infty$, so the integral diverges.

8.9.10 $\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2x}\right)\Big|_0^b = \lim_{b\to\infty}\frac{1}{2}\left(1 - e^{-2b}\right) = \frac{1}{2}.$

8.9.11 $\int_0^\infty e^{-ax}\,dx = \lim_{b\to\infty}\int_0^b e^{-ax}\,dx = \lim_{b\to\infty}\frac{e^{-ax}}{-a}\Big|_0^b = \lim_{b\to\infty}\left(-\frac{1}{ae^{ab}} + \frac{1}{a}\right) = \frac{1}{a}.$

8.9.12 $\lim_{b\to-\infty}\int_b^{-1}\frac{dx}{\sqrt[3]{x}} = \lim_{b\to-\infty}\frac{3}{2}x^{2/3}\Big|_b^{-1} = \lim_{b\to-\infty}\frac{3}{2}\left(1 - b^{2/3}\right) = -\infty$. This integral diverges.

8.9.13 $\int_0^\infty \cos x\,dx = \lim_{b\to\infty}\int_0^b \cos x\,dx = \lim_{b\to\infty}\sin x\Big|_0^b = \lim_{b\to\infty}\sin b$, which does not exist so the integral diverges.

8.9.14 $\lim_{b\to-\infty}\int_b^{-1}\frac{dx}{x^3} = \lim_{b\to-\infty} -\left(\frac{1}{2x^2}\right)\Big|_b^{-1} = \lim_{b\to-\infty}\left(-\frac{1}{2} + \frac{1}{2b^2}\right) = -\frac{1}{2}.$

8.9.15 The given integral is equal to $\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+100} + \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+100}$, if both limits exist.

$$\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+100} = \lim_{b\to-\infty}\frac{1}{10}\tan^{-1}\frac{x}{10}\Big|_b^0 = \frac{1}{10}\left(0 + \frac{\pi}{2}\right) = \frac{\pi}{20}.$$

$$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+100} = \lim_{b\to\infty}\frac{1}{10}\tan^{-1}\frac{x}{10}\Big|_0^b = \frac{1}{10}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{20}.$$

So the given integral has value $\frac{\pi}{20} + \frac{\pi}{20} = \frac{\pi}{10}$.

8.9.16 The given integral is equal to $\lim\limits_{b\to-\infty}\int_b^0 \frac{dx}{x^2+a^2} + \lim\limits_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2}$, if both limits exist.

$$\lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+a^2} = \lim_{b\to-\infty} \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_b^0 = \frac{1}{a}\left(0+\frac{\pi}{2}\right) = \frac{\pi}{2a}.$$

$$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2} = \lim_{b\to\infty} \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^b = \frac{1}{a}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{2a}.$$

So the given integral has value $\frac{\pi}{2a}+\frac{\pi}{2a}=\frac{\pi}{a}$.

8.9.17 $\lim\limits_{b\to\infty}\int_7^b \frac{dx}{(x+1)^{1/3}} = \lim\limits_{b\to\infty}\frac{3}{2}(x+1)^{2/3}\Big|_7^b = \lim\limits_{b\to\infty}\left(\frac{3}{2}(b+1)^{2/3}-\frac{3}{2}7^{2/3}\right)=\infty$.
The integral diverges.

8.9.18 $\int_2^\infty \frac{dx}{(x+2)^2} = \lim\limits_{b\to\infty}\int_2^b \frac{dx}{(x+2)^2} = \lim\limits_{b\to\infty} -\frac{1}{x+2}\Big|_2^b = \lim\limits_{b\to\infty} -\frac{1}{b+2}+\frac{1}{4}=\frac{1}{4}$.

8.9.19 Let $u = x^3+x$ so that $du=(3x^2+1)\,dx$ Substituting gives

$$\int_2^\infty \frac{1}{u}\,du = \lim_{b\to\infty}\int_2^b \frac{1}{u}\,du = \lim_{b\to\infty}\ln u\,\Big|_2^b = \lim_{b\to\infty}(\ln b-\ln 2)=\infty.$$

The given integral diverges.

8.9.20 $\int_1^\infty 2^{-x}\,dx = \lim\limits_{b\to\infty}\int_1^b 2^{-x}\,dx = \lim\limits_{b\to\infty}\left(\frac{-1}{\ln 2(2^x)}\right)\Big|_1^b = \lim\limits_{b\to\infty}\left(\frac{-1}{\ln 2(2^b)}+\frac{1}{2\ln 2}\right)=\frac{1}{2\ln 2}$.

8.9.21 $\int_2^\infty \frac{\cos(\pi/x)}{x^2}\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{\cos(\pi/x)}{x^2}\,dx = \lim\limits_{b\to\infty}\left(-\frac{1}{\pi}\sin(\pi/x)\right)\Big|_2^b = \lim\limits_{b\to\infty}\frac{1}{\pi}(1-\sin(\pi/b))=\frac{1}{\pi}$.

