➤ A more general form of the Mean Value Theorem states that if f and g are continuous on $[a, b]$ with $g(x) \ge 0$ on $[a, b]$, then there exists a number c in (a, b) such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

QUICK CHECK 3 Explain why $f(x) = 0$ for at least one point of (a, b) if f is continuous and $\int_a^b f(x)\,dx = 0$. ◄

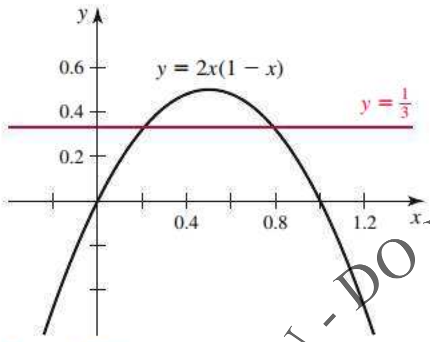


Figure 5.57

Combining these observations, we have

$$f(c) = \frac{1}{b - a}\int_a^b f(t)\,dt,$$

where c is a point in (a, b).

EXAMPLE 3 **Average value equals function value** Find the point(s) (0, 1) at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.

SOLUTION The average value of f on $[0, 1]$ is

$$\bar{f} = \frac{1}{1 - 0}\int_0^1 2x(1 - x)\,dx = \left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 = \frac{1}{3}.$$

We must find the points on $(0, 1)$ at which $f(x) = \frac{1}{3}$ (**Figure 5.57**). Usin formula, the two solutions of $f(x) = 2x(1 - x) = \frac{1}{3}$ are

$$\frac{1 - \sqrt{1/3}}{2} \approx 0.211 \quad \text{and} \quad \frac{1 + \sqrt{1/3}}{2} \approx 0.789.$$

These two points are located symmetrically on either side of $x = \frac{1}{2}$. The 0.211 and 0.789, are the same for $f(x) = ax(1 - x)$ for any nonzero va (Exercise 53).

Related

SECTION 5.4 EXERCISES

Getting Started

1. If f is an odd function, why is $\int_{-a}^{a} f(x)\,dx = 0$?

2. If f is an even function, why is $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$?

3. Suppose f is an even function and $\int_{-8}^{8} f(x)\,dx = 18$.
   a. Evaluate $\int_0^8 f(x)\,dx$. **b.** Evaluate $\int_{-8}^{8} xf(x)\,dx$.

4. Suppose f is an odd function, $\int_0^4 f(x)\,dx = 3$, and $\int_0^8 f(x)\,dx = 9$.
   a. Evaluate $\int_{-4}^{8} f(x)\,dx$. **b.** Evaluate $\int_{-8}^{4} f(x)\,dx$.

5. Use symmetry to explain why
$$\int_{-4}^{4}(5x^4 + 3x^3 + 2x^2 + x + 1)\,dx = 2\int_0^4 (5x^4 + 2x^2 + 1)\,dx.$$

6. Use symmetry to fill in the blanks:
$\int_{-\pi}^{\pi}(\sin x + \cos x)\,dx = \_\_\_\_\_ \int_0^{\pi} \_\_\_\_\_\ dx.$

7. Is x^{12} an even or odd function? Is $\sin x^2$ an even or odd function?

8. Explain how to find the average value of a function on an interval $[a, b]$ and why this definition is analogous to the definition of the average of a set of numbers.

9. Explain the statement that a continuous function on an interval $[a, b]$ equals its average value at some point on (a, b).

10. Sketch the function $y = x$ on the interval $[0, 2]$ and let R be the region bounded by $y = x$ and the x-axis on $[0, 2]$. Now sketch a rectangle in the first quadrant whose base is $[0, 2]$ and whose area equals the area of R.

Practice Exercises

11–24. Symmetry in integrals *Use symmetry to eva integrals.*

11. $\int_{-2}^{2} x^9\,dx$

12. $\int_{-200}^{200} 2x^5\,dx$

13. $\int_{-2}^{2}(3x^8 - 2)\,dx$

14. $\int_{-\pi/4}^{\pi/4} \cos x$

15. $\int_{-2}^{2}(x^2 + x^3)\,dx$

16. $\int_{-\pi}^{\pi} t^2 \sin t$

17. $\int_{-2}^{2}(x^9 - 3x^5 + 2x^2 - 10)\,dx$

18. $\int_{-\pi/2}^{\pi/2} 5\sin\theta\,d\theta$

19. $\int_{-\pi/4}^{\pi/4} \sin^5 t$

20. $\int_{-1}^{1}(1 - |x|)\,dx$

21. $\int_{-\pi/4}^{\pi/4} \sec^2 x$

22. $\int_{-\pi/4}^{\pi/4} \tan\theta\,d\theta$

23. $\int_{-2}^{2} \frac{x^3 - 4x}{x^2 + 1}$

24. $\int_{-2}^{2}(1 - |x|^3)\,dx$

25–32. Average values *Find the average value of th tions on the given interval. Draw a graph of the func the average value.*

25. $f(x) = x^3$ on $[-1, 1]$

26. $f(x) = x^2$ -

27. $f(x) = \frac{1}{x^2 + 1}$ on $[-1, 1]$

28. $f(x) = 1/x$

29. $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

30. $f(x) = x(1$

31. $f(x) = x^n$ on $[0, 1]$, for any positive integer n

32. $f(x) = x^{1/n}$ on $[0, 1]$, for any positive integer n

33. Average distance on a parabola What is the average distance between the parabola $y = 30x(20 - x)$ and the x-axis on the interval $[0, 20]$?

