8.3 Trigonometric Integrals

8.3.1 The half-angle identities for sine and cosine:

$$\sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2},$$

which are conjugates.

8.3.2 The Pythagorean Identities:

$$\cos^2 x + \sin^2 x = 1 \text{ and } 1+\tan^2 x = \sec^2 x \text{ and } \cot^2 x + 1 = \csc^2 x,$$

where the second follows from the first by dividing through by $\cos^2 x$, and the third follows from the first by dividing through by $\sin^2 x$.

8.3.3 To integrate $\sin^3 x$, write $\sin^3 x = \sin x \sin^2 x = \sin x(1-\cos^2 x)$, and let $u = \cos x$ so that $du = -\sin x\,dx$.

8.3.4 To integrate $\sin^m x \cos^n x$ where m is even and n is odd, write $\sin^m x\cos^n x = \sin^m x \cos^{n-1} x \cos x = \sin^m x(1-\sin^2 x)^{(n-1)/2}\cos x$ and let $u = \sin x$ so that $du = \cos x\,dx$.

8.3.5 A reduction formula is a recursive formula involving integrals. Using it, one can rewrite an integral of a certain type in a simpler form – which can then perhaps be evaluated or further reduced.

8.3.6 One would compute this integral by writing $\cos^2 x \sin^3 x$ as $\cos^2 x(\sin^2 x)\sin x = \cos^2 x(1-\cos^2 x)\sin x$ and then performing the ordinary substitution $u = \cos x$.

8.3.7 One would compute this integral by letting $u = \tan x$, so that $du = \sec^2 x\,dx$. This substitution leads to the integral $\int u^{10}\,du$, which can easily be evaluated.

8.3.8 One would compute this integral by letting $u = \sec x$, so that $du = \sec x \tan x\,dx$. This substitution leads to the integral $\int u^{11}\,du$, which can easily be evaluated.

8.3.9 $\int \cos^3 x\,dx = \int \cos x(1-\sin^2 x)\,dx$Let $u = \sin x$. Then $du = \cos x\,dx$. Substituting gives

$$\int (1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

8.3.10 $\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx$.. Let $u = \cos x$. Then $du = -\sin x\,dx$. Substituting gives

$$\int (u^2-1)\,du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

8.3.11 $\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C = \frac{x}{2} - \frac{1}{12}\sin 6x + C.$

8.3.12

$$\begin{aligned}\int \cos^4 2\theta\,d\theta &= \int \left(\frac{1+\cos 4\theta}{2}\right)^2 d\theta = \int \frac{1}{4}(1+2\cos 4\theta + \cos^2 4\theta)\,d\theta \\ &= \frac{1}{4}\int \left(1 + 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right) d\theta = \frac{1}{4}\int\left(\frac{3}{2} + 2\cos 4\theta + \frac{1}{2}\cos 8\theta\right) d\theta \\ &= \frac{1}{4}\left(\frac{3\theta}{2} + \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right) + C = \frac{1}{8}\left(3\theta + \sin 4\theta + \frac{1}{8}\sin 8\theta\right) + C.\end{aligned}$$

8.3.13 $\int \sin^5 x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$-\int (1-u^2)^2\,du = \int (-u^4+2u^2-1)\,du = \frac{-u^5}{5} + \frac{2u^3}{3} - u + C = \frac{-\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C.$$

8.3.14 $\int \cos^3 20x\,dx = \int \cos^2 20x \cos 20x\,dx = \int (1-\sin^2 20x)\cos 20x\,dx$. Let $u = \sin 20x$, so that $du = 20\cos 20x\,dx$. Substituting yields

$$\frac{1}{20}\int (1-u^2)\,du = \frac{1}{20}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{20}\left(\sin 20x - \frac{\sin^3 20x}{3}\right) + C.$$

8.3.15 $\int \sin^3 x \cos^2 x\,dx = \int \sin x(1-\cos^2 x)(\cos^2 x)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting gives

$$\int (u^4-u^2)\,du = u^5/5 - u^3/3 + C = (\cos^5 x)/5 - (\cos^3 x)/3 + C.$$

8.3.16 $\int \sin^2\theta \cos^5\theta\,d\theta = \int (\sin^2\theta)(1-\sin^2\theta)^2\cos\theta\,d\theta$. Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Substituting gives

$$\int u^2(1-u^2)^2\,du = \int (u^2-2u^4+u^6)\,du = u^3/3 - 2u^5/5 + u^7/7 + C = (\sin^3\theta)/3 - 2(\sin^5\theta)/5 + (\sin^7\theta)/7 + C.$$