8.9.22 Note that $\int \frac{1}{z^2}\sin(\pi/z)\,dz = -\frac{1}{\pi}\int \sin u\,du = \frac{1}{\pi}\cos u + C = \frac{1}{\pi}\cos(\pi/z)+C$. Thus,

$$\lim_{b\to-\infty}\int_b^{-2}\frac{1}{z^2}\sin(\pi/z)\,dz = \lim_{b\to-\infty}\frac{1}{\pi}\cos(\pi/z)\Big|_b^{-2} = \lim_{b\to-\infty}\frac{1}{\pi}(0-\cos(\pi/b)) = -\frac{1}{\pi}.$$

8.9.23 $\int_0^\infty \frac{e^u}{e^{2u}+1}\,du = \lim\limits_{b\to\infty}\int_0^b \frac{e^u}{e^{2u}+1}\,du$. Let $u=e^u$ so that $du=e^u\,du$. After substitution we have $\lim\limits_{b\to\infty}\int_1^{e^b}\frac{1}{u^2+1}\,du = \lim\limits_{b\to\infty}\left(\tan^{-1}(u)\right)\Big|_1^{e^b} = \lim\limits_{b\to\infty}(\tan^{-1}(e^b)-\tan^{-1}(1)) = \pi/2-\pi/4=\pi/4$.

8.9.24 $\int_{-\infty}^a \sqrt{e^x}\,dx = \lim\limits_{b\to-\infty}\int_b^a e^{x/2}\,dx = \lim\limits_{b\to-\infty}\left(2e^{x/2}\right)\Big|_b^a = \lim\limits_{b\to-\infty}2(e^{a/2}-e^{b/2})=2e^{a/2}$.

8.9.25 $\int_{-\infty}^\infty \frac{e^{3x}}{1+e^{6x}}\,dx = \lim\limits_{a\to-\infty}\int_a^0 \frac{e^{3x}}{1+e^{6x}}\,dx + \lim\limits_{b\to\infty}\int_0^b \frac{e^{3x}}{1+e^{6x}}\,dx$. Let $u=e^{3x}$ so that $du=3e^{3x}$. Note that

$$\int \frac{e^{3x}}{1+e^{6x}}\,dx = \frac{1}{3}\int\frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}e^{3x}+C.$$

So we have

$$\frac{1}{3}\lim_{a\to-\infty}\left(\tan^{-1}1-\tan^{-1}e^{3a}\right)+\frac{1}{3}\lim_{b\to\infty}\left(\tan^{-1}e^{3b}-\tan^{-1}1\right) = \frac{1}{3}\left(\frac{\pi}{4}-0\right)+\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{6}.$$

8.9.26 $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\, dx = \lim_{a\to-\infty} \int_a^0 \frac{x}{(x^2+1)^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{x}{(x^2+1)^2}\, dx$. Let $u = x^2+1$ so that $du = 2x\, dx$. Note that

$$\int \frac{x}{(x^2+1)^2}\, dx = \frac{1}{2}\int u^{-2}\, du = -\frac{1}{2}\cdot\frac{1}{u} + C = -\frac{1}{2}\cdot\frac{1}{x^2+1} + C.$$

Then our integral is given by

$$\lim_{a\to-\infty} -\frac{1}{2}\left(1 - \frac{1}{a^2}\right) + \lim_{b\to\infty} -\frac{1}{2}\left(\frac{1}{b^2+1} - 1\right) = -\frac{1}{2} + \frac{1}{2} = 0.$$

8.9.27

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-x^2}\, dx &= \lim_{b\to-\infty}\int_b^0 xe^{-x^2} + \lim_{b\to\infty}\int_0^b xe^{-x^2}\\ &= \lim_{b\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_b^0 + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^b\\ &= \lim_{b\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-b^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{2} = 0.\end{aligned}$$

8.9.28 $\int_1^{\infty} \frac{\tan^{-1} s}{s^2+1}\, ds = \lim_{b\to\infty}\int_1^b \frac{\tan^{-1} s}{s^2+1}\, ds = \lim_{b\to\infty}\left(\frac{1}{2}(\tan^{-1} s)^2\right)\Big|_1^b = \lim_{b\to\infty}\frac{1}{2}\left((\tan^{-1} b)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{3\pi^2}{32}$.

8.9.29 By symmetry, the given integral is equal to $2\int_0^{\infty} \frac{(\tan^{-1} t)^2}{t^2+1}\, dt$ if the integral exists. Let $u = \tan^{-1} t$ so that $du = \frac{dt}{t^2+1}$. We have

$$2\lim_{b\to\infty}\int_0^{\tan^{-1} b} u^2\, du = 2\lim_{b\to\infty}\frac{u^3}{3}\Big|_0^{\tan^{-1} b} = 2\lim_{b\to\infty}\frac{(\tan^{-1} b)^3}{3} = \frac{2(\pi/2)^3}{3} = \frac{\pi^3}{12}.$$

8.9.30 $\int_{-\infty}^0 e^x\, dx = \lim_{b\to-\infty}\int_b^0 e^x\, dx = \lim_{b\to-\infty} e^x\Big|_b^0 = \lim_{b\to-\infty}\left(1 - e^b\right) = 1.$

8.9.31 $\int_1^{\infty} \frac{1}{v(v+1)}\, dv = \lim_{b\to\infty}\int_1^b \left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \lim_{b\to\infty} \ln\left|\frac{v}{v+1}\right|\Big|_1^b = \lim_{b\to\infty}\ln\left|\frac{b}{b+1}\right| - \ln(1/2) = 0 - \ln(1/2) = \ln 2.$

8.9.32 Write

$$\int_1^{\infty} \frac{1}{x^2(x-1)}\, dx = \int_1^2 \frac{1}{x^2(x-1)}\, dx + \int_2^{\infty}\frac{1}{x^2(x-1)} dx,$$

if both integrals converge.

