Contents

Chapter 8 involves various integration techniques, i.e. techniques for finding indefinite integrals. **The only way to get reasonably good at finding indefinite integrals is by doing a lot of examples.**

8.1 §8.1 Basic Approaches

Before plunging into new integration techniques, we devote this section to two practical goals. The first is to review what you learned about the **substitution** method in Section 5.5. The other is to introduce several basic **simplifying procedures** that are worth keeping in mind for any integral that you might be working on.

To simplify the integration procedure, we need to memorize some typical integrals. The following is the list of antiderivatives that **must be memorized**.

Note: D_x denotes $\frac{d}{dx}$.

$$D_x(x^n) = nx^{n-1} \implies \int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$$

$$D_x(e^x) = e^x \implies \int e^x \, dx = e^x + C$$

$$D_x(a^x) = a^x \ln a \implies \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$D_x(\ln|x|) = \frac{1}{x} \implies \int \frac{dx}{x} = \ln|x| + C$$

$$D_x(\sin x) = \cos x \implies \int \cos x \, dx = \sin x + C$$

$$D_x(\cos x) = -\sin x \implies \int \sin x \, dx = -\cos x + C$$

$$D_x(\tan x) = \sec^2 x \implies \int \sec^2 x \, dx = \tan x + C$$

$$D_x(\sec x) = \sec x \tan x \implies \int \sec x \tan x \, dx = \sec x + C$$

$$D_x(\cot x) = -\csc^2 x \implies \int \csc^2 \, dx = -\cot x + C$$

$$D_x(\csc x) = -\csc x \cot x \implies \int \csc x \cot x \, dx = -\csc x + C$$

$$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \implies \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \implies \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C = -\sin^{-1} x + C$$

$$D_x(\tan^{-1} x) = \frac{1}{1+x^2} \implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$D_x(\cot^{-1} x) = \frac{-1}{1+x^2} \implies \int \frac{-dx}{1+x^2} = \cot^{-1} x + C = -\tan^{-1} x + C$$

$$D_x(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \implies \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

$$D_x(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} \implies \int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}|x| + C = -\sec^{-1}|x| + C$$

$$D_x(\ln|\sec x|) = \tan x \implies \int \tan x \, dx = \ln|\sec x| + C$$

$$D_x(\ln|\sin x|) = \cot x \implies \int \cot x \, dx = \ln|\sin x| + C$$

$$D_x(\ln|\sec x + \tan x|) = \sec x \implies \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$D_x(-\ln|\csc x + \cot x|) = \csc x \implies \int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

Example 1.(Substitution review) Evaluate $\int \tan ax \; dx, a \neq 0$

Solution.

$\square$

Example 2. (Multiplication by 1) Derive the integral formula

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

Solution. We derived this before.

$\square$

Example 3. (Subtle substitution) Evaluate $\int \frac{dx}{e^x + e^{-x}}$.

Solution.

$\square$

Example 4. (Split up fractions) Evaluate $\int \frac{\cos x + \sin^3 x}{\sec x} \; dx$.

Solution.

$\square$

Example 5. (Division with rational functions) Evaluate $\int \frac{x^2+2x-1}{x+4}\,dx$.

Solution.

$\square$

Example 6. (Complete the square) Evaluate $\int \frac{dx}{\sqrt{-x^2-8x-7}}$.

Solution.

$\square$

8.2 §8.2 Integration by Parts

This is an anti-differentiation technique that follows from the **Product Rule** for derivatives.

$$
\begin{aligned}
(uv)' &= u'v + uv' \\
&\Longrightarrow \boxed{\int u\, v' dx = uv - \int v\, u' dx} \\
&\Longrightarrow \boxed{\int u\, dv = uv - \int v\, du}.
\end{aligned}
$$

This is the formula method for **Integration by Parts**. There is a "diagram method" for integration by parts that often makes things easier, particularly, when we have to do repeated applications of Integration by Parts.

(1) **Formula Method**: $\boxed{\int u\, dv = uv - \int v\, du.}$

(2) **Diagram Method**:

diff.		int.	
u		v'	
	$\searrow$		Diagonal arrows = products
u'	$(-) \rightarrow$	v	Horizontal arrows = prod.& int.

Remark 1. $v = \int v'\, dx$ should be easy to find.

Remark 2. Integration by parts is very useful in deriving reduction formulas, i.e. reduce an integrand to a simpler case. However, there are no general rules when integration by parts should be used. Usually integrands involving $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$ are done using integration by parts.

In the following, we will run through the examples in the textbook using the **Diagram Method**.

Example 1. Find $\int xe^x \ dx$.

Solution.

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
x & & e^x \\
 & \searrow & \\
1 & (-) \rightarrow & e^x
\end{array}$$

Therefore,

$$\int xe^x \ dx = xe^x - \int e^x \ dx = xe^x - e^x + C.$$

Example 2. Find $\int x \sin x \ dx$.

Solution. $\int x \sin x \ dx$

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
u & & v' \\
 & \searrow & \\
u' & (-) \rightarrow & v
\end{array}
=
\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
x & & \sin x \\
 & \searrow & \\
1 & (-) \rightarrow & -\cos x
\end{array}$$

Therefore,

$$\begin{aligned}
\int x \sin x \ dx &= -x \cos x - \int -\cos \ dx \\
&= -x \cos x + \int \cos x \ dx = -x \cos x + \sin x + C.
\end{aligned}$$

$\square$

Remark. If the integration is still hard to get after one step, we can apply the technique of "Integration by Parts" again.

Example 3.

a. Find $\int x^2 e^x \, dx$.

Solution.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^2 & & e^x \\ & \searrow & \\ 2x & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx.$$

We apply the integration by parts to $\int xe^x \, dx$,

$$\int xe^x \, dx = \int x \, de^x = xe^x - \int e^x dx = xe^x - e^x + C.$$

Therefore,

$$\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x) + C = x^2 e^x - 2xe^x + 2e^x + C.$$

Remark. Repeated application of Integration by Parts can be made easier by the Diagram Method, but be careful to the signs.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^2 & & e^x \\ & \searrow & \\ 2x & & e^x \\ & (-)\searrow & \\ 2 & (+)\rightarrow & e^x \end{array}$$

$\square$

b. How would you evaluate $\int x^n e^x \, dx$, where n is a positive integer?

Solution.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^n & & e^x \\ & \searrow & \\ nx^{n-1} & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

We see that integration by parts reduces the power of the variable in the integrand.

Remark. Sometimes we simply go back to the original integration if we apply the integration by parts over and over. This can be easily checked by the diagram method. We must be careful because it might seem that we are going in circles BUT we are not. The following is an example of this.

Example 4. Find $\int e^{2x} \sin x \, dx$.

Solution.

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
\sin x & & e^{2x} \\
 & \searrow & \\
\cos x & & \frac{1}{2}e^{2x} \\
 & (-) \searrow & \\
-\sin x & (+) \rightarrow & \frac{1}{4}e^{2x}
\end{array}$$

Now, we have

$$\begin{aligned}
\int e^{2x} \sin x \, dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x \, dx \\
\implies \frac{5}{4}\int e^{2x}\sin x \, dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x \\
\implies \int e^{2x}\sin x \, dx &= \frac{e^{2x}}{5}\Big[2\sin x - \cos x\Big] + C.
\end{aligned}$$

□

Theorem (Integration by Parts for Definite Integrals). Let u and v be differentiable. Then $\boxed{\int_a^b u(x)v^{'}(x)\ dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u^{'}(x)\ dx.}$

Remark. Or we can first find the anti-derivative of $u(x)v^{'}(x)$ and then find the definite integral.

Example. Find $\int_4^9 \frac{\ln y}{\sqrt{y}}\ dy$.

Solution.

$$\int \frac{\ln y}{\sqrt{y}}\ dy = 2\sqrt{y}\ln y - 2\int y^{-1/2}\ dy = 2\sqrt{y}\ln y - 4\sqrt{y} + C.$$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \ln y & & y^{-\frac{1}{2}} \\ & \searrow & \\ \frac{1}{y} & (-)\rightarrow & 2\sqrt{y} \end{array}$$

Therefore,

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\ dy = (2\sqrt{y}\ln y - 4\sqrt{y})\Big|_4^9$$

□

More examples

Example. Find $\int \ln x\ dx$.

