so

$$\int_a^b f(t)\,dt = f(c)(b-c+c-a) = f(c)(b-a),$$

so

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c).$$

5.4.60

a. The left Riemann sum is given by $\displaystyle\frac{\pi}{2n}\sum_{k=0}^{n-1}\sin((k\pi)/(2n))$.

b.

$$\begin{aligned}\lim_{\theta\to0}\theta\left(\frac{\cos\theta+\sin\theta-1}{2(1-\cos\theta)}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right) &= \lim_{\theta\to0}\frac{\theta}{2}\left(\frac{(1+\cos\theta)(\cos\theta+\sin\theta-1)}{\sin^2\theta}\right)\\ &= \left(\frac{1}{2}\lim_{\theta\to0}\frac{\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1}\right)\left(\lim_{\theta\to0}\frac{\cos\theta-1}{\sin\theta}+\lim_{\theta\to0}\frac{\sin\theta}{\sin\theta}\right)\\ &= \frac{1}{2}\cdot1\cdot2\left(\lim_{\theta\to0}\frac{\frac{\cos-1}{\theta}}{\frac{\sin\theta}{\theta}}+1\right) = 1(0+1) = 1.\end{aligned}$$

c. Using the previous result, the left Riemann sum is given by $\displaystyle\frac{\pi}{2n}\left(\frac{\cos(\pi/(2n))+\sin(\pi/(2n))-1}{2(1-\cos(\pi/(2n)))}\right)$. Let $\theta=\dfrac{\pi}{2n}$. Then as $n\to\infty$, $\theta\to0$, and the limit of the left Riemann sum as $n\to\infty$ is 1.

5.5 Substitution Rule

5.5.1 It is based on the Chain Rule for differentiation.

5.5.2 After making a substitution, one obtains an integral in terms of a different variable, so the variable has "changed."

5.5.3 Typically u is substituted for the inner function, so $u=g(x)$.

5.5.4 One can either let $u=\tan x$, which is a good choice because the derivative is then $\sec^2 x$ which is a factor of the integrand, or one can let $u=\sec x$, because then the derivative is $\tan x\sec x$ which is also a factor of the integrand.

5.5.5 The new integral is $\displaystyle\int_{g(a)}^{g(b)} f(u)\,du$.

5.5.6 The new limits of integration are $2^2-4=0$ and $4^2-4=12$.

5.5.7 Because $u=x^2+1$, $du=2x\,dx$. Substituting yields $\displaystyle\int u^4\,du=\frac{u^5}{5}+C=\frac{(x^2+1)^5}{5}+C$.

5.5.8 Because $u=4x^2+3$, $du=8x\,dx$. Substituting yields $\displaystyle\int\cos u\,du=\sin u+C=\sin(4x^2+3)+C$.

5.5.9 Because $u=\sin x$, $du=\cos x\,dx$. Substituting yields $\displaystyle\int u^3\,du=\frac{u^4}{4}+C=\frac{\sin^4(x)}{4}+C$.

5.5.10 Because $u=3x^2+x$, $du=6x+1\,dx$. Substituting yields $\displaystyle\int\sqrt{u}\,du=\frac{2}{3}\cdot u^{3/2}+C=\frac{2}{3}\cdot\sqrt{(3x^2+x)^3}+C$.

5.5.11 Let $u = x+1$. Then $du = dx$, and $\int (x+1)^{12}\,dx = \int u^{12}\,du = \frac{u^{13}}{13} + C = \frac{(x+1)^{13}}{13} + C$. Check: $\frac{d}{dx}\left(\frac{(x+1)^{13}}{13} + C\right) = (x+1)^{12}$.

5.5.12 Let $u = 3x+1$. Then $du = 3dx$, and $\int e^{3x+1}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{e^{3x+1}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{e^{3x+1}}{3} + C\right) = e^{3x+1}$.

5.5.13 Let $u = 2x+1$. Then $du = 2dx$ and $\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{(2x+1)^{3/2}}{3} + C\right) = \frac{3}{2}\cdot\frac{1}{3}\cdot(2x+1)^{1/2}\cdot 2 = \sqrt{2x+1}$.

5.5.14 Let $u = 2x+5$. Then $du = 2dx$, and $\int \cos(2x+5)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{\sin(2x+5)}{2} + C$. Check: $\frac{d}{dx}\left(\frac{\sin(2x+5)}{2} + C\right) = \cos(2x+5)$.

