Contents

Question. What is a definite integral? Is it just an area as previously discussed? We shall see that it can be many more things.

Chapter 6 involves explanations via graphs. You are strongly encouraged to study these diagrams that the professional drawers have prepared for you in the textbook. They are well prepared with shading and accuracy. You are encouraged to use a TI-83/84 calculator to draw any functions in an exam.

6.1 §6.1 The Net change Theorem

This section is left for your own reading.

Theorem (The Net change Theorem)**.** The integral of the rate of change is the net change:

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

Example. If $f(x)$ is the slope of a trail at a distance x from the start, what does $\int_3^5 f(x)\ dx$ represent?

Solution. Suppose that the elevation of the trail at x is $e(x)$. Then the slope of the trail at x is $e'(x) = f(x)$.

By the Net Change Theorem, $\int_3^5 f(x)\ dx = \int_3^5 e'(x)\ dx = e(5) - e(3)$.

□

The following lists some other examples.

• If $V(t)$ is the volume of water in a reservoir at time t, then $V'(t)$ is the rate at which the water is flowing into/out of the reservoir at time t. Thus,

$$\int_{t_1}^{t_2} V'(t)\ dt = V(t_2) - V(t_1)$$

is the net change in the amount of water in the reservoir between time t_1 and t_2.

• If $[C](t)$ is the concentration of a product due to a chemical reaction at time t, then the rate of reaction is $D_x([C](t))$ and so

$$\int_{t_1}^{t_2} D_x([C](t)\ dt = [C](t_2) - [C](t_1)$$

is the net change in the concentration of C from time t_1 to t_2.

• If the mass of a rod measured from the left end to a point x is given by $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\ dx = m(b) - m(a)$$

is the mass of the rod between the points $x = a$ and $x = b$.

• If the rate of growth of a population at time t is $dn(t)/dt$, then

$$\int_{t_1}^{t_2} D_x(n(t))\ dt = n(t_2) - n(t_1)$$

is the net change in the population over time the time period $[t_1, t_2]$.

• If $C(x)$ is the cost of producing x units of a commodity, then $C'(x)$ is the marginal cost. So

$$\int_{t_1}^{t_2} C'(x)\ dx = C(t_2) - C(t_1)$$

is the increase in cost when production is increased from t_1 units to t_2 units.

If an object moves along a straight line with an nondecreasing position function $s(t)$, measured from left to right, then its velocity is $\boxed{v(t) = s'(t) \geq 0}$. So

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1) = \text{ total distance traveled.}$$

Definition. Here, $s(t_2) - s(t_1)$ is generally called the **displacement** of the object between $t = t_1$ and $t = t_2$.

When we do not assume $v(t) = s'(t) \geq 0$, then

$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1) = \boxed{\text{net change}}$ of position or the **displacement of the object**

and the total **distance travelled** is computed as

$$\int_a^b |v(x)|\, dx = \boxed{\text{total distance}} \text{ traveled.}$$

Theorem (Theorem 6.1 Position from Velocity)**.** Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x)\, dx.$$

Example 2 (Position from velocity). A block hangs at rest from a massless spring at the origin ($s = 0$). At $t = 0$, the block is pulled downward 1/4 m to its initial position $s(0) = -1/4$ and released (Figure 6.4). Its velocity (in m/s) is given by $v(t) = 0.25 \sin t$, for $t \geq 0$. Assume that the upward direction is positive.

a. Find the position of the block, for $t \geq 0$.

b. Graph the position function, for $0 \leq t \leq 3\pi$.

c. When does the block move through the origin for the first time?

d. When does the block reach its highest point for the first time and what is its position at that time? When does the block return to its lowest point?

Solution.