Solution.

$$\begin{array}{cccc} \text{diff.} & & \text{int.} & \\ \ln x & & 1 & \\ & \searrow & & \\ \frac{1}{x} & (-)\rightarrow & x & \text{Easy Integration} \end{array}$$

Thus,

$$\boxed{\int \ln x\ dx = x\ln x - \int 1\ dx = x\ln x - x + C}.$$

□

Example. Find $\int (\ln x)^2\ dx$.

Solution.

$\square$

Example. Find $\int x^3 \tan^{-1} x \, dx$.

Solution. $\int x^3 \tan^{-1} x \, dx$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \tan^{-1} x & & x^3 \\ & \searrow & \\ \frac{1}{1+x^2} & (-) \rightarrow & \frac{x^4}{4} \end{array}$$

Now, by long division, we have

$$\frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2}.$$

Thus,

$$\begin{aligned} \int x^3 \tan^{-1} x \, dx &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \, dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \left[\frac{x^3}{3} - x + \tan^{-1} x \right] + C. \end{aligned}$$

$\square$

Example. Find $\int \sin^{-1} x \, dx$.

Solution. Think of this integral as $\int (\sin^{-1} x)(1) \, dx$ and stop the diagram after one step.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \sin^{-1} x & & 1 \\ & \searrow & \\ \frac{1}{\sqrt{1-x^2}} & (-) \rightarrow & x \end{array}$$

Now, put things together with integration by Substitution to get

$$\begin{aligned} \int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx \text{ (Set } u = 1 - x^2) \\ &= x \sin^{-1} x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int u^{-\frac{1}{2}} \, du \\ &= x \sin^{-1} x + \frac{1}{2}(2) u^{\frac{1}{2}} + C = x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

$\square$

Example. Find $\int s2^s \; ds$.

Solution. Recall that $\int 2^s \; ds = 2^s/\ln 2$.

$$\int s2^s \; ds = \frac{1}{\ln 2}s2^s - \frac{1}{\ln 2}\int 2^s \; ds = \frac{1}{\ln 2}s2^s - \frac{1}{(\ln 2)^2}2^s.$$

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
s & & e^{2^s} \\
 & \searrow & \\
1 & (-)\rightarrow & 2^s/\ln 2
\end{array}$$

$\square$

Example. Find $\int x^2 \cos mx \; dx$.

Solution.

$$\begin{aligned}
\int x^2 \cos mx \; dx =& \frac{x^2 \sin mx}{m} + \frac{2x \cos mx}{m^2} - \frac{2}{m^2}\int \cos mx \; dx \\
=& \frac{x^2 \sin mx}{m} + \frac{2x \cos mx}{m^2} - \frac{2 \sin mx}{m^3} + C.
\end{aligned}$$

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
x^2 & & \cos mx \\
 & \searrow & \\
2x & & \sin mx/m \\
 & (-)\searrow & \\
2 & (+)\rightarrow & -\cos mx/m^2
\end{array}$$

$\square$

8.3 §8.3 Trigonometric Integrals

At the moment, our inventory of integrals involving trigonometric functions is rather limited. The goal of this section is to develop techniques for evaluating integrals involving trigonometric functions. We must memorize the trigonometric identities:

TR1. $\sin^2 x + \cos^2 x = 1$.

TR2. $\tan^2 x + 1 = \sec^2 x$.

TR3. $\cot^2 x + 1 = \csc^2 x$.

TR4. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$.

TR5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

TR6. $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$.

TR7. $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$, since $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

TR8. $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$.

TR9. $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$.

TR10. $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$.

TR11. $\sin x \cos x = \frac{1}{2}[\sin(2x)]$, since $\sin(2x) = \sin(x+x) = 2\sin x \cos x$.

Example. Find $\int \sin^6 x \cos^3 x \; dx$.

Solution. Since the exponent on the cosine function is odd we keep one of these to construct our differential and rewrite the others as powers of the sine function.

$$\sin^6 x \cos^3 x \; dx = \sin^6 \cos^2 x(\cos x \; dx) = \sin^6 x(1 - \sin^2 x)d(\sin x).$$

Let $u = \sin x$. Then

$$\int \sin^6 x \cos^3 x \; dx = \int u^6(1-u^2) \; du = \int (u^6 - u^8) \; du = \frac{u^7}{7} - \frac{u^9}{9} = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C.$$

This example gives us a glimpse at how to tackle these integrals.

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Then integrals involving trigonometric functions can sometimes be classified as follows:

Integrand	**Restriction on m**	**Restriction on n**	**Procedure**
$\sin mx \sin nx$	$\neq 0$	$\neq 0$	Use TR6
$\cos mx \cos nx$	$\neq 0$	$\neq 0$	Use TR8
$\sin mx \cos nx$	$\neq 0$	$\neq 0$	Use TR10
$\sin^m x \cos^n x$	Odd in $\mathbb{Z}_{>0}$	None	$\underbrace{-\sin^{m-1} x}_{-(1-\cos^2 x)^{\frac{m-1}{2}}} \cos^n x \; d(\cos x)$
$\sin^m x \cos^n x$	None	Odd in $\mathbb{Z}_{>0}$	$\sin^m x \underbrace{\cos^{n-1} x}_{(1-\sin^2 x)^{\frac{n-1}{2}}} \; d(\sin x)$
$\sin^m x \cos^n x$	Even in $\mathbb{Z}_{\geq 0}$	Even in $\mathbb{Z}_{\geq 0}$	Use TR7 & TR9 $\rightarrow \cos(2x)$
$\tan^m x \sec^n x$	None	Even in $\mathbb{Z}_{>0}$	Use: $\sec^n x \; dx =$ $(\tan^2 x + 1)^{\frac{n-2}{2}} \; d(\tan x)$
$\tan^m x \sec^n x$	in $\mathbb{Z}_{\geq 2}$	$= 0$	Reduce power using: $\tan^m x = \tan^{m-2}(\sec^2 x - 1)$ $= \tan^{m-2} \; d(\tan x) - \tan^{m-2} x = \cdots$
$\tan^m x \sec^n x$	$= 0$	Odd in $\mathbb{Z}_{>0}$	If $n = 1,$ use foumula. If not, use repeated int. by parts.
$\tan^m x \sec^n x$	Odd in $\mathbb{Z}_{>0}$	Odd in $\mathbb{Z}_{>0}$	Simplify by using: $\tan^m x \sec^n x$ $= \tan^{m-1} x \sec^{n-1} x \; d\sec x$ $= (\sec^2 - 1)^{\frac{m-1}{2}} \sec^{n-1} x \; d(\sec x)$
$\tan^m x \sec^n x$	Even in $\mathbb{Z}_{>0}$	Odd in $\mathbb{Z}_{>0}$	Use: $\tan^m x = (\sec^2 x - 1)^{\frac{m}{2}}$
$\cot^m x \csc^n x$			Use: Similar to $\tan^m x \sec^n x$

Example 1.

a. $\int \cos^5 x \, dx$

Solution.

$\square$

b. $\int \sin^4 x \, dx$

Solution.

$\square$

Example 2.

a. $\int \sin^4 x \cos^2 x \, dx$

Solution.

$\square$

b. $\int \sin^3 x (\cos x)^{-2} \ dx$

Solution.

$\square$

Example. $\int \cos^2 x \ dx$.

Solution. The Outline says that the integral $\int \sin^m x \cos^n x \ dx$ with m and n both even is done using TR7 and TR9.

$$\begin{aligned}\int \cos^2 x \ dx &= \int \frac{1}{2}(1+\cos 2x) \ dx \\ &= \frac{x}{2} + \frac{1}{2}\int \cos 2x \ dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C\end{aligned}$$

$\square$

Remark. Try $\int \cos^4 x \ dx$ and $\int \cos^6 x \ dx$.