8.3.17 $\int \cos^3 x\sqrt{\sin x}\,dx = \int \cos x(1-\sin^2 x)\sqrt{\sin x}\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Substituting gives

$$\int (1-u^2)u^{1/2}\,du = \int (u^{1/2}-u^{5/2})\,du = 2u^{3/2}/3 - 2u^{7/2}/7 + C = 2(\sin^{3/2} x)/3 - 2(\sin^{7/2} x)/7 + C.$$

8.3.18 $\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \int (\sin\theta)\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)\,d\theta$. Let $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. Substituting gives

$$\int \frac{u^2-1}{u^2}\,du = \int (1-u^{-2})\,du = u + \frac{1}{u} + C = \cos\theta + \sec\theta + C.$$

8.3.19 $\int_0^{\pi/3} \sin^5 x \cos^{-2} x\,dx = \int_0^{\pi/3} (\sin x)\left(\frac{(1-\cos^2 x)^2}{\cos^2 x}\right)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$\begin{aligned}-\int_1^{1/2} \frac{(1-u^2)^2}{u^2}\,du &= \int_1^{1/2} (-u^{-2}+2-u^2)\,du = \left(\frac{1}{u} + 2u - \frac{u^3}{3}\right)\Bigg|_1^{1/2} \\ &= \left(2+1-\frac{1}{24} - \left(1+2-\frac{1}{3}\right)\right) = \frac{7}{24}.\end{aligned}$$

8.3.20 $\int \sin^{-3/2} x \cos^3 x\,dx = \int \sin^{-3/2}\cos^2 x \cos x\,dx = \int (\sin^{-3/2} x)(1-\sin^2 x)\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Then a substitution yields

$$\int u^{-3/2}(1-u^2)\,du = \int u^{-3/2} - u^{1/2}\,du = \frac{-2}{u^{1/2}} - \frac{2u^{3/2}}{3} + C = \frac{-2}{\sqrt{\sin x}} - \frac{2\sin^{3/2} x}{3} + C.$$

8.3.21 $\int_0^{\pi/2} \cos^3 x\sqrt{\sin^3 x}\,dx = \int_0^{\pi/2}(1-\sin^2 x)\sin^{3/2} x\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. Substituting gives

$$\int_0^1 (1-u^2)u^{3/2}\,du = \int_0^1 (u^{3/2}-u^{7/2})\,du = \left(\frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2}\right)\Big|_0^1 = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}.$$

8.3.22 $\int_{\pi/4}^{\pi/2} \sin^2 2x\cos^3 2x\,dx = \int_{\pi/4}^{\pi/2} \sin^2 2x\cos^2 2x\cos 2x\,dx = \int_{\pi/4}^{\pi/2} \sin^2 2x(1-\sin^2 2x)\cos 2x\,dx$. Let $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. Substituting gives

$$\frac{1}{2}\int_1^0 (u^2-u^4)\,du = \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_1^0 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{1}{15}.$$

8.3.23

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int 1-\cos^2 2x\,dx \\ &= \frac{1}{4}\int\left(1 - \frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C.\end{aligned}$$

8.3.24 $\int \sin^3 x\cos^5 x\,dx = \int \sin^2 x\cos^5 x\sin x\,dx = \int (1-\cos^2 x)\cos^5 x\sin x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting gives

$$-\int (u^5-u^7)\,du = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C.$$

8.3.25 $\int \sin^2 x\cos^4 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int(1-\cos^2 2x)(1+\cos 2x)\,dx = \frac{1}{8}\int 1 + \cos 2x - \cos^2 2x - \cos^3 2x\,dx = \frac{1}{8}\int 1+\cos 2x - \frac{1+\cos 4x}{2} - \cos^3 2x\,dx = \frac{1}{8}\int \frac{1}{2} + \cos 2x - \frac{1}{2}\cos 4x\,dx - \frac{1}{8}\int(1-\sin^2 2x)\cos 2x\,dx = \frac{x}{16} + \frac{\sin 2x}{16} - \frac{\sin 4x}{64} - \frac{1}{8}\int(1-\sin^2 2x)\cos 2x\,dx$. To compute this last integral, we let $u = \sin 2x$, so that $du = 2\cos 2x\,dx$. Then

$$\int(1-\sin^2 2x)\cos 2x\,dx = \frac{1}{2}\int(1-u^2)\,du = \frac{1}{2}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C.$$

Thus, our original given integral is equal to

$$\frac{x}{16} + \frac{\sin 2x}{16} - \frac{\sin 4x}{64} - \frac{1}{8}\left(\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)\right) + C = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

8.3.26 $\int \sin^3 x\cos^{3/2} x\,dx = \int(\sin x)(1-\cos^2 x)\cos^{3/2} x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Then substituting yields

$$-\int(1-u^2)u^{3/2}\,du = \int u^{7/2} - u^{3/2}\,du = \frac{2u^{9/2}}{9} - \frac{2u^{5/2}}{5} + C = \frac{2\cos^{9/2} x}{9} - \frac{2\cos^{5/2} x}{5} + C.$$

8.3.27 $\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$.