$\square$

• Given the acceleration $a(t)$ of an object moving along a line with velocity $v(t)$, $\boxed{a(t) = v^{'}(t)}$.

$$\int_{t_1}^{t_2} a(t)\ dt = v(t_2) - v(t_1) = \boxed{\text{net change}} \text{ of velocity}$$

Theorem (Theorem 6.2 Velocity from Acceleration). Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\ dx.$$

□

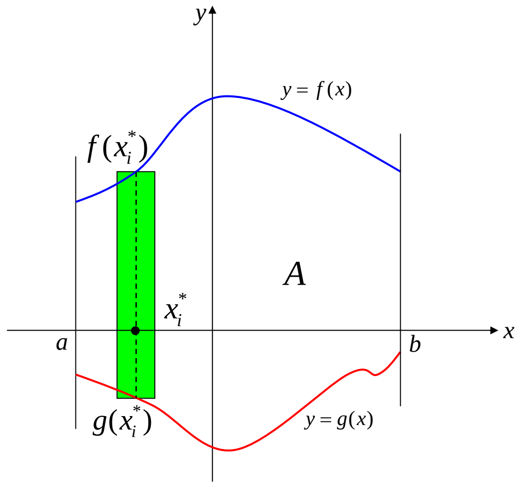
Example 4 (Motion in a gravitational field). An artillery shell is fired directly upward with an initial velocity of $300m/s$ from a point $30m$ above the ground. Assume that only the force of gravity acts on the shell and it produces an acceleration of $9.8m/s^2$. Find the velocity of the shell while it is in the air.

Solution.

□

6.2 §6.2 Regions Between Curves

Again, we rely on the slice-and-sum strategy (Section 5.2) for finding areas by Riemann sums.



$$A = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]\,dx$$

Theorem. The area A bounded by $x = a$, $x = b$ with $a < b$ and the curves $y = f(x)$ and $y = g(x)$ is given by

$$\text{Area} \;=\; \int_a^b |f(x) - g(x)|\; dx.$$

Remark 1. If $f(x) \geq g(x)$ for all x in $[a, b]$, then the integrand becomes $f(x) - g(x)$ (see derivation of the formula using approximating rectangles in the text);

Remark 2. The key to this section is to **sketch the graphs of the curves** involved so that we can determine the upper curve and the lower curve. If the curves **cross** each other, we must set up more than one integral to compute the area. For example, we want to find the area bounded by $x = a$, $x = b$ and the curves $y = f(x)$ and $y = g(x)$. Suppose further that $f(x) \geq g(x)$ on $[a, c]$ and that $g(x) \geq f(x)$ on $[c, b]$. Then the diagram in this case is something like the figure in Example 5 and the required integrals are

$$\begin{aligned}\text{Area} &= \int_a^c \Big(\text{upper curve} - \text{lower curve}\Big)\; dx + \int_c^b \Big(\text{upper curve} - \text{lower curve}\Big)\; dx \\ &= \int_a^c [f(x) - g(x)]\; dx + \int_c^b [g(x) - f(x)]\; dx. \quad \square\end{aligned}$$

Example 1. Find the area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$.

Solution.

$\square$

Example 2. Find the area of the region bounded by the graphs of $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$.

Solution.

$\square$

Example Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

Solution. We begin by sketching the graphs of $y = \sin x$ and $y = \cos x$ above the interval $[0, \frac{\pi}{2}]$.