Example. $\int \cos \pi x \cos 4\pi x \; dx$.

Solution. We first simplify the integrand by substitution and then use Identity TR8: $\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$.

$$\int \cos \pi x \cos 4\pi x \; dx = \frac{1}{\pi}\int \cos \pi x \cos 4\pi x \; d(\pi x) = \frac{1}{\pi}\int \underbrace{\cos u}_{\cos A}\underbrace{\cos 4u}_{\cos B} \; du$$
$$= \frac{1}{\pi}\int \frac{1}{2}[\underbrace{\cos(1-4)u}_{\cos(A-B)} + \underbrace{\cos(1+4)u}_{\cos(A+B)}] \; du.$$

Therefore,

$$\begin{aligned}
\int \cos \pi x \cos 4\pi x \; dx &= \frac{1}{2\pi}\int [\cos(-3u) + \cos 5u] \; du \\
&= \frac{1}{2\pi}\Big[\int \cos(3u) \; du + \int \cos 5u \; du\Big] \\
&= \frac{1}{2\pi}\Big[\frac{1}{3}\int \cos 3u \; d(3u) + \frac{1}{5}\int \cos 5u \; d(5u)\Big] \\
&= \frac{1}{2\pi}\Big[\frac{1}{3}\sin 3u + \frac{1}{5}\sin 5u\Big] + C = \frac{1}{10\pi}\sin 5u + \frac{1}{6\pi}\sin 3u + C \\
&= \frac{1}{10\pi}\sin 5\pi x + \frac{1}{6\pi}\sin 3\pi x + C.
\end{aligned}$$

□

Example (reading). The Outline says that the integral $\int \tan^m x \sec^n x \; dx$ with $m = 0$ and n odd is done by repeated use of Integration by Parts. As a special case we consider

$$\int \sec^3 x \; dx.$$

Solution. This requires Integration by Parts. In order to apply Integration by Parts, we rewrite the integrand

$$\int \sec^3 x \; dx = \int \sec x \sec^2 x \; dx.$$

Again, we apply Integration by Parts.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \sec x & & \sec^2 x \\ & \searrow & \\ \sec x \tan x & (-) \rightarrow & \tan x \end{array}$$

$$\begin{aligned} \int \sec^3 x \; dx &= \sec x \tan x - \int \sec x \tan^2 x \; dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \; dx \\ &= \sec x \tan x - \int (\sec^3 x - \sec x) \; dx \\ &= \sec x \tan x - \int \sec^3 x \; dx + \int \sec x \; dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sec^3 x \; dx &= \frac{1}{2}\big(\sec x \tan x + \int \sec x \; dx \big) \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C. \end{aligned}$$

□

Example 4. Evaluate the following integrals.

a. $\int \tan^3 x \sec^4 x \; dx$

Solution.

□

b. $\int \tan^2 x \sec x \, dx$

Solution.

$\square$

Reduction Formulas

Some reduction formulas have been developed to ease the trigonometric integration workload. These formulas are not required to be memorized.

Assume that n is a positive integer.

1. $$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$
2. $$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$
3. $$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$
4. $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \; n \neq 1$$

Example 3. Evaluate $\int \tan^4 dx$.

Method 1 Sol.

$\square$

Method 2 (using reduction formula).

$\square$

Example. $\int \cot^5 x \sin^4 x \; dx$.

Solution.

$$\begin{aligned}\int \cot^5 x \sin^4 x \; dx &= \int \frac{\cos^5 x}{\sin^5 x} \sin^4 x \; dx = \int \frac{\cos^5 x}{\sin x} \; dx \\ &= \int \frac{(\cos^2 x)^2}{\sin x} \cos x \; dx = \int \frac{(1-\sin^2 x)^2}{\sin x} \cos x \; dx.\end{aligned}$$

Set $u = \sin x$ then $du = \cos x \; dx$. Therefore,

$$\begin{aligned}\int \cot^5 x \sin^4 x \; dx &= \int \frac{(1-u^2)^2}{u} \; du = \int \frac{1-2u^2+u^4}{u} \; du \\ &= \int \frac{1}{u} \; du - \int 2u \; du + \int u^3 \; du \\ &= \ln|u| - u^2 + \frac{u^4}{4} + C \\ &= \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C.\end{aligned}$$

$\square$

8.4 §8.4 Trigonometric Substitutions

Integrals involving a **radical** (integer power sometimes) of certain forms are often solved by a trigonometric (inverse) substitution:

$$\int f(g(x))g^{'}(x)\,dx = \int f(u)\,du \overbrace{\Longrightarrow}^{Let\ x=g(t)} \boxed{\int f(x)\,dx = \int f(g(t))g^{'}(t)\,dt}$$

When a constant $c > 0$ and the integrand involves one of

(i) $\sqrt{c^2 - x^2}$. Substitute: $x = c\sin\theta$. Then

$$\sqrt{c^2 - x^2} = \sqrt{c^2(1-\sin^2\theta)} = c|\cos\theta| = c\cos\theta \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(ii) $\sqrt{x^2 + c^2}$. Substitute: $x = c\tan\theta$. Then

$$\sqrt{x^2 + c^2} = \sqrt{c^2(\tan^2\theta + 1)} = c\sec\theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(iii) $\sqrt{x^2 - c^2}$. Substitute: $x = c\sec\theta$. Then

$$\sqrt{x^2 - c^2} = \sqrt{c^2(\sec^2\theta - 1)} = c\tan\theta \text{ for } 0 \le \theta < \frac{\pi}{2} \text{ or } \pi \le \theta < \frac{3\pi}{2}.$$

Example 1. Verify that the area of a circle of radius r is πr^2.

Solution.

□

Example 2.(similar to Example 1) Evaluate $\int \dfrac{dx}{(16-x^2)^{3/2}}$.

Solution.

$\square$

Example 3. Evaluate $\int_0^2 \sqrt{1+4x^2}\, dx$.

Solution.

$\square$

Example 4.(similar to Example 3) Evaluate $\int \frac{1}{(1+x^2)^2}\, dx$.

Solution.

$\square$

Example 5. (similar to Example 3) Evaluate $\int \frac{1}{\sqrt{x^2+4}}\, dx$.

Solution.

$\square$

Exercise 19. Evaluate $\int \frac{dx}{\sqrt{x^2-81}}\,dx$.

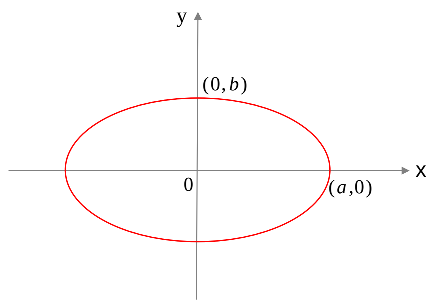
Solution.

$\square$

Example (reading). Show that the area enclosed by the following ellipse is πab

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution.



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \implies y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

The ellipse can be thought of as four parts of equal area and so we have

$$\text{Area} = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx. \qquad \begin{cases} \text{Set } x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{cases}$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta}\, (a\cos\theta\, d\theta) \qquad \begin{cases} \text{Since } \theta = \sin^{-1}\frac{x}{a} \\ \theta = 0 \text{ when } x = 0 \\ \theta = \frac{\pi}{2} \text{ when } x = a \end{cases}$$

$$= 4\int_0^{\frac{\pi}{2}} ab\sqrt{1 - \sin^2\theta}\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (ab)\cos^2\theta\, d\theta = (4ab)\int_0^{\frac{\pi}{2}} \frac{1}{2}\big(1 + \cos(2\theta)\big)\, d\theta$$

$$= 2ab\Big[\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, d(2\theta)\Big]$$

$$= (2ab)\Big[\theta + \frac{1}{2}\sin(2\theta)\Big]_0^{\frac{\pi}{2}} = (2ab)\Big[\frac{\pi}{2} - 0 + \frac{1}{2}(\sin\pi - \sin 0)\Big]$$

$$= (2ab)\frac{\pi}{2} = \pi ab.$$

□

Sometimes the form of the expression under the radical does not fit into a standard form. But we can still apply the substitution directly. However, we need "**to Complete the Squares**" before we use the substitution.