8.3.28 $\int 6\sec^4 x\,dx = \int 6\sec^2 x(\tan^2 x + 1)\,dx$. Let $u = \tan x$, so that $du = \sec^2 x\,dx$. Then we have

$$6\int(u^2+1)\,du = 2u^3 + 6u + C = 2\tan^3 x + 6\tan x + C.$$

8.3.29 $\int \cot^4 x\,dx = \int \cot^2 x(\csc^2 x - 1)\,dx = \int (\cot^2 x \csc^2 x - (\csc^2 x - 1))\,dx = \int \cot^2 x \csc^2 x\,dx +$ $\cot x + x$. Let $u = \cot x$ so that $du = -\csc^2 x\,dx$. Substituting gives

$$-\int u^2\,du + \cot x + x = -u^3/3 + \cot x + x + C = -(\cot^3 x)/3 + \cot x + x + C.$$

8.3.30 $\int \tan^3\theta\,d\theta = \int \tan\theta(\sec^2\theta - 1)\,d\theta = \int (\tan\theta\sec^2\theta - \tan\theta)\,d\theta = \int \tan\theta\sec^2\theta\,d\theta + \ln|\cos\theta|$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Substituting gives

$$\int u\,du + \ln|\cos\theta| = u^2/2 + \ln|\cos\theta| + C = (\tan^2\theta)/2 + \ln|\cos\theta| + C.$$

Note that this can also be written as $\sec^2\theta/2 + \ln|\cos\theta| + C$, because $\sec^2\theta$ and $\tan^2\theta$ differ by a constant.

8.3.31

$$\begin{aligned}\int 20\tan^6 x\,dx &= 20\int (\tan^4 x)(\sec^2 x - 1)\,dx = 20\int ((\tan^4 x)\sec^2 x - (\tan^2 x)(\sec^2 x - 1))\,dx\\ &= 20\int (\tan^4 x\sec^2 x - \tan^2 x\sec^2 x + \sec^2 x - 1)\,dx.\end{aligned}$$

Let $u = \tan x$ so that $du = \sec^2 x\,dx$. We have

$$\begin{aligned}20\left(\int (u^4 - u^2)\,du + \tan x - x\right) + C &= 4u^5 - \frac{20u^3}{3} + 20\tan x - 20x + C\\ &= 4\tan^5 x - \frac{20\tan^3 x}{3} + 20\tan x - 20x + C.\end{aligned}$$

8.3.32

$$\begin{aligned}\int \cot^5 3x\,dx &= \int (\cot^3 3x)(\csc^2 3x - 1)\,dx = \int (\cot^3 3x\csc^2 3x - (\cot 3x)(\csc^2 3x - 1))\,dx\\ &= \cot^3 3x\csc^2 3x - \cot 3x\csc^2 3x\,dx + \frac{\ln|\sin 3x|}{3} + C.\end{aligned}$$

Now let $u = \cot 3x$, so that $du = -3\csc^2 3x\,dx$. Substituting gives

$$\begin{aligned}-\frac{1}{3}\int (u^3 - u)\,du + \frac{\ln|\sin 3x|}{3} + C &= -\frac{u^4}{12} + \frac{u^2}{6} + \frac{\ln|\sin 3x|}{3} + C\\ &= -\frac{\cot^4 3x}{12} + \frac{\cot^2 3x}{6} + \frac{\ln|\sin 3x|}{3} + C.\end{aligned}$$

8.3.33 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives $\int 10u^9\,du = u^{10} + C = \tan^{10} x + C$.

8.3.34 $\int \tan^9 x(\tan^2 x + 1)\sec^2 x\,dx$. Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int u^9(u^2 + 1)\,du = \int (u^{11} + u^9)\,du = u^{12}/12 + u^{10}/10 + C = (\tan^{12} x)/12 + (\tan^{10} x)/10 + C.$$

8.3.35 Let $u = \sec x$ so that $du = \sec x\tan x\,dx$. Substituting gives

$$\int u^2\,du = u^3/3 + C = (\sec^3 x)/3 + C.$$

8.3.36 $\int \tan 4x \sec^{3/2} 4x\, dx = \int \sec^{1/2} 4x \sec 4x \tan 4x\, dx$. Let $u = \sec 4x$. Then $du = 4\sec 4x \tan 4x$. Substituting gives

$$\frac{1}{4}\int u^{1/2}\, du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{6}\sec^{3/2} 4x + C.$$