However,

$$\begin{aligned}\lim_{c\to1^+}\int_c^2 \frac{1}{x^2(x-1)}\, dx &= \lim_{c\to1^+}\int_c^2 \left(-\frac{1}{x^2} + \frac{1}{x-1} - \frac{1}{x}\right) dx\\ &= \lim_{c\to1^+}\left(\frac{1}{x} + \ln\left|\frac{x-1}{x}\right|\Big|_c^2\right) = \lim_{c\to1^+}\left(\frac{1}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{c} - \ln\left(\frac{c-1}{c}\right)\right)\end{aligned}$$

which diverges. Therefore, $\int_1^{\infty} \frac{1}{x^2(x-1)}\, dx$ diverges.

8.9.33 $\displaystyle\int_2^\infty \frac{dy}{y\ln y} = \lim_{b\to\infty}\int_2^b \frac{dy}{y\ln y} = \lim_{b\to\infty}(\ln(\ln y))\Big|_2^b = \lim_{b\to\infty}(\ln(\ln b)-\ln(\ln 2)) = \infty$, so the integral diverges.

8.9.34 $\displaystyle\int_{-\infty}^{-4/\pi}\frac{\sec^2(1/x)}{x^2}\,dx = \lim_{b\to-\infty}\int_b^{-4/\pi}\frac{\sec^2(1/x)}{x^2}\,dx$. Let $u=1/x$ so that $du=-1/x^2\,dx$. Then we have $\displaystyle\lim_{b\to-\infty}\int_{1/b}^{-\pi/4}(-\sec^2 u)\,du = -\lim_{b\to-\infty}\tan u\Big|_{1/b}^{-\pi/4} = -\lim_{b\to-\infty}(-1-\tan(1/b)) = 1.$

8.9.35

$$\int_{-\infty}^0\frac{dx}{\sqrt[3]{2-x}} = \lim_{b\to-\infty}\int_b^0(2-x)^{-1/3}\,dx = \lim_{b\to-\infty}\left(-\frac{3}{2}(2-x)^{2/3}\right)\Big|_b^0$$
$$= \lim_{b\to-\infty}\left(\frac{3}{2}\left(-2^{2/3}+(2-b)^{2/3}\right)\right)=\infty,$$

so the integral diverges.

8.9.36

$$\int_{e^2}^\infty\frac{dx}{x\ln^p x} = \lim_{b\to\infty}\int_{e^2}^b\frac{dx}{x\ln^p x} = \lim_{b\to\infty}\left(\frac{1}{1-p}\ln^{1-p}x\right)\Big|_{e^2}^b = \lim_{b\to\infty}\frac{1}{p-1}\left(2^{1-p}-\ln^{1-p}b\right) = \frac{1}{(p-1)2^{p-1}}.$$

8.9.37 $\displaystyle\int_0^8\frac{dx}{\sqrt[3]{x}} = \lim_{c\to0^+}\int_c^8 x^{-1/3}\,dx = \lim_{c\to0^+}\left(\frac{3}{2}x^{2/3}\right)\Big|_c^8 = \frac{3}{2}\lim_{c\to0^+}(4-c^{2/3}) = 6.$

8.9.38 $\displaystyle\lim_{c\to1^+}\int_c^2\frac{1}{\sqrt{x-1}}\,dx = \lim_{c\to1^+}\left(2\sqrt{x-1}\right)\Big|_c^2 = \lim_{c\to1^+}\left(2-2\sqrt{c-1}\right)=2.$

8.9.39 $\displaystyle\int_0^{\pi/2}\tan\theta\,d\theta = \lim_{c\to\pi/2^-}\int_0^c\tan\theta\,d\theta = \lim_{c\to\pi/2^-}(\ln\sec\theta)\Big|_0^c = \lim_{c\to\pi/2^-}\ln\sec c=\infty$, so the integral diverges.

8.9.40 $\displaystyle\lim_{c\to-3^+}\int_c^1(2x+6)^{-2/3}\,dx = \lim_{c\to-3^+}\left(\frac{3}{2}\sqrt[3]{2x+6}\right)\Big|_c^1 = \lim_{c\to-3^+}\frac{3}{2}\left(2-\sqrt[3]{2c+6}\right)=3.$

8.9.41 $\displaystyle\lim_{b\to(\pi/2)^-}\int_0^b\sec x\tan x\,dx = \lim_{b\to(\pi/2)^-}(\sec x)\Big|_0^b = \lim_{b\to(\pi/2)^-}(\sec b-1)=\infty$. The given integral diverges.

8.9.42 $\displaystyle\lim_{c\to3^+}\int_c^4(z-3)^{-3/2}\,dz = \lim_{c\to3^+}\left(-2(z-3)^{-1/2}\right)\Big|_c^4 = \lim_{c\to3^+}\left(-2-(-2(c-3)^{-1/2}\right)=\infty$. The given integral diverges.

8.9.43 Note that $\displaystyle\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}+C$. Thus,

$$\lim_{c\to0^+}\int_c^1\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{c\to0^+}\left(2e^{\sqrt{x}}\right)\Big|_c^1 = \lim_{c\to0^+}\left(2e-2e^{\sqrt{c}}\right)=2e-2.$$

8.9.44 Use the substitution $u=e^y-1$ so that $du=e^y\,dy$. Then $y=0$ corresponds to $u=0$ and $y=\ln 3$ corresponds to $u=2$, so we have

$$\int_0^{\ln 3}\frac{e^y}{(e^y-1)^{2/3}}\,dy = \int_0^2 u^{-2/3}\,du = \lim_{b\to0^+}\int_b^2 u^{-2/3}\,du = \lim_{b\to0^+}3u^{1/3}\Big|_b^2 = \lim_{b\to0^+}(3\cdot2^{1/3}-3b^{1/3}) = 3\cdot2^{1/3}.$$

8.9.45 $\int_0^1 \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\left(\frac{1}{4}\ln|x^4-1|\right)\Big|_0^c = \frac{1}{4}\lim_{c\to 1^-}\ln|c^4-1| = -\infty$, so the integral diverges.