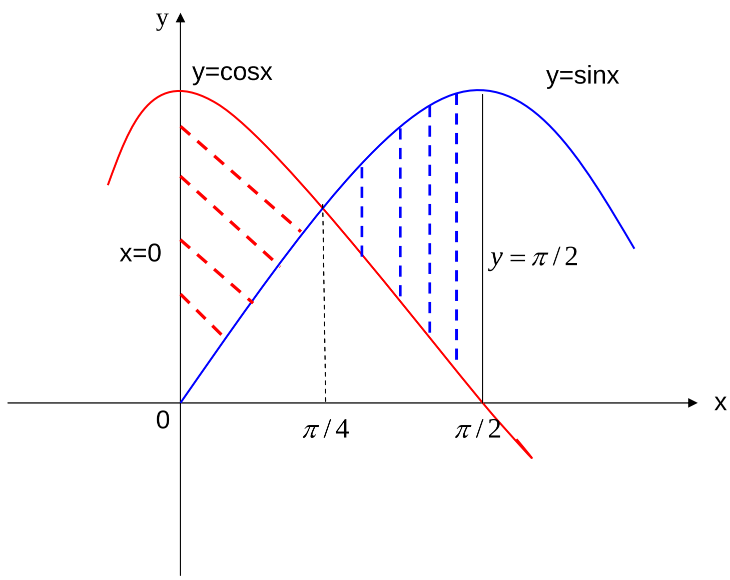


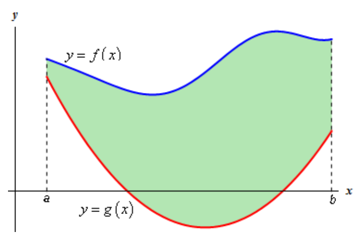
Figure 1: Figure of the Example

From the diagram below we see that they cross at $x = \frac{\pi}{4}$ with $\cos x \geq \sin x$ on $[0, \frac{\pi}{4}]$ and $\sin x \geq \cos x$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$. Thus,

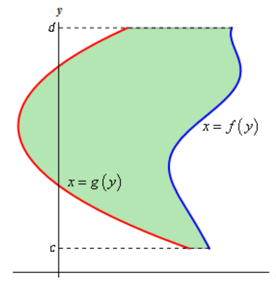
$$\begin{aligned}
\text{Area } &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\ dx \\
&= \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} + \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right] + \left[-1 - 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 4\frac{1}{\sqrt{2}} - 2 = 2\sqrt{2} - 2.
\end{aligned}$$

□

Remark. Sometimes it is better to integrate with respect to the y variable.



$$A = \int_a^b [f(x) - g(x)]dx$$



$$A = \int_c^d [f(y) - g(y)]dy$$

Theorem. The area A bounded by $y = c$, $y = d$ with $c < d$ and the curves $x = f(y)$ and $x = g(y)$ is given by

$$\text{Area} = \int_c^d |f(y) - g(y)| \, dy.$$

Remark. If $f(y) \geq g(y)$ for all y in $[c, d]$ (that is, $f(y)$ is to the right of $g(y)$), then the integrand becomes $f(y) - g(y)$.

Example 3. Find the area of the region R bounded by the graphs of $y = x^3, y = x + 6$, and the x-axis.

Solution.

□

Example 4. Find the area of the region R in the first quadrant bounded by the curves $y = x^{2/3}$ and $y = x - 4$.

Solution.

$\square$

More examples

Example 5 Find the area of the shaded region.

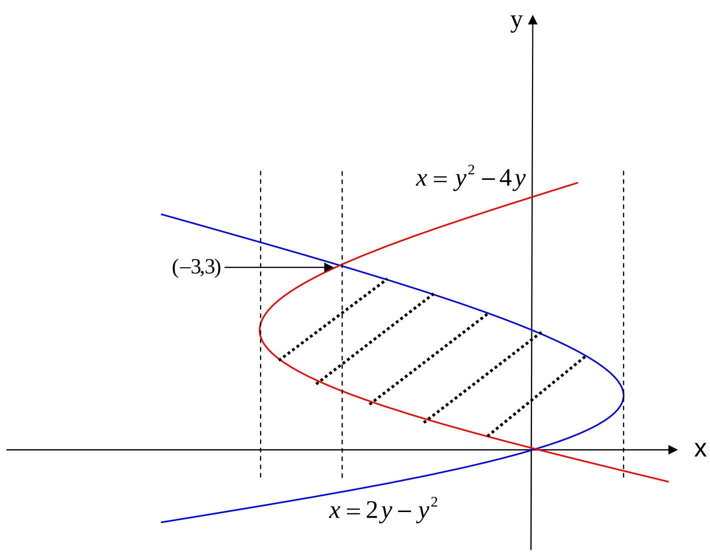


Figure 2: Figure of Example 5

Solution. Method 1. Integrating with respect to x. We must first represent the two curves in terms of y being a function of x.

