Gateway Exam Review

Monday, October 18, 2021 10:54 AM

Q1. Apply the Power Rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

You may need some algebra

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

1.

$$\int\left(3x^5 - x^2 + \sqrt{3} - 4x^{-3}\right)dx = 3\int x^5\,dx - \int x^2\,dx - \sqrt{3}\int 1\,dx - 4\int x^{-3}\,dx$$

$$= 3\cdot\frac{x^6}{6} - \frac{x^3}{3} - \sqrt{3}x - 4\cdot\frac{x^{-2}}{-2} + C$$

1.

$$\int\left(x^{100} + \frac{\overset{x^{\frac{1}{2}}}{\sqrt{x}} - x^{3/2}}{x}\right)dx$$

$$= \int x^{100}\,dx + \int x^{-\frac{1}{2}}\,dx - \int x^{\frac{1}{2}}\,dx$$

Q2. substitution method.

2.

$$\int (x^5)\left(\sqrt[3]{x^6+1}\right)dx = \int x^5\,(x^6+1)^{\frac{1}{3}}\,dx$$

$u = x^6 + 1,\quad du = 6x^5\,dx$

$\Rightarrow\ x^5\,dx = \frac{1}{6}\,du$

2.
$$\int \tan x \; sec^3 x \, dx$$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \sec x \tan x$

$u = \sec x, \quad du = \sec x \tan x \, dx$

$$= \int \sec^2 x \; \sec x \tan x \, dx$$

Q3. Integration by parts

$$\int u v' \, dx$$

diff		int
u		v'
u'	$\xrightarrow[(-)]{}$	v

3.
$$\int x e^x dx$$

diff		int
x		e^x
1	$\xrightarrow[(-)]{}$	e^x

$$= xe^x - \int e^x \, dx$$

$$= xe^x - e^x + C$$

Q4. partial fractions $\dfrac{P(x)}{Q(x)}$

with $Q(x)$ linear factors only

4.

$$\int \frac{5x-2}{(x+4)(3x+1)}dx$$

4.

$$\int \frac{100}{x^2-1}dx = \int \frac{100}{(x-1)(x+1)}dx$$

4.

$$\int \frac{5x-2}{(x+4)(3x+1)}dx$$

Sol. $\frac{5x-2}{(x+4)(3x+1)} = \frac{A}{x+4} + \frac{B}{3x+1}$

$$= \frac{A(3x+1) + B(x+4)}{(x+4)(3x+1)}$$

$$5x-2 = A(3x+1) + B(x+4)$$

let $x=-4 \Rightarrow -22 = -11A \Rightarrow A = 2$

let $x = -\frac{1}{3} \Rightarrow \frac{-5}{3} - 2 = B(-\frac{1}{3}+4)$

$$\Rightarrow -\frac{11}{3} = B \cdot \frac{11}{3} \Rightarrow B = -1$$

Method 2: check posted solution.

$$\Rightarrow \int \frac{5x-2}{(x+4)(3x+1)}\,dx = 2\int \frac{dx}{x+4} - \int \frac{dx}{3x+1}$$

$u = 3x+1$
$dx = \frac{1}{3}du$

$$= 2\ln|x+4| - \frac{1}{3}\ln|3x+1| + C$$

Q5 Trigonometric substitution, sometimes you can use $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$

5.

$$\int \frac{1}{\sqrt{49-x^2}}\,dx = \frac{1}{\sqrt{49}}\int \frac{dx}{\sqrt{1-\left(\frac{x}{7}\right)^2}}$$

$u = \frac{x}{7}$

Method 2. $x = 7\sin\theta$.

5.

$$\int \frac{10}{\sqrt{10-100x^2}}\,dx = \frac{10}{\sqrt{10}}\int \frac{dx}{\sqrt{1-(\sqrt{10}\,x)^2}}$$

$u = \sqrt{10}\,x$

Q6. partial fractions $\frac{P(x)}{Q(x)}$

With $Q(x)$ has a linear term and quadratic term.

6.

$$\int \frac{6x^2+2x+8}{(x^2+2)(x+1)}dx$$

풀이. $\dfrac{6x^2+2x+8}{(x^2+2)(x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2}$

$$= \frac{A(x^2+2)+(Bx+C)(x+1)}{(x+1)(x^2+2)}$$

let $x=-1$, $6-2+8 = 3A \Rightarrow 12 = 3A \Rightarrow A=4$

let $x=0 \Rightarrow 8 = 2A + C \Rightarrow C=0$

$ 8$

let $x=1 \Rightarrow 16 = 3A + 2B \Rightarrow 2B = 4 \Rightarrow B=2$

$ 12$

$$\Rightarrow \int \frac{6x^2+2x+8}{(x^2+2)(x+1)}dx = 4\int\frac{dx}{x+1} + \int \frac{2x}{x^2+2}dx$$

$$= 4\ln|x+1| + \ln|x^2+2| + C$$

$u = x^2+2$

$du = 2x\,dx$

Remark. $\displaystyle\int \frac{2}{x^2+2}dx$ $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

$$= 2\cdot\frac{1}{2}\int\frac{dx}{1+\left(\frac{x}{\sqrt{2}}\right)^2}$$

$x =$

$u = \frac{x}{\sqrt{2}}$

$dx = \sqrt{2}\,du$

$\sqrt{2} \int \frac{du}{1+u^2}$

Q7. Definite Integral, with substitution

7.

$$\int_0^1 x^2 + 7x - 3\,dx$$

7.

$$\int_0^1 \frac{6x}{(x^2+1)^5}\,dx$$

Q8. Integration by Parts twice or more

8.

$$\int x^2 \cos x\;dx$$

$$\int x^2 e^x\,dx$$

8.

$$\int x^4 \sin x\,dx$$

$$\left\{\begin{array}{l}\int e^x \sin x\,dx \\ \int e^x \cos x\,dx\end{array}\right.$$

"loop"

8.

$$\int x^4 \sin x \, dx$$

diff	Int
x^4	$\sin x$
$4x^3$	$-\cos x$
$12x^2$	$-\sin x$
$24x$	$\cos x$
24	$\sin x$

(−) (−) (+)

$\int \sin x \, dx = -\cos x$

$$= -x^4\cos x + 4x^3 \sin x + 12x^2 \cos x - 24x\sin x + 24(-\cos x) + C$$

Q9. Q10 Improper Integrals

① Find the value if it converges

② Justify it if it diverges

9.

$$\int_1^{\infty} \frac{1}{\sqrt{x+2}} dx$$

$$\int \frac{1}{\sqrt{x+2}} dx = \int (x+2)^{-\frac{1}{2}} dx$$

$$= \frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x+2} + C$$

Thus, $\int_1^{\infty} \frac{1}{\sqrt{x+2}} dx$

$= \lim_{t \to \infty} \int_1^t \frac{1}{\sqrt{x+2}} dx$

$= \lim_{t \to \infty} 2\sqrt{x+2} \Big|_1^t = 2\left[\lim_{t \to \infty} \sqrt{t+2} - \sqrt{3}\right]$

$= \infty$ divergent

9.
$$\int_1^{\infty} \frac{2}{(x+1)^{3/2}} dx$$ ← converges

10.
$$\int_3^7 \frac{1}{x-3} dx$$

10.
$$\int_1^2 \frac{1}{(x-1)^{1/2}} dx$$