8.3.37 Let $u = \ln\theta$ so that $du = \frac{1}{\theta}\, d\theta$. Substituting yields $\int \sec^4 u\, du = \int (\sec^2 u)(1+\tan^2 u)\, du$. Let $w = \tan u$ so that $dw = \sec^2 u\, du$. Substituting again gives

$$\int (1+w^2)\, dw = w + \frac{w^3}{3} + C = \tan(\ln(\theta)) + \frac{\tan^3(\ln(\theta))}{3} + C.$$

8.3.38 $\int \tan^5\theta \sec^4\theta\, d\theta = \int (\tan^5\theta)(\tan^2\theta+1)(\sec^2\theta)\, d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\, d\theta$. Substituting yields $\int u^7 + u^5\, du = \frac{u^8}{8} + \frac{u^6}{6} + C = \frac{\tan^8\theta}{8} + \frac{\tan^6\theta}{6} + C$.

8.3.39

$$\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\, d\theta = 2\int_0^{\pi/3} \sqrt{\sec^2\theta - 1}\, d\theta = 2\int_0^{\pi/3} \tan\theta\, d\theta$$
$$= -2\ln|\cos\theta|\,\Big|_0^{\pi/3} = -2\ln(1/2) + 2\ln(1) = 2\ln 2.$$

8.3.40 $\int_0^{\pi/6} \tan^5 2x \sec 2x\, dx = \int_0^{\pi/6} (\tan^2 2x)^2 \tan 2x \sec 2x\, dx = \int_0^{\pi/6} (\sec^2 2x - 1)^2 \tan 2x \sec 2x\, dx$. Let $u = \sec 2x$. Then $du = 2\sec 2x \tan 2x\, dx$. Substituting gives

$$\frac{1}{2}\int_1^2 (u^2-1)^2\, du = \frac{1}{2}\int_1^2 (u^4 - 2u^2 + 1)\, du = \frac{1}{2}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right)\Big|_1^2$$
$$= \frac{1}{2}\left(\frac{32}{5} - \frac{16}{3} + 2 - \left(\frac{1}{5} - \frac{2}{3} + 1\right)\right) = \frac{19}{15}.$$

8.3.41 $\int_0^{\pi/4} \sec^7 x \sin x\, dx = \int_0^{\pi/4} \sec^6 x \tan x\, dx = \int_0^{\pi/4} \sec^5 x \sec x \tan x\, dx$. Let $u = \sec x$ so that $du = \sec x \tan x\, dx$. Substituting gives

$$\int_1^{\sqrt{2}} u^5\, du = \frac{u^6}{6}\Big|_1^{\sqrt{2}} = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.$$

8.3.42 $\int \sqrt{\tan x}\sec^4 x\, dx = \int \sqrt{\tan x}(\tan^2 x + 1)(\sec^2 x)\, dx$. Let $u = \tan x$ so that $du = \sec^2 x\, dx$. Substituting gives

$$\int \sqrt{u}(u^2+1)\, du = \int (u^{5/2} + u^{1/2})\, du = 2u^{7/2}/7 + 2u^{3/2}/3 + C = 2(\tan^{7/2} x)/7 + 2(\tan^{3/2} x)/3 + C.$$

8.3.43

$$\int \tan^3 4x\, dx = \int (\tan 4x)(\sec^2 4x - 1)\, dx = \int (\tan 4x)\sec^2 4x\, dx - \int \tan 4x\, dx$$
$$= \int (\tan 4x)\sec^2 4x\, dx + \frac{\ln|\cos 4x|}{4} + C.$$

Let $u = \tan 4x$ so that $du = 4\sec^2 4x\, dx$. Substituting gives

$$\frac{1}{4}\int u\, du + \frac{\ln|\cos 4x|}{4} + C = \frac{u^2}{8} + \frac{\ln|\cos 4x|}{4} + C = \frac{\tan^2 4x}{8} + \frac{\ln|\cos 4x|}{4} + C.$$

8.3.44 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Substituting gives

$$\int u^{-5}\,du = -\frac{u^{-4}}{4} + C = -\frac{1}{4\tan^4 x} + C.$$

8.3.45 Let $u = \tan x$ so that $du = \sec^2 x\,dx$. Then

$$\int \sec^2 x \tan^{1/2} x\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\tan^{3/2} x + C.$$

8.3.46

$$\int \sec^{-2} x \tan^3 x\,dx = \int \sec^{-2}(\sec^2 x - 1)\tan x\,dx = \int (\tan x - \sec^{-2} x \tan x)\,dx$$
$$= \int \left(\frac{\sin x}{\cos x} - \cos x \sin x\right)\,dx.$$

