Homework 6 Solution

Wednesday, October 20, 2021 10:55 AM

Recall. Suppose $f(x) \geq 0$, $g(x) \geq 0$, $g(x) \leq f(x)$.

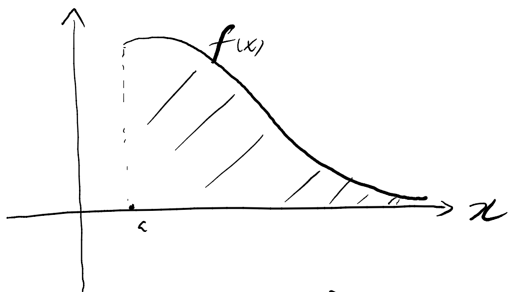
$$\begin{cases} \int_a^\infty f(x)\,dx \\ \int_a^\infty g(x)\,dx \end{cases}$$

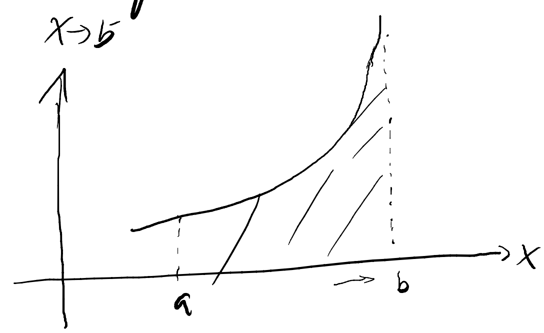
$$\int_a^b f(x)\,dx$$

$$\int_a^b g(x)\,dx$$

$(f(x), g(x)$ continuous$)$

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} g(x) = \infty$$





convergence of $\int_a^\infty f(x)\,dx$ is depending on how fast $f(x) \to 0$ as $x \to \infty$

convergence of $\int_a^b f(x)\,dx$ is depending on how slow $f(x) \to \infty$ as $x \to b^-$

comparison test:

i) If $\begin{cases} \int_a^\infty f(x)\,dx \text{ converges,} \\ \int_a^b f(x)\,dx \text{ converges} \end{cases}$ then $\begin{cases} \int_a^\infty g(x)\,dx \text{ converges} \\ \int_a^b g(x)\,dx \text{ converges} \end{cases}$

ii) If $\begin{cases} \int_a^\infty g(x)\,dx \text{ diverges,} \\ \int_a^b g(x)\,dx \text{ diverges} \end{cases}$ then $\begin{cases} \int_a^\infty f(x)\,dx \text{ diverges} \\ \int_a^b f(x)\,dx \text{ diverges} \end{cases}$

Limit comparison test:

If $\begin{cases} \lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{finite number} \\ \lim_{x\to b^-} \frac{f(x)}{g(x)} = \text{finite number} \end{cases}$, then

convergence of $\int_a^\infty f(x)dx$ (or $\int_a^b f(x)dx$) and convergence of $\int_a^\infty g(x)dx$ (or $\int_a^b g(x)dx$) are equivalent

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Assignment Homework6 due 10/20/2021 at 11:59pm EDT

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1. **(2 points)** Library/Wiley/setAnton_Section_7.8/Anton_7_8_Q2.pg

In each part, determine all values of p for which the integral is improper. Enter in interval notation or "none" if there are no relevant values of p.

(a) $\int_1^5 \frac{dx}{x^p}$

p values that make integral improper none

(b) $\int_2^3 \frac{dx}{x-p}$

p values that make integral improper [2, 3]

$\frac{1}{x-p}$ on [2, 3] discontinuous

$\Leftarrow$ $x-p=0$ on [2, 3]

(c) $\int_{-3}^1 e^{-px}dx$

p values that make integral improper none

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2. **(2 points)** Library/WHFreeman/Rogawski_Calculus_Early_Transcendentals_Second_Edition/7_Techniques_of_Integration/7.6_Improper_Integrals/7.6.63.pg

Determine if the improper integral converges and, if so, evaluate it.