5.5.15

a. $\int e^{10x}\,dx = \frac{1}{10}e^{10x} + C$.

b. $\int \sec 5x \tan 5x\,dx = \frac{1}{5}\sec 5x + C$.

c. $\int \sin 7x\,dx = -\frac{1}{7}\cos 7x + C$.

d. $\int \cos\frac{x}{7}\,dx = 7\sin\frac{x}{7} + C$.

e. $\int \frac{dx}{81+9x^2} = \frac{1}{9}\int\frac{dx}{9+x^2} = \frac{1}{27}\tan^{-1}\frac{x}{3} + C$.

f. $\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\frac{x}{6} + C$.

5.5.16

a. $\int_0^1 10^x\,dx = \frac{1}{\ln 10}\cdot 10^x\Big|_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$.

b. $\int_0^{\pi/40} \cos 20x\,dx = \frac{1}{20}\sin 20x\Big|_0^{\pi/40} = \frac{1}{20}(1-0) = \frac{1}{20}$.

c. $\int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\frac{x}{3}\Big|_{3\sqrt{2}}^6 = \frac{1}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{36}$.

d. $\int_0^{\pi/16} \sec^2 4x\,dx = \frac{1}{4}\tan 4x\Big|_0^{\pi/16} = \frac{1}{4}(1-0) = \frac{1}{4}$.

5.5.17 Let $u = x^2 - 1$. Then $du = 2x\,dx$. Substituting yields $\int u^{99}\,du = \frac{u^{100}}{100} + C = \frac{(x^2-1)^{100}}{100} + C$.

5.5.18 Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$. Substituting yields $\frac{1}{2}\int e^u\,du = \frac{1}{2}\cdot e^u + C = \frac{1}{2}\cdot e^{x^2} + C$.

5.5.19 Let $u = 1 - 4x^3$. Then $du = -12x^2\,dx$, so $-\frac{1}{6}du = 2x^2\,dx$. Substituting yields $-\frac{1}{6}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{3}\cdot\sqrt{u} + C = -\frac{1}{3}\cdot\sqrt{1-4x^3} + C$

5.5.20 Let $u = \sqrt{x} + 1$. Then $du = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $\int u^4\,du = \frac{u^5}{5} + C = \frac{(\sqrt{x}+1)^5}{5} + C$.

5.5.21 Let $u = x^2 + x$. Then $du = (2x+1)\,dx$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(x^2+x)^{11}}{11} + C$.

5.5.22 Let $u = 10x - 3$. Then $du = 10\,dx$, so $\frac{1}{10}du = dx$. Substituting yields $\frac{1}{10}\int \frac{1}{u}\,du = \frac{1}{10}\cdot\ln|u| + C = \frac{1}{10}\ln|10x-3| + C$.

5.5.23 Let $u = x^4 + 16$. Then $du = 4x^3\,dx$, so $\frac{1}{4}du = x^3\,dx$. Substituting yields $\frac{1}{4}\int u^6\,du = \frac{1}{4}\cdot\frac{u^7}{7} + C = \frac{(x^4+16)^7}{28} + C$.

5.5.24 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(\sin\theta)^{11}}{11} + C$.

5.5.25 $\int \frac{dx}{\sqrt{36-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9-x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{3} + C$ by equation 10 in Table 5.6.

5.5.26 $\int \frac{dx}{\sqrt{1-(3x)^2}} = \frac{1}{3}\sin^{-1}3x + C$, by equation 10 in Table 5.6.

5.5.27 Let $u = x^3$. Then $du = 3x^2\,dx$. Then $\int 6x^2 4^{x^3}\,dx = 2\int 4^u\,du = 2\cdot\frac{4^u}{\ln 4} + C = 2\cdot\frac{4^{x^3}}{\ln 4} + C = \frac{4^{x^3}}{\ln 2} + C$.

5.5.28 Let $u = x^{10}$. Then $du = 10x^9\,dx$, so $\frac{1}{10}du = x^9 dx$. Substituting yields $\frac{1}{10}\int \sin u\,du = -\frac{1}{10}\cos u + C = -\frac{1}{10}\cos x^{10} + C$.

5.5.29 Let $u = x^6 - 3x^2$. Then $du = (6x^5 - 6x)\,dx$, so $\frac{1}{6}du = (x^5 - x)\,dx$. Substituting yields $\frac{1}{6}\int u^4\,du = \frac{1}{6}\cdot\frac{u^5}{5} + C = \frac{(x^6-3x^2)^5}{30} + C$.