Let $u = \cos x$ so that $du = -\sin x\,dx$. Then we have

$$\int \left(-\frac{1}{u} + u\right)\,du = -\ln|u| + \frac{u^2}{2} + C = -\ln|\cos x| + \frac{1}{2}\cos^2 x + C.$$

8.3.47 $\displaystyle\int \frac{\csc^4 x}{\cot^2 x}\,dx = \int (\csc^2 x)\left(\frac{\cot^2 x + 1}{\cot^2 x}\right)\,dx$. Let $u = \cot x$ so that $du = -\csc^2 x\,dx$. Substituting gives

$$-\int \frac{u^2+1}{u^2}\,du = \int -1 - u^{-2}\,du = -u + \frac{1}{u} + C = -\cot x + \tan x + C.$$

8.3.48 Let $u = \csc x$ so that $du = -\csc x \cot x\,dx$. Substituting gives

$$-\int u^9\,du = -u^{10}/10 + C = -(\csc^{10} x)/10 + C.$$

8.3.49 Let $u = \cot 5w$. Then $du = -5\csc^2 5w$. Substituting gives

$$-\frac{1}{5}\int_1^0 u^4\,du = -\frac{1}{25}u^5\Big|_1^0 = -\frac{1}{25}(0-1) = \frac{1}{25}.$$

8.3.50 $\displaystyle\int \csc^{10} x \cot^3 x\,dx = \int \csc^9 x \cot^2 x \csc x \cot x\,dx = \int \csc^9 x(\csc^2 x - 1)\csc x \cot x\,dx$. Let $u = \csc x$ so that $du = -\csc x \cot x\,dx$. Substituting gives

$$-\int u^9(u^2-1)\,du = -\int (u^{11} - u^9)\,du = -(u^{12}/12 - u^{10}/10) + C = \frac{-\csc^{12} x}{12} + \frac{\csc^{10} x}{10} + C.$$

8.3.51 $\displaystyle\int (\csc^2 x + \csc^4 x)\,dx = \int (1 + \csc^2 x)\csc^2 x\,dx$. Using the identity $\csc^2 x = 1 + \cot^2 x$, we can write this integral as $\displaystyle\int (2 + \cot^2 x)\csc^2 x\,dx$. Substituting $u = \cot x$ so that $du = -\csc^2 x\,dx$ gives

$$-\int (2 + u^2)\,du = -2u - \frac{u^3}{3} + C = -2\cot x - \frac{\cot^3 x}{3} + C.$$

8.3.52

$$\int_0^{\pi/8} (\tan 2x + \tan^3 2x)\,dx = \int_0^{\pi/8} (1 + \tan^2 2x)\tan 2x\,dx = \int_0^{\pi/8} \sec^2 2x \tan 2x\,dx$$
$$= \int_0^{\pi/8} \sec 2x \sec 2x \tan 2x\,dx.$$

Let $u = \sec 2x$. Then $du = 2\sec 2x \tan 2x\,dx$. Substituting gives

$$\frac{1}{2}\int_1^{\sqrt{2}} u\,du = \frac{1}{4}u^2\Big|_1^{\sqrt{2}} = \frac{1}{4}(2-1) = \frac{1}{4}.$$

8.3.53 $\int_0^{\pi/4} \sec^4\theta\,d\theta = \int_0^{\pi/4} (\sec^2\theta)(1+\tan^2\theta)\,d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Note that when $\theta = 0$ we have $u = 0$ and when $\theta = \dfrac{\pi}{4}$ we have $u = 1$. So the original integral is equal to

$$\int_0^1 (1+u^2)\,du = \left(u + \frac{u^3}{3}\right)\Big|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

8.3.54 $\int_0^{\sqrt{\pi/2}} x\sin^3(x^2)\,dx = \dfrac{1}{2}\int_0^{\pi/2} \sin^3 u\,du$, where $u = x^2$ and $du = 2x\,dx$. Now $\dfrac{1}{2}\int_0^{\pi/2} \sin^3 u\,du = \dfrac{1}{2}\int_0^{\pi/2} (\sin u)(1-\cos^2 u)\,du$. Now let $w = \cos u$ so that $dw = -\sin u\,du$. We then have

$$-\frac{1}{2}\int_1^0 (1-w^2)\,dw = \frac{1}{2}\left(w - \frac{w^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