8.9.46 This integral is improper at both limits, so we split it as $\int_1^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^2 \frac{dx}{\sqrt[3]{x-1}} + \int_2^\infty \frac{dx}{\sqrt[3]{x-1}}$. If we let $u = x-1$ in the second integral on the right we obtain

$$\int_2^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^\infty \frac{du}{u^{1/3}},$$

which diverges using the result in Example 2. Therefore the original integral diverges.

8.9.47

$$\int_0^{10}\frac{dx}{\sqrt[4]{10-x}} = \lim_{c\to 10^-}\int_0^c (10-x)^{-1/4}\,dx = \lim_{c\to 10^-}\left(-\frac{4}{3}(10-x)^{3/4}\right)\Big|_0^c$$
$$= \frac{4}{3}\lim_{c\to 10^-}\left(10^{3/4}-(10-c)^{3/4}\right) = \frac{4}{3}10^{3/4}.$$

8.9.48 This integral is improper at the point $x=3$, so we split it as $\int_1^{11}\frac{dx}{(x-3)^{2/3}} = \int_1^3 \frac{dx}{(x-3)^{2/3}} + \int_3^{11}\frac{dx}{(x-3)^{2/3}}$ and evaluate each integral separately:

$$\int_1^3 \frac{dx}{(x-3)^{2/3}} = \lim_{c\to 3^-}\int_1^c (x-3)^{-2/3}\,dx = \lim_{c\to 3^-}\left(3(x-3)^{1/3}\right)\Big|_1^c = 3\lim_{c\to 3^-}\left(2^{1/3}-(3-c)^{1/3}\right) = 3\cdot 2^{1/3},$$

and

$$\int_3^{11}\frac{dx}{(x-3)^{2/3}} = \lim_{c\to 3^+}\int_c^{11}(x-3)^{-2/3}\,dx = \lim_{c\to 3^+}\left(3(x-3)^{1/3}\right)\Big|_c^{11} = 3\lim_{c\to 3^+}\left(8^{1/3}-(c-3)^{1/3}\right) = 6,$$

so $\int_1^{11}\frac{dx}{(x-3)^{2/3}} = 6 + 3\cdot 2^{1/3}$.

8.9.49 The given integrand isn't defined for $x=1$, so we write the given integral as

$$\int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}.$$

We will show that the given integral diverges by showing that $\int_0^1 \frac{dx}{(x-1)^2}$ diverges. We have

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^2} = \lim_{b\to 1^-} -\frac{1}{x-1}\Big|_0^b = \lim_{b\to 1^-}\left(-\frac{1}{b-1}+1\right) = \infty.$$

8.9.50 We can write this as

$$\int_0^1 \frac{1}{(x-1)^{1/3}}\,dx + \int_1^9 \frac{1}{(x-1)^{1/3}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^{1/3}}\,dx + \lim_{c\to 1^+}\int_c^9 \frac{1}{(x-1)^{1/3}}\,dx$$
$$= \lim_{b\to 1^-}\frac{3}{2}\left((x-1)^{2/3}\right)\Big|_0^b + \lim_{c\to 1^+}\frac{3}{2}\left((x-1)^{2/3}\right)\Big|_c^9 = -\frac{3}{2}+6 = \frac{9}{2}.$$

8.9.51 By symmetry,

$$\begin{aligned}\int_{-2}^{2}\frac{dp}{\sqrt{4-p^2}} &= 2\int_0^2\frac{dp}{\sqrt{4-p^2}} = 2\lim_{c\to 2^-}\int_0^c\frac{dp}{\sqrt{4-p^2}}\\ &= 2\lim_{c\to 2^-}\left(\sin^{-1}(p/2)\right)\Big|_0^c = 2\lim_{c\to 2^-}\left(\sin^{-1}(c/2)-\sin^{-1}0\right)\\ &= 2(\sin^{-1}1-0)=\pi.\end{aligned}$$

8.9.52 Integration by parts gives $\int xe^{-x}\,dx = -(x+1)e^{-x}+C$, so $\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(-(x+1)e^{-x}\right)\big|_0^b = \lim_{b\to\infty}(1-(b+1)e^{-b})=1$.

8.9.53 Integration by parts gives $\int \ln x\,dx = x\ln x - x + C$, so

$$\int_0^1 \ln x\,dx = \lim_{c\to 0^+}\int_c^1 \ln x\,dx = \lim_{c\to 0^+}(x\ln x - x)\Big|_c^1 = \lim_{c\to 0^+}(-1-(c\ln c - c)) = -1,$$

because $\lim_{c\to 0^+} c\ln c = 0$.

8.9.54 Integration by parts gives $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x + 1}{x} + C$, so

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b\frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln x+1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{\ln b+1}{b}\right)=1,$$

because $\lim_{b\to\infty}\frac{\ln b+1}{b} = 0$.

