**8.5.96** Let $u^2 = \tan x$ so that $2u\,du = \sec^2 x\,dx = 1 + u^4\,dx$. Then we can write

$$\int \sqrt{\tan x}\,dx = \int \frac{2u^2}{1+u^4}\,du = \int \frac{u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)}\,du.$$

If we write $\dfrac{u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)} = \dfrac{Au+B}{u^2-\sqrt{2}u+1} + \dfrac{Cu+D}{u^2+\sqrt{2}u+1}$ then we have $u^2 = (Au + B)(u^2+\sqrt{2}u+1) + (Cu+D)(u^2-\sqrt{2}u+1)$. Multiplying out the right-hand side, equating coefficients, and solving for the unknowns yields $A = \dfrac{\sqrt{2}}{4}$, $B = 0$, $C = -\dfrac{\sqrt{2}}{4}$, and $D = 0$. Now

$$\begin{aligned} \frac{u^2}{1+u^4} &= \frac{\sqrt{2}}{4}\left(\frac{u}{u^2-\sqrt{2u}+1} - \frac{u}{u^2+\sqrt{2}u+1}\right) = \frac{\sqrt{2}}{8}\left(\frac{2u}{u^2-\sqrt{2}u+1} - \frac{2u}{u^2+\sqrt{2}u+1}\right) \\ &= \frac{\sqrt{2}}{8}\left(\frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1} + \frac{\sqrt{2}}{u^2-\sqrt{2}u+1} - \frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{\sqrt{2}}{u^2+\sqrt{2}u+1}\right). \end{aligned}$$

Then

$$\begin{aligned} 2\int \frac{u^2}{1+u^4}\,du &= \frac{\sqrt{2}}{4}\left(\ln|u^2-\sqrt{2}u+1| - \ln|u^2+\sqrt{2}u+1|\right) \\ &\quad + \frac{1}{2}\int\left(\frac{1}{(u^2-\sqrt{2}u+1)} + \frac{1}{(u^2+\sqrt{2}u+1)}\right)du. \end{aligned}$$

This last integral can be written as

$$\int\left(\frac{1}{(\sqrt{2}u+1)^2+1} + \frac{1}{(1-\sqrt{2}u)^2+1}\right)du = \frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2}u+1) - \tan^{-1}(1-\sqrt{2}u)\right) + C.$$

Putting this all together and replacing $u$ by $\sqrt{\tan x}$ yields

$$\begin{aligned} &\frac{\sqrt{2}}{4}\left(\ln|\tan x - \sqrt{2\tan x}+1| - \ln|\tan x + \sqrt{2\tan x}+1|\right) \\ &+ \frac{1}{\sqrt{2}}\left(\tan^{-1}(1+\sqrt{2\tan x}) - \tan^{-1}(1-\sqrt{2\tan x})\right) + C. \end{aligned}$$

Thus,

$$\int_0^{\pi/4} \sqrt{\tan x}\,dx = \frac{\sqrt{2}}{4}\left(\ln(2-\sqrt{2}) - \ln(2+\sqrt{2}\right) + \frac{1}{\sqrt{2}}\left(\tan^{-1}(1+\sqrt{2}) - \tan^{-1}(1-\sqrt{2})\right) \approx 0.4875.$$

## 8.6 Integration Strategies

**8.6.1** Because the integrand is a product of a polynomial and a trigonometric function, it makes sense to integrate by parts.

**8.6.2** Because $\dfrac{d}{dx}(1+\tan x) = \sec^2 x$, use a $u$-substitution of $u = 1 + \tan x$.

**8.6.3** The presence of $64 - x^2$ in the integrand suggests the trigonometric substitution $x = 8\sin\theta$.

**8.6.4** One approach is to rewrite the integrand by replacing $\tan^2 x + 1$ with $\sec^2 x$ and then proceed with with a $u$-substitution of $u = \tan x$. Another approach is to split up the fraction and simplify the integrand by writing

$$\int \frac{\tan^2 x+1}{\tan x}dx = \int\left(\frac{\tan^2 x}{\tan x} + \frac{1}{\tan x}\right)dx = \int(\tan x + \cot x)dx.$$

Then use the integral formulas for $\tan x$ and $\cot x$.

**8.6.5** The method of partial fractions is appropriate because the integrand is a proper rational function.

**8.6.6** Using the identity $1 - \sin^2 x = \cos^2 x$, rewrite the integrand as $\dfrac{\cos^5 x \sin^4 x}{\cos^2 x}$, or $\cos^3 x \sin^4 x$. Then split off a factor of $\cos x$, express the remaining powers in terms of $\sin x$, and make the substitution $u = \sin x$.

**8.6.7** Let $u = \cos\theta$, which implies that $du = -\sin\theta\, d\theta$, or $-du = \sin\theta\, d\theta$. The lower limit of integration becomes $u = 1$ and the upper limit becomes $u = 0$. So we have

$$\int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta} d\theta = -\int_1^0 \frac{1}{1+u^2} = \int_0^1 \frac{1}{1+u^2} = \tan^{-1} u\Big|_0^1 = \frac{\pi}{4}.$$

**8.6.8** Using the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ with $\theta = 10x$, we have

$$\int \cos^2 10x\, dx = \int \frac{1+\cos 20x}{2} dx = \frac{x}{2} + \frac{\sin 20x}{40} + C.$$

**8.6.9** After completing the square of $8x - x^2$, we make a substitution of $u = x - 4$. The lower limit of integration becomes $u = 0$ and the upper limit of integration becomes $u = 2$. Integrating, we have

$$\int_4^6 \frac{dx}{\sqrt{8x-x^2}} = \int_4^6 \frac{dx}{\sqrt{16-(x-4)^2}} = \int_0^2 \frac{dx}{\sqrt{16-u^2}} = \sin^{-1}\frac{u}{4}\Big|_0^2 = \frac{\pi}{6}.$$

**8.6.10** Because $\sin x$ and $\cos x$ both have odd powers, we have a choice of either splitting off a factor $\sin x$ or a factor of $\cos x$. In this case, you can verify that it is easier to split off $\cos x$ by rewriting $\cos^3 x$ as $\cos^2 x \cos x$. Here is the integration process:

$$\begin{aligned}
\int \sin^9 x \cos^3 x\, dx &= \int \sin^9 x \cos^2 x \cos x\, dx && \text{Split off } \cos x.\\
&= \int \sin^9 x(1-\sin^2 x)\cos x\, dx && \text{Pythagorean identity}\\
&= \int (\sin^9 x - \sin^{11} x)\cos x\, dx && \text{Expand.}\\
&= \int (u^9 - u^{11}) du && \text{Let } u = \sin x;\ du = \cos\, dx.\\
&= \frac{u^{10}}{10} - \frac{u^{12}}{12} + C && \text{Integrate.}\\
&= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C. && \text{Replace } u \text{ with } \sin x.
\end{aligned}$$

**8.6.11** The integral is evaluated after expanding the integrand.

$$\begin{aligned}
\int_0^{\pi/4} (\sec x - \cos x)^2\, dx &= \int_0^{\pi/4} (\sec^2 x - 2\underbrace{\sec x \cos x}_{1} + \cos^2 x)\, dx && \text{Expand integrand.}\\
&= \int_0^{\pi/4} \left(\sec^2 x - 2 + \frac{1+\cos 2x}{2}\right) dx && \cos^2 x = \frac{1+\cos 2x}{2}\\
&= \int_0^{\pi/4} \left(\sec^2 x - \frac{3}{2} + \frac{1}{2}\cos 2x\right) dx && \text{Simplify integrand.}\\
&= \left(\tan x - \frac{3}{2}x + \frac{1}{4}\sin 2x\right)\Big|_0^{\pi/4} = \frac{5}{4} - \frac{3\pi}{8} && \text{Evaluate.}
\end{aligned}$$

**8.6.12** Letting $u = e^x$, it follows that $du = e^x dx$. So we have

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.$$

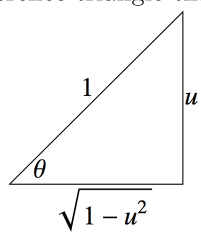
**8.6.13** Let $u = e^x$, which implies that $du = e^x dx$ and $dx = \frac{du}{e^x} = \frac{du}{u}$. The integrand is then expressed in terms of $u$:

$$\int \frac{dx}{e^x\sqrt{1-e^{2x}}} = \int \frac{du}{u^2\sqrt{1-u^2}}.$$

The expression $1-u^2$ suggests the trigonometric substitution $u = \sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. It follows that $du = \cos\theta\, d\theta$ and

$$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Using the equation $u = \sin\theta$, we create a reference triangle that helps us evaluate the integral.