5.5.30 Let $u = 2x$, so that $du = 2dx$. Substituting yields $\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}2x + C$.

5.5.31 Let $u = 5x$ so that $du = 5dx$. Substituting yields $\frac{3}{5}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{3}{5}\sin^{-1}u + C = \frac{3}{5}\sin^{-1}x + C$.

5.5.32 Let $u = 2x$, so that $du = 2dx$. Substituting yields $2\int \frac{1}{u\sqrt{u^2-1}}\,du = 2\sec^{-1}u + C = 2\sec^{-1}2x + C$.

5.5.33 The integral can be rewritten as $\int \frac{e^w}{36+(e^w)^2}\,dw$. Let $u = e^w$, so that $du = e^w\,dw$. Substituting yields $\int \frac{du}{36+u^2} = \frac{1}{6}\tan^{-1}\frac{u}{6} + C = \frac{1}{6}\tan^{-1}\frac{e^w}{6} + C$.

5.5.34 Let $u = 2x^2 + 3x$, so that $du = (4x+3)\,dx = \frac{1}{2}(8x+6)\,dx$. Substituting yields $2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln|2x^2+3x| + C$.

5.5.35 Let $u = x^2$ so that $du = 2x\,dx$. Substitution yields $\frac{1}{2}\int \csc u \cot u\,du = -\frac{1}{2}\csc u + C = -\frac{1}{2}\csc x^2 + C$.

5.5.36 Let $u = 4w$. Then $du = 4\,dw$. Substituting yields $\frac{1}{4}\int \sec u \tan u\,du = \frac{1}{4}\sec u + C = \frac{1}{4}\sec 4w + C$.

5.5.37 Let $u = 10x+7$ so that $du = 10\,dx$. Substituting yields $\frac{1}{10}\int \sec^2 u\,du = \frac{1}{10}\tan u + C = \frac{1}{10}\tan(10x+7) + C$.

5.5.38 Let $u = \tan^{-1} w$ so that $du = \frac{1}{1+w^2}\,dw$. Substituting yields $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\tan^{-1} w\right)^2 + C$.

5.5.39 Let $u = 4t+1$ so that $du = 4dt$. Substituting yields $\frac{1}{4}\int 10^u\,du = \frac{1}{4}\cdot\frac{10^u}{\ln 10} + C = \frac{10^u}{4\ln 10} + C$.

5.5.40 Let $u = \sin x$. Then $du = \cos x\,dx$. Substituting yields $\int u^5 + 3u^3 - u\,du = \frac{u^6}{6} + \frac{3u^4}{4} - \frac{u^2}{2} + C = \frac{\sin^6 x}{6} + \frac{3\sin^4 x}{4} - \frac{\sin^2 x}{2} + C$.

5.5.41 Let $u = \cot x$. Then $du = -\csc^2 x\,dx$. Substituting yields $-\int u^{-3}\,du = \frac{1}{2u^2} + C = \frac{1}{2\cot^2 x} + C$.

5.5.42 Let $u = x^{3/2} + 8$. Then $du = \frac{3}{2}\cdot\sqrt{x}\,dx$. Substituting gives $\frac{2}{3}\int u^5\,du = \frac{2}{3}\frac{u^6}{6} + C = \frac{(x^{3/2}+8)^6}{9} + C$.

5.5.43 Note that $\sin x \sec^8 x = \frac{\sin x}{\cos^8 x}$. Let $u = \cos x$, so that $du = -\sin x\,dx$. Substituting yields $-\int u^{-8}\,du = \frac{1}{7u^7} + C = \frac{1}{7\cos^7 x} + C = \frac{\sec^7 x}{7} + C$.

5.5.44 Let $u = e^{2x} + 1$. Then $du = 2e^{2x}\,dx$. Substituting yields $\frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln|u|}{2} + C = \frac{\ln(e^{2x}+1)}{2} + C$.

5.5.45 $\displaystyle\int_0^{\pi/8} \cos 2x\,dx = \left(\frac{\sin 2x}{2}\right)\Big|_0^{\pi/8} = \frac{\sqrt{2}/2 - 0}{2} = \frac{\sqrt{2}}{4}$.

5.5.46 $\displaystyle\int_0^1 10e^{2x}\,dx = \left(5e^{2x}\right)\Big|_0^1 = 5(e^2-1)$.