Completing the Squares. To find the roots of $ax^2 + bx + c$ when $a \neq 0$, we first Complete the Squares

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}) \\ &= a\Big[(x + \frac{\mathbf{b}}{\mathbf{2a}})^2 - (\frac{b^2 - 4ac}{4a^2})\Big]. \end{aligned}$$

When $ax^2 + bx + c$ is under a radical as in $\sqrt{ax^2 + bx + c}$ then we need to study three cases to simplify the expression $\sqrt{ax^2 + bx + c}$.

$$\begin{aligned} \sqrt{ax^2 + bx + c} &= \sqrt{a\Big[(x + \frac{b}{2a})^2 - (\frac{b^2 - 4ac}{4a^2})\Big]} \\ &= \begin{cases} \sqrt{u^2 - d^2} & \text{if } a > 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) > 0 \\ \sqrt{u^2 + d^2} & \text{if } a > 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) < 0 \\ \sqrt{d^2 - u^2} & \text{if } a < 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) > 0 \end{cases}, \end{aligned}$$

where $u = x + \frac{b}{2a}$ is the variable in substitution and d is a constant.

(i) $\sqrt{ax^2 + bx + c} = \sqrt{u^2 - d^2}$. Substitute: $u = d\sec\theta$.

(ii) $\sqrt{ax^2 + bx + c} = \sqrt{u^2 + d^2}$. Substitute: $u = d\tan\theta$.

(iii) $\sqrt{ax^2 + bx + c} = \sqrt{d^2 - u^2}$. Substitute: $u = d\sin\theta$.

Example 6. Evaluate $\int_1^4 \dfrac{\sqrt{x^2+4x-5}}{x+2}\,dx$.

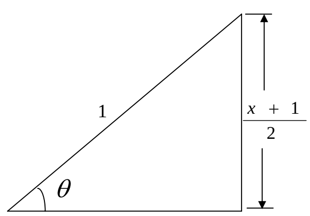
Solution.

□

More examples:

Example. Find $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution. $3-2x-x^2 = 2^2-(x+1)^2$. Now, set $x+1 = 2\sin\theta$. Notice that $\sin\theta = \frac{x+1}{2}$, so set up a triangle that describes this substitution.



Replace the various terms in the integrand of $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ by the their equivalent expressions given by

$$x = 2\sin\theta - 1,\ dx = 2\cos\theta\ d\theta,$$
$$\sqrt{3-2x-x^2} = \sqrt{2^2-(x+1)^2} = \sqrt{2^2-2^2\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$$

Now,

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{2\sin\theta-1}{2\cos\theta}(2\cos\theta\ d\theta) = \int (2\sin\theta-1)\ d\theta$$
$$= -2\cos\theta - \theta + C = -2\frac{\sqrt{2^2-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

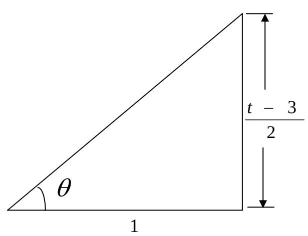
□

Example (reading). Find $\int \frac{dt}{\sqrt{t^2-6t+13}}\, dt$.

Solution. First, we complete the squares

$$\begin{aligned} t^2 - 6t + 13 &= t^2 - 6t + 3^2 - 3^3 + 13 = (t-3)^2 + 4 \\ &= (t-3)^2 + 4 = (t-3)^2 + 2^2 = u^2 + c^2. \\ \sqrt{t^2 - 6t + 13} &= \sqrt{(t-3)^2 + 2^2} \\ &= \sqrt{u^2 + c^2} \implies \text{ Substitution } t - 3 = u = c\tan\theta = 2\tan\theta. \end{aligned}$$

Now set $t - 3 = 2\tan\theta$ ($\tan\theta = \frac{t-3}{2}$)and set up a triangle that describes this substitution.



Replace the various terms in the integrand of $\int \frac{dt}{\sqrt{t^2-6t+13}}$ by the their equivalent expressions given by

$$\begin{aligned} &t = 3 + 2\tan\theta, dt = 2\sec^2\theta\ d\theta, \\ &\sqrt{t^2 - 6t + 13} = \sqrt{(t-3)^2 + 2^2} = \sqrt{2^2\tan^2\theta + 2^2} = 2\sqrt{\tan^2\theta + 1} = 2\sec\theta. \end{aligned}$$

Now,

$$\begin{aligned} \int \frac{dt}{\sqrt{t^2 - 6t + 13}} &= \int \frac{2\sec^2\theta\ d\theta}{2\sec\theta} = \int \sec\theta\ d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\frac{\sqrt{(t-3)^2 + 2^2}}{2} + \frac{t-3}{2}| + C \\ &= \ln|\sqrt{(t-3)^2 + 2^2} + (t-3)| + C. \end{aligned}$$

□

8.5 §8.5 Integration of Rational Functions by Partial Fractions

The idea behind partial fractions is to "undo" or reverse the addition of rational functions (that is, decomposition of the rational functions). For example,

$$\frac{3x}{x^2+2x-8} \overset{\text{method of partial fractions}}{=\!=\!=\!=\!=} \frac{1}{x-2}+\frac{2}{x+4}.$$

A rational function is the ratio of two polynomials. A **polynomial** is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. Such a polynomial is said to have degree n,

$$\deg P(x) = n, \quad \text{provided } a_n \neq 0.$$

A **rational function** is a function that can be expressed as the ratio of two polynomials.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}.$$

The Method of Partial Fractions consists of three steps to simplify the integrand and then do the integration after this is done.

Step 1. If $\deg P(x) < \deg Q(x)$, go to step 2; If $\deg P(x) \geq \deg Q(x)$, we divide $Q(x)$ by $P(x)$ (long division) and represent $f(x)$ as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \text{ where } R(x) = 0 \text{ or } \deg R(x) < \deg Q(x).$$

Step 2. Factor $Q(x)$ as far as possible (there are two types of factors: linear and quadratic),

$$Q(x) = (\alpha_1 x + \beta_1)^{\epsilon_1} \cdots (\alpha_k x + \beta_k)^{\epsilon_k} (a_1 x^2 + b_1 x + c_1)^{e_1} \cdots (a_\ell x^2 + b_\ell x + c_\ell)^{e_\ell},$$

where each quadratic polynomial has no real roots, i.e.

$$a_i x^2 + b_i x + c_i \text{ has the property that } b_i^2 - 4a_i c_i < 0.$$

Step 3. Using this factorization of (x), express $\frac{R(x)}{Q(x)}$ as the sum of terms of the form

$$\frac{A_{i1}}{(\alpha_i x+\beta_i)}+\cdots+\frac{A_{i\epsilon_i}}{(\alpha_i x+\beta_i)^{\epsilon_i}} \text{ for } i=1,\ldots,k \tag{$\star$}$$

plus terms of the form

$$\frac{A_{i1}x+B_{i1}}{(a_i x^2+b_i x+c_i)}+\cdots+\frac{A_{ie_i}x+B_{ie_i}}{(a_i x^2+b_i x+c_i)^{e_1}} \text{ for } i=1,\ldots,\ell \tag{$\star\star$}$$

The expressions in $(\star)$ are easy to integrate by substitution but those in $(\star\star)$ are more complicated and might involve completing the squares.

SUMMARY Partial Fraction Decompositions

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\frac{A_3}{(x-r)^3}+\cdots+\frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor** (See Exercises 83–86.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

Example (reading). Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$$

Solution. Step 1. Divide the denominator by the numerator (use long division) to get

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

Step 2. Factor the denominator. This often involve <u>trial and error</u> while looking for a root.

x	$x^3 - x^2 - x + 1$
0	1
1	0
-1	0

$\Longrightarrow (x-1)(x+1)$ divides $x^3 - x^2 - x + 1$.

