Q1 (8 marks). Use the **comparison or limit comparison test** to determine whether the following improper integral converges or diverges. **Show your justification.**

$$\int_1^\infty \frac{\tan^{-1}x}{x^2+1}\,dx$$

Method 1 $\lim_{x\to\infty} \frac{1/x^2}{\tan^{-1}x/(x^2+1)} = \lim_{x\to\infty} \frac{x^2+1}{x^2\tan^{-1}x}$

$= \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)\cdot\frac{1}{\tan^{-1}x} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

and $\int_1^\infty \frac{1}{x^2}dx$ converges by the p-test

$\Rightarrow \int_1^\infty \frac{\tan^{-1}x}{x^2+1}dx$ converges by the limit comparison test.

Method 2. $|\tan^{-1}x| < \frac{\pi}{2} \Rightarrow \frac{\tan^{-1}x}{x^2+1} < \frac{\frac{\pi}{2}}{x^2+1} < \frac{\pi/2}{x^2}$

and $\int_1^\infty \frac{\pi/2}{x^2}\,dx$ converges by the p-test

$\Rightarrow \int_1^\infty \frac{\tan^{-1}x}{x^2+1}dx$ converges by the comparison test.

Q2 (8 marks). Use the **comparison or limit comparison test** to determine whether the following improper integral converges or diverges. **Show your justification.**

$$\int_0^1 \frac{1}{\sqrt{x^{1/2}+x^{100}}}\,dx$$

Method 1

$$\frac{1}{\sqrt{x^{1/2}+x^{100}}} < \frac{1}{\sqrt{x^{\frac{1}{2}}}} = \frac{1}{x^{\frac{1}{4}}} \text{ on } (0,1]$$

and $\int_0^1 \frac{1}{x^{1/4}}\,dx$ converges by the p-test

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x^{1/2}+x^{100}}}\,dx$ converges by the comparison test.

Method 2

$$\lim_{x\to 0} \frac{1/\sqrt{x^{1/2}}}{1/\sqrt{x^{1/2}+x^{100}}} = \lim_{x\to 0} \sqrt{\frac{x^{1/2}+x^{100}}{x^{1/2}}}$$

$$= \lim_{x\to 0} \sqrt{1+x^{100-1/2}} = 1$$

and $\int_0^1 \frac{1}{\sqrt{x^{1/2}}}\,dx = \int_0^1 \frac{1}{x^{\frac{1}{4}}}\,dx$ converges by the p-test

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x^{1/2}+x^{100}}}\,dx$ converges by the limit comparison test.

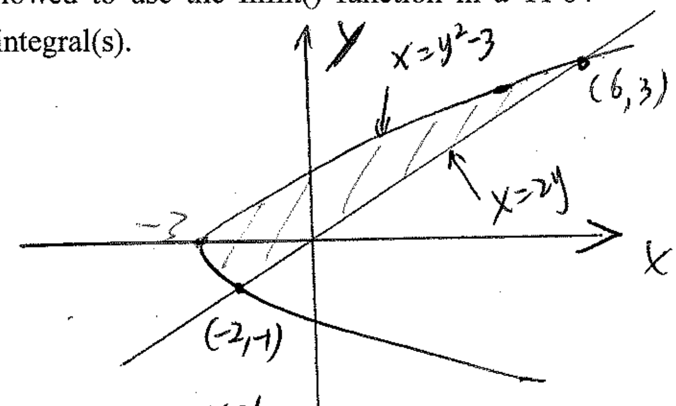
Q3 (8 marks).

$y^2 = 3 + x$

(1) Draw the region between the two curves $y = \frac{x}{2}$ and $x = y^2 - 3$. $\Rightarrow$ $y = \pm\sqrt{3+x}$

(2) Use algebraic method find all intersections of the curves and label the intersections in the graph. Show your work.

(3) Find the area of the region. You are allowed to use the fnInt() function in a TI-84 calculator to find the value of the definite integral(s).