$\int_5^\infty \frac{dx}{\sqrt{x}-2}$.

$f(x) = \frac{1}{\sqrt{x}-2}$ [5, ∞)

- A. 0
- B. 5

$\int_5 \quad \sqrt{x}-2$.

- A. 0
- B. 5
- C. Diverges
- D. 1

$$f(x) = \frac{1}{\sqrt{x}-2} \qquad [5, \infty)$$

$$g(x) = \frac{1}{\sqrt{x}}$$

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{1/\sqrt{x}}{1/(\sqrt{x}-2)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}-2}{\sqrt{x}} = 1$$

and $\int_5^\infty \frac{1}{x^{\frac{1}{2}}}\,dx$ diverges

Solution:
Since $\sqrt{x} \geq \sqrt{x}-2$, we have (for $x > 5$)
$\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-2}$.

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3. **(2 points)** Library/UCSB/Stewart5_7_8/Stewart5_7_8_48.pg

Let $g(x) = \dfrac{1}{\sqrt{x}-1}$. $\quad \int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$ diverges

(a) Use a calculator or computer algebra system to evaluate $\int_2^t g(x)\,dx$ for t = 5, 10, 100, 1000, and 10000. Make sure each answer is correct to three decimal places.

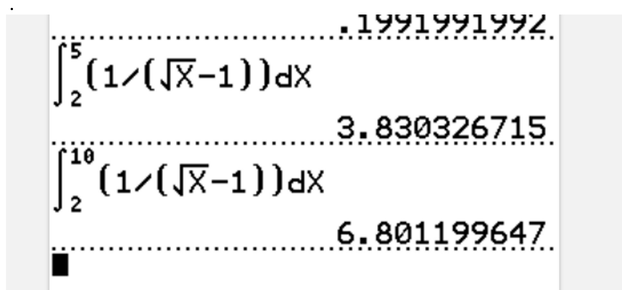
t = 5: ______________

t = 10: ______________

t = 100: ______________

t = 1000: ______________

t = 10000: ______________



4. **(2 points)** `Library/UCSB/Stewart5_7_8/Stewart5_7_8_52.pg`

Use the Comparison Theorem to determine whether the following integral is convergent or divergent.

[?]1. $\int_1^\infty \frac{1x}{\sqrt{1+x^6}}\,dx$

Correct Answers:

- C

$$\frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$$

$$\lim_{x\to\infty} \frac{1/x^2}{x/\sqrt{1+x^6}} = \lim_{x\to\infty} \frac{\sqrt{1+x^6}}{x^3}$$

$$= \lim_{x\to\infty} \sqrt{\frac{1}{x^6}+1} = 1$$

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges by the p-test

$\Rightarrow \int_1^\infty \frac{x\,dx}{\sqrt{1+x^6}}$ converges

5. **(2 points)** `Library/Rochester/setIntegrals18Improper/S07.08.ImproperIntegrals.PTP18.pg`

The improper integral $\int_{-\infty}^{\infty} x\,dx$ is

$$= \int_{-\infty}^{0} x\,dx + \int_0^\infty x\,dx \Leftarrow\Rightarrow \int_0^\infty \frac{1}{x^{-1}}\,dx$$

(p-test)

$$\lim_{x\to\infty} x = \infty$$

or

$\Rightarrow \int_0^\infty x\,dx = \lim_{t\to\infty} \frac{x^2}{2}\Big|_0^t = \infty$

$\Rightarrow \int_{-\infty}^{\infty} x\,dx$ diverges

• E. divergent since both integrals $\int^0 x\,dx$ and $\int^\infty x\,dx$ are divergent

- E. divergent since both integrals $\int_{-\infty}^{0} x\,dx = -\infty$ and $\int_{0}^{\infty} x\,dx = +\infty$ are divergent.