5.5.47 Let $u = 4 - x^2$. Then $du = -2x\,dx$. Also, when $x = 0$ we have $u = 4$ and when $x = 1$ we have $u = 3$. Substituting yields $\displaystyle -\int_4^3 u\,du = \int_3^4 u\,du = \left(\frac{u^2}{2}\right)\Big|_3^4 = 8 - 4.5 = 3.5$.

5.5.48 Let $u = x^2 + 1$. Then $du = 2x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = 2$ we have $u = 5$. Substituting yields $\displaystyle\int_1^5 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}$.

5.5.49 Let $u = 2^x + 4$ so that $du = 2^x \ln 2$. Also, when $x = 1$, $u = 6$ and when $x = 3$, $u = 12$. Substituting yields

$$\frac{1}{\ln 2}\int_6^{12} \frac{du}{u} = \frac{1}{\ln 2}\ln u\Big|_6^{12} = \frac{1}{\ln 2}(\ln 12 - \ln 6) = \frac{1}{\ln 2}(\ln 2) = 1.$$

5.5.50 Let $u = \dfrac{\theta}{8}$ so that $du = \dfrac{1}{8}\,d\theta$. Also, when $\theta = -2\pi$, $u = -\dfrac{\pi}{4}$, and when $\theta = 2\pi$, $u = \dfrac{\pi}{4}$. Substituting yields

$$8\int_{-\pi/4}^{\pi/4} \cos u\,du = 8\sin u\Big|_{-\pi/4}^{\pi/4} = 8\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = 8\sqrt{2}.$$

5.5.51 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Also, when $\theta = 0$ we have $u = 0$ and when $\theta = \pi/2$ we have $u = 1$. Substituting yields $\displaystyle\int_0^1 u^2\,du = \left(\frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.52 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $\displaystyle -\int_1^{\sqrt{2}/2} \frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^1 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_{\sqrt{2}/2}^1 = -1 + \frac{2}{\sqrt{2}} = \sqrt{2} - 1$.

5.5.53 Let $u = e^w$. Then $du = e^w\,dw$. Also, when $w = \ln\pi/4$, $u = \pi/4$, and when $w = \ln\pi/2$, $u = \pi/2$. Substituting yields

$$\int_{\pi/4}^{\pi/2} \cos u\,du = \sin u\Big|_{\pi/4}^{\pi/2} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}.$$

5.5.54 Let $u = 4x$ so that $du = 4\,dx$. Also, when $x = \pi/16$, $u = \pi/4$ and when $x = \pi/8$, $u = \pi/2$. Substituting yields

$$2\int_{\pi/4}^{\pi/2} \csc^2 u\,du = -2\cot u\Big|_{\pi/4}^{\pi/2} = -2(0-1) = 2.$$

5.5.55 Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Also, when $x = -1$ we have $u = 0$ and when $x = 2$ we have $u = 9$. Substituting yields $\displaystyle\frac{1}{3}\int_0^9 e^u\,du = \left(\frac{e^u}{3}\right)\Big|_0^9 = \frac{e^9 - 1}{3}$.

5.5.56 Let $u = 9 + p^2$. Then $du = 2p\,dp$. Also, when $p = 0$ we have $u = 9$ and when $p = 4$ we have $u = 25$. Substituting yields $\displaystyle\frac{1}{2}\int_9^{25} u^{-1/2}\,du = \sqrt{u}\,\Big|_9^{25} = 5 - 3 = 2$.

5.5.57 Let $u = \sin x$. Then $du = \cos x\,dx$. Also, when $x = \pi/4$ we have $u = \sqrt{2}/2$ and when $x = \pi/2$ we have $u = 1$. Substituting yields $\displaystyle\int_{\sqrt{2}/2}^1 \frac{1}{u^2}\,du = \left(\frac{-1}{u}\right)\Big|_{\sqrt{2}/2}^1 = \left(-1 - \left(-\frac{2}{\sqrt{2}}\right)\right) = \sqrt{2} - 1$.

5.5.58 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $\displaystyle -\int_1^{\sqrt{2}/2} \frac{1}{u^3}\,du = \int_{\sqrt{2}/2}^1 u^{-3}\,du = \left(-\frac{1}{2u^2}\right)\Big|_{\sqrt{2}/2}^1 = -\frac{1}{2} + 1 = \frac{1}{2}$.