Summarizing what we've done so far and then evaluating the integral, we have

$$\begin{aligned}
\int \frac{dx}{e^x\sqrt{1-e^{2x}}} &= \int \frac{du}{u^2\sqrt{1-u^2}} && u\text{-substitution}\\
&= \int \frac{\cos\theta}{\sin^2\theta\cos\theta}\,d\theta && \text{Trigonometric substitution}\\
&= \int \csc^2\theta\, d\theta && \text{Simplify.}\\
&= -\cot\theta + C && \text{Integrate.}\\
&= -\frac{\sqrt{1-u^2}}{u} + C && \text{Use the reference triangle.}\\
&= -\frac{\sqrt{1-e^{2x}}}{e^x} + C. && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

**8.6.14** We start by eliminating the negative powers in the integrand by multiplying the numerator and denominator by $x^3$. Then we split the integral into a sum of integrals:

$$\int \frac{x^{-2}+x^{-3}}{x^{-1}+16x^{-3}}\cdot\frac{x^3}{x^3}\,dx = \int \frac{x+1}{x^2+16}dx = \int \frac{x}{x^2+16}dx + \int \frac{1}{x^2+16}dx.$$

The second integral on the right equals $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$. For the first integral on the right, we let $u = x^2+16$. It follows that $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$. Therefore

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+16) + C.$$

Adding the integrals, we have

$$\int \frac{x^{-2}+x^{-3}}{x^{-1}+16x^{-3}}\,dx = \frac{1}{2}\ln(x^2+16) + \frac{1}{4}\tan^{-1}\frac{x}{4} + C.$$

**8.6.15** We let $u = \sqrt{x}$, which implies that $du = \dfrac{dx}{2\sqrt{x}}$ or $2du = \dfrac{dx}{\sqrt{x}}$. The lower limit of integration is $u = 1$ and the upper limit is $u = 2$. Therefore

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 2^u du = \frac{2\cdot 2^u}{\ln 2}\bigg|_1^2 = \frac{4}{\ln 2}.$$

**8.6.16** The denominator factors as $x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$. Therefore the appropriate form of the partial fraction decomposition is

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}.$$

Multiplying both sides of the equation by $(x^2+1)(x+1)(x-1)$ leads to

$$1 = (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1).$$

We can use this equation to find the constants A, B, C, and D.

$$x = 1 \Rightarrow 1 = 4D \text{ or } D = \frac{1}{4}$$

$$x = -1 \Rightarrow 1 = -4C \text{ or } C = -\frac{1}{4}$$

$$x = 0 \Rightarrow 1 = -B + \frac{1}{4} + \frac{1}{4} \text{ or } B = -\frac{1}{2}$$

$$\text{Equating coefficients of } x^3 \Rightarrow 0 = A + C + D \Rightarrow A = 0$$

So the partial fraction decomposition is

$$\frac{1}{x^4-1} = -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}.$$

Using this decomposition, integration is straightforward:

$$\int \frac{dx}{x^4-1} = -\int \frac{dx}{2(x^2+1)} - \int \frac{dx}{4(x+1)} + \int \frac{dx}{4(x-1)} = -\frac{1}{2}\tan^{-1}x - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C.$$

**8.6.17** Let's evaluate the corresponding indefinite integral first. Let $z = w^2$ which implies that $dz = 2w\,dw$, or $\dfrac{dz}{2} = w\,dw$. We then have

$$\int w^3 e^{w^2}\,dw = \frac{1}{2}\int ze^z\,dz.$$

For the integral on the right, use integration by parts with the following choices.

$$u = z \qquad dv = e^z\,dz$$

$$du = dz \quad v = e^z$$

Integrating by parts, we have

$$\int w^3 e^{w^2} dw = \frac{1}{2}\int ze^z dz = \frac{1}{2}\left(ze^z - \int e^z\,dz\right) = \frac{e^z}{2}(z-1) + C$$

After replacing $z$ with $w^2$, we evaluate the definite integral:

$$\int_1^2 w^3 e^{w^2} dx = \left(\frac{1}{2}e^{w^2}(w^2-1)\right)\bigg|_1^2 = \frac{3e^4}{2}.$$

**8.6.18** Letting $u = x - 5$, it follows that $du = dx$ and $x = u + 5$. The lower limit of integration is $u = 0$ and the upper limit is $u = 1$.

$$\int_5^6 x(x-5)^{10}\,dx = \int_0^1 (u+5)u^{10}\,du = \int_0^1 (u^{11} + 5u^{10})\,du = \left(\frac{u^{12}}{12} + \frac{5u^{11}}{11}\right)\Bigg|_0^1 = \frac{71}{132}.$$

**8.6.19** The integral has an odd power of $\sin x$, so we start by splitting off a factor of $\sin x$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^7 x\,dx &= \int_0^{\pi/2} (\sin^2 x)^3 \sin x\,dx && \text{Split off } \sin x.\\
&= \int_0^{\pi/2} (1-\cos^2 x)^3 \sin x\,dx && \sin^2 x = 1 - \cos^2 x\\
&= \int_1^0 -(1-u^2)^3\,du && \text{Let } u = \cos x;\, du = -\sin x\,dx.\\
& && x = 0 \Rightarrow u = 1;\, x = \frac{\pi}{2} \Rightarrow u = 0\\
&= \int_0^1 (-u^6 + 3u^4 - 3u^2 + 1)\,du && \text{Expand.}\\
&= \left(-\frac{u^7}{7} + \frac{3}{5}u^5 - u^3 + u\right)\Bigg|_0^1 && \text{Integrate.}\\
&= \frac{16}{35}. && \text{Evaluate.}
\end{aligned}$$

**8.6.20** Let $u = \sqrt{t}$ which implies that $du = \frac{dt}{2\sqrt{t}}$, $t = u^2$, and $\dfrac{dt}{\sqrt{t}} = 2du$. The lower limit of integration becomes $u = 1$ and the upper limit of integration becomes $u = \sqrt{3}$. Changing variables, we have

$$\int_1^3 \frac{dt}{\sqrt{t}(t+1)} = 2\int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2\tan^{-1} u\Big|_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}.$$

**8.6.21** We use integration by parts with the following choices.

$$u = \ln 3x \qquad dv = x^9\,dx$$

$$du = \frac{1}{x}dx \qquad v = \tfrac{x^{10}}{10}$$

Integrating by parts, we have

$$\int x^9 \ln 3x\,dx = \frac{x^{10}}{10}\ln 3x - \frac{1}{10}\int x^9\,dx = \frac{x^{10}}{10}\ln 3x - \frac{x^{10}}{100} + C.$$

**8.6.22** The denominator consists of two distinct linear terms $x - a$ and $x - b$ because $a \neq b$. An appropriate form of the partial fraction decomposition is

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Multiplying both sides of this equation by $(x-a)(x-b)$ gives us

$$1 = A(x-b) + B(x-a).$$

Choosing $x = b$, we have $1 = B(b - a)$ or $B = 1/(b - a)$ and if $x = a$, $1 = A(a - b)$ or $A = 1/(a - b)$. Substituting these values for $A$ and $B$ in the partial fraction decomposition, we carry out integration.

$$\begin{aligned}
\int \frac{dx}{(x-a)(x-b)} &= \frac{1}{a-b}\int \frac{dx}{x-a} + \frac{1}{b-a}\int \frac{dx}{x-b} && \text{Partial fractions}\\
&= \frac{1}{a-b}\ln|x-a| + \frac{1}{b-a}\ln|x-b| + C && \text{Integrate.}\\
&= \frac{\ln|x-a| - \ln|x-b|}{a-b} + C. && \text{Simplify.}
\end{aligned}$$

**8.6.23** Letting $u = \cos x$ and $du = -\sin x\,dx$, or $-du = \sin x\,dx$, we have

$$\int \frac{\sin x}{\cos^2 x + \cos x}dx = -\int \frac{du}{u^2+u} = -\int \frac{du}{u(u+1)}.$$

To evaluate the integral on the right, we first find the constants $A$ and $B$ in the partial fraction decomposition

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

Multiplying both sides of this equation by $u(u+1)$, we have

$$1 = A(u+1) + Bu = (A+B)u + A.$$

It follows that $A = 1$ and $B = -1$. Using these constants in the partial fraction decomposition, integration is straightforward.

$$\begin{aligned}
\int \frac{\sin x}{\cos^2 x + \cos x}dx &= -\int \frac{du}{u(u+1)} && u = \cos x;\ du = -\sin x\,dx\\
&= -\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du && \text{Partial fractions}\\
&= -(\ln|u| - \ln|u+1|) + C && \text{Integrate.}\\
&= \ln(\cos x + 1) - \ln|\cos x| + C. && \text{Replace } u \text{ with } \cos x.
\end{aligned}$$

**8.6.24** The degree of the polynomial in the numerator is greater the the degree of the polynomial in the denominator, so we perform long division to obtain

$$\frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w} = 3w^2 + 2w + \frac{8w^2 - 12w - 32}{w^3 - 4w}.$$

The first part of the right side of the equation is easy to integrate, so we focus on the fractional part. Factoring the denominator and then setting up the partial fraction decomposition, we have

$$\frac{8w^2 - 12w - 32}{w(w+2)(w-2)} = \frac{A}{w} + \frac{B}{w+2} + \frac{C}{w-2}.$$

We multiply both sides by $w(w+2)(w-2)$:

$$8w^2 - 12w - 32 = A(w+2)(w-2) + Bw(w-2) + Cw(w+2).$$

Now we find the values of unknown constants:

$$w = 0 \Rightarrow -32 = -4A \text{ or } A = 8$$

$$w = -2 \Rightarrow 24 = 8B \text{ or } B = 3$$

$$w = 2 \Rightarrow -24 = 8C \text{ or } C = -3$$

Substituting these values into decomposition and evaluating the integral, we have

$$\int \frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w}\,dw = \int \left(3w^2 + 2w + \frac{8}{w} + \frac{3}{w+2} - \frac{3}{w-2}\right) dw$$

$$= w^3 + w^2 + 8\ln|w| + 3\ln|w+2| - 3\ln|w-2| + C.$$

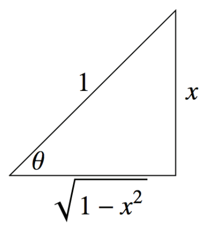
**8.6.25** One approach for evaluating this integral is to use a trigonometric substitution. The expression $1-x^2$ suggests the substitution $x = \sin\theta$ with $-\pi/2 < \theta < \pi/2$. With this substitution, $dx = \cos\theta\, d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Substituting these values into the integral and simplifying, we have

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + C.$$

Given that $\sin\theta = x$, a geometric description of the relationship between $x$ and $\theta$ is given by the following reference triangle.