8.9.55 Write the given integral as $\lim_{a\to 0^+}\int_a^{\ln 2}\frac{e^x}{\sqrt{e^{2x}-1}}\,dx$ and then let $u=e^x$ so that $du = e^x\,dx$. Substituting gives $\lim_{a\to 0^+}\int_{e^a}^2\frac{du}{\sqrt{u^2-1}}$. Then we let $u=\sec\theta$ with $du=\sec\theta\tan\theta\,d\theta$. Substituting again gives

$$\begin{aligned}\lim_{a\to 0^+}\int_{\sec^{-1}e^a}^{\pi/3}\sec\theta\,d\theta &= \lim_{a\to 0^+}\ln|\sec\theta+\tan\theta|\Big|_{\sec^{-1}e^a}^{\pi/3}\\ &= \lim_{a\to 0^+}\left(\ln(2+\sqrt{3})-\ln\left(e^a+\sqrt{e^{2a}-1}\right)\right) = \ln(2+\sqrt{3}).\end{aligned}$$

8.9.56 Write the given integral as $\lim_{a\to 0^+}\int_a^1\frac{dx}{x+\sqrt{x}} = \lim_{a\to 0^+}\int_a^1\frac{dx}{(\sqrt{x}+1)\sqrt{x}}$. Let $u=\sqrt{x}+1$ so that $du = \frac{1}{2\sqrt{x}}\,dx$. Substituting gives

$$\lim_{a\to 0^+}\int_{1+\sqrt{a}}^2\frac{2du}{u} = \lim_{a\to 0^+}2\ln u\Big|_{1+\sqrt{a}}^2 = \lim_{a\to 0^+}\left(2\ln 2 - 2\ln(1+\sqrt{a})\right). = 2\ln 2.$$

8.9.57 The function $e^{-|x|}$ is even, so $\int_{-\infty}^\infty e^{-|x|}\,dx = 2\int_0^\infty e^{-x}\,dx$. We have

$$2\lim_{b\to\infty}\int_0^b e^{-x}\,dx = 2\lim_{b\to\infty}-e^{-x}\Big|_0^b = 2\lim_{b\to\infty}\left(-e^{-b}+1\right)=2.$$

8.9.58 Note that $x^2 + 2x + 5 = (x^2 + 2x + 1) + 4 = (x + 1)^2 + 4$. Also recall that $\int \frac{1}{(x+1)^2 + 4}\, dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+4} &= \lim_{b\to-\infty} \int_b^0 \frac{dx}{(x+1)^2+4} + \lim_{b\to\infty} \int_0^b \frac{dx}{(x+1)^2+4} \\
&= \lim_{b\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\bigg|_b^0 + \lim_{b\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\bigg|_0^b \\
&= \lim_{b\to-\infty} \frac{1}{2}\left(\tan^{-1}(1/2) - \tan^{-1}\left(\frac{b+1}{2}\right)\right) + \lim_{b\to\infty} \frac{1}{2}\left(\tan^{-1}\left(\frac{b+1}{2}\right) - \tan^{-1}(1/2)\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}.
\end{aligned}$$

8.9.59 We have the relation $B = I\int_0^\infty e^{-rt}\, dt = \frac{I}{r}$, using the result that $\int_0^\infty e^{-ax}\, dx = \frac{1}{a}$ for $a > 0$. Therefore $B = \frac{5000}{0.12} = \$41{,}666.67$.

8.9.60 The total amount of water drained is $W = 100\int_0^\infty e^{-0.05t}\, dt = \frac{100}{0.05} = 2000\,\text{gal}$ (here we use the fact that $\int_0^\infty e^{-ax}\, dx = \frac{1}{a}$ for $a > 0$).

8.9.61 As in Example 6, we have

$$\text{AUC}_i = \int_0^\infty C_i(t)\, dt = 250\int_0^\infty e^{-0.08t}\, dt = \frac{250}{0.08} = 3125$$

and

$$\text{AUC}_o = \int_0^\infty C_0(t)\, dt = 200\int_0^\infty (e^{-0.08t} - e^{-1.8t})\, dt = 200\left(\frac{1}{0.08} - \frac{1}{1.8}\right) = \frac{21{,}500}{9} \approx 2389,$$

(here we use the fact that $\int_0^\infty e^{-ax}\, dx = \frac{1}{a}$ for $a > 0$). Therefore the bioavailability of the drug is

$$F = \frac{\text{AUC}_o}{\text{AUC}_i} = \frac{21{,}500}{9\cdot 3125} \approx 0.764.$$

8.9.62

a. We will make use of the result $\int_b^\infty e^{-at}\, dt = \frac{e^{-ab}}{a}$ for $a > 0$ (this is derived in problem 73). The probability that a chip fails after 15,000 hours of operation (or equivalently, lasts at least 15,000 hours) is $p = 0.00005\int_{15{,}000}^\infty e^{-0.00005t}\, dt = e^{-0.00005\cdot 15{,}000} \approx 0.472$.

b. The probability that a chip fails after 30,000 hours of operation is $p = 0.00005\int_{30{,}000}^\infty e^{-0.00005t}\, dt = e^{-0.00005\cdot 30{,}000} \approx 0.223$. Of the chips that are still operating at 15,000 hours, the fraction that operate for at least another 15,000 hours is $\approx 0.223/0.472 = 0.472$.

c. From part a, we see that $0.00005\int_0^\infty e^{-0.00005t}\, dt = 1$, which can be interpreted as meaning that all the chips are working initially.