Divide $x^3 - x^2 - x + 1$ by $x^2 - 1$.

$$\frac{x^3 - x^2 - x + 1}{x^2 - 1} = x - 1 \implies x^3 - x^2 - x + 1 = (x-1)^2(x+1).$$

Step 3. Now we must represent $\frac{4x}{x^3 - x^2 - x - 1}$ in the form of $(\star)$.

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiply through by the denominator on the left hand side to get

$$\frac{4x}{(x-1)^2(x+1)}(x-1)^2(x+1) = \frac{A}{x-1}(x-1)^2(x+1) + \frac{B}{(x-1)^2}(x-1)^2(x+1) + \frac{C}{x+1}(x-1)^2(x+1)$$

and so we must solve

$$(\bullet) \qquad 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

for A, B, and C. Since $(\bullet)$ is valid for all possible value of x we substitute values in for x if this is helpful.

$$x = 1 \implies 4(1) = A((1)-1)((1)+1) + B((1)+1) + C((1)-1)^2 = 2B$$
$$\implies 4 = 2B \implies B = 2$$

Put this into (•) and simplify to get

$$
\begin{aligned}
2x-2 = 2(x-1) &= A(x-1)(x+1) + C(x-1)^2 \\
(\bullet\bullet) \qquad 2 &= A(x+1) + C(x-1) \\
x = 1 &\implies 2 = 2A \implies A = 1 \\
x = -1 &\implies 2 = -2C \implies C = -1
\end{aligned}
$$

Or we can expand the right hand side of (•) and compare the corresponding coefficients. Thus,

$$
\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}
$$

and

$$
\begin{aligned}
&\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int \left[x+1+\frac{4x}{x^3-x^2-x+1}\right] dx \\
&= \int \left[x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right] dx \\
&= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K,
\end{aligned}
$$

where K is any constant. $\square$

Example 1. Evaluate $\int f(x)\, dx = \int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}\, dx$

Solution.

$\square$

Example 3.(repeated linear factors) Evaluate $\int f(x)\ dx = \int \frac{5x^2-3x+2}{x^3-2x^2}\ dx$

Solution.

$\square$

Example 5.(irreducible quadrtic factors) Evaluate $\int f(x)\,dx = \int \dfrac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$

Solution.

$\square$

8.6 §8.6 Integration Strategies

The only way to get to the point that you can do indefinite integrals is by experience. **DO PROBLEMS**. After you have done **a lot** of problems then you will see that the process follows these guideline.

1. **Simplify the integrand.** This might involve algebraic manipulation or the use of trigonometric identities which makes the method of integration obvious.
2. **Substitution.** See if there is a substitution that either puts the integrand in a standard form or which makes the method of integration obvious.
3. **Determine the form of the integrand.**
   (a) Rational Function
   (b) Trigonometric Function
   (c) Contains Radical
   (d) Integration by Parts (Does the integrand contain and inverse trigonometric function or ln?)

Remark. Some Standard Simplifying Substitutions.

(a) Integrals involving $\sqrt[n]{ax+b}$. Substitute: $u = \sqrt[n]{ax+b}$ or $u = ax+b$.

(b) Integrals involving $\sqrt{ax^2+bx+c}$. Complete the squares to get one of

  (i) $\sqrt{ax^2+bx+c} = \sqrt{p^2-u^2}$. Substitute: $u = p\sin\theta$

  (ii) $\sqrt{ax^2+bx+c} = \sqrt{u^2-p^2}$. Substitute: $u = p\sec\theta$

  (iii) $\sqrt{ax^2+bx+c} = \sqrt{u^2+p^2}$. Substitute: $u = p\tan\theta$

(c) Transform rational functions of trigonometric functions into rational functions in variable u by substituting: $u = \tan\frac{x}{2}$ so that (**Exercise 8.4.86**)

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}, \qquad dx = \frac{2du}{1+u^2}.$$

Example 1. Evaluate $\int \frac{\sin x + 1}{\cos^2 x}\, dx$

Solution.

$\square$

Example 2. Evaluate $\int \frac{x^2}{\sqrt{4 - x^6}}$.

Solution.

$\square$

Example (Reading). Find

$$\int \frac{dx}{2\sqrt{x+3} + x}$$

Solution. Using the idea in Part (a). Set $u = \sqrt{x+3}$. Then

$$u = \sqrt{x+3} \implies u^2 = x+3 \implies x = u^2 - 3 \implies dx = 2udu.$$

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2udu}{2u+u^2-3} = \int \frac{2udu}{u^2+2u-3} = \int \frac{2udu}{(u+3)(u-1)}.$$

Apply the Method of Partial Fractions.

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{(u+3)} + \frac{B}{(u-1)} \implies 2u = A(u-1) + B(u+3)$$

$$\implies A = \frac{3}{2}, B = \frac{1}{2}$$

$$\int \frac{2udu}{(u+3)(u-1)} = \frac{3}{2}\int \frac{du}{u+3} + \frac{1}{2}\int \frac{du}{u-1} = \frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1| + C$$

$$= \frac{3}{2}\ln|\sqrt{x+3}+3| + \frac{1}{2}\ln|\sqrt{x+3}-1| + C.$$

□

Example 3. Evaluate the following integrals.

a. $\int \frac{4-3x^2}{x(x^2-4)}\, dx$

Solution.

□

b. (Reading) $\int xe^{\sqrt{1+x^2}}\, dx$

c. $\int \ln(1+x^2)\ dx$

Solution.

$\square$

Example 4. Evaluate $\int xe^{\sqrt{x}}\ dx$

Solution.

$\square$

8.7 §8.7 Other Methods of Integration

Omit - Using Tables and Computer Algebra Systems such as Mathematica and Sage-Math.

8.8 §8.9 Improper Integrals

Definition (**Definition of Improper Integrals over Infinite Intervals).** The integrals of *unbounded domain* $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ are defined in the following way.

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$ and the limit $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists and is finite, then the **improper integral** $\int_a^\infty f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$ and the limit $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ exists and is finite, then the **improper integral** $\int_{-\infty}^b f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

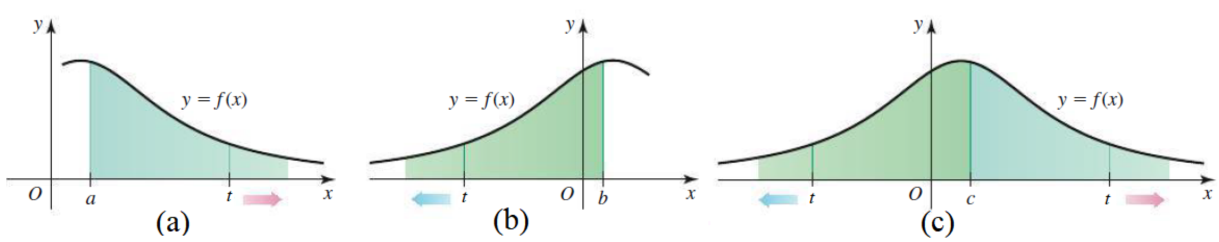
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

(c) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then the **improper integral** $\int_{-\infty}^\infty f(x)\,dx$ is defined to be value of this limit

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

where a is any real number.

An improper integral that is not convergent, is said to be **divergent**. (Convergence or divergence depends only on the "tail".



Remark. To simplify the calculation of the definite integral $\int_a^t f(x)\,dx$ or $\int_t^b f(x)\,dx$, we can find the indefinite integral $\int f(x)\,dx$ first.

Example 1.

a. $\int_0^\infty e^{-x}\,dx$

Solution.

□

b. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to-\infty}\Big[\tan^{-1}x\Big]_t^0 + \lim_{t\to\infty}\Big[\tan^{-1}x\Big]_0^t \\
&= \lim_{t\to-\infty}\Big[\tan^{-1}0 - \tan^{-1}t\Big] + \lim_{t\to\infty}\Big[\tan^{-1}t - \tan^{-1}0\Big] \\
&= \Big[0-(-\frac{\pi}{2})\Big] + \Big[\frac{\pi}{2}-0\Big] = \pi.
\end{aligned}$$

□

Example 2 - The p-Test for Improper Integrals.