With the help of the reference triangle, we continue:

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln|\csc\theta + \cot\theta| + C = -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C = \ln\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C.$$

**8.6.26** Let $u = \ln x$ which implies that $du = \dfrac{dx}{x}$. Using this substitution, the lower limit of integration is $u = -1$ and the upper limit of integration is $u = 0$. After making these substitutions and then completing the square, we evaluate the integral.

$$\int_{1/e}^{1} \frac{dx}{x(\ln^2 x + 2\ln x + 2)} = \int_{-1}^{0} \frac{du}{u^2 + 2u + 2} = \int_{-1}^{0} \frac{du}{(u+1)^2 + 1} = \tan^{-1}(u+1)\Big|_{-1}^{0} = \frac{\pi}{4}$$

**8.6.27** Because we have an even power of $\sin\frac{x}{2}$, we use the half-angle formula for $\sin^2\dfrac{x}{2}$ to rewrite the integrand:

$$\int \sin^4\frac{x}{2}\,dx = \int \Bigg(\underbrace{\frac{1-\cos x}{2}}_{\sin^2\frac{x}{2}}\Bigg)^2 dx = \frac{1}{4}\int \left(1 - 2\cos x + \cos^2 x\right) dx.$$

Using the half-angle formula for $\cos^2 x$, the evaluation may be completed:

$$\int \sin^4 \frac{x}{2}\, dx = \frac{1}{4}\int \big(1 - 2\cos x + \underbrace{\frac{1+\cos 2x}{2}}_{\cos^2 x}\big)\, dx = \frac{1}{4}\int \left(\frac{3}{2} - 2\cos x + \frac{\cos 2x}{2}\right) dx$$
$$= \frac{3x}{8} - \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C.$$

**8.6.28** The denominator factors as $(x^2+1)^2$, which is repeated irreducible quadratic factor. So the form of the partial fraction decomposition is

$$\frac{3x^2+2x+3}{x^4+2x^2+1} = \frac{3x^2+2x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Multiplying by $(x^2+1)^2$, we have

$$3x^2+2x+3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + B + D.$$

Comparing coefficients of equal powers, starting with the highest power, we find that $A = 0$, $B = 3$, $C = 2$, and $D = 0$. Now we use the partial fraction decomposition, with these constant values, to integrate:

$$\begin{aligned}
\int \frac{3x^2+2x+3}{x^4+2x^2+1} dx &= \int \left(\frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}\right) dx && \text{Partial fractions} \\
&= \int \frac{3}{x^2+1} dx + \int \frac{2x}{(x^2+1)^2} dx && \text{Split into 2 integrals.} \\
&= 3\tan^{-1} x + \int \frac{du}{u^2} && \text{Evaluate first integral;} \\
& && \text{Second integral: } u = x^2+1,\ du = 2x\, dx \\
&= 3\tan^{-1} x - \frac{1}{x^2+1} + C. && \text{Evaluate and replace } u \text{ with } x^2+1.
\end{aligned}$$

**8.6.29** Before integrating, let's write $\cot x$ in terms of $\sin x$ and $\cos x$ and then simplify the integrand by multiplying the numerator and denominator by $\sin x$.

$$\frac{2\cos x + \overbrace{\frac{\cos x}{\sin x}}^{\cot x}}{1+\sin x} = \frac{2\sin x\cos x + \cos x}{\sin x + \sin^2 x}.$$

The integral is evaluated with the help of a $u$-substitution.

$$\begin{aligned}
\int \frac{2\cos x + \cot x}{1+\sin x}\, dx &= \int \frac{2\sin x\cos x + \cos x}{\sin x + \sin^2 x} dx && \text{Write in terms of } \sin x \text{ and } \cos x\text{; simplify.} \\
&= \int \frac{du}{u} && \text{Let } u = \sin x + \sin^2 x;\ du = (\cos x + 2\sin x\cos x)\, dx. \\
&= \ln|u| + C && \text{Integrate.} \\
&= \ln|\sin x + \sin^2 x| + C. && \text{Substitute } \sin x + \sin^2 x \text{ for } u.
\end{aligned}$$

**8.6.30** Using the trigonometric substitution $v = 5\sin\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, it follows that $dv = 5\cos\theta\, d\theta$ and

$$\sqrt{25 - v^2} = \sqrt{25 - 25\sin^2\theta} = |5\cos\theta| = 5\cos\theta.$$

The lower limit $v = \frac{5}{2}$ implies that $\sin\theta = \frac{1}{2}$ and it follows that $\theta = \frac{\pi}{6}$. Similarly, if $v = \frac{5\sqrt{3}}{2}$ then $\sin\theta = \frac{\sqrt{3}}{2}$ and so $\theta = \frac{\pi}{3}$.

$$\int_{5/2}^{5\sqrt{3}/2} \frac{dv}{v^2\sqrt{25-v^2}} = \int_{\pi/6}^{\pi/3} \frac{5\cos\theta}{(25\sin^2\theta)(5\cos\theta)}d\theta = \frac{1}{25}\int_{\pi/6}^{\pi/3} \csc^2\theta\, d\theta = -\frac{1}{25}\cot\theta\Big|_{\pi/6}^{\pi/3}$$
$$= -\frac{1}{25}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = \frac{2}{25\sqrt{3}}.$$

**8.6.31** Removing a factor of 3 from the square root, we have

$$\int \sqrt{36-9x^2}\, dx = \int \sqrt{9(4-x^2)}\, dx = 3\int \sqrt{4-x^2}\, dx.$$

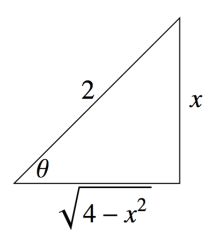
The expression $4-x^2$ suggests the substitution $x = 2\sin\theta$, for $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It follows that $dx = 2\cos\theta\, d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2|\cos\theta| = 2\cos\theta.$$

Making these substitutions, we have

$$\begin{aligned}
\int \sqrt{36-9x^2}dx &= 3\int \sqrt{4-x^2}\, dx && \text{Factor.}\\
&= 12\int \cos^2\theta\, d\theta && \text{Trigonometric substitution}\\
&= 6\int (1+\cos 2\theta)\, d\theta && \cos^2\theta = \frac{1+\cos 2\theta}{2}\\
&= 6\theta + 3\sin 2\theta + C && \text{Integrate.}\\
&= 6\theta + 6\sin\theta\cos\theta + C. && \sin 2\theta = 2\sin\theta\cos\theta
\end{aligned}$$

From the equation $\sin\theta = \frac{x}{2}$ we have $\theta = \sin^{-1}\dfrac{x}{2}$. We use the following reference triangle to observe that $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.



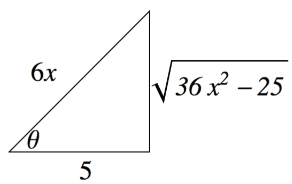

Therefore

$$\int \sqrt{36-9x^2}dx = 6\underbrace{\sin^{-1}\frac{x}{2}}_{\theta} + 6\underbrace{\left(\frac{x}{2}\right)}_{\sin\theta}\underbrace{\left(\frac{\sqrt{4-x^2}}{2}\right)}_{\cos\theta} + C = 6\sin^{-1}\frac{x}{2} + \frac{3}{2}x\sqrt{4-x^2} + C.$$

**8.6.32** We factor a 6 out of the square root, then proceed with a trigonometric substitution of $x = \frac{5}{6}\sec\theta$, where $0 < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2-25}} &= \int \frac{dx}{6\sqrt{x^2-\frac{25}{36}}} && \text{Factor.}\\
&= \int \frac{\frac{5}{6}\sec\theta\tan\theta}{6\sqrt{\frac{25}{36}\sec^2\theta-\frac{25}{36}}}d\theta && \text{Trigonometric substitution}\\
&= \int \frac{\frac{5}{6}\sec\theta\tan\theta}{5\tan\theta}d\theta && \sqrt{\frac{25}{36}\sec^2\theta-\frac{25}{36}}=\frac{5}{6}|\tan\theta|=\frac{5}{6}\tan\theta\\
&= \frac{1}{6}\int \sec\theta\, d\theta && \text{Simplify.}\\
&= \frac{1}{6}\ln(\sec\theta+\tan\theta)+C && \text{Integrate.}
\end{aligned}$$

From our trigonometric substitution, we know that $\sec\theta = \frac{6x}{5}$ which is represented by the following reference triangle.