8.3.55 $\int_{\pi/6}^{\pi/3} \cot^3\theta\,d\theta = \int_{\pi/6}^{\pi/3} (\cot\theta)(\csc^2\theta - 1)\,d\theta = \int_{\pi/6}^{\pi/3} \cot\theta\csc^2\theta\,d\theta - \int_{\pi/6}^{\pi/3} \dfrac{\cos\theta}{\sin\theta}\,d\theta$. For the first integral, let $u = \cot\theta$ so that $du = -\csc^2\theta\,d\theta$. For the second integral, let $w = \sin\theta$ so that $dw = \cos\theta\,d\theta$. Substituting gives

$$\begin{aligned}-\int_{\sqrt{3}}^{1/\sqrt{3}} u\,du - \int_{1/2}^{\sqrt{3}/2} \frac{1}{w}\,dw &= -\frac{u^2}{2}\Big|_{\sqrt{3}}^{1/\sqrt{3}} - \ln w\Big|_{1/2}^{\sqrt{3}/2} \\ &= -\frac{1}{2}\left(\frac{1}{3} - 3\right) - \left(\ln\sqrt{3} - \ln 2 - \ln 1 + \ln 2\right) = \frac{4}{3} - \frac{\ln 3}{2}.\end{aligned}$$

8.3.56 $\int_0^{\pi/4} \tan^3\theta\sec^2\theta\,d\theta$. Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. Substituting yields $\int_0^1 u^3\,du = \dfrac{u^4}{4}\Big|_0^1 = \dfrac{1}{4}$.

8.3.57 $\int_0^{\pi} (1-\cos 2x)^{3/2}\,dx = \int_0^{\pi} (2\sin^2 x)^{3/2}\,dx = 2\sqrt{2}\int_0^{\pi} \sin^3 x\,dx = 2\sqrt{2}\int_0^{\pi} (\sin x)(1-\cos^2 x)\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. Substituting yields

$$-2\sqrt{2}\int_1^{-1} (1-u^2)\,du = 2\sqrt{2}\int_{-1}^1 (1-u^2)\,du = 4\sqrt{2}\int_0^1 (1-u^2)\,du = 4\sqrt{2}\left(u - \frac{u^3}{3}\right)\Big|_0^1 = \frac{8\sqrt{2}}{3}.$$

8.3.58 $\int_{-\pi/4}^{\pi/4} \sqrt{1+\cos 4x}\,dx = 2\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = 2\sqrt{2}\int_0^{\pi/4} \cos 2x\,dx$. Let $u = 2x$, so that $du = 2\,dx$. Substituting yields

$$\sqrt{2}\int_0^{\pi/2} \cos u\,du = \sqrt{2}\sin u\Big|_0^{\pi/2} = \sqrt{2}.$$

8.3.59 $\int_0^{\pi/2} \sqrt{1-\cos 2x}\,dx = \sqrt{2}\int_0^{\pi/2} \sin x\,dx = -\sqrt{2}\cos x\Big|_0^{\pi/2} = \sqrt{2}$.

8.3.60 $\int_0^{\pi/8} \sqrt{1-\cos 8x}\,dx = \sqrt{2}\int_0^{\pi/8} \sin 4x\,dx = -\sqrt{2}\cdot\dfrac{\cos 4x}{4}\Big|_0^{\pi/8} = \dfrac{\sqrt{2}}{4}$.

8.3.61 $\int_0^{\pi/4} (1+\cos 4x)^{3/2}\,dx = \int_0^{\pi/4} (2\cos^2 2x)^{3/2}\,dx = 2\sqrt{2}\int_0^{\pi/4} \cos^3 2x\,dx = 2\sqrt{2}\int_0^{\pi/4} (\cos 2x)(1-\sin^2 2x)\,dx$. Let $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. Substituting gives

$$\sqrt{2}\int_0^1 (1-u^2)\,du = \sqrt{2}\left(u - \frac{u^3}{3}\right)\Big|_0^1 = \frac{2\sqrt{2}}{3}.$$

8.3.62 If $y = \ln(\sec x)$ then $\dfrac{dy}{dx} = \tan x$. Thus,

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2}+1).$$

8.3.63

a. True. We have $\int_0^{\pi} \cos^{2m+1} x\, dx = \int_0^{\pi} (\cos^2 x)^m \cos x\, dx = \int_0^{\pi} (1-\sin^2 x)^m \cos x\, dx$. Let $u = \sin x$ so that $du = \cos x\, dx$. Substituting yields $\int_0^0 (1-u^2)^m\, du = 0$.

b. False. For example, suppose $m = 1$. Then $\int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -(-1-1) = 2 \neq 0$.

8.3.64 Using the disk method, we have

$$\frac{V}{\pi} = \int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} (1-\cos 2x)\, dx = \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi} \cos 2x\, dx = \frac{\pi}{2} - \left(\frac{\sin 2x}{4}\right)\Big|_0^{\pi} = \frac{\pi}{2}.$$

Thus, $V = \dfrac{\pi^2}{2}$.