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\text{diverges, if } p \le 1 \\ \text{converges, if } p > 1\end{cases}}$$

Proof.

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases}\ln t & \text{if } p = 1 \\ \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} & \text{if } p \ne 1\end{cases}$$

Now,

$$\lim_{t\to\infty} \ln t = \infty \implies \int_1^\infty \frac{1}{x^p}\,dx \text{ divergent when } p = 1 \text{ and}$$

$$\lim_{t\to\infty} \frac{1}{1-p}(t^{1-p}-1) = \begin{cases} \infty & \text{if } p < 1 \\ -\frac{1}{1-p} & \text{if } p > 1. \end{cases} \qquad \square$$

Example. Evaluate

$$\int_{-\infty}^0 xe^x\,dx$$

Solution. First we find an antiderivative by Integration by Parts.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x & & e^x \\ & \searrow & \\ 1 & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^0 xe^x\,dx &= \lim_{t\to-\infty} \int_t^0 xe^x\,dx \\ &= \lim_{t\to-\infty} \Big[xe^x - e^x\Big]_t^0 = \lim_{t\to-\infty} \Big[-1-(te^t-e^t)\Big] \end{aligned}$$

Certainly,

$$\lim_{t\to-\infty} e^t = \lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

We apply L'Hospital Rule to evaluate $\lim\limits_{t\to-\infty} te^t$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to\infty} \frac{-t}{e^t} = \lim_{t\to\infty} \frac{-1}{e^t} = 0.$$

Now we put these parts together to get

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \Big[-1-(te^t-e^t)\Big] = -1.$$

$\square$

Definition (**Definition of Improper Integrals of Unbounded(Discontinuous) Integrand).** If $f(x)$ has *discontinuous* point on $[a, b]$, we define $\int_a^b$ in what follows.

(a) If f is continuous on $(a, b]$, discontinuous at a, but the limit $\lim\limits_{t \to a^+} \int_t^b f(x)\, dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\, dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx.$$

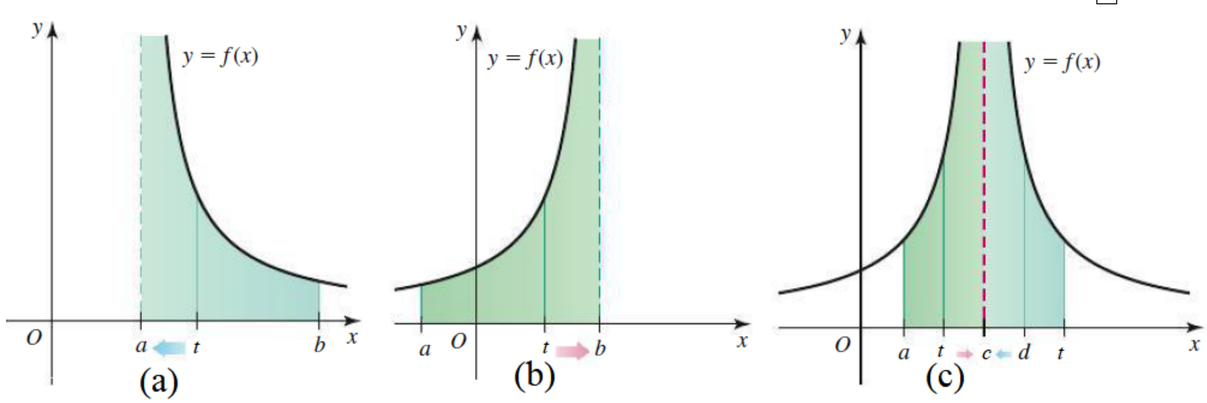
(b) If f is continuous on $[a, b)$, discontinuous at b, but the limit $\lim\limits_{t \to b^-} \int_a^t f(x)\, dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\, dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx.$$

(c) If f is continuous on $[a, c)$ and on $(c, b]$ with $a < c < b$, discontinuous at c, but the limit $\lim\limits_{t \to c^-} \int_a^t f(x)\, dx$ and $\lim\limits_{t \to c^+} \int_t^b f(x)\, dx$ both exist and are finite, then the **improper integral** $\int_a^b f(x)\, dx$ is defined to be

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$\square$



Example 4. Find the area of the region R between the graph of $f(x) = \frac{1}{\sqrt{9-x^2}}$ and the x-axis on the interval $(-3, 3)$ (if it exists).

Solution.

$\square$

Example 5. Evaluate $\int_1^{10} \frac{1}{(x-2)^{1/3}}\, dx$

Solution.

$\square$

The p-test for Improper Integrals of Unbounded Integrand

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \geq 1 \\ \text{converges, if } p < 1 \end{cases}}$$

Proof. Exercise 94

Example. Find $\int_0^3 \frac{dx}{x-1}$, if convergent.

Solution. Be careful. The following calculation is **NOT valid**.

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln 2 - \ln|-1| = \ln 2.$$

This is an improper integral due to the fact that $\frac{1}{x-1}$ is discontinuous at $c = 1$ with $0 < 1 < 3$. This means that we must check the convergence of both

$$\int_0^1 \frac{dx}{x-1} \qquad \text{and} \qquad \int_1^3 \frac{dx}{x-1}.$$

$$\begin{aligned} \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} &= \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t \\ &= \lim_{t\to 1^-} \Big[\ln|t-1| - \ln|0-1|\Big] = \lim_{t\to 1^-} \ln(1-t) \\ &= \lim_{s\to 0^+} \ln s = -\infty \end{aligned}$$

Therefore, $\int_0^3 \frac{dx}{x-1}$ is divergent. $\square$

The Comparison Test

There are some cases in which we cannot determine whether a given improper integral converges, simply because it is impossible to compute an anti-derivative.

Theorem (Thm 8.2: Comparison Test for Type I Improper integrals)**.** Suppose that f and g are continuous functions on the interval $[a, \infty)$ with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\ dx$ is convergent, then $\int_a^\infty g(x)\ dx$ is convergent.

(b) If $\int_a^\infty g(x)\ dx$ is divergent, then $\int_a^\infty f(x)\ dx$ is divergent.

Theorem (Limit Comparison Theorem - Type I)**.** Let $f(x)$ and $g(x)$ be two **positive continuous** functions on $[a, \infty)$. Assume that the limit

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and finite. That is, $0 < L < \infty$. Then,

$$\int_a^\infty f(x)\ dx \text{ converges } \Leftrightarrow \int_a^\infty g(x)\ dx \text{ converges}$$

Remark. Similar theorems are true for the other two cases of improper integrals over infinite intervals.

Example 7. Determine whether the following integrals converge.

a. $\int_0^\infty e^{-x^2}\ dx$

Solution. We need only test the "tail". $\int_1^\infty e^{-x^2}\ dx$ since

$$\int_0^\infty e^{-x^2}\ dx = \underbrace{\int_0^1 e^{-x^2}\ dx}_{\text{finite number}} + \int_1^\infty e^{-x^2}\ dx$$

On the interval $[1, \infty)$ we have $e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{e^x} = e^{-x}$ and so we test $\int_1^\infty e^{-x}\ dx$ and apply the Comparison Test to get convergence.

$$\int_1^\infty e^{-x}\ dx = \lim_{t\to\infty} \int_1^t e^{-x}\ dx = \lim_{t\to\infty} \Big[-e^{-x}\Big]_1^t = \lim_{t\to\infty} \Big[-e^{-t} + e^{-1}\Big]_1^t = \frac{1}{e}$$

This means $\int_1^\infty e^{-x^2}\ dx$ is convergent by the convergence test. $\square$

b. $\int_1^\infty \dfrac{1}{\sqrt[3]{x^2-0.5}}\,dx$

Solution.

$\square$

More examples

Example.

Test $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ for convergence or divergence.