5.5.59 Let $u = 5x$, so that $du = 5\,dx$. Also, when $x = 2/(5\sqrt{3})$ we have $u = 2/\sqrt{3}$ and when $x = 2/5$ we have $u = 2$. Substituting yields $\displaystyle\int_{2/\sqrt{3}}^2 \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u\,\Big|_{2/\sqrt{3}}^2 = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

5.5.60 Let $u = v^4 + 4v + 4$, so that $du = (4v^3 + 4)\,dv$, so that $\dfrac{1}{4}\cdot du = (v^3+1)\,dv$. Also, when $v = 0$ we have $u = 4$ and when $v = 1$ we have $u = 9$. Substituting yields $\displaystyle\frac{1}{4}\int_4^9 u^{-1/2}\,du = \frac{1}{4}\left(2\sqrt{u}\right)\Big|_4^9 = \frac{1}{2}(3-2) = \frac{1}{2}$.

5.5.61 Let $u = x^2 + 1$, so that $du = 2x\,dx$. Substituting yields $\displaystyle\frac{1}{2}\int_1^{17} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\,\Big|_1^{17} = \frac{\ln 17}{2}$.

5.5.62 Let $u = 1 - 16x^2$, so that $du = -32x\,dx$. Substituting yields $\displaystyle -\frac{1}{32}\int_1^0 \frac{1}{\sqrt{u}}\,du = \frac{1}{16}\,\sqrt{u}\,\Big|_0^1 = \frac{1}{16}$.

5.5.63 Let $u = 3x$, so that $du = 3\,dx$. Substituting yields $\frac{4}{3}\int_1^{3/\sqrt{3}} \frac{1}{u^2+1}\,du = \frac{4}{3}\tan^{-1} u \Big|_1^{3/\sqrt{3}} = \frac{4}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{4}{3}\cdot\frac{\pi}{12} = \frac{\pi}{9}$.

5.5.64 Let $u = 3 + 2e^x$, so that $du = 2e^x\,dx$. Substituting yields $\frac{1}{2}\int_5^{11} \frac{1}{u}\,du = \frac{1}{2}\ln|u| \Big|_5^{11} = \frac{\ln(11/5)}{2}$.

5.5.65 Let $u = 1 - x^2$. Then $du = -2x\,dx$. Also note that when $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 0$. Substituting yields $-\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \left(\frac{u^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.66 Let $u = \ln p$. Then $du = \frac{1}{p}\,dp$. Also note that when $p = 1$ we have $u = 0$, and when $p = e^2$ we have $u = 2$. Substituting yields $\int_0^2 u\,du = \left(\frac{u^2}{2}\right)\Big|_0^2 = 2$.

5.5.67 Let $u = x^2 - 1$, so that $du = 2x\,dx$. Also note that when $x = 2$ we have $u = 3$, and when $x = 3$ we have $u = 8$. Substituting yields $\frac{1}{2}\int_3^8 u^{-1/3}\,du = \frac{1}{2}\left(\frac{3u^{2/3}}{2}\right)\Big|_3^8 = \frac{3}{4}\left(4 - \sqrt[3]{9}\right)$.

5.5.68 Let $u = 5x/6$ so that $du = \frac{5}{6}\,dx$. Also note that when $x = 0$ we have $u = 0$ and when $x = 6/5$ we have $u = 1$. Substituting yields $\frac{6}{5\cdot 36}\int_0^1 \frac{1}{u^2+1}\,du = \frac{1}{30}\left(\tan^{-1} u\right)\Big|_0^1 = \frac{\pi}{120}$.

5.5.69 Let $u = 16 - x^4$. Then $du = -4x^3\,dx$. Also note that when $x = 0$ we have $u = 16$, and when $x = 2$ we have $u = 0$. Substituting yields $\frac{1}{4}\int_0^{16} \sqrt{u}\,du = \frac{1}{4}\left(\frac{2u^{3/2}}{3}\right)\Big|_0^{16} = \frac{32}{3}$.

5.5.70 Let $u = x^2 - 2x$. Then $du = 2(x-1)\,dx$. Also note that when $x = -1$ we have $u = 3$ and when $x = 1$ we have $u = -1$. Substituting yields $\frac{1}{2}\int_3^{-1} u^7\,du = \frac{1}{16}\left(u^8\right)\Big|_3^{-1} = \frac{1}{16}\left(1 - 3^8\right) = -\frac{6560}{16} = -410$.