Using this triangle, we complete the integral:

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2-25}} &= \frac{1}{6}\ln(\sec\theta+\tan\theta)+C\\
&= \frac{1}{6}\ln\left(\frac{6x}{5}+\frac{\sqrt{36x^2-25}}{5}\right)+C\\
&= \frac{1}{6}\ln\left(6x+\sqrt{36x^2-25}\right)-\frac{\ln 5}{6}+C\\
&= \frac{1}{6}\ln\left(6x+\sqrt{36x^2-25}\right)+C.
\end{aligned}$$

**8.6.33** Letting $u=e^x$, we have $e^{2x}=(e^x)^2=u^2$ and $du=e^x dx$. Provided that $a\neq 0$,

$$\int \frac{e^x}{a^2+e^{2x}}dx=\int\frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C \;=\frac{1}{a}\tan^{-1}\frac{e^x}{a}+C.$$

**8.6.34** Let $u=\cos 3x+1$ which implies that $du=-3\sin 3x\,dx$, or $-\frac{1}{3}du=\sin 3x\,dx$.

$$\int_0^{\pi/9}\frac{\sin 3x}{\cos 3x+1}\,dx=-\frac{1}{3}\int_2^{3/2}\frac{du}{u}=-\frac{1}{3}\left(\ln u\right)\Big|_2^{3/2}=\frac{\ln 2-\ln\frac{3}{2}}{3}=\frac{1}{3}\ln\frac{4}{3}.$$

**8.6.35** Let $u=\tan\theta$ which implies that $du=\sec^2\theta\,d\theta$. The lower limit of integration becomes $u=\tan 0=0$ and the upper limit becomes $u=\tan\pi/4=1$. Changing variables, we have

$$\int_0^{\pi/4}(\tan^2\theta+\tan\theta+1)\sec^2\theta\,d\theta=\int_0^1(u^2+u+1)du=\left(\frac{u^3}{3}+\frac{u^2}{2}+u\right)\Big|_0^1=\frac{1}{3}+\frac{1}{2}+1=\frac{11}{6}$$

**8.6.36** We apply integration by parts with the following choices for our variables.

$$\begin{array}{ll} u = x & dv = 10^x\,dx \\ du = dx & v = \frac{10^x}{\ln 10} \end{array}$$

We then have

$$\int x \cdot 10^x\,dx = \frac{x \cdot 10^x}{\ln 10} - \int \frac{10^x}{\ln 10} dx = \frac{x \cdot 10^x}{\ln 10} - \frac{10^x}{\ln^2 10} + C = \frac{10^x}{\ln 10}\left(x - \frac{1}{\ln 10}\right) + C.$$

**8.6.37** We start by multiplying the integrand by $\dfrac{1+\sin 2x}{1+\sin 2x}$.

$$\begin{aligned} \int_0^{\pi/6} \frac{1}{1-\sin 2x} \cdot \frac{1+\sin 2x}{1+\sin 2x}\,dx &= \int_0^{\pi/6} \frac{1+\sin 2x}{1-\sin^2 2x}\,dx && \text{Simplify integrand.} \\ &= \int_0^{\pi/6} \frac{1+\sin 2x}{\cos^2 2x}\,dx && 1-\sin^2 2x = \cos^2 2x. \\ &= \int_0^{\pi/6} \big(\sec^2 2x + \sec 2x \tan 2x\big)\,dx && \text{Simplify integrand.} \\ &= \frac{1}{2}\big(\tan 2x + \sec 2x\big)\Big|_0^{\pi/6} && \text{Integrate.} \\ &= \frac{1}{2}\big(\sqrt{3} + 2 - 1\big) = \frac{\sqrt{3}+1}{2}. && \text{Evaluate.} \end{aligned}$$

**8.6.38** One approach for evaluating $\displaystyle\int_{\pi/6}^{\pi/2} \cos x \ln(\sin x)\,dx$ is to use a $u$-substitution by letting $u = \sin x$ and to then apply the integral formula for $\ln u$. Another approach that we use here is to use integration by parts with the following variable choices.

$$\begin{array}{ll} u = \ln(\sin x) & dv = \cos x\,dx \\ du = \dfrac{\cos x}{\sin x} dx & v = \sin x \end{array}$$

Integrating by parts, we have

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x)\Big|_{\pi/6}^{\pi/2} - \int_{\pi/6}^{\pi/2} \cos x\,dx &= -\frac{1}{2}\ln\frac{1}{2} - \sin x\Big|_{\pi/6}^{\pi/2} \\ &= -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2} = \frac{\ln 2 - 1}{2}. \end{aligned}$$

**8.6.39** We integrate by parts with the following variable choices.

$$\begin{array}{ll} u = \ln(\sin x) & dv = \sin x\,dx \\ du = \dfrac{\cos x}{\sin x} dx & v = -\cos x \end{array}$$

It follows that

$$\begin{aligned}
\int \sin x \ln(\sin x)\,dx &= -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x} && \text{Integration by parts}\\
&= -\cos x \ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx && \cos^2 x = 1-\sin^2 x\\
&= -\cos x \ln(\sin x) + \int \Big(\csc x - \sin x\Big)\,dx && \text{Simplify integrand.}\\
&= -\cos x \ln(\sin x) - \ln|\csc x + \cot x| + \cos x + C. && \text{Evaluate.}
\end{aligned}$$

**8.6.40** We integrate by parts with the following variable choices.

$$u = \ln(\sin x) \qquad dv = \sin 2x\,dx$$

$$du = \frac{\cos x}{\sin x}dx \quad v = -\tfrac{1}{2}\cos 2x$$

Then we have

$$\begin{aligned}
\int \sin 2x \ln(\sin x)\,dx &= -\frac{1}{2}\cos 2x \ln(\sin x) + \int \frac{\cos x \cos 2x}{2\sin x}dx && \text{Integration by parts}\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \int \frac{\cos x(1-2\sin^2 x)}{2\sin x}dx && 1-2\sin^2 x = \cos 2x\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\int \cot x\ dx - \int \sin x \cos x dx && \text{Expand the integral.}\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\int \cot x\ dx - \int \frac{1}{2}\sin 2x dx && \sin x\cos x = \frac{1}{2}\sin 2x\\
&= -\frac{1}{2}\cos 2x \ln(\sin x) + \frac{1}{2}\ln|\sin x| + \frac{1}{4}\cos 2x + C. && \text{Evaluate.}
\end{aligned}$$

**8.6.41** Split $\csc^4 x$ and use the identity $\csc^2 x = 1 + \cot^2 x$:

$$\begin{aligned}
\int \cot^{3/2} x \csc^4 x\,dx &= \int \cot^{3/2} x \csc^2 x \csc^2 x\,dx && \csc^4 x = \csc^2 x \csc^2 x\\
&= \int \cot^{3/2} x (1+\cot^2 x)\csc^2 x\,dx && \csc^2 x = 1+\cot^2 x\\
&= -\int u^{3/2}(1+u^2)\,du && u = \cot x,\ du = -\csc^2 x\,dx\\
&= -\int \big(u^{3/2} + u^{7/2}\big)\,du && \text{Expand integrand.}\\
&= -\frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C && \text{Evaluate.}\\
&= -\frac{2}{5}\cot^{5/2} x - \frac{2}{9}\cot^{9/2} x + C. && \text{Replace } u \text{ with } \cot x.
\end{aligned}$$

**8.6.42** Let $u = \sin^{-1} x$, which implies that $du = \dfrac{dx}{\sqrt{1-x^2}}$. We also change the limits of integration.

$$x = 0 \Rightarrow u = \sin^{-1} 0 = 0$$
$$x = \frac{1}{2} \Rightarrow u = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Integrate:

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} u\, du = \left.\frac{u^2}{2}\right|_0^{\pi/6} = \frac{\pi^2}{72}.$$

**8.6.43** The integral $\int \frac{x^9}{\sqrt{1-x^{20}}}\, dx$ can be expressed as $\int \frac{x^9}{\sqrt{1-\left(x^{10}\right)^2}}\, dx$, so we let $u = x^{10}$ which implies that $du = 10x^9 dx$, or $x^9 dx = \frac{du}{10}$. Therefore

$$\int \frac{x^9}{\sqrt{1-\left(x^{10}\right)^2}}\, dx = \frac{1}{10}\int \frac{du}{\sqrt{1-u^2}} = \frac{\sin^{-1} u}{10} + C = \frac{\sin^{-1} x^{10}}{10} + C.$$

**8.6.44** Factoring $x^3 - x^2$, an appropriate form of the partial fraction decomposition is

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiplying both sides by $x^2(x-1)$ leads to

$$1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B.$$

Equating coefficients of equal powers of $x$, we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\int \frac{dx}{x^3 - x^2} = -\int \frac{1}{x} dx - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln|x| + \frac{1}{x} + \ln|x-1| + C.$$

**8.6.45** Letting $u = 1 + e^x$, it follows that $du = e^x\, dx$, $e^x = u - 1$, and $dx = \frac{du}{e^x} = \frac{du}{u-1}$. The lower limit of integration is $u = 2$ and the upper limit is $u = 3$. We have

$$\int_0^{\ln 2} \frac{dx}{(1+e^x)^2} = \int_2^3 \frac{du}{u^2(u-1)}.$$

Partial fractions is used to evaluate the integral on the right. An appropriate form for the partial fraction decomposition of the integrand is

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}.$$

Multiplying both sides by $u^2(u-1)$ leads to

$$1 = Au(u-1) + B(u-1) + Cu^2 = (A+C)u^2 + (B-A)u - B.$$

Equating coefficients of equal powers of $u$, we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\begin{aligned}\int_0^{\ln 2} \frac{dx}{(1+e^x)^2} = \int_2^3 \frac{du}{u^2(u-1)} = \int_2^3 \left(-\frac{1}{u} - \frac{1}{u^2} + \frac{1}{u-1}\right) du &= \left.\left(-\ln|u| + \frac{1}{u} + \ln|u-1|\right)\right|_2^3 \\ &= \ln\frac{2}{3} - \frac{1}{6} + \ln 2 = \ln\frac{4}{3} - \frac{1}{6}.\end{aligned}$$

**8.6.46** Letting $u = e^x$, we have $du = e^x\, dx$, $dx = \frac{du}{e^x} = \frac{du}{u}$, and $\int \frac{dx}{e^{2x}+1} = \int \frac{du}{u(u^2+1)}$. The form of the partial fraction decomposition of $\frac{1}{u(u^2+1)}$ is

$$\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu + C}{u^2+1}.$$

Multiplying both sides by $u(u^2+1)$, we have

$$1 = A(u^2+1) + (Bu+C)u = (A+B)u^2 + Cu + A.$$

Comparing coefficients of equal powers, we find that $A = 1$, $B = -1$, and $C = 0$.