8.9.63 Evaluate the improper integral

$$\int_0^\infty te^{-at}\,dt = \lim_{b\to\infty}\int_0^b te^{-at}\,dt = \lim_{b\to\infty}\left(-\frac{e^{-at}(at+1)}{a^2}\right)\Big|_0^b = \frac{1}{a^2}\lim_{b\to\infty}\left(1-e^{-ab}(ab+1)\right) = \frac{1}{a^2},$$

provided $a>0$. Therefore $0.00005\int_0^\infty te^{-0.00005t}\,dt = \frac{0.00005}{0.00005^2} = 20{,}000\,\text{hrs}$.

8.9.64 The maximum distance is

$$D = 10\int_0^\infty (t+1)^{-2}\,dt = 10\lim_{b\to\infty}\int_0^b (t+1)^{-2}\,dt = 10\lim_{b\to\infty}\left(-(t+1)^{-1}\right)\Big|_0^b = 10\,\text{mi}.$$

8.9.65 Using the result from Example 2, we see that the volume is given by $V = \pi\int_1^\infty x^{-4}\,dx = \frac{\pi}{4-1} = \frac{\pi}{3}$.

8.9.66 $V = \pi\int_2^\infty \frac{dx}{x^2+1} = \pi\lim_{b\to\infty}\int_2^b \frac{dx}{x^2+1} = \pi\lim_{b\to\infty}\left(\tan^{-1}x\right)\Big|_2^b = \pi\left(\frac{\pi}{2}-\tan^{-1}2\right)\approx 1.457.$

8.9.67 Using the result from Example 2, we see that the volume is given by

$$V = \pi\int_1^\infty\left(\frac{1}{x^2}+\frac{1}{x^3}\right)dx = \frac{\pi}{2-1}+\frac{\pi}{3-1} = \frac{3\pi}{2}.$$

8.9.68 $V = 2\pi\int_0^\infty \frac{x}{(x+1)^3}\,dx$; using the method of partial fractions, we see that $\frac{x}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$, so

$$V = 2\pi\left(\int_0^\infty \frac{dx}{(x+1)^2} - \int_0^\infty \frac{dx}{(x+1)^3}\right).$$

Now make the substitution $u = x+1$ and use the result from Example 2 to obtain

$$V = 2\pi\left(\int_1^\infty \frac{du}{u^2} - \int_1^\infty \frac{du}{u^3}\right) = 2\pi\left(\frac{1}{2-1}-\frac{1}{3-1}\right) = \pi.$$

8.9.69 $V = \pi\int_2^\infty \frac{dx}{x(\ln x)^2} = \pi\lim_{b\to\infty}\int_2^b \frac{dx}{x(\ln x)^2} = \pi\lim_{b\to\infty}\left(-\frac{1}{\ln x}\right)\Big|_2^b = \pi\lim_{b\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln b}\right) = \frac{\pi}{\ln 2}.$

8.9.70 The volume is given by

$$\begin{aligned} V &= \pi\int_0^\infty \frac{x}{(x^2+1)^{2/3}}\,dx = \pi\lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^{2/3}}\,dx \\ &= \pi\lim_{b\to\infty}\left(\frac{3}{2}(x^2+1)^{1/3}\right)\Big|_0^b = \pi\lim_{b\to\infty}\frac{3}{2}\left((b^2+1)^{1/3}-1\right) = \infty, \end{aligned}$$

so the volume is infinite.

8.9.71 Using disks, we have

$$V = \pi\int_1^2 (x-1)^{-1/2}\,dx = \pi\lim_{c\to 1^+}\int_c^2 (x-1)^{-1/2}\,dx = \pi\lim_{c\to 1^+}\left(2(x-1)^{1/2}\right)\Big|_c^2 = 2\pi\lim_{c\to 1^+}(1-\sqrt{c-1}) = 2\pi.$$

8.9.72 Using disks, we have

$$V = \pi\int_{-1}^1\left(1+(x+1)^{-3/2}\right)^2 dx = \pi\int_0^2 (1+u^{-3/2})^2\,du = \pi\int_0^2 (1+2u^{-3/2}+u^{-3})\,du$$

via the substitution $u = x+1$. Therefore

$$V = \pi\lim_{c\to 0+}\int_c^2 (1+2u^{-3/2}+u^{-3})\,du = \pi\lim_{c\to 0^+}\left(u-4u^{-1/2}-\frac{1}{2}u^{-2}\right)\Big|_c^2 = \infty,$$

so the volume is infinite.

8.9.73 $V = \pi \int_0^{\pi/2} \tan^2 x\, dx = \lim_{a \to \frac{\pi}{2}^-} \pi \int_0^a \tan^2 x\, dx = \lim_{a \to \frac{\pi}{2}^-} \pi \int_0^a (\sec^2 x - 1)\, dx = \lim_{a \to \frac{\pi}{2}^-} \pi\,(\tan x - x)\Big|_0^a =$ $\lim_{a \to \frac{\pi}{2}^-} \pi\,(\tan a - a) = \infty$, so the volume doesn't exist.

8.9.74 First note that $\int \ln^2 x\, dx = 2x + x\ln^2 x - 2x\ln x$. This is obtained by integration by parts with $u = \ln x$ and $dv = \ln x\, dx$. This gives $du = \frac{1}{x}\, dx$ and $v = x\ln x - x$. Then we have $\int \ln^2 x\, dx = x\ln^2 x - x\ln x - \int (\ln x - 1)\, dx = x\ln^2 x - x\ln x - (x\ln x - x - x) + C = x\ln^2 x - 2x\ln x + 2x + C$.

