SUMMARY Partial Fraction Decompositions

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x - r}$.

2. **Repeated linear factor** A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m}.$$

3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax + B}{ax^2 + bx + c}.$$

4. **Repeated irreducible quadratic factor** (See Exercises 61–64.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}.$$

SECTION 8.5 EXERCISES

Getting Started

1. What kinds of functions can be integrated using partial fraction decomposition?

2. Give an example of each of the following.
   a. A simple linear factor
   b. A repeated linear factor
   c. A simple irreducible quadratic factor
   d. A repeated irreducible quadratic factor

3. What term(s) should appear in the partial fraction decomposition of a proper rational function with each of the following?
   a. A factor of $x - 3$ in the denominator
   b. A factor of $(x - 4)^3$ in the denominator
   c. A factor of $x^2 + 2x + 6$ in the denominator

4. What is the first step in integrating $\dfrac{x^2 + 2x - 3}{x + 1}$?

5–16. *Set up the appropriate form of the partial fraction decomposition for the following expressions. Do not find the values of the unknown constants.*

5. $\dfrac{4x}{x^2 - 9x + 20}$

6. $\dfrac{4x + 1}{4x^2 - 1}$

7. $\dfrac{x + 3}{(x - 5)^2}$

8. $\dfrac{2}{x^3 - 2x^2 + x}$

9. $\dfrac{4}{x^5 - 5x^3 + 4x}$

10. $\dfrac{20x}{(x - 1)^2(x^2 + 1)}$

11. $\dfrac{1}{x(x^2 + 1)}$

12. $\dfrac{2x^2 + 3}{(x^2 - 8x + 16)(x^2 + 3x + 4)}$

13. $\dfrac{x^4 + 12x^2}{(x - 2)^2(x^2 + x + 2)^2}$

14. $\dfrac{6x^4 - 4x^3 + 15x^2 - 5x + 7}{(x - 2)(2x^2 + 3)^2}$

15. $\dfrac{x}{(x^4 - 16)^2}$

16. $\dfrac{x^2 + 2x + 6}{x^3(x^2 + x + 1)^2}$

Practice Exercises

17–22. *Give the partial fraction decomposition for the following expressions.*

17. $\dfrac{5x - 7}{x^2 - 3x + 2}$

18. $\dfrac{11x - 10}{x^2 - x}$

19. $\dfrac{6}{x^2 - 2x - 8}$

20. $\dfrac{x^2 - 4x + 11}{(x - 3)(x - 1)(x + 1)}$

21. $\dfrac{2x^2 + 5x + 6}{x^2 + 3x + 2}$ (*Hint:* Use long division first.)