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6. (2 points) `Library/Michigan/Chap7Sec8/Q13.pg`

For each of the following improper integrals, carefully use the comparison test to decide if the integral converges or diverges. Give a reasonable "best" comparison function that you use in the comparison (by "best", we mean that the comparison function has known integral convergence properties, and is a reasonable upper or lower bound for the integrand we are evaluating).

1. $\int_5^9 \frac{6}{\sqrt{t-5}}\,dt$ $\quad \frac{6}{\sqrt{t-5}}$ discontinuous at 5

This integral

- A. converges
- B. diverges

$$\int_5^9 \frac{6}{\sqrt{t-5}}\,dt \overset{x=t-5}{=\!=} \int_0^4 \frac{6}{\sqrt{x}}\,dx$$

by the p-test

2. $\int_{-4}^{5} \frac{dt}{(t+4)^2}$ $\quad \underset{dx=dt}{\overset{x=t+4}{=\!=}} \int_0^9 \frac{dx}{x^2}$

This integral

- A. converges
- B. diverges

3. $\int_6^{\infty} \frac{d\theta}{\sqrt{\theta^3+4}}$ $\quad \frac{1}{\sqrt{\theta^3}} = \frac{1}{\theta^{\frac{3}{2}}}$ by p-test

This integral

- A. converges
- B. diverges

$$\lim_{\theta\to\infty} \frac{1/\theta^{\frac{3}{2}}}{1/\sqrt{\theta^3+4}} = 1$$

4. $\int_4^{\infty} \frac{dz}{e^z+5^z}$

This integral

In calculus I, $e^x >> x^p$

$e^z + 5^z > e^z$

$\Rightarrow \frac{1}{e^{z+5z}} < \frac{1}{e^z}$

and $\int_4^\infty \frac{1}{e^z}\,dz$ converges

- A. converges
- B. diverges

5. $\int_4^\infty \frac{3+\sin z}{z}\,dz$

This integral

- A. converges
- B. diverges

$-1 \le \sin z \le 1$

$\frac{3+\sin z}{z} > \frac{3-1}{z} = \frac{2}{z}$

$\int_4^\infty \frac{2}{z}\,dz$ diverges by p-test

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7. **(2 points)** `Library/Rochester/setIntegrals18Improper/S07.08.ImproperIntegrals.PTP17.pg`

For each of the improper integrals below, if the comparison test applies, enter either A or B followed by one letter from C to K that best applies, and if the comparison test does not apply, enter only L. For example, one possible answer is BF, and another one is L.

Hint: $0 < e^{-x} \le 1$ for $x \ge 1$.

___1. $\displaystyle\int_1^\infty \frac{6+\sin(x)}{\sqrt{x-0.7}}\,dx$

___2. $\displaystyle\int_1^\infty \frac{1}{x^2+5}\,dx$

___3. $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^6+5}}\,dx$

___4. $\displaystyle\int_1^\infty \frac{e^{-x}}{x^2}\,dx$

___5. $\displaystyle\int_1^\infty \frac{\cos^2(x)}{x^2+5}\,dx$

$-1 \le \sin x \le 1$

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

$\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$

4. $f(x) = \frac{e^{-x}}{x^2} = \frac{1}{e^x x^2} < \frac{1}{x^2}$

4. $f(x) = \frac{e}{x^2} = \frac{1}{e^x x^2} < \frac{1}{x^2}$

or $< \frac{1}{e^x}$

converges

$0 \le \cos^2 x \le 1$

5. $f(x) = \frac{\cos^2 x}{x^2+5} \Longleftrightarrow \frac{1}{x^2}$

converges because $\int_1^\infty \frac{1}{x^2}\,dx$ converges

$$\boxed{\begin{array}{l} \int_1^\infty \frac{1}{x^p}\,dx \text{ converges} \Leftrightarrow p > 1 \\ \int_0^1 \frac{1}{x^p}\,dx \text{ converges} \Leftrightarrow p < 1 \\ \int_1^0 \frac{1}{x^p}\,dx \end{array}}$$