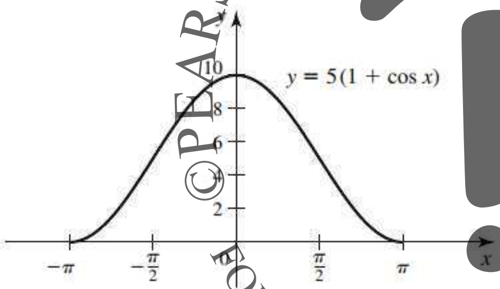
T 34. Average elevation The elevation of a path is given by $f(x) = x^3 - 5x^2 + 30$, where x measures horizontal distance. Draw a graph of the elevation function and find its average value, for $0 \le x \le 4$.

35. Average velocity The velocity in m/s of an object moving along a line over the time interval [0, 6] is $v(t) = t^2 + 3t$. Find the average velocity of the object over this time interval.

36. Average velocity A rock is launched vertically upward from the ground with a speed of 64 ft/s. The height of the rock (in ft) above the ground after t seconds is given by the function $s(t) = -16t^2 + 64t$. Find its average velocity during its flight.

37. Average height of an arch The height of an arch above the ground is given by the function $y = 10 \sin x$, for $0 \le x \le \pi$. What is the average height of the arch above the ground?

38. Average height of a wave The surface of a water wave is described by $y = 5(1 + \cos x)$, for $-\pi \le x \le \pi$, where $y = 0$ corresponds to a trough of the wave (see figure). Find the average height of the wave above the trough on $[-\pi, \pi]$.



39–44. Mean Value Theorem for Integrals *Find or approximate all points at which the given function equals its average value on the given interval.*

39. $f(x) = 8 - 2x$ on $[0, 4]$ **40.** $f(x) = e^x$ on $[0, 2]$

41. $f(x) = 1 - \dfrac{x^2}{a^2}$ on $[0, a]$, where a is a positive real number

T 42. $f(x) = \dfrac{\pi}{4} \sin x$ on $[0, \pi]$ **43.** $f(x) = 1 - |x|$ on $[-1, 1]$

44. $f(x) = \dfrac{1}{x}$ on $[1, 4]$

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If f is symmetric about the line $x = 2$, then $\int_0^4 f(x)\,dx = 2\int_0^2 f(x)\,dx$.

b. If f has the property $f(a + x) = -f(a - x)$, for all x, where a is constant, then $\int_{a-2}^{a+2} f(x)\,dx = 0$.

c. The average value of a linear function on an interval $[a, b]$ is the function value at the midpoint of $[a, b]$.

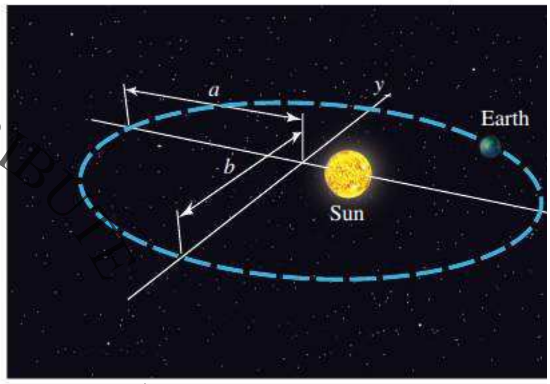
d. Consider the function $f(x) = x(a - x)$ on the interval $[0, a]$, for $a > 0$. Its average value on $[0, a]$ is $\frac{1}{2}$ of its maximum value.

46. Planetary orbits The planets orbit the Sun in elliptica the Sun at one focus (see Section 12.4 for more on elli equation of an ellipse whose dimensions are $2a$ in the and $2b$ in the y-direction is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

a. Let d^2 denote the square of the distance from a pla the center of the ellipse at $(0, 0)$. Integrate over the $[-a, a]$ to show that the average value of d^2 is $\dfrac{a^2}{}$

b. Show that in the case of a circle ($a = b = R$), the value in part (a) is R^2.

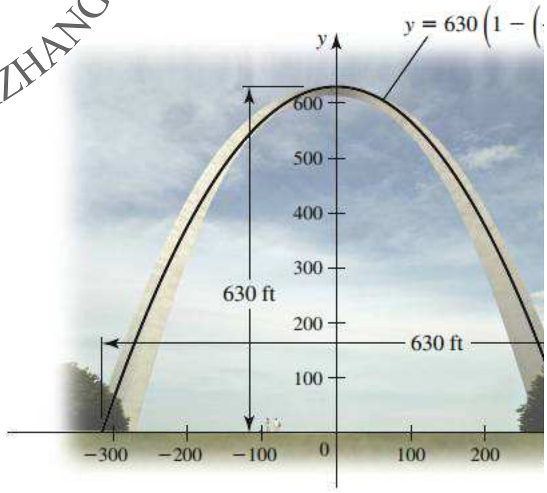
c. Assuming $0 < b < a$, the coordinates of the Sun $(\sqrt{a^2 - b^2}, 0)$. Let D^2 denote the square of the di the planet to the Sun. Integrate over the interval [− show that the average value of D^2 is $\dfrac{4a^2 - b^2}{3}$.



47. Gateway Arch The Gateway Arch in St. Louis is 630 has a 630-ft base. Its shape can be modeled by the par

$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right).$$

Find the average height of the arch above the ground.



T 48. Comparing a sine and a quadratic function Consid tions $f(x) = \sin x$ and $g(x) = \dfrac{4}{\pi^2} x(\pi - x)$.

a. Carefully graph f and g on the same set of axes. Ve both functions have a single local maximum on the $[0, \pi]$ and that they have the same maximum value

b. On the interval $[0, \pi]$, which is true: $f(x) \ge g(x)$, $g(x) \ge f(x)$, or neither?

c. Compute and compare the average values of f and