5.5.71 Let $u = 2 + \cos x$ so that $du = -\sin x\,dx$. Note that when $x = -\pi$, $u = 1$ and when $x = 0$, $u = 3$. Substituting yields $\int_1^3 -\frac{1}{u}\,du = (-\ln|u|)\Big|_1^3 = -(\ln 3 - \ln 1) = -\ln 3$.

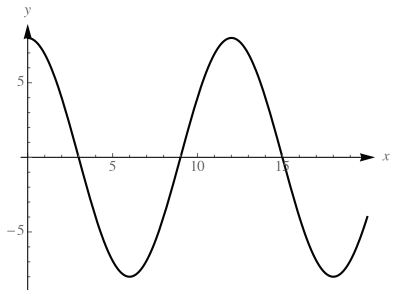
5.5.72 Let $u = 2v^3 + 9v^2 + 12v + 36$, so that $du = (6v^2 + 18v + 12)\,dv = 6(v+1)(v+2)\,dv$. Note that $u = 36$ when $v = 0$ and $u = 59$ when $v = 1$. Substituting yields $\frac{1}{6}\int_{36}^{59} \frac{1}{u}\,du = \frac{1}{6}\left(\ln|u|\right)\Big|_{36}^{59} = \frac{1}{6}(\ln 59 - \ln 36) = \frac{1}{6}\ln(59/36)$.

5.5.73 Let $u = 3x + 1$ so that $du = 3\,dx$. Note that $9x^2 + 6x + 1 = (3x+1)^2 = u^2$, and also that when $x = 1$, $u = 4$ and when $x = 2$, $u = 7$. Substituting yields $\frac{4}{3}\int_4^7 \frac{1}{u^2}\,du = \frac{4}{3}\left(-\frac{1}{u}\right)\Big|_4^7 = \frac{4}{3}\left(-\frac{1}{7} - \left(-\frac{1}{4}\right)\right) = \frac{4}{3}\left(\frac{3}{28}\right) = \frac{1}{7}$.

5.5.74 Let $u = \sin^2 x$, so that $du = 2\sin x\cos x\,dx = \sin 2x\,dx$. Note that when $x = 0$, $u = 0$, and when $x = \pi/4$, $u = 1/2$. Substituting yields $\int_0^{1/2} e^u\,du = e^u\Big|_0^{1/2} = \sqrt{e} - 1$.

5.5.75 The average velocity is given by $\frac{1}{10-0}\int_0^{10} (8\sin\pi t + 2t)\,dt = \frac{1}{10}\left(-\frac{8}{\pi}\cos\pi t + t^2\right)\Big|_0^{10} = \frac{1}{10}\left(-\frac{8}{\pi}\cos 10\pi + 100 + \frac{8}{\pi}\cos 0 - 0\right) = 10$.

5.5.76

a. 

b. $\int_0^t 8\cos(\pi y/6)\,dy = \left(\frac{48}{\pi}\sin(\pi y/6)\right)\Big|_0^t = \frac{48}{\pi}\sin(\pi t/6)$.

c. The period is $\frac{2\pi}{\pi/6} = 12$.

5.5.77

a. $\int_0^4 \frac{200}{(t+1)^2}\,dt = \left(\frac{-200}{t+1}\right)\Big|_0^4 = -40+200 = 160$.

b. $\int_0^6 \frac{200}{(t+1)^3}\,dt = \left(\frac{-200}{2(t+1)^2}\right)\Big|_0^6 = \frac{-100}{49} + 100 = \frac{4800}{49}$.

c. $\Delta P = \int_0^T \frac{200}{(t+1)^r}\,dt$. This decreases as r increases, because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$.

d. Suppose $\int_0^{10} \frac{200}{(t+1)^r}\,dt = 350$. Then $\left(\frac{200(t+1)^{-r+1}}{1-r}\right)\Big|_0^{10} = 350$, so $11^{1-r} - 1 = \frac{350(1-r)}{200}$, and thus $\frac{11}{11^r} = \frac{7-7r}{4} + \frac{4}{4} = \frac{11-7r}{4}$, and $11^r = \frac{44}{11-7r}$. Using trial and error to find r, we arrive at $r \approx 1.278$.

e. $\int_0^T \frac{200}{(t+1)^3}\,dt = \left(-\frac{200}{2(t+1)^2}\right)\Big|_0^T = -\frac{100}{(T+1)^2} + 100$. As $T \to \infty$, this expression $\to 100$, so in the long run, the bacteria approaches a finite limit.