8.3.65 $V = \int_0^{\pi/2} \pi \sin^4 x \cos^3 x\, dx = \pi \int_0^{\pi/2} \sin^4 x(1-\sin^2 x)\cos x\, dx = \int_0^{\pi/2} (\sin^4 x - \sin^6 x)\cos x\, dx$. Let $u = \sin x$. Then $du = \cos x\, dx$ and substitution gives

$$\pi \int_0^1 (u^4 - u^6)\, du = \pi\left(\frac{u^5}{5} - \frac{u^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35}.$$

8.3.66 $s(t) = s(0) + \int_0^t \sec^4\left(\frac{\pi x}{12}\right) dx = 0 + \int_0^t \left(\tan^2\left(\frac{\pi x}{12}\right) + 1\right)\sec^2\left(\frac{\pi x}{12}\right) dx$. Let $u = \tan\left(\frac{\pi x}{12}\right)$. Then $du = \frac{\pi}{12}\sec^2\left(\frac{\pi x}{12}\right)$. Substituting gives

$$\frac{12}{\pi}\int_0^{\tan(\pi t/12)} (u^2+1)\, du = \frac{12}{\pi}\left(\frac{\tan^3\left(\frac{\pi t}{12}\right)}{3} + \tan\left(\frac{\pi t}{12}\right)\right) = \frac{4}{\pi}\left(\tan^3\left(\frac{\pi t}{12}\right) + 3\tan\left(\frac{\pi t}{12}\right)\right).$$

8.3.67 $\int \sin 3x \cos 7x\, dx = \frac{1}{2}\left(\int \sin(-4x)\, dx + \int \sin 10x\, dx\right) = \frac{1}{2}\left(\frac{\cos(-4x)}{4} - \frac{\cos 10x}{10}\right) + C = \frac{\cos 4x}{8} - \frac{\cos 10x}{20} + C$.

8.3.68 $\int \sin 5x \sin 7x\, dx = \frac{1}{2}\left(\int \cos(-2x)\, dx - \int \cos 12x\, dx\right) = \frac{1}{2}\left(\int \cos 2x\, dx - \int \cos 12x\, dx\right) = \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 12x}{12}\right) + C = \frac{\sin 2x}{4} - \frac{\sin 12x}{24} + C$.

8.3.69 $\int \sin 3x \sin 2x\, dx = \frac{1}{2}\left(\int \cos x\, dx - \int \cos 5x\, dx\right) = \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$.

8.3.70 $\int \cos x \cos 2x\, dx = \frac{1}{2}\left(\int \cos(-x)\, dx + \int \cos 3x\, dx\right) = \frac{\sin x}{2} + \frac{\sin 3x}{6} + C$.

8.3.71

a. $\int_0^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \cos(m-n)x\, dx - \int_0^{\pi} \cos(m+n)x\, dx\right) =$
$\frac{1}{2}\left(\frac{1}{m-n}\int_0^{(m-n)\pi} \cos u\, du - \frac{1}{m+n}\int_0^{(m+n)\pi} \cos v\, dv\right)$ where $u = (m-n)x$ and $v = (m+n)x$. But this yields $\frac{1}{2}\left(\frac{1}{m-n}\sin u\,\Big|_0^{(m-n)\pi} - \frac{1}{m+n}\sin v\,\Big|_0^{(m+n)\pi}\right) = \frac{1}{2}(0-0) = 0.$

b. $\int_0^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \cos(m-n)x\, dx + \int_0^{\pi} \cos(m+n)x\, dx\right) = 0$ by the previous part of this problem,

c. $\int_0^{\pi} \sin mx \cos nx\, dx = \frac{1}{2}\left(\int_0^{\pi} \sin(m-n)x\, dx + \int_0^{\pi} \sin(m+n)x\, dx\right) =$
$\frac{1}{2}\left(\frac{1}{m-n}\int_0^{(m-n)\pi} \sin u\, du + \frac{1}{m+n}\int_0^{(m+n)\pi} \sin v\, dv\right)$ where $u = (m-n)x$ and $v = (m+n)x$. This quantity is equal to $\frac{-1}{2}\left(\frac{1}{m-n}\cos u\,\Big|_0^{(m-n)\pi} + \frac{1}{m+n}\cos v\,\Big|_0^{(m+n)\pi}\right) =$
$\frac{-1}{2}\left(\frac{1}{m-n}(\cos(m-n)\pi - 1) + \frac{1}{m+n}(\cos(m+n)\pi - 1)\right) =$
$\begin{cases} 0 & \text{if } m \text{ and } n \text{ are both even or both odd;} \\ \frac{1}{m-n} + \frac{1}{m+n} = \frac{2m}{m^2-n^2} & \text{otherwise.} \end{cases}$

8.3.72 $\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$. Let $u = \sin^{n-1} x$ and $dv = \sin x\, dx$. Then we have $du = (n-1)\sin^{n-2} x \cos x\, dx$ and $v = -\cos x$. We have $\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x)(1 - \sin^2 x)\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$. Adding the appropriate quantity to both sides of this last equation gives