$$\begin{aligned}
x = y^2 - 4y &\iff y^2 - 4y - x = 0 \\
&\iff y = \frac{4 \pm \sqrt{16+4x}}{2} \\
&\iff y = 2 \pm \sqrt{4+x} \\
x = 2y - y^2 &\iff y^2 - 2y + x = 0 \\
&\iff y = \frac{2 \pm \sqrt{4-4x}}{2} \\
&\iff y = 1 \pm \sqrt{1-x}
\end{aligned}$$

Next we must find the left-most point of our interval and the right-most point of our interval. These can be found by computing where the graphs have vertical tangents, i.e. where $\frac{dx}{dy} = 0$. These occur for the curve $x = y^2 - 4y$ at the point $(-4, 2)$ and for the curve $x = 2y - y^2$ at the point $(1, 1)$.

Now, we must pay attention to the intervals and to upper curves and lower curves.

$$\begin{aligned}
\text{Area } &= \int_{-4}^{-3} \Big((2 + \sqrt{4+x}) - (2 - \sqrt{4+x}) \Big)\, dx \\
&\quad + \int_{-3}^{0} \Big((1 + \sqrt{1-x}) - (2 - \sqrt{4+x}) \Big)\, dx \\
&\quad + \int_{0}^{1} \Big((1 + \sqrt{1-x}) - (1 - \sqrt{1-x}) \Big)\, dx.
\end{aligned}$$

Too complex! This is not the way to do this problem!!!

Method 2. Integrating with respect to y. In this case, we draw our approximating rectangles horizontally.

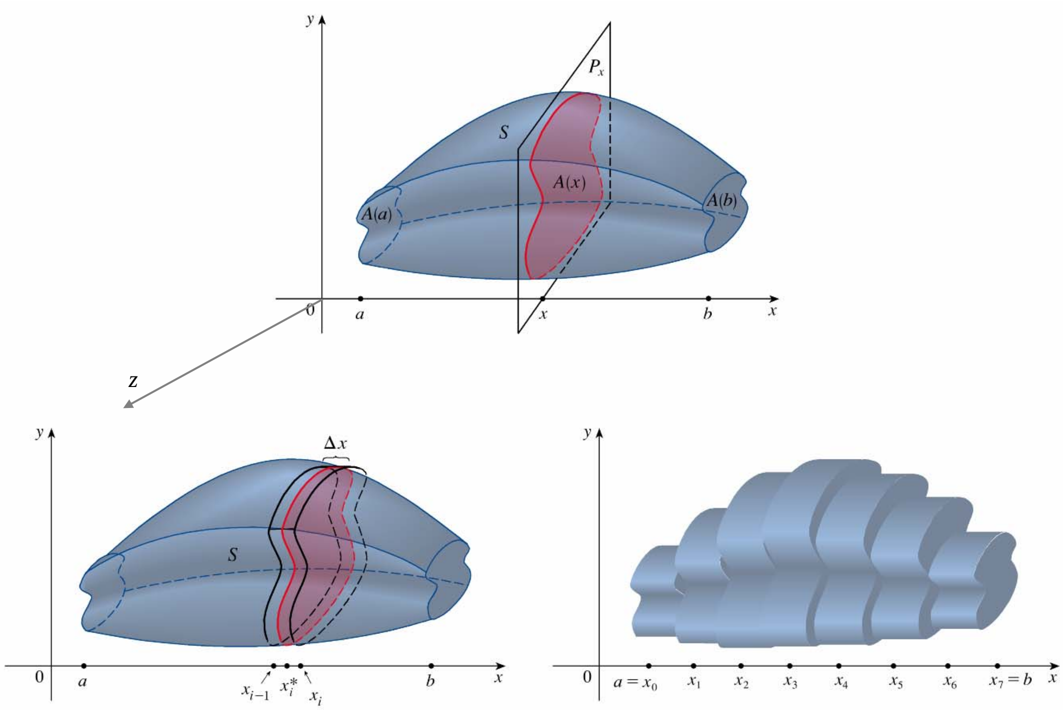
$$\begin{aligned}
\text{Area } &= \int_0^3 \Big((\text{right curve}) - (\text{left curve}) \Big)\, dy \\
&= \int_0^3 \Big((2y - y^2) - (y^2 - 4y) \Big)\, dy = \int_0^3 (-2y^2 + 6y)\, dy \\
&= \Big[-\frac{2}{3}y^3 + 3y^2 \Big]_0^3 = -(2)(9) + (3)(9) = 9.
\end{aligned}$$