5.5.78 Let $u = x-2$, so that $u+2 = x$. Then $du = dx$. Substituting yields $\int \frac{u+2}{u}\,du = \int \left(1 + \frac{2}{u}\right)\,du = u + 2\ln|u| + D = x - 2 + 2\ln|x-2| + D$. The constant $-2+D$ could be renamed as a different constant C, yielding $x + 2\ln|x-2| + C$.

5.5.79 Let $u = x-4$, so that $u+4 = x$. Then $du = dx$. Substituting yields $\int \frac{u+4}{\sqrt{u}}\,du =$ $\int \left(\frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}}\right)\,du = \int u^{1/2} + 4u^{-1/2}\,du = \frac{2}{3}u^{3/2} + 8u^{1/2} + C = \frac{2}{3}\cdot(x-4)^{3/2} + 8\sqrt{x-4} + C$.

5.5.80 Let $u = y+1$, so that $u-1 = y$. Then $du = dy$. Substituting yields $\int \frac{(u-1)^2}{u^4}\,du =$ $\int \frac{u^2-2u+1}{u^4}\,du = \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)\,du = -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} + C = -\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{1}{3(y+1)^3} + C$.

5.5.81 Let $u = x+4$, so that $u-4 = x$. Then $du = dx$. Substituting yields

$$\begin{aligned}\int \frac{u-4}{\sqrt[3]{u}}\,du = \int \left(u^{2/3} - 4u^{-1/3}\right)\,du &= \frac{3}{5}u^{5/3} + -6u^{2/3} + C \\ &= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C.\end{aligned}$$

5.5.82 Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\,dx$. Substituting yields $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + e^{-x}) + C$.

5.5.83 Let $u = 2x+1$. Then $du = 2dx$ and $x = \frac{u-1}{2}$. Substituting yields $\displaystyle\frac{1}{2}\int \frac{u-1}{2}\cdot\sqrt[3]{u}\,du = \frac{1}{4}\int(u^{4/3} - u^{1/3})\,du = \frac{1}{4}\left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right) + C = \frac{3(2x+1)^{7/3}}{28} - \frac{3(2x+1)^{4/3}}{16} + C = (2x+1)^{4/3}\left(\frac{3(2x+1)}{28} - \frac{3}{16}\right) = \frac{3}{112}(2x+1)^{4/3}(8x+4-7) = \frac{3}{112}(2x+1)^{4/3}(8x-3)$.

5.5.84 Let $u = 3z+2$. Then $du = 3dz$ and $z = \frac{u-2}{3}$. Substituting yields $\displaystyle\frac{1}{3}\int \frac{u+1}{3}\cdot\sqrt{u}\,du = \frac{1}{9}\int(u^{3/2} + u^{1/2})\,du = \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{2(3z+2)^{5/2}}{45} + \frac{2(3z+2)^{3/2}}{27} + C. = \frac{2}{9}(3z+2)^{3/2}\left(\frac{3z+2}{5} + \frac{1}{3}\right) = \frac{2}{9}(3z+2)^{3/2}\left(\frac{9z+6+5}{15}\right) = \frac{2}{135}(3z+2)^{3/2}(9z+11)$.

5.5.85 Let $u = x+10$. Then $du = dx$ and $x = u - 10$. Substituting gives $\displaystyle\int(u-10)u^9\,du = \int(u^{10} - 10u^9)\,du = \frac{u^{11}}{11} - u^{10} + C = \frac{1}{11}(x+10)^{11} - (x+10)^{10} + C = (x+10)^{10}\left(\frac{x+10}{11} - 1\right) + C = \frac{(x+10)^{10}(x-1)}{11} + C$.

5.5.86 Using Table 5.6: $\displaystyle\int_0^{\sqrt{3}} \frac{3dx}{9+x^2} = \tan^{-1}(x/3)\Big|_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3}/3) = \pi/6$.

5.5.87 $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \left(x + \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \pi$.

5.5.88 $\displaystyle\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

5.5.89 $\displaystyle\int \sin^2\left(\theta + \frac{\pi}{6}\right)\,d\theta = \frac{1}{2}\int\left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right)\,d\theta = \frac{\theta}{2} - \frac{\sin\left(2\theta + \frac{\pi}{3}\right)}{4} + C$.

5.5.90 $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\,d\theta = \int_0^{\pi/4} \frac{1+\cos 16\theta}{2}\,d\theta = \left(\frac{\theta}{2} + \frac{\sin 16\theta}{32}\right)\Big|_0^{\pi/4} = \frac{\pi}{8}$.