Inserting these constants into our partial fraction decomposition implies that

$$\int \frac{du}{u(u^2+1)} = \int \frac{du}{u} - \int \frac{u}{u^2+1} du.$$

In the second integral, we let $v = u^2+1$. Then $dv = 2u\,du$, or $\frac{1}{2}dv = u\,du$. It follows that

$$\int \frac{u}{u^2+1} du = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + C = \frac{1}{2}\ln(u^2+1) + C.$$

Assembling all our pieces, we have

$$\begin{aligned}\int \frac{dx}{e^{2x}+1} &= \int \frac{du}{u} - \int \frac{u}{u^2+1} du \\ &= \ln|u| - \frac{1}{2}\ln(u^2+1) + C \\ &= \ln e^x - \frac{1}{2}\ln(e^{2x}+1) + C \\ &= x - \frac{1}{2}\ln(e^{2x}+1) + C.\end{aligned}$$

**8.6.47** Apply long division to obtain

$$\frac{2x^3+x^2-2x-4}{x^2-x-2} = 2x+3+\frac{5x+2}{x^2-x-2}.$$

Factor $x^2-x-2$ and then find the partial fraction decomposition of $\dfrac{5x+2}{x^2-x-2}$.

$$\frac{5x+2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiply both sides by $(x-2)(x+1)$.

$$5x+2 = A(x+1) + B(x-2).$$

Note that $x = 2$ implies that $A = 4$ and $x = -1$ implies that $B = 1$. Therefore

$$\begin{aligned}\int \left(\frac{2x^3+x^2-2x-4}{x^2-x-2}\right)dx &= \int \left(2x+3+\frac{4}{x-2}+\frac{1}{x+1}\right)dx \\ &= x^2+3x+4\ln|x-2| + \ln|x+1| + C.\end{aligned}$$

.

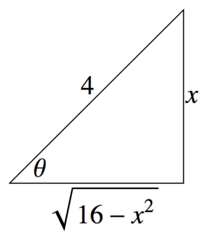
**8.6.48** Let $x = 4\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$. This substitution implies that $dx = 4\cos\theta\,d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = |4\cos\theta| = 4\cos\theta.$$

With these substitutions, we have

$$\begin{aligned}\int \frac{\sqrt{16-x^2}}{x^2}\,dx &= \int \frac{4\cos\theta}{16\sin^2\theta}\,4\cos\theta\,d\theta \\ &= \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C.\end{aligned}$$

Given that $\sin\theta = \frac{x}{4}$, we can form the following right triangle that helps us evaluate the integral.




Therefore

$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx = -\cot\theta - \theta + C = -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\frac{x}{4} + C.$$

**8.6.49** We start by splitting off factors of $\tan x$ and $\sec x$ and then we replace $\tan^2 x$ with $\sec^2 x - 1$:

$$\begin{aligned}\int \tan^3 x\sec^9 x\,dx &= \int \tan^2 x\sec^8 x\sec x\tan x\,dx && \text{Split off } \tan x \text{ and } \sec x. \\ &= \int (\sec^2 x - 1)\sec^8 x\sec x\tan x\,dx && \tan^2 x = \sec^2 x - 1 \\ &= \int (\sec^{10} x - \sec^8 x)\sec x\tan x\,dx && \text{Expand integrand.} \\ &= \int (u^{10} - u^8)du && u = \sec x,\ du = \sec x\tan x\,dx \\ &= \frac{u^{11}}{11} - \frac{u^9}{9} + C && \text{Integrate.} \\ &= \frac{\sec^{11} x}{11} - \frac{\sec^9 x}{9} + C. && \text{Replace } u \text{ with } \sec x.\end{aligned}$$

**8.6.50** With an even power of $\sec x$, we split off a factor of $\sec^2 x$ and prepare the integral for a substitution of $u = \tan x$:

$$\begin{aligned}
\int \tan^7 x \sec^4 x\,dx &= \int \tan^7 x \sec^2 x \sec^2 x\,dx && \text{Split off } \sec^2 x.\\
&= \int \tan^7 x(\tan^2 x+1)\sec^2 x\,dx && \sec^2 x = \tan^2 x+1\\
&= \int u^7(u^2+1)\,du && u=\tan x,\ du=\sec^2 x\,dx\\
&= \int (u^9+u^7)\,du && \text{Expand.}\\
&= \frac{u^{10}}{10}+\frac{u^8}{8}+C && \text{Integrate.}\\
&= \frac{\tan^{10} x}{10}+\frac{\tan^8 x}{8}+C. && \text{Replace } u \text{ with } \tan x.
\end{aligned}$$

**8.6.51** Rewrite $\sec^{7/4} x$ as $\sec^{3/4} x \sec x$ and let $u = \sec x$. Then $du = \sec x \tan$ and after making this substitution, the lower limit of integration is $u = \sec 0 = 1$ and the upper limit is $u = \sec\frac{\pi}{3} = 2$. Integrating, we have

$$\int_0^{\pi/3} \tan x \sec^{7/4} x\,dx = \int_0^{\pi/3} \sec^{3/4} x \sec x \tan x\,dx = \int_1^2 u^{3/4} du = \frac{4}{7}u^{7/4}\bigg|_1^2 = \frac{4}{7}\left(2^{7/4}-1\right).$$

**8.6.52** We apply the method of integration by parts using the following choices.

$$\begin{aligned} u &= t^2 & dv &= e^{3t}\,dt\\ du &= 2t\,dt & v &= \tfrac{e^{3t}}{3}\end{aligned}$$

The integral then becomes

$$\int t^2 e^{3t}\,dt = \frac{t^2e^{3t}}{3} - \frac{2}{3}\int te^{3t}dt.$$

Now we apply integration by parts again to the integral $\int te^{3t}dt$ with the following choices.

$$\begin{aligned} u &= t & dv &= e^{3t}dt\\ du &= dt & v &= \tfrac{e^{3t}}{3}\end{aligned}$$

We then have

$$\begin{aligned}
\frac{t^2e^{3t}}{3} - \frac{2}{3}\int te^{3t}dt &= \frac{t^2e^{3t}}{3} - \frac{2}{3}\left(\frac{te^{3t}}{3} - \frac{1}{3}\int e^{3t}dt\right)\\
&= \frac{t^2e^{3t}}{3} - \frac{2te^{3t}}{9} + \frac{2e^{3t}}{27} + C\\
&= e^{3t}\left(\frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27}\right) + C.
\end{aligned}$$

**8.6.53** We begin with the substitution $u = e^x$, $du = e^x\,dx$, and then split apart the integrand.

$$\int e^x \cot^3 e^x\,dx = \int \cot^3 u\,du = \int \cot^2 u \cot u\,du = \int (\csc^2 u - 1)\cot u\,du = \int \csc^2 u \cot u\,du - \int \cot u\,du$$

Using the substitution $v = \cot u$ in the first integral with $dv = -\csc^2 u\,du$ or $-dv = \csc^2 u\,du$, we find that

$$\int \csc^2 u \cot u\,du = -\int v\,dv = -\frac{v^2}{2} + C = -\frac{\cot^2 u}{2} + C.$$

Therefore

$$\int e^x \cot^3 e^x\,dx = \int \csc^2 u \cot u\,du - \int \cot u\,du = -\frac{\cot^2 u}{2} - \ln|\sin u| + C = -\frac{\cot^2 e^x}{2} - \ln|\sin e^x| + C.$$

**8.6.54** The denominator of the integrand consists of a linear function and an irreducible quadratic factor. The form of the partial fraction decomposition is

$$\frac{2x^2+3x+26}{(x-2)(x^2+16)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+16}.$$

Multiplying both sides of this equation by $(x-2)(x^2+16)$ leads to

$$2x^2+3x+26 = A(x^2+16) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (16A-2C)$$

Equating coefficients of equal powers of $x$ results in the equations

$$A+B = 2, C-2B = 3, \text{ and } 16A-2C = 26.$$

The solution to this system of equations is $A = 2, B = 0,$ and $C = 3$. Therefore

$$\int \frac{2x^2+3x+26}{(x-2)(x^2+16)}\,dx = \int \frac{2}{x-2}dx + \int \frac{3}{x^2+16}dx = 2\ln|x-2| + \frac{3}{4}\tan^{-1}\frac{x}{4} + C.$$

**8.6.55** The denominator factors as $x^3 + x = x(x^2+1)$ and therefore the form of the partial fraction decomposition is

$$\frac{3x^2+3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying both sides by $x(x^2+1)$ leads to

$$3x^2+3x+1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

By equating coefficients of equal powers of $x$, we find that $A = 1$, $B = 2$, and $C = 3$. The original integral can be written as

$$\int \frac{3x^2+3x+1}{x^3+x}\,dx = \int \frac{dx}{x} + \int \frac{2x+3}{x^2+1}dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx.$$

Using a substitution if $u = x^2+1$ in the second integral with $du = 2x\,dx$, we find that $\int \frac{2x}{x^2+1}dx = \ln(x^2+1) + C$. Completing the integration process we have

$$\int \frac{3x^2+3x+1}{x^3+x}\,dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx = \ln|x| + \ln(x^2+1) + 3\tan^{-1}x + C.$$

**8.6.56** Using the trigonometric identity $\sin 2x = 2\sin x\cos x$ the integral equals $\displaystyle\int_\pi^{3\pi/2} 2\sin x \cos x\, e^{\sin^2 x}\,dx$. Making the substitution $u = \sin^2 x$, we have $du = 2\sin x\cos x\,dx$; it is also the case that $x = \pi \Rightarrow u = \sin^2 0 = 0$ and $x = 3\pi/2 \Rightarrow u = \sin^2\left(\frac{3\pi}{2}\right) = 1$.