Solution. We claim that it is convergent. We only need to check the "tail", i.e. $\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$. First observe that $\frac{x}{(x^2+2)^2} < \frac{1}{x^2}$ on $[1,\infty)$ because

$$x \cdot x^2 = x^3 < x^4 + 4x^2 + 4 = (x^2+2)^2 \implies \frac{x}{(x^2+2)^2} < \frac{1}{x^2}.$$

Now observe that

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent .}$$

By the p-Test and then by the Comparison Test $\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$ is convergent and hence $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ is convergent. We can actually find the value of $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_0^t \frac{d(x^2+2)}{(x^2+2)^2} = \lim_{t\to\infty} \frac{1}{2}\Big[-(x^2+2)^{-1}\Big]_0^t = \frac{1}{4}.$$

$\square$

Example. Test $\int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx$ for convergence or divergence.

Solution. Notice that we are not actually asked to evaluate this integral but just determine convergence or divergence.

Method I: We apply the Test for Convergence. First observe that $\tan^{-1}x \le \frac{\pi}{2} < 2$. Therefore,

$$\frac{x\tan^{-1}x}{(x^2+1)^2} \le \frac{2x}{(x^2+1)^2} \text{ and so}$$
$$\text{if } \int_0^\infty \frac{2x}{(x^2+1)^2}\,dx \text{ converges then } \int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx \text{ converges.}$$

Now,

$$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \int_0^\infty \frac{d(x^2+1)}{(x^2+1)^2} = \int_1^\infty \frac{1}{u^2}\,du \text{ (Convergent by the } p\text{-Test)}$$
$$\therefore \int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx \text{ (Convergent by the Comparison Test.)}$$

Method II Evaluate the improper integral... $\square$

Theorem (Comparison Test for Type II Improper integrals)**.** Suppose that the two function $f(x) \ge g(x) \ge 0$ on the interval $(a,b]$, but $f(x)$ and $g(x)$ are discontinuous (unbounded) at a.

(a) If $\int_a^b f(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.

(b) If $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Theorem (Limit Comparison Theorem - Type II)**.** Let $f(x)$ and $g(x)$ be two **positive continuous** functions on $(a,b]$, but $f(x)$ and $g(x)$ are discontinuous (unbounded) at a. Assume that the limit

$$L = \lim_{x\to a^+} \frac{f(x)}{g(x)}$$

exits and finite. That is, $0 < L < \infty$. Then,

$$\int_a^b f(x)\,dx \text{ converges } \Leftrightarrow \int_a^b g(x)\,dx \text{ converges}$$

Remark. Similar theorems are true for the other two cases of improper integrals of unbounded integrand.

Exercise 8.9.85. Determine whether the following integral converges or diverges.

$$\int_0^1 \frac{dx}{\sqrt{x^{1/3}+x}}$$

Solution.

Exercise 8.9.86. Determine whether the following integral converges or diverges.

$$\int_0^1 \frac{1+\sin x}{x^5}\,dx$$

Solution.

8.9 §8.8 Numerical Integration

The two primary cases when we want to use approximate **numerical integration** are: i) when the antiderivative is too difficult or impossible to find, for example $\int e^{x^2}\ dx$ and $\int \sqrt{1+x^3}\ dx$; ii) when we are involved in a mathematical model in which the formula of the function is unknown.

Because numerical methods do not typically produce exact results, we should be concerned about the accuracy of approximations, which leads to the ideas of **absolute** and **relative error**.

Definition (Error of Approximation)**.** Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absoluate error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c-x|}{|x|} \quad (if\ x \neq 0).$$

Now, the only method is to find a numerical approximation for the definite integral

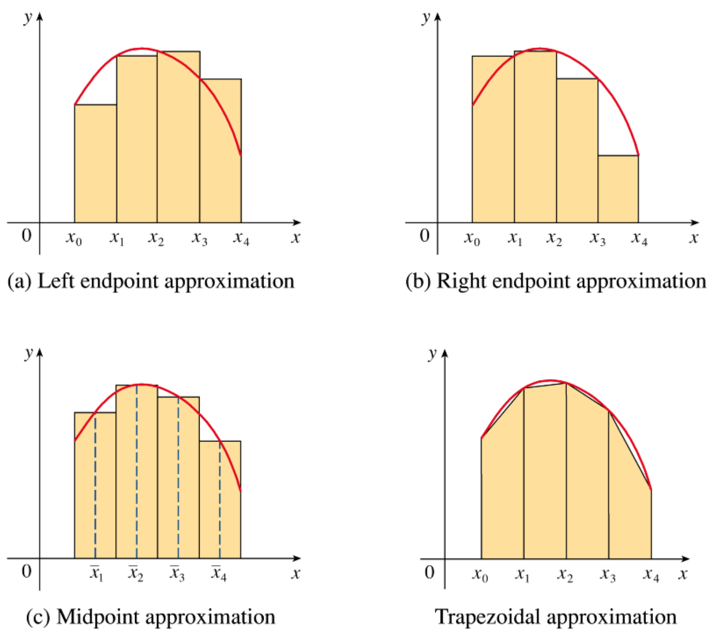
$$\int_a^b f(x)\ dx$$

by partitioning the interval $[a, b]$ into n subintervals of equal length

$$a = x_0 < x_1 = a + \frac{b-a}{n} < \cdots < x_i = a + i\frac{b-a}{n} < \cdots < x_n = a + n\frac{b-a}{n} = b.$$

Note that

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\frac{b-a}{n}.$$



Integral approximations

We consider three types of Approximate Integration.

(1). A Riemann Sum Approximation - Midpoint Rule,

(2). Trapezoidal Rule, and

(3). Simpson's Rule.

MIDPOINT RULE.

$$\boxed{M_n = \int_a^b f(x)\,dx \approx \Delta x\big[f(\bar{x}_1) + \cdots + f(\bar{x}_{n-1}) + f(\bar{x}_n)\big]},$$

where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$ is the midpoint of the subinterval. $\square$

Moreover, if $|f''(x)| \le K$ on $[a,b]$ and E_M is the error of the Midpoint Approximation, then

$$|E_M| = \left|\int_a^b f(x)\,dx - M_n\right| \le \frac{K(b-a)^3}{24n^2}.$$

$\square$

A trapezoid is a quadrilateral with one pair of opposite side parallel. In our case, two sides will be parallel and perpendicular to the base. If such a trapezoid has base of width Δx and parallel side of length $f(x_{i-1})$ and $f(x_i)$, then

$$\text{Area of Trapezoid } = \Delta x \frac{f(x_{i-1}) + f(x_i)}{2}.$$

The integral is approximated by

$$\sum_{i=1}^{n} \Delta x \frac{f(x_{i-1}) + f(x_i)}{2} = \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1 - 1) + \cdots + 2f(x_{n-1}) + f(x_n)\big].$$

TRAPEZOIDAL RULE.

$$\boxed{T_n = \int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big].}$$

Moreover, if $|f''(x)| \leq K$ on $[a, b]$ and E_T is the error of the approximation, then

$$|E_T| = \Big|\int_a^b f(x)\ dx - T_n\Big| \leq \frac{K(b-a)^3}{12n^2}.$$

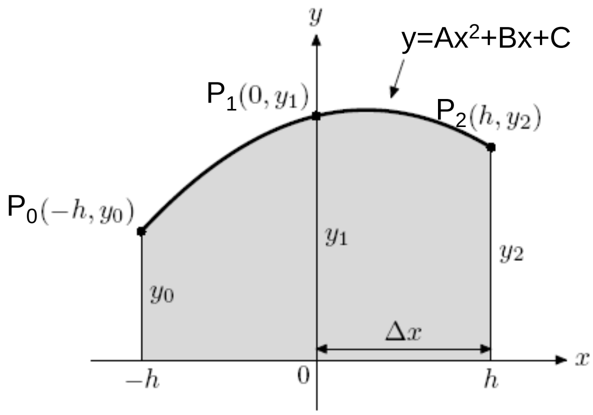
□

Simpson's Rule is somewhat more involved. On subintervals of length $2\Delta x$, we approximate the curve with parabolas and use the sum of the integrals of these approximating parabolas to approximate the integral. This requires that the number of parts n of our partition is **even** $\underline{n = 2k}$. On the interval $[x_{2(\ell)}, x_{2\ell+2}]$ the approximating parabola is required to pass through the points

$$P_0 = \big(x_{2\ell}, f(x_{2\ell})\big), \quad P_1 = \big(x_{2\ell+1}, f(x_{2\ell+1})\big), \quad P_2 = \big(x_{2\ell+2}, f(x_{2\ell+2})\big)$$