Then the volume is $V = \pi \int_0^1 (-\ln x)^2\, dx = \lim_{a \to 0^+} \pi \int_a^1 \ln^2 x\, dx = \lim_{a \to 0^+} \pi \left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_a^1$ $= \lim_{a \to 0^+} \pi \left(2 - (a\ln^2 a - 2a\ln a + 2a)\right) = 2\pi$. Note that

$$\lim_{a \to 0^+} a\ln a = \lim_{a \to 0^+} \frac{\ln a}{1/a} = \lim_{a \to 0^+} \frac{1/a}{-1/a^2} = \lim_{a \to 0^+} -a = 0,$$

by l'Hôpital's rule. Also

$$\lim_{a \to 0^+} a\ln^2 a = \lim_{a \to 0^+} \frac{\ln^2 a}{1/a} = \lim_{a \to 0^+} \frac{2\ln a/a}{-1/a^2} = \lim_{a \to 0^+} -2a\ln a = 0.$$

8.9.75 Using shells, we have

$$V = 2\pi \int_0^4 x(4-x)^{-1/3}\, dx = 2\pi \int_0^4 (4-u)u^{-1/3}\, du = 2\pi \int_0^4 (4u^{-1/3} - u^{2/3})\, du$$

via the substitution $u = 4 - x$. Therefore

$$V = 2\pi \lim_{c \to 0+} \int_c^4 (4u^{-1/3} - u^{2/3})\, du = 2\pi \lim_{c \to 0^+} \left(6u^{2/3} - \frac{3}{5}u^{5/3}\right)\Big|_c^4 = \frac{72 \cdot 2^{1/3}\pi}{5}.$$

8.9.76 Using shells, we have

$$\begin{aligned} V &= 2\pi \int_1^2 x(x^2-1)^{-1/4}\, dx = 2\pi \lim_{c \to 1^+} \int_c^2 x(x^2-1)^{-1/4}\, dx \\ &= \pi \lim_{c \to 1^+} \left(\frac{4}{3}(x^2-1)^{3/4}\right)\Big|_c^2 = \frac{4}{3}\pi \lim_{c \to 1^+} (3^{3/4} - (c^2-1)^{3/4}) = \frac{4\pi}{3^{1/4}}. \end{aligned}$$

8.9.77 $x^3 + 1 > x^3$ for $x \ge 1$, so $\frac{1}{x^3+1} < \frac{1}{x^3}$ for $x \ge 1$. Because $\int_1^\infty \frac{1}{x^3}\, dx$ converges, $\int_1^\infty \frac{1}{x^3+1}\, dx$ converges as well by the Comparison Test. Note that the fact that $\int_1^\infty \frac{1}{x^3}\, dx$ converges to $\frac{1}{2}$ follows from Example 2, Case 1.

8.9.78 $e^x + x + 1 > e^x$ for $x \ge 1$, so $\frac{1}{e^x + x + 1} < \frac{1}{e^x}$ for $x \ge 1$. Note that

$$\int_0^\infty \frac{dx}{e^x} = \lim_{b \to \infty} \int_0^b e^{-x}\, dx = \lim_{b \to \infty} -e^{-x}\Big|_0^b = \lim_{b \to \infty} \left(-e^{-b} + 1\right) = 1.$$

Thus $\int_0^\infty \frac{dx}{e^x + x + 1}$ converges by the Comparison Test.

8.9.79 For $x \ge 3$, $0 < \ln x \le x$, so $0 < \frac{1}{x} \le \frac{1}{\ln x}$. Note that

$$\int_3^\infty \frac{dx}{x} = \lim_{b \to \infty} \int_3^b \frac{dx}{x} = \lim_{b \to \infty} \ln x\Big|_3^b = \lim_{b \to \infty} (\ln b - \ln 3) = \infty.$$

Using the Comparison Test, because $\int_3^\infty \frac{dx}{x}$ diverges, $\int_3^\infty \frac{dx}{\ln x}$ diverges as well.

8.9.80 For $x \geq 2$, $0 < x^4 - x - 1 \leq x^4 - 1$, so $\dfrac{x^3}{x^4 - x - 1} \geq \dfrac{x^3}{x^4 - 1} > 0$. We will show that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - 1}\,dx$ is divergent, and then conclude by the Comparison Test that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - x - 1}\,dx$ is divergent. In order to show that $\displaystyle\int_2^\infty \frac{x^3}{x^4 - 1}\,dx$ is divergent, we let $u = x^4 - 1$ so that $du = 4x^3\,dx$. Then we have

$$\lim_{b\to\infty} \int_{15}^{b^4-1} \frac{1}{4}\left(\frac{1}{u}\right) du = \lim_{b\to\infty} \frac{1}{4} \ln u \Big|_{15}^{b^4-1} = \frac{1}{4} \lim_{b\to\infty} \left(\ln(b^4 - 1) - \ln 15\right) = \infty.$$

8.9.81 $0 \leq \sin^2 x \leq 1$, so $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ for $x \geq 1$. By Example 2, $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges. Therefore $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges as well by the Comparison Test.

8.9.82 $e^x > 1$ for $x \geq 1$, so $e^x(1 + x^2) \geq 1 + x^2$ for $x \geq 1$. It follows that

$$0 \leq \frac{1}{e^x(1 + x^2)} \leq \frac{1}{1 + x^2},$$

for $x \geq 1$. We will show that $\displaystyle\int_1^\infty \frac{1}{1 + x^2}\,dx$ converges to $\dfrac{\pi}{4}$ and then conclude that $\displaystyle\int_1^\infty \frac{1}{e^x(1 + x^2)}\,dx$ converges. We have

$$\lim_{b\to\infty} \int_1^b \frac{1}{1 + x^2}\,dx = \lim_{b\to\infty} \tan^{-1} x \Big|_1^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

8.9.83 $2 + \cos x \geq 1$, so $\dfrac{2 + \cos x}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}} \geq 0$. By Example 2, $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges, so $\displaystyle\int_1^\infty \frac{2 + \cos x}{\sqrt{x}}\,dx$ diverges by the Comparison Test.