Therefore

$$\int_\pi^{3\pi/2} \sin 2x\, e^{\sin^2 x}\,dx = \int_\pi^{3\pi/2} 2\sin x\cos x\, e^{\sin^2 x}\,dx = \int_0^1 e^u du = e^u\Big|_0^1 = e - 1.$$

**8.6.57** Some observations are in order before we start integrating. Notice that

$$\int \sin\sqrt{x}\,dx = \int \sqrt{x}\,\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx.$$

Also observe that $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\cos\sqrt{x} + C$ because $\dfrac{d}{dx}\big(-2\cos\sqrt{x}\big) = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$. These facts will help us evaluate the integral using integration by parts with the following variable choices.

$$\begin{array}{ll} u = \sqrt{x} & dv = \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx \\ du = \frac{dx}{2\sqrt{x}} & v = -2\cos\sqrt{x} \end{array}$$

Integrating by parts, we have

$$\begin{aligned} \int \sin\sqrt{x}dx &= -2\sqrt{x}\cos\sqrt{x} + \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C. \end{aligned}$$

**8.6.58** We apply integration by parts with the following choices for our variables.

$$\begin{array}{ll} u = \tan^{-1} w & dv = w^2\,dw \\ du = \frac{dw}{1+w^2} & v = \frac{w^3}{3} \end{array}$$

The integral then becomes

$$\int w^2 \tan^{-1} w\,dw = \frac{w^3}{3}\tan^{-1} w - \frac{1}{3}\int \frac{w^3}{1+w^2}dw.$$

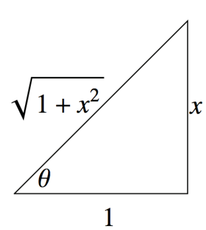
Using long division to obtain the equality $\dfrac{w^3}{1+w^2} = w - \dfrac{w}{1+w^2}$ and a $u$-substitution of $u = 1 + w^2$ to evaluate $\displaystyle\int \frac{w}{1+w^2}dw$, the integral is straightforward to complete:

$$\begin{aligned} \int w^2 \tan^{-1} w\,dw &= \frac{1}{3}\left(w^3 \tan^{-1} w - \int w\,dw + \int \frac{w}{1+w^2}dw\right) \\ &= \frac{1}{3}\left(w^3\tan^{-1} w - \frac{w^2}{2} + \frac{1}{2}\ln(1+w^2)\right) + C. \end{aligned}$$

**8.6.59** Factor the denominator and make the trigonometric substitution $x = \tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{dx}{x^4+x^2} = \int \frac{dx}{x^2(x^2+1)} = \int \frac{\sec^2\theta}{\tan^2\theta\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}}d\theta = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C.$$

Because $\tan\theta = x$, we use the following reference triangle and the substitution $\theta = \tan^{-1} x$ to complete the integration.



$$\int \frac{dx}{x^4 + x^2} = -\cot\theta - \theta + C = -\frac{1}{x} - \tan^{-1} x + C.$$

**8.6.60** We evaluate the indefinite integral first. Apply integration by parts with the following variable choices.

$$\begin{array}{ll} u = e^{-3x} & dv = \cos x\, dx \\ du = -3e^{-3x} dx & v = \sin x \end{array}$$

The corresponding integral then becomes

$$\int e^{-3x} \cos x\, dx = e^{-3x} \sin x + 3 \int e^{-3x} \sin x\, dx.$$

Integration by parts is applied again to the integral on the right side of the equation using the following choices.

$$\begin{array}{ll} u = e^{-3x} & dv = \sin x\, dx \\ du = -3e^{-3x} dx & v = -\cos x \end{array}$$

We then have

$$\int e^{-3x} \cos x\, dx = e^{-3x} \sin x + 3 \int e^{-3x} \sin x\, dx = e^{-3x} \sin x + 3\Big( -e^{-3x} \cos x - 3 \int e^{-3x} \cos x\, dx\Big).$$

Simplifying the equation, we find that

$$\int e^{-3x} \cos x\, dx = e^{-3x} \sin x - 3e^{-3x} \cos x - 9 \int e^{-3x} \cos x\, dx.$$

The indefinite integral is then found by solving for $\int e^{-3x} \cos x\, dx$ and adding a constant of integration:

$$\int e^{-3x} \cos x\, dx = \frac{e^{-3x}}{10}\big(\sin x - 3\cos x\big) + C.$$

Using this result, we evaluate the definite integral:

$$\int_0^{\pi/2} e^{-3x} \cos x\, dx = \frac{e^{-3x}}{10}\big(\sin x - 3\cos x\big)\Big|_0^{\pi/2} = \frac{e^{-3\pi/2} + 3}{10}.$$

**8.6.61** Letting $u = \sin^{-1} x$ we have $x = \sin u$ and $dx = \cos u\, du$. In this case, $x = 0 \Rightarrow u = \sin^{-1} 0 = 0$ and $x = \frac{\sqrt{2}}{2} \Rightarrow u = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. So the integral becomes

$$\int_0^{\sqrt{2}/2} e^{\sin^{-1} x}\, dx = \int_0^{\pi/4} e^u \cos u\, du.$$

Using integration by parts twice (similar to the solution in the previous problem), it can be shown that

$$\int e^u \cos u\, du = \frac{e^u}{2}\big(\cos u + \sin u\big) + C.$$

Now we summarize what we've done so far and then complete the exercise:

$$\begin{aligned}
\int_0^{\sqrt{2}/2} e^{\sin^{-1} x}\, dx &= \int_0^{\pi/4} e^u \cos u\, du && u\text{-substitution} \\
&= \frac{e^u}{2}\big(\cos u + \sin u\big)\Big|_0^{\pi/4} && \text{Integraton by parts (twice)} \\
&= \frac{e^{\pi/4}}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \frac{1}{2} && \text{Evaluate.} \\
&= \frac{\sqrt{2}e^{\pi/4} - 1}{2}. && \text{Simplify.}
\end{aligned}$$

**8.6.62** Rearranging the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

and replacing $\theta$ with $\theta/2$, we have $\sqrt{1+\cos\theta} = \sqrt{2\cos^2\frac{\theta}{2}} = \sqrt{2}\cos\frac{\theta}{2}$ (the last equality is justified because $\cos\frac{\theta}{2} \geq 0$ for $0 \leq \theta \leq \pi/2$). Therefore

$$\int_0^{\pi/2} \sqrt{1+\cos\theta}\, d\theta = \sqrt{2}\int_0^{\pi/2} \cos\frac{\theta}{2}\, d\theta = 2\sqrt{2}\sin\frac{\theta}{2}\Big|_0^{\pi/2} = 2.$$

**8.6.63** We apply integration by parts with the following choices for our variables.

$$\begin{aligned}
u &= \ln x & dv &= x^a\, dx \\
du &= \tfrac{1}{x}dx & v &= \frac{x^{a+1}}{a+1}
\end{aligned}$$

The integration process is straightforward:

$$\begin{aligned}
\int x^a \ln x\, dx &= \frac{x^{a+1}\ln x}{a+1} - \int \frac{x^a}{a+1}dx && \text{Integration by parts} \\
&= \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C && \text{Evaluate.} \\
&= \frac{x^{a+1}}{a+1}\left(\ln x - \frac{1}{a+1}\right) + C. && \text{Factor.}
\end{aligned}$$

**8.6.64** Letting $u = \ln ax$ so that $du = \frac{1}{ax}a\, dx = \frac{dx}{x}$, we have

$$\int \frac{\ln ax}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\ln^2 ax}{2} + C.$$

**8.6.65** By factoring the denominator, we can apply the formula $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$:

$$\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}} = \int_0^{1/6} \frac{dx}{\sqrt{9(\frac{1}{9}-x^2)}} = \frac{1}{3}\int_0^{1/6} \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} = \frac{1}{3}\sin^{-1}(3x)\Big|_0^{1/6} = \frac{\pi}{18}.$$

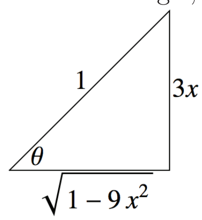
**8.6.66** Although trigonometric substitution works here, it's easier to use a $u$-substitution of $u = 1-9x^2$. This implies that $du = -18x\, dx$ or $-\frac{1}{18}du = x\, dx$. The substitution may now be done:

$$\int \frac{x}{\sqrt{1-9x^2}}dx = -\frac{1}{18}\int \frac{du}{\sqrt{u}} = -\frac{1}{9}\sqrt{u} + C = -\frac{1}{9}\sqrt{1-9x^2} + C.$$

**8.6.67** $\sqrt{1-9x^2} = \sqrt{1-(3x)^2}$ suggests the change of variables $3x = \sin\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. This substitution implies that $3\,dx = \cos\theta\,d\theta$ and

$$\sqrt{1-9x^2} = \sqrt{1-(3x)^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

Using these substitutions and the following reference triangle, the integral is evaluated as follows.