To get an approximation on the interval $x_{2\ell} \leq x_{2\ell+1} \leq x_{2\ell+2}$, we simplify things by shifting the graph so that $x_{2\ell} = -h$, $x_{2\ell+1} = 0$ and $x_{2\ell+2} = h$. Then

$$P_0 = \big(-h, f(-h)\big), \quad P_1 = \big(0, f(0)\big), \quad P_2 = \big(h, f(h)\big)$$



Now, our approximating polynomial on $[-h, h]$ has the form $Ax^2 + Bx + C$ and so the approximating integral is

$$\begin{aligned}\int_{-h}^{h} (Ax^2 + Bx + C)\ dx &= \left[A\frac{x^3}{3} + B\frac{x^2}{2} + Cx\right]_{-h}^{h} \\ &= \left[A\frac{h^3}{3} + B\frac{h^2}{2} + Ch\right] - \left[A\frac{-h^3}{3} + B\frac{h^2}{2} - Ch\right] \\ &= 2A\frac{h^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).\end{aligned}$$

Now, we want to represent this integral using the values $y_0 = f(-h)$, $y_1 = f(0)$, $y_2 = f(h)$ and so we solve the system for the expression $2Ah^2 + 6C$.

$$\begin{aligned} y_0 &= f(-h) = Ah^2 - Bh + C \\ y_1 &= f(0) = C \qquad \Longrightarrow\ y_0 + y_2 = 2Ah^2 + 2C \\ y_2 &= f(h) = Ah^2 + Bh + C \end{aligned}$$

$$\begin{aligned} &\Longrightarrow\ y_0 + 4y_1 + y_2 = 2Ah^2 + 6C \\ &\Longrightarrow\ \int_{-h}^{h} (Ax^2 + Bx + C)\ dx = \frac{h}{3}[y_0 + 4y_1 + y_2]. \end{aligned}$$

Finally, since the shift that we made to move the interval $[x_{2\ell}, x_{2\ell+2}]$ to the interval $[-h, h]$ does not affect the height of the graph at corresponding points and the value of the approximating integral depends only on the values of $y_0 = f(x_{2\ell})$, $y_1 = f(x_{2\ell+1})$

and $y_2 = f(x_{2\ell+2})$, we see that

$$\int_{x_{2\ell}}^{x_{2\ell+2}} \underbrace{Ax^2 + Bx + C}_{\text{Approx. Parabola}} \quad dx = \frac{h}{3}[f(x_{2\ell}) + 4f(x_{2\ell+1}) + f(x_{2\ell+2})].$$

Note that here $n = 2k$ and $h = \Delta x = \frac{b-a}{n} = \frac{b-a}{2k}$ and so

$$\int_{x_{2\ell}}^{x_{2\ell+2}} \underbrace{Ax^2 + Bx + C}_{\text{Approx. Parabola}} \quad dx = \frac{\Delta x}{3}[f(x_{2\ell}) + 4f(x_{2\ell+1}) + f(x_{2\ell+2})].$$

Simpson's Rule is an approximation obtained by adding together all the approximations over the k intervals of length $2\Delta x$:

$$[a, a + 2\Delta x], \cdots, [a + 2(k-1)\Delta x, a + 2k\Delta x]$$

SIMPSON'S RULE. Partition $[a, b]$ into $n = 2k$ subintervals using the end points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \cdots, x_n = a + 2k\Delta x = b.$$

Then

$$S_n = \int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big].$$

□

Moreover, if $|f^{(4)}(x)| \leq K$ for all x in $[a, b]$ and E_S is the error of the Simpson's Rule approximation, then

$$|E_S| = \Big|\int_a^b f(x)\ dx - S_n\Big| \leq \frac{K(b-a)^5}{180n^4}.$$

□

Example. Set $n = 10$ and use each of the Midpoint Rule, Trapezoidal Rule and Simpson's Rule to approximate

$$I = \int_0^1 e^{x^2}\ dx.$$

Also, use the bound on the error term in each case to get an estimate of the accuracy of the approximation.

Solution. First we bound the errors. In the cases of the Midpoint Approximation and the Trapezoidal Rule Approximation, we need to bound the second derivative on $[0, 1]$ with K and in the case of the Simpson's Rule Approximation, we need to bound the fourth derivative on $[0, 1]$ with $\hat{K}$.

$$\begin{aligned}
f(x) &= e^{x^2} \\
f'(x) &= 2xe^{x^2} \\
f''(x) &= (2 + 4x^2)e^{x^2} \\
f^{(3)}(x) &= (12x + 8x^3)e^{x^2} > 0 \text{ on } (0, 1] \implies f''(x) \text{ incr. on } [0.1] \\
&\implies K = f''(1) = (6)e = 16.3096 \\
f^{(4)} &= (12 + 48x^2 + 16x^4)e^{x^2} \\
f^{(5)}(x) &= (120x + 112x^2 + 32x^5)e^{x^2} > 0 \text{ on } (0, 1] \implies f^{(4)}(x) \text{ incr. on } [0.1] \\
&\implies \hat{K} = f^{(4)}(1) = (76)e = 206.589
\end{aligned}$$

Now,

	Error Bound Formula	Error Bound Value
Midpoint Rule	$\frac{K(b-a)^3}{24*n^2}$	$\frac{(16.3096)(1-0)^3}{24*10^2} = .00679$
Trapezoidal Rule	$\frac{K(b-a)^3}{12*n^2}$	$\frac{(16.3096)(1-0)^3}{12*10^2} = .013591$
Simpson's Rule Rule	$\frac{\hat{K}(b-a)^5}{180*n^4}$	$\frac{(206.589)(1-0)^5}{180*10^4} = .00011477$

Computing the values needed in the approximations is done easily using a spread sheet such as Excel. We used this software for the following values.

i	x_i	$\bar{x}_i$	Midpt. Approx.	Trap. Approx.	Simp. Approx.
0	0			$f(x_0) = 1$	$f(x_0) = 1$
1	.1	.05	$f(\bar{x}_1) = 1.0025$	$2f(x_1) = 2.0201$	$4f(x_1) = 4.0402$
2	.2	.15	$f(\bar{x}_2) = 1.022755$	$2f(x_2) = 2.0816$	$2f(x_2) = 2.0816$
3	.3	.25	$f(\bar{x}_3) = 1.06449$	$2f(x_3) = 2.1883$	$4f(x_3) = 4.37669$
4	.4	.35	$f(\bar{x}_4) = 1.13031$	$2f(x_4) = 2.347$	$2f(x_4) = 2.347$
5	.5	.45	$f(\bar{x}_5) = 1.2244$	$2f(x_5) = 2.568$	$4f(x_5) = 5.1361$
6	.6	.55	$f(\bar{x}_6) = 1.3532$	$2f(x_6) = 2.8666$	$2f(x_6) = 2.8666$
7	.7	.65	$f(\bar{x}_7) = 1.52577$	$2f(x_7) = 3.2646$	$4f(x_7) = 6.5292$
8	.8	.75	$f(\bar{x}_8) = 1.755$	$2f(x_8) = 3.7929$	$2f(x_8) = 3.7929$
9	.9	..85	$f(\bar{x}_9) = 2.05957$	$2f(x_9) = 4.4958$	$4f(x_9) = 8.9916$
10	1	.95	$f(\bar{x}_{10}) = 2.4657$	$f(x_{10}) = 2.718$	$f(x_{10}) = 2.718$
			$\Delta x(Sum)$	$\frac{\Delta x}{2}(Sum)$	$\frac{\Delta x}{3}(Sum)$
			$= 1.46039$	$= 1.4672$	$= 1.4627$

It is interesting to note how close these values are and how different are the error bounds.