8.9.84 $0 \leq 2 + \cos x \leq 3$, which implies that $0 \leq \dfrac{2 + \cos x}{x^2} \leq \dfrac{3}{x^2}$. By Example 2, $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{2 + \cos x}{x^2}\,dx$ converges as well by the Comparison Test.

8.9.85 $\sqrt{x^{1/3} + x} \geq \sqrt{x^{1/3}} = x^{1/6}$. So for $0 \leq x \leq 1$, $0 < \dfrac{1}{\sqrt{x^{1/3} + x}} \leq \dfrac{1}{x^{1/6}}$. We will show that $\displaystyle\int_0^1 \frac{dx}{x^{1/6}}$ converges and conclude that $\displaystyle\int_0^1 \frac{1}{\sqrt{x^{1/3} + x}}\,dx$ converges by the Comparison Test. We have

$$\lim_{a\to0^+} \int_a^1 \frac{dx}{x^{1/6}} = \lim_{a\to0^+} \frac{6}{5} x^{5/6} \Big|_a^1 = \lim_{a\to0^+} \frac{6}{5}\left(1 - a^{5/6}\right) = \frac{6}{5}.$$

8.9.86 $\sin x + 1 \geq 1$, so $\dfrac{\sin x + 1}{x^5} \geq \dfrac{1}{x^5} > 0$ for all $x > 0$. We will show that $\displaystyle\int_0^1 \frac{1}{x^5}$ diverges, and then conclude that $\displaystyle\int_0^1 \frac{\sin x + 1}{x^5}\,dx$ diverges as well by the Comparison Test. We have

$$\lim_{a\to0^+} \int_a^1 \frac{1}{x^5}\,dx = \lim_{a\to0^+} -\frac{1}{4} \cdot \frac{1}{x^4} \Big|_a^1 = \lim_{a\to0^+} -\frac{1}{4}\left(1 - \frac{1}{a^4}\right) = \infty.$$

8.9.87

a. True. The area under the curve $y = f(x)$ from 0 to ∞ is less than the area under $y = g(x)$ on this interval, which by assumption is finite.

b. False. For example, take $f(x) = 1$; then $\int_0^\infty f(x)\,dx = \infty$.

c. False. For example, take $p = 1/2$ and $q = 1$.

d. True. The area under the curve $y = x^{-q}$ from 1 to ∞ is less than the area under $y = x^{-p}$ on this interval, which by assumption is finite.

e. True. Using the result in Example 2, we see that this integral exists if and only if $3p + 2 > 1$, which is equivalent to $p > -1/3$.

8.9.88

a The fundamental theorem cannot be applied because the function $\frac{1}{x}$ is not continuous or bounded on $[-1, 1]$.

b. We will show that $\int_0^1 \frac{dx}{x}$ diverges, and then we may conclude that $\int_{-1}^1 \frac{dx}{x}$ diverges as well. We have

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \ln x\bigg|_a^1 = \lim_{a\to 0^+} (\ln 1 - \ln a) = \infty.$$

8.9.89 The region R has area $A = \int_0^\infty e^{-bx}\,dx - \int_0^\infty e^{-ax}\,dx = \frac{1}{b} - \frac{1}{a}$.

8.9.90 The region R has area $A = \int_1^\infty x^{-p}\,dx - \int_1^\infty x^{-q}\,dx = \frac{1}{p-1} - \frac{1}{q-1}$, where we use the result in Example 2.

8.9.91

a. We have

$$A(a, b) = \int_b^\infty e^{-ax}\,dx = \lim_{c\to\infty} \int_b^c e^{-ax}\,dx = \lim_{c\to\infty} \left(-\frac{1}{a}e^{-ax}\right)\bigg|_b^c = \frac{1}{a}\lim_{c\to\infty}(e^{-ab} - e^{-ac}) = \frac{e^{-ab}}{a}.$$

b. Solving $e^{-ab} = 2a$ for b gives $b = g(a) = -\frac{1}{a}\ln 2a$.

c. The function g has $g'(x) = \frac{1}{x^2}\ln 2x - \frac{1}{x^2} = \frac{\ln 2x - 1}{x^2}$, so g has a critical point at $x = e/2$, and the first derivative test shows that g takes a minimum at this point. Hence $b^* = g(e/2) = -\frac{2}{e}$.

8.9.92 We first rewrite the integrand as $e^{-x\ln x}(\ln x + 1)$, and note that if $u = x\ln x$, then $du = (\ln x + 1)\,dx$. Note that as $x \to 0^+$, $u \to 0^-$, and as $x \to \infty$, $u \to \infty$, and when $x = 1$, $u = 0$. We consider the integral as

$$\int_0^1 x^{-x}(\ln x + 1)\,dx = \int_0^0 e^{-u}du = 0.$$

Now we consider

$$\int_1^\infty x^{-x}(\ln x + 1)\,dx = \int_0^\infty e^{-u}\,du = \lim_{b\to\infty} \int_0^b e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\bigg|_0^b = -\lim_{b\to\infty} \left(e^{-b} - 1\right) = 1.$$