1. $\int\left(\frac{\sqrt[3]{x}+x^2+2\sqrt{x}}{x^2}\right)dx = \int\left(x^{\frac{1}{3}-2}+1+2x^{\frac{1}{2}-2}\right)dx$

$= \int\left(x^{-5/3}+1+2x^{-\frac{3}{2}}\right)dx = \frac{x^{-\frac{5}{3}+1}}{-\frac{5}{3}+1}+x+2\cdot\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}+C$

$= -\frac{3}{2}x^{-\frac{2}{3}}+x-4x^{-\frac{1}{2}}+C$

2. $\int\frac{\sin(\sqrt{x})}{\sqrt{x}}dx$ Let $u=\sqrt{x}$. Then $\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \frac{dx}{\sqrt{x}} = 2du$

$\Rightarrow \int\frac{\sin(\sqrt{x})}{\sqrt{x}}dx = \int 2\sin u\,du = -2\cos u + C$

$= -2\cos(\sqrt{x})+C$

3. $\int\frac{\ln x}{x^6}\,dx$

Integration by parts

diff	Int
$\ln x$	x^{-6}
$\frac{1}{x}$	$\frac{x^{-6+1}}{-6+1} = \frac{-1}{5}x^{-5}$

$(-)$

$+\int\frac{1}{5}x^{-5-1}dx$

$= \frac{-1}{5}\ln x\, x^{-5}+\frac{1}{5}\int x^{-6}dx$

$= \frac{-1}{5}\ln x\cdot x^{-5}+\frac{1}{5}\frac{x^{-6+1}}{-6+1}+C$

$= \frac{-\ln x}{5x^5}+\frac{(-1)}{25}\cdot\frac{1}{x^5}+C$

4.

$$\int \frac{x}{x^2+4x+4}dx = \int \frac{x}{(x+2)^2}dx \quad \text{Let } \frac{x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$= \frac{A(x+2)+B}{(x+2)^2}$$

$$\Rightarrow x = A(x+2)+B$$

$$\begin{cases} x=-2 & \Rightarrow B=-2 \\ x=0 & \Rightarrow 2A-2=0 \Rightarrow A=1 \end{cases}$$

$$\Rightarrow \int \frac{x\,dx}{x^2+4x+4} = \int \frac{dx}{x+2} - 2\int \frac{dx}{(x+2)^2}$$

$$= \ln|x+2| - 2\frac{(x+2)^{-2+1}}{-2+1} + C$$

$$= \ln|x+2| + \frac{2}{x+2} + C$$

5.

$$\int \frac{1}{\sqrt{4-2x^2}}dx$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{1-\frac{x^2}{2}}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}$$

$$\xrightarrow[dx=\sqrt{2}du]{u=x/\sqrt{2}} \frac{1}{2}\cdot\sqrt{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{\sqrt{2}}{2}\arcsin u + C = \frac{\sqrt{2}}{2}\text{arsin}\left(\frac{x}{\sqrt{2}}\right) + C$$

6.

$$\int \frac{8x^2 + 3x + 45}{(x^2 + 9)(x + 1)} dx$$

$$\frac{8x^2+3x+45}{(x^2+9)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)(x+1)}{(x^2+9)(x+1)}$$

$$\Rightarrow 8x^2+3x+45 = A(x^2+9)+(Bx+C)(x+1)$$

i) Let $x=-1$. $8-3+45 = 10A \Rightarrow 50 = 10A \Rightarrow A=5$

ii) Let $x=0$, $45 = 9A + C = 45 + C \Rightarrow C = 0$

iii) Let $x=1$ $8+3+45 = 10A + 2B \Rightarrow 56 = 50 + 2B \Rightarrow B = 3$

$$\Rightarrow \int \frac{8x^2+3x+45}{(x^2+9)(x+1)} dx = 5\int \frac{dx}{x+1} + 3\int \frac{x}{x^2+9} dx$$

$$= 5 \ln|x+1| + 3\int \frac{x}{x^2+9} dx$$

$$\underset{du=2x\,dx}{\overset{u=x^2+9}{=\!=}} 5 \ln|x+1| + \frac{3}{2}\int \frac{du}{u}$$

$$= 5 \ln|x+1| + \frac{3}{2} \ln|x^2+9| + C$$

7.

$$\int_0^{1/3} \frac{6\sin^{-1} x}{\sqrt{1-x^2}}\, dx \qquad \left(u = \sin^{-1}x,\ du = \frac{dx}{\sqrt{1-x^2}}\right)$$

$$= 6\int_{\sin^{-1}0}^{\sin^{-1}(\frac{1}{3})} u\,du = 6\int_0^{\sin^{-1}(\frac{1}{3})} u\,du$$

$$= 6 \cdot \frac{u^2}{2}\Big|_0^{\sin^{-1}(\frac{1}{3})} = 3\left[\left(\sin^{-1}\left(\tfrac{1}{3}\right)\right)^2 - 0\right]$$

$$= 3\left[\sin^{-1}\left(\tfrac{1}{3}\right)\right]^2$$

Integration by parts

8.

$$\int x^2 e^{-2x}\, dx$$

diff	Int
x^2	e^{-2x}
$2x$	$-\frac{1}{2}e^{-2x}$
2	$(-)$ $\frac{1}{4}e^{-2x}$
	$(+)$

$$= \frac{-x^2}{2}e^{-2x} - \frac{2x}{4}e^{-2x}$$

$$+ \frac{2}{4}\int e^{-2x}\, dx$$

$$= \frac{-x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2x} + C$$

$$= -e^{-2x}\left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}\right) + C$$

For question 9 and 10, DO NOT use comparison or limit comparison test.

9.

$$\int_1^\infty \frac{9}{2x-3}\,dx$$

First, $\displaystyle\int \frac{9}{2x-3}\,dx \;\xrightarrow[du=2dx]{u=2x-3}\; \frac{9}{2}\int \frac{du}{u}$

$$= \frac{9}{2}\ln|u| + c = \frac{9}{2}\ln|2x-3| + C$$

$$\Rightarrow \int_1^\infty \frac{9\,dx}{2x-3} = \lim_{t\to\infty}\int_1^t \frac{9\,dx}{2x-3}$$

$$= \frac{9}{2}\lim_{t\to\infty} \ln|2x-3|\Big|_1^t = \frac{9}{2}\left[\lim_{t\to\infty}\ln|2t-3| - 0\right]$$

$$= \frac{9}{2}\lim_{t\to\infty}\ln|2t-3| = +\infty$$

10.

$$\int_{1/3}^1 \frac{1}{\sqrt{3x-1}}\,dx$$

First, $\displaystyle\int \frac{dx}{\sqrt{3x-1}} = \int (3x-1)^{-\frac{1}{2}}\,dx \quad \left(\begin{array}{l} u = 3x-1 \\ dx = \frac{1}{3}du \end{array}\right)$

$$= \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{1}{3}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{2}{3}(3x-1)^{\frac{1}{2}} + C$$

$$\Rightarrow \int_{1/3}^1 \frac{dx}{\sqrt{3x-1}} = \lim_{t\to\frac{1}{3}^+}\int_t^1 \frac{dx}{\sqrt{3x-1}}$$

$$= \frac{2}{3}\lim_{t\to\frac{1}{3}^+}(3x-1)^{\frac{1}{2}}\Big|_t^1 = \frac{2}{3}\left[2^{\frac{1}{2}} - \lim_{t\to\frac{1}{3}^+}(3t-1)^{\frac{1}{2}}\right]$$

$$= \frac{2}{3}\left(\sqrt{2} - 0\right) = \frac{2\sqrt{2}}{3}$$