5.5.91 $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta = \left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4} = \frac{\pi}{4}$.

5.5.92 Let $u = x^2$, so that $du = 2x\,dx$. Substituting yields

$$\begin{aligned}\frac{1}{2}\int \cos^2 u\,du = \frac{1}{2}\int \frac{1+\cos 2u}{2}\,du &= \frac{1}{4}\left(u + \frac{\sin 2u}{2}\right) + C\\ &= \frac{x^2}{4} + \frac{\sin 2x^2}{8} + C.\end{aligned}$$

5.5.93 Let $u = \sin^2 y + 2$ so that $du = 2\sin y\cos y\,dy = \sin 2y\,dy$. Substituting yields $\displaystyle\int_2^{9/4} \frac{1}{u}\,du = (\ln|u|)\Big|_2^{9/4} = \ln(9/4) - \ln 2 = \ln(9/8)$.

5.5.94 Because $\sin^4\theta = (\sin^2\theta)^2 = \left(\frac{1-\cos 2\theta}{2}\right)^2 = \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}$, we have

$$\int \sin^4\theta\,d\theta = \int \left(\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\right)d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{4}\int \cos^2 2\theta\,d\theta.$$

Because $\frac{1}{4}\cos^2 2\theta = \frac{1+\cos 4\theta}{8}$, we have

$$\int \sin^4\theta\,d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{8}\theta + \frac{\sin 4\theta}{32} = \frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}.$$

Thus, $\int_0^{\pi/2} \sin^4\theta\,d\theta = \left(\frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right)\Big|_0^{\pi/2} = \frac{3\pi}{16}$.

5.5.95

a. True. This follows by substituting $u = f(x)$ to obtain the integral $\int u\,du = \frac{u^2}{2} + C = \frac{f(x)^2}{2} + C$.

b. True. Again, this follows from substituting $u = f(x)$ to obtain the integral $\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$ where $n \neq -1$.

c. False. If this were true, then $\sin 2x$ and $2\sin x$ would have to differ by a constant, which they do not. In fact, $\sin 2x = 2\sin x\cos x$.

d. False. The derivative of the right hand side is $(x^2+1)^9 \cdot 2x$ which is not the integrand on the left hand side.

e. False. If we let $u = f'(x)$, then $du = f''(x)\,dx$. Substituting yields $\int_{f'(a)}^{f'(b)} u\,du = \left(\frac{u^2}{2}\right)\Big|_{f'(a)}^{f'(b)} = \frac{(f'(b))^2}{2} - \frac{(f'(a))^2}{2}$.

5.5.96 $A(x) = \int_4^5 \frac{x}{\sqrt{x^2-9}}\,dx$. Let $u = x^2 - 9$, so that $du = 2x\,dx$. Also, when $x = 4$ we have $u = 7$ and when $x = 5$ we have $u = 16$. Substituting yields $\frac{1}{2}\int_7^{16} u^{-1/2}\,du = \sqrt{u}\,\Big|_7^{16} = 4 - \sqrt{7}$.

5.5.97 $A(x) = \int_0^{\sqrt{\pi}} x\sin x^2\,dx$. Let $u = x^2$, so that $du = 2x\,dx$. Also, when $x = 0$ we have $u = 0$ and when $x = \sqrt{\pi}$ we have $u = \pi$. Substituting yields $\frac{1}{2}\int_0^{\pi} \sin u\,du = \frac{1}{2}\left(-\cos u\right)\Big|_0^{\pi} = 1$.

5.5.98 $A(x) = \int_2^6 (x-4)^4\,dx = \frac{(x-4)^5}{5}\Big|_2^6 = \frac{2^5}{5} - \left(-\frac{(2)^5}{5}\right) = \frac{64}{5}$.

5.5.99 $A(a) = \int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3}\right)dx = \left(\frac{x}{a} - \frac{x^3}{3a^3}\right)\Big|_0^a = 1 - \frac{1}{3} = \frac{2}{3}$. This is a constant function.

5.5.100

a. Let $u = x^2$, so that $du = 2x\,dx$. Note that when $x = 1$ or $x = -1$, we have $u = 1$. Substituting gives $\frac{1}{2}\int_1^1 f(u)\,du = 0$. Alternatively, we could note that when f is even, $xf(x^2)$ is odd, so $\int_{-1}^1 xf(x^2)\,dx = 0$.