22. $\dfrac{x^4 + 2x^3 + x}{x^2 - 1}$

23–64. Integration *Evaluate the following integrals.*

23. $\displaystyle\int \frac{3}{(x - 1)(x + 2)}\,dx$

24. $\displaystyle\int \frac{8}{(x - 2)(x + 6)}\,dx$

25. $\displaystyle\int \frac{6}{x^2 - 1}\,dx$

26. $\displaystyle\int_0^1 \frac{dt}{t^2 - 9}$

27. $\int \frac{8x-5}{3x^2-5x+2}\,dx$

28. $\int_1^2 \frac{7x-2}{3x^2-2x}\,dx$

29. $\int_{-1}^2 \frac{5x}{x^2-x-6}\,dx$

30. $\int \frac{21x^2}{x^3-x^2-12x}\,dx$

31. $\int \frac{6x^2}{x^4-5x^2+4}\,dx$

32. $\int \frac{4x-2}{x^3-x}\,dx$

33. $\int \frac{3x^2+4x-6}{x^2-3x+2}\,dx$

34. $\int \frac{2z^3+z^2-6z+7}{z^2+z-6}\,dz$

35. $\int \frac{x^2+12x-4}{x^3-4x}\,dx$

36. $\int \frac{z^2+20z-15}{z^3+4z^2-5z}\,dz$

37. $\int \frac{dx}{x^4-10x^2+9}$

38. $\int_0^5 \frac{2}{x^2-4x-32}\,dx$

39. $\int \frac{81}{x^3-9x^2}\,dx$

40. $\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

41. $\int_{-1}^1 \frac{x}{(x+3)^2}\,dx$

42. $\int \frac{dx}{x^3-2x^2-4x+8}$

43. $\int \frac{2}{x^3+x^2}\,dx$

44. $\int_1^2 \frac{2}{t^3(t+1)}\,dt$

45. $\int \frac{x-5}{x^2(x+1)}\,dx$

46. $\int \frac{x^2}{(x-2)^3}\,dx$

47. $\int \frac{x^3-10x^2+27x}{x^2-10x+25}\,dx$

48. $\int \frac{x^3+2}{x^3-2x^2+x}\,dx$

49. $\int \frac{x^2-4}{x^3-2x^2+x}\,dx$

50. $\int \frac{8(x^2+4)}{x(x^2+8)}\,dx$

51. $\int \frac{x^2+x+2}{(x+1)(x^2+1)}\,dx$

52. $\int \frac{x^2+3x+2}{x(x^2+2x+2)}\,dx$

53. $\int \frac{2x^2+5x+5}{(x+1)(x^2+2x+2)}\,dx$

54. $\int \frac{z+1}{z(z^2+4)}\,dz$

55. $\int \frac{20x}{(x-1)(x^2+4x+5)}\,dx$

56. $\int \frac{2x+1}{x^2+4}\,dx$

57. $\int \frac{x^3+5x}{(x^2+3)^2}\,dx$

58. $\int \frac{x^4+1}{x^3+9x}\,dx$

59. $\int \frac{x^3+6x^2+12x+6}{(x^2+6x+10)^2}\,dx$

60. $\int \frac{dy}{(y^2+1)(y^2+2)}$

61. $\int \frac{2}{x(x^2+1)^2}\,dx$

62. $\int \frac{dx}{(x+1)(x^2+2x+2)^2}$

63. $\int \frac{9x^2+x+21}{(3x^2+7)^2}\,dx$

64. $\int \frac{9x^5+6x^3}{(3x^2+1)^3}\,dx$

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. To evaluate $\int \frac{4x^6}{x^4+3x^2}\,dx$, the first step is to find the partial fraction decomposition of the integrand.

b. The easiest way to evaluate $\int \frac{6x+1}{3x^2+x}\,dx$ is with a partial fraction decomposition of the integrand.

c. The rational function $f(x)=\frac{1}{x^2-13x+42}$ has an irreducible quadratic denominator.

d. The rational function $f(x)=\frac{1}{x^2-13x+43}$ has an irreducible quadratic denominator.

T 66–68. Areas of regions *Find the area of the following regions.*

66. The region bounded by the curve $y=\frac{x-x^2}{(x+1)(x^2+1)}$ and the x-axis from $x=0$ to $x=1$

67. The region bounded by the curve $y=\frac{10}{x^2-2x-24}$, the x-axis, and the lines $x=-2$ and $x=2$

68. The region in the first quadrant bounded by the curves $y=\frac{3x^2+2x+1}{x(x^2+x+1)}$, $y=\frac{2}{x}$, and $x=2$

69–72. Volumes of solids *Find the volume of the following solids.*

69. The region bounded by $y=\frac{x}{x+1}$, the x-axis, and $x=4$ is revolved about the x-axis.

70. The region bounded by $y=\frac{1}{x^2(x^2+2)^2}$, $y=0$, $x=1$, and $x=2$ is revolved about the y-axis.

71. The region bounded by $y=\frac{1}{\sqrt{x(3-x)}}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

72. The region bounded by $y=\frac{3x^2+25}{x^2(x^2+25)}$, $y=0$, $x=5$, and $x=10$ is revolved about the y-axis.

73. Two methods Evaluate $\int \frac{dx}{x^2-1}$, for $x>1$, in two ways: using partial fractions and a trigonometric substitution. Reconcile your two answers.

T 74–75. Finding constants with a computer algebra system *Give the appropriate form of the partial fraction decomposition of the expression, and then use a computer algebra system to find the unknown constants.*

74. $\frac{3x^2+2x+1}{(x+1)^3(x^2+x+1)^2}$

75. $\frac{x^4+3x^2+1}{x(x^2+1)^2(x^2+x+4)^2}$

76–83. Preliminary steps *The following integrals require a preliminary step such as a change of variables before using the method of partial fractions. Evaluate these integrals.*

76. $\int \frac{\cos\theta}{(\sin^3\theta-4\sin\theta)}\,d\theta$

77. $\int \frac{e^x}{(e^x-1)(e^x+2)}\,dx$

78. $\int \frac{dy}{y(\sqrt{a}-\sqrt{y})}$, $a>0$ (*Hint:* Let $u=\sqrt{y}$.)

79. $\int \frac{\sec t}{1+\sin t}\,dt$

80. $\int \sqrt{e^x+1}\,dx$ (*Hint:* Let $u=\sqrt{e^x+1}$.)

81. $\int \frac{(e^{3x}+e^{2x}+e^x)}{(e^{2x}+1)^2}\,dx$

82. $\int \frac{dx}{x\sqrt{1+2x}}$