$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-(3x)^2}}dx &= \frac{1}{27}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\,d\theta && \text{Trigonometric substitution}\\
&= \frac{1}{27}\int \sin^2\theta\,d\theta && \text{Simplify.}\\
&= \frac{1}{54}\int (1-\cos 2\theta)\,d\theta && \sin^2\theta = \frac{1-\cos 2\theta}{2}\\
&= \frac{1}{54}(\theta - \frac{1}{2}\sin 2\theta) + C && \text{Evaluate.}\\
&= \frac{1}{54}(\theta - \sin\theta\cos\theta) + C && \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta\\
&= \frac{1}{54}\left(\sin^{-1}3x - 3x\sqrt{1-9x^2}\right) + C && \text{Reference triangle}
\end{aligned}$$

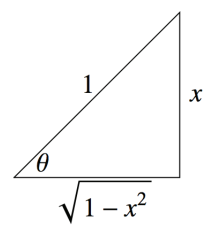
**8.6.68** We begin by making a $u$-substitution and then we complete the square in the denominator.

$$\begin{aligned}
\int \frac{e^x}{e^{2x}+2e^x+17}\,dx &= \int \frac{du}{u^2+2u+17} && u = e^x,\ du = e^x dx\\
&= \int \frac{du}{(u+1)^2+16} && \text{Complete the square.}\\
&= \frac{1}{4}\tan^{-1}\frac{u+1}{4} + C && \text{Evaluate.}\\
&= \frac{1}{4}\tan^{-1}\frac{e^x+1}{4} + C && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

**8.6.69** The presence of the expression $1-x^2$ in the integrand suggests a trigonometric substitution of $x = \sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. For this choice, $dx = \cos\theta\,d\theta$, $1-x^2 = 1-\sin^2\theta = \cos^2\theta$ and $\sqrt{1-x^2} = |\cos\theta| = \cos\theta$. This trigonometric substitution gets the integration process started.

$$\begin{aligned}
\int \frac{dx}{1-x^2+\sqrt{1-x^2}} &= \int \frac{\cos\theta}{\cos^2\theta+\cos\theta}d\theta && \text{Trigonometric substitution}\\
&= \int \frac{d\theta}{\cos\theta+1} && \text{Simplify.}\\
&= \int \frac{1}{\cos\theta+1}\cdot\frac{1-\cos\theta}{1-\cos\theta}\,d\theta && \text{Multiply by } \frac{1-\cos\theta}{1-\cos\theta}.\\
&= \int \frac{1-\cos\theta}{\sin^2\theta}d\theta && \text{Simplify.}\\
&= \int (\csc^2\theta-\csc\theta\cot\theta)d\theta && \frac{1-\cos\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}=\csc^2\theta-\csc\theta\cot\theta\\
&= -\cot\theta+\csc\theta+C && \text{Evaluate.}
\end{aligned}$$

Given that $\sin\theta = x$, a geometric description of the relationship between $x$ and $\theta$ is given by the following reference triangle.




Therefore

$$\int \frac{dx}{1-x^2+\sqrt{1-x^2}} = -\cot\theta+\csc\theta = -\frac{\sqrt{1-x^2}}{x}+\frac{1}{x}+C = \frac{1-\sqrt{1-x^2}}{x}+C.$$

**8.6.70** We use integration by parts with the following choices for the variables.

$$\begin{aligned}
u &= \ln(x^2+a^2) & dv &= dx\\
du &= \tfrac{2x}{x^2+a^2}dx & v &= x
\end{aligned}$$

Applying integration by parts, we have

$$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2\int \frac{x^2}{x^2+a^2}dx.$$

Using long division, it can be shown that $\dfrac{x^2}{x^2+a^2} = 1-\dfrac{a^2}{x^2+a^2}$. Therefore

$$\begin{aligned}
\int \ln(x^2+a^2)\,dx &= x\ln(x^2+a^2) - 2\int\left(1-\frac{a^2}{x^2+a^2}\right)dx\\
&= x\ln(x^2+a^2) - 2\int dx + 2a^2\int \frac{dx}{x^2+a^2}\\
&= x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C.
\end{aligned}$$

**8.6.71** We start by multiplying the numerator and the denominator of the integrand by $(1-\cos x)$:

$$\begin{aligned}
\frac{1-\cos x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} &= \frac{1-2\cos x+\cos^2 x}{1-\cos^2 x} && \text{Multiply by } \frac{1-\cos x}{1-\cos x}.\\
&= \frac{1-2\cos x+\cos^2 x}{\sin^2 x} && \text{Trigonometric Identity.}\\
&= \csc^2 x - 2\cot x\csc x + \cot^2 x && \text{Simplify.}\\
&= \csc^2 x - 2\cot x\csc x + \csc^2 x - 1 && \text{Replace } \cot^2 x \text{ with } \csc^2 x - 1.\\
&= 2\csc^2 x - 2\cot x\csc x - 1 && \text{Simplify.}
\end{aligned}$$

With the transformed integrand, the integral is easier to evaluate:

$$\int \frac{1-\cos x}{1+\cos x}\,dx = \int (2\csc^2 x - 2\cot x\csc x - 1)dx = -2\cot x + 2\csc x - x + C.$$

**8.6.72** We use integration by parts with the following choices for the variables.

$$\begin{aligned}
u &= x^2 & dv &= \sinh x\,dx\\
du &= 2x\,dx & v &= \cosh x
\end{aligned}$$

$$\int x^2\sinh x\,dx = x^2\cosh x - \int 2x\cosh x\,dx$$

We apply integration by parts again with the following choices for the variables.

$$\begin{aligned}
u &= 2x & dv &= \cosh x\,dx\\
du &= 2\,dx & v &= \sinh x
\end{aligned}$$

$$\begin{aligned}
\int x^2\sinh x\,dx &= x^2\cosh x - \int 2x\cosh x\,dx\\
&= x^2\cosh x - 2x\sinh x + \int 2\sinh x\,dx + C\\
&= x^2\cosh x - 2x\sinh x + 2\cosh x + C.
\end{aligned}$$

**8.6.73** Letting $u = 1+\sqrt{x}$ it follows that $du = \frac{dx}{2\sqrt{x}}$. Rearranging these equations, we have $\sqrt{x} = u-1$ and $dx = 2\sqrt{x}\,du = 2(u-1)du$. After the substitution is made, the lower limit of integration is $u = 4$ and the upper limit is $u = 5$. Therefore

$$\begin{aligned}
\int_9^{16}\sqrt{1+\sqrt{x}}\,dx &= 2\int_4^5 \sqrt{u}(u-1)du\\
&= 2\int_4^5 (u^{3/2} - u^{1/2})du\\
&= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\bigg|_4^5\\
&= 4u^{3/2}\left(\frac{u}{5} - \frac{1}{3}\right)\bigg|_4^5\\
&= \frac{4u^{3/2}}{15}(3u-5)\bigg|_4^5\\
&= \frac{40\sqrt{5}}{3} - \frac{224}{15}.
\end{aligned}$$

**8.6.74** If $u = e^x - 1$, then $du = e^x dx$, $e^x = u + 1$, and $e^{2x} = (u+1)^2$. Therefore

$$\begin{aligned}
\int \frac{e^{3x}}{e^x - 1}\,dx &= \int \frac{e^{2x}e^x}{e^x - 1}\,dx \\
&= \int \frac{(u+1)^2}{u} du \\
&= \int \left(u + 2 + \frac{1}{u}\right) du \\
&= \frac{u^2}{2} + 2u + \ln|u| + C \\
&= \frac{(e^x - 1)^2}{2} + 2(e^x - 1) + \ln|e^x - 1| + C \\
&= \frac{e^{2x}}{2} - e^x + \frac{1}{2} + 2e^x - 2 + \ln|e^x - 1| + C \\
&= \frac{e^{2x}}{2} + e^x + \ln|e^x - 1| + C.
\end{aligned}$$

**8.6.75** Letting $u = \tan^{-1}\sqrt{x}$, it follows that $du = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} dx = \frac{dx}{2(\sqrt{x} + x^{3/2})}$. Then $2du = \frac{dx}{\sqrt{x} + x^{3/2}}$. The lower limit of integration is $u = \tan^{-1} 1 = \frac{\pi}{4}$ and the upper limit is $u = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. Therefore

$$\int_1^3 \frac{\tan^{-1}\sqrt{x}}{x^{1/2} + x^{3/2}}\,dx = \int_{\pi/4}^{\pi/3} 2u\,du = u^2\Big|_{\pi/4}^{\pi/3} = \frac{7\pi^2}{144}.$$

**8.6.76** By completing the square of the denominator of the integrand and then making a $u$-substitution we have

$$\begin{aligned}
\int \frac{x}{x^2 + 6x + 18}\,dx &= \int \frac{x}{(x+3)^2 + 9}\,dx && \text{Complete the square.} \\
&= \int \frac{u - 3}{u^2 + 9}\,du && u = x + 3,\ du = dx,\ x = u - 3 \\
&= \int \frac{u}{u^2 + 9}\,du - \int \frac{3}{u^2 + 9} du && \text{Split into two integrals.}
\end{aligned}$$

The second integral equals $\tan^{-1}\frac{u}{3}$; the first integral is evaluated by making a substitution of $w = u^2 + 9$, which implies that $dw = 2u\,du$ or $\frac{1}{2}dw = u\,du$:

$$\int \frac{u}{u^2 + 9} du = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(u^2 + 9) + C.$$

Therefore,

$$\int \frac{x}{x^2 + 6x + 18}\,dx = \frac{1}{2}\ln(u^2 + 9) - \tan^{-1}\frac{u}{3} + C = \frac{1}{2}\ln\left((x+3)^2 + 9\right) - \tan^{-1}\frac{x+3}{3} + C.$$