$y = \frac{x}{2} \Rightarrow x = 2y$

$\begin{cases} x = 2y \\ x = y^2 - 3 \end{cases} \Rightarrow y^2 - 3 = 2y$

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y-3)(y+1) = 0 \Rightarrow \begin{cases} x = -2 \\ y = -1 \end{cases}$ and $\begin{cases} x = 6 \\ y = 3 \end{cases}$

$(3)\ A = \int_{-1}^{3} [\text{right curve} - \text{left curve}]\, dy$

$= \int_{-1}^{3} [2y - (y^2 - 3)]\, dy$

$= [y^2 - \frac{y^3}{3} + 3y]\Big|_{-1}^{3} = (3^2 - \frac{3^3}{3} + 9) - (1 + \frac{1}{3} - 3)$

$= 18 - 9 + 2 - \frac{1}{3} = 9 + 2 - \frac{1}{3} = \frac{32}{3} \approx 10.667$

Method 2: Draw the region bounded by

$\begin{cases} x = \frac{y}{2} \\ y = x^2 - 3 \end{cases}$ or $\begin{cases} y = 2x \\ y = x^2 - 3 \end{cases}$

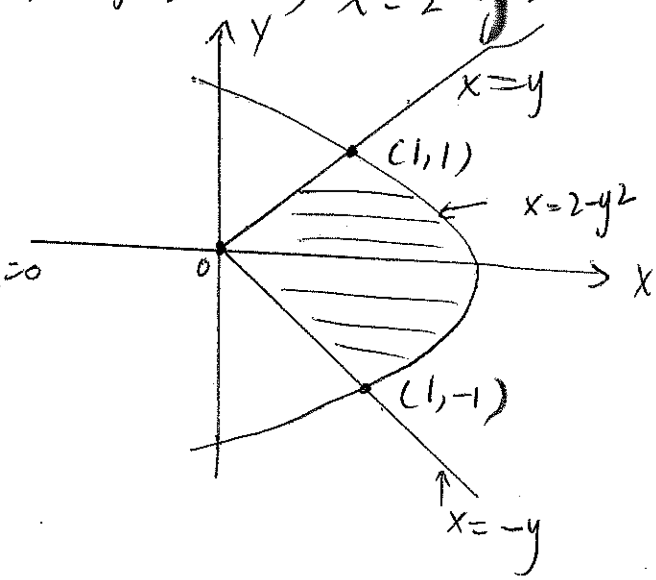
Q4 (8 marks).

(1) Draw the region between the two curves $x = |y|$ and $x + y^2 - 2 = 0$ (**Hint**: regarding x as function of y makes graphing easier!)

(2) Use algebraic method find all intersections of the curves and label the intersections in the graph. Show your work.

(3) Find the area of the region. You are allowed to use the fnInt() function in a TI-84 calculator to find the value of the definite integral(s).

Sol. $x = |y| = \begin{cases} y, & y \geq 0 \\ -y, & y < 0 \end{cases}$ $\quad x + y^2 - 2 = 0 \Rightarrow x = 2 - y^2$



i) $\begin{cases} x = y, \ y \geq 0 \\ x = 2 - y^2 \end{cases} \Rightarrow y = 2 - y^2$

$\Rightarrow y^2 + y - 2 = 0 \Rightarrow (y+2)(y-1) = 0$

$\Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}, \quad y \geq 0$

ii) $\begin{cases} x = -y, \ y < 0 \\ x = 2 - y^2 \end{cases} \Rightarrow -y = 2 - y^2$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0 \Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}, \quad y < 0$

$$\Rightarrow A = \int_{-1}^{1} [\text{right curve} - \text{left curve}]\, dy$$

$$= \int_{-1}^{0} (2 - y^2 + y)\, dy + \int_{0}^{1} (2 - y^2 - y)\, dy$$

$$\overset{\text{symmetry of graph}}{=} 2\int_0^1 (2 - y^2 - y)\, dy = 2\left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]_0^1 = \frac{14}{6} \approx 2.33$$

Method 2: Draw the region $\begin{cases} y = |x| \\ y + x^2 - 2 = 0 \end{cases}$