$\square$

6.3 §6.3 Volume by Slicing

We have seen that integration is used to compute the area of two-dimensional regions bounded by curves. Integrals are also used to find the volume of three-dimensional regions (or solids). Once again, the **slice-and-sum** method is the key to solving these problems.

In this section, we first analyze the problem of finding the volume of a solid that extends in the x-direction from $x = a$ to $x = b$. For each value of x between a and b we know the cross sectional area is given by $A(x)$. Study illustrations pp. 425-426 (see the following two figures as well).



A cross-section and "slabs" of a solid *S*

Recall that the volume of a cylinder is given by

$$\text{Volume} = Ah = \text{Area of base } \times \text{ Height} .$$

For example, for a circular cylinder $A = \pi r^2$.

We see that the volume of one of our cross sectional slabs is given by

$$\boxed{\text{Volume of the } i^{th} \text{ slab } = A(x_i^*)\Delta x.}$$

Theorem (General Slicing Method)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x (the plane through x and perpendicular to the x-axis) is $A(x)$, then the **volume** of S is given by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx.$$

Remark. The key is to find the cross sectional area $A(x)$ at x. Both the **Disc Method** and the **Washer Method** are examples of methods involving a known cross sectional area.

Example 1.[Volume of a "parabolic cube"] Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base R, and cross sections through the solid perpendicular to the base and parallel to the y-axis are **squares** (Figure 6.25a). Find the volume of the solid.

Solution.

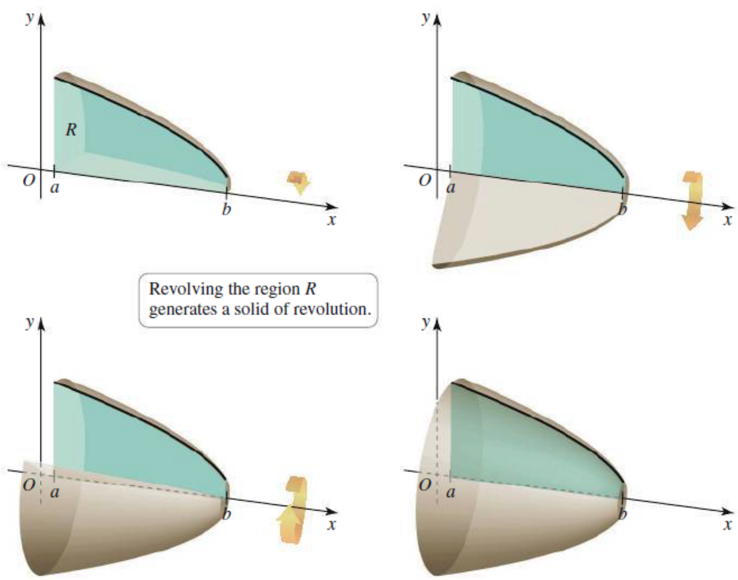
$\square$

The Disk Method

The volume of a $\boxed{\textbf{disc}}$ of height Δx and radius R is

$$\text{Volume of Disc } = \pi R^2 \Delta x.$$

We now consider a specific type of solid known as a **solid of revolution**.



Theorem (Disk Method about the x-Axis