**8.6.77** We use integration by parts with the following choices for the variables:

$$u = \cos^{-1} x \qquad dv = dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \qquad v = x$$

Now we apply integration by parts followed by integration by substitution.

$$\begin{aligned}
\int \cos^{-1} x\, dx &= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx && \text{Integration by parts} \\
&= x\cos^{-1} x - \frac{1}{2}\int u^{-1/2} du && u = 1-x^2;\ du = -2x\,dx \\
&= x\cos^{-1} x - u^{1/2} + C && \text{Integrate.} \\
&= x\cos^{-1} x - \sqrt{1-x^2} + C. && \text{Replace } u \text{ with } 1-x^2.
\end{aligned}$$

**8.6.78** Start with integration by parts, using the following variable choices.

$$u = \left(\cos^{-1} x\right)^2 \qquad dv = dx$$

$$du = -\frac{2\cos^{-1} x}{\sqrt{1-x^2}} dx \qquad v = x$$

It follows that

$$\int (\cos^{-1} x)^2\, dx = x(\cos^{-1} x)^2 + 2\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}} dx.$$

Apply integration by parts again to the integral on the right using the following choices for the variables.

$$u = \cos^{-1} x \qquad dv = \frac{x}{\sqrt{1-x^2}} dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \qquad v = -\sqrt{1-x^2}$$

The second integral becomes

$$\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2}\cos^{-1} x - \int dx = -\sqrt{1-x^2}\cos^{-1} x - x + C.$$

Assembling our work, we have

$$\begin{aligned}
\int (\cos^{-1} x)^2\, dx &= x(\cos^{-1} x)^2 + 2\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}} dx \\
&= x(\cos^{-1} x)^2 + 2\left(-\sqrt{1-x^2}\cos^{-1} x - x\right) + C \\
&= x(\cos^{-1} x)^2 - 2\sqrt{1-x^2}\cos^{-1} x - 2x + C.
\end{aligned}$$

**8.6.79** We apply integration by parts with the following choices for the variables.

$$u = \sin^{-1} x \qquad dv = \frac{dx}{x^2}$$

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = -\frac{1}{x}$$

Using these substitutions with integration by parts, along with the answer to Exercise 25, we evaluate the integral.

$$\int \frac{\sin^{-1} x}{x^2}\,dx = -\frac{\sin^{-1} x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} = -\frac{\sin^{-1} x}{x} + \ln\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C.$$

**8.6.80** Start by completing the square of $-4x - x^2$ and then make a $u$-substitution of $u = x+2$. The lower limit of integration becomes 0 and the upper limit becomes 1.

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_{-2}^{-1} \sqrt{4-(x+2)^2}\,dx = \int_0^1 \sqrt{4-u^2}\,du$$

Using the trigonometric substitution $u = 2\sin\theta$, it follows that $du = 2\cos\theta\,d\theta$. With this substitution, $u = 0$ implies that $\theta = 0$ and $u = 1$ implies $\theta = \frac{\pi}{6}$. So we have

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_0^1 \sqrt{4-u^2}\,du = \int_0^{\pi/6} 4\cos^2\theta\,d\theta.$$

We evaluate the integral with the help of the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

$$\int_{-2}^{-1} \sqrt{-4x-x^2}\,dx = \int_0^{\pi/6} 2(1+\cos 2\theta)\,d\theta = \big(2\theta + \sin 2\theta\big)\Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

**8.6.81** Factoring the denominator, we have

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2+1)^2.$$

So the partial fraction decomposition of the integrand has the form

$$\frac{x^4 + 2x^3 + 5x^2 + 2x + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

To find the constants, we expand the right hand side of equation and group equal powers of $x$:

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.$$

Comparing coefficients of equal powers, we see that $A = 1$, $B = 0$, $C = 2$, $D = 3$, and $E = 0$. Using these values, the integral is evaluated with the help of a $u$-substitution in the third integral.

$$\begin{aligned}
\int \frac{x^4+2x^3+5x^2+2x+1}{x(x^2+1)^2}dx &= \int \frac{dx}{x} + \int \frac{2}{x^2+1}dx + \int \frac{3x}{(x^2+1)^2}dx && \text{Partial fractions.}\\
&= \int \frac{dx}{x} + \int \frac{2}{x^2+1}dx + \frac{3}{2}\int \frac{du}{u^2} && u = x^2+1,\ \frac{1}{2}du = x\,dx\\
&= \ln|x| + 2\tan^{-1}x - \frac{3}{2u} + C && \text{Evaluate.}\\
&= \ln|x| + 2\tan^{-1}x - \frac{3}{2(x^2+1)} + C. && \text{Replace } u \text{ with } x^2+1.
\end{aligned}$$

**8.6.82** Letting $u = \tan x$ it follows that $du = \sec^2 x\,dx$, and

$$dx = \frac{du}{\sec^2 x} = \frac{du}{\tan^2 x + 1} = \frac{du}{u^2+1}.$$

With this change of variable,

$$\int \frac{dx}{1+\tan x} = \int \frac{du}{(u+1)(u^2+1)}.$$

Because the denominator on the right side of the equation contains a linear term and an irreducible quadratic factor, the form of the partial fraction decomposition is

$$\frac{1}{(u+1)(u^2+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

Multiplying by $(u+1)(u^2+1)$ leads to

$$1 = A(u^2+1) + (Bu+C)(u+1) = (A+B)u^2 + (B+C)u + A + C.$$

Comparing coefficients of equal powers, we obtain three equations with three unknowns:

$$A+B=0, B+C=0, \text{ and } A+C=1.$$

Solving this system gives

$$A = \frac{1}{2}, B = -\frac{1}{2}, \text{ and } C = \frac{1}{2}.$$

Substituting these values into the partial fraction decomposition, the integral can be evaluated as follows:

$$\begin{aligned}
\int \frac{dx}{1+\tan x} &= \int \frac{du}{(u+1)(u^2+1)} && u\text{-substitution}\\
&= \frac{1}{2}\int \frac{du}{u+1} - \frac{1}{2}\int \frac{u}{u^2+1}du + \frac{1}{2}\int \frac{du}{u^2+1} && \text{Partial fractions}\\
&= \frac{1}{2}\ln|u+1| - \frac{1}{4}\ln(u^2+1) + \frac{1}{2}\tan^{-1}u + C && \text{Integrate.}\\
&= \frac{1}{2}\ln|\tan x+1| - \frac{1}{4}\ln(\tan^2 x+1) + \frac{1}{2}\tan^{-1}\big(\tan x\big) + C && \text{Replace } u \text{ with } \tan x.\\
&= \frac{1}{2}\ln|\tan x+1| - \frac{1}{4}\ln(\sec^2 x) + \frac{x}{2} + C. && 1+\tan^2 x = \sec^2 x
\end{aligned}$$

**8.6.83** We simplify the integrand first using the substitution $u = e^x$, so that $du = e^x dx$ and

$$\int e^x \sin^{998}(e^x)\cos^3(e^x)\,dx = \int \sin^{998} u \cos^3 u\,du.$$

We then continue the integration process, by first splitting $\cos^3 u$.

$$\begin{aligned}
\int e^x \sin^{998}(e^x)\cos^3(e^x)\,dx &= \int \sin^{998} u \cos^3 u\,du && u = e^x,\ du = e^x dx\\
&= \int \sin^{998} u \cos^2 u \cos u\,du && \text{Split } \cos^3 u.\\
&= \int \sin^{998} u(1-\sin^2 u)\cos u\,du && \cos^2 x = 1-\sin^2 x\\
&= \int (\sin^{998} u - \sin^{1000} u)\cos u\,du && \text{Expand.}\\
&= \int (w^{998} - w^{1000})\,dw && w = \sin u,\ dw = \cos u\,du\\
&= \frac{w^{999}}{999} - \frac{w^{1001}}{1000} + C && \text{Integrate.}\\
&= \frac{\sin^{999} u}{999} - \frac{\sin^{1001} u}{1001} + C && \text{Replace } w \text{ with } \sin u.\\
&= \frac{\sin^{999}(e^x)}{999} - \frac{\sin^{1001}(e^x)}{1001} + C && \text{Replace } u \text{ with } e^x.
\end{aligned}$$

**8.6.84** Let $u = \tan\theta$ which implies that $du = \sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{\tan\theta + \tan^3\theta}{(1+\tan\theta)^{50}}\,d\theta &= \int \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan\theta)^{50}}\,d\theta\\
&= \int \frac{\tan\theta\sec^2\theta}{(1+\tan\theta)^{50}}\,d\theta = \int \frac{u}{(1+u)^{50}}\,du.
\end{aligned}$$

Now let $w = u+1$ so that $dw = du$ and $u = w-1$. We then have

$$\begin{aligned}
\int \frac{w-1}{w^{50}}\,dw &= \int \left(w^{-49} - w^{-50}\right)\,dw\\
&= -\frac{w^{-48}}{48} + \frac{w^{-49}}{49} + C = \frac{(u+1)^{-49}}{49} - \frac{(u+1)^{-48}}{48} + C\\
&= \frac{1}{49(\tan\theta+1)^{49}} - \frac{1}{48(\tan\theta+1)^{48}} + C.
\end{aligned}$$

**8.6.85**

a True. For example, the method of partial fractions or a trigonometric substitution can be used to evaluate the integral.

b. True. See Example 4.

c. False. It is easiest to use the substitution $u = \tan 3x$.

d. False. The integral $\int \frac{u^2}{u-1}\,du$ would imply that $du = dx$, which is not the case here.