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx,$$

so $\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$

Thus,

$$\int \sin^6 x\, dx = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6}\left(\frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\right) + C.$$

8.3.73 For $n \neq 1$, $\int \tan^n x\, dx = \int (\tan^{n-2} x)(\sec^2 x - 1)\, dx = \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$. Let $u = \tan x$ so that $du = \sec^2 x\, dx$. Then substituting in the first of these last two integrals yields

$$\int u^{n-2}\, du - \int \tan^{n-2} x\, dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

Thus $\int_0^{\pi/4} \tan^3 x\, dx = \frac{\tan^2 x}{2}\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx = \frac{1}{2} + \ln|\cos x|\,\Big|_0^{\pi/4} = \frac{1}{2} - \frac{\ln 2}{2}.$

8.3.74 $\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$. Let $u = \sec^{n-2} x$ and $dv = \sec^2 x\, dx$. Then $du = (n-2)\sec^{n-2} x \tan x\, dx$ and $v = \tan x$. Integration by Parts gives us $\sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx =$

$$\sec^{n-2} x \tan x - (n-2) \int (\sec^{n-2} x)(\sec^2 x - 1)\, dx = \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx.$$

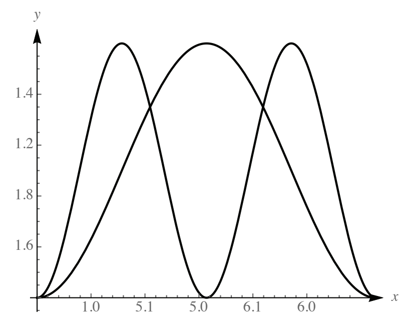
Combining like terms then gives $(n-1) \int \sec^n x\, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x\, dx$, so as long as $n \neq 1$ we have

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx.$$

8.3.75

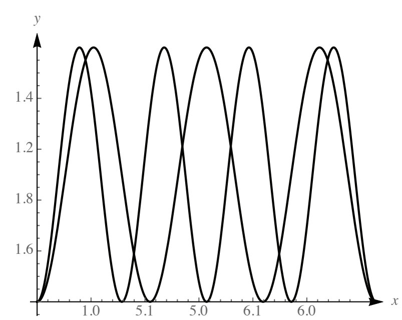
a. $\displaystyle\int_0^\pi \sin^2 x\, dx = \frac{1}{2} \int_0^\pi (1 - \cos 2x)\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$

$\displaystyle\int_0^\pi \sin^2 2x\, dx = \frac{1}{2} \int_0^\pi (1 - \cos 4x)\, dx = \frac{1}{2}\left(x - \frac{\sin 4x}{4}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$



b. $\displaystyle\int_0^\pi \sin^2 3x\, dx = \frac{1}{2} \int_0^\pi (1 - \cos 6x)\, dx = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$

$\displaystyle\int_0^\pi \sin^2 4x\, dx = \frac{1}{2} \int_0^\pi (1 - \cos 8x)\, dx = \frac{1}{2}\left(x - \frac{\sin 8x}{8}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$



c. $\displaystyle\int_0^\pi \sin^2 nx\, dx = \frac{1}{2} \int_0^\pi (1 - \cos 2nx)\, dx = \frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$

d. Yes. $\displaystyle\int_0^\pi \cos^2 nx\, dx = \frac{1}{2} \int_0^\pi (1 + \cos 2nx)\, dx = \frac{1}{2}\left(x + \frac{\sin 2nx}{2n}\right)\Bigg|_0^\pi = \frac{\pi}{2}.$

e. Claim: The corresponding integrals are all equal to $\frac{3\pi}{8}$. Proof:

$$\begin{aligned}
\int_0^\pi \sin^4 nx\, dx &= \int_0^\pi \left(\frac{1 - \cos 2nx}{2}\right)^2 dx \\
&= \int_0^\pi \frac{1 - 2\cos 2nx + \cos^2 2nx}{4}\, dx = \int_0^\pi \frac{1}{4}\, dx - \frac{1}{2} \int_0^\pi \cos 2nx\, dx + \frac{1}{4} \int_0^\pi \cos^2 2nx\, dx \\
&= \frac{\pi}{4} - \frac{1}{2}\left(\frac{\sin 2nx}{2n}\right)\Bigg|_0^\pi + \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{4} - 0 + \frac{\pi}{8} = \frac{3\pi}{8}.
\end{aligned}$$

8.4 Trigonometric Substitutions

8.4.1 This would suggest $x = 3\sec\theta$, because then $\sqrt{x^2 - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta$, for $\theta \in [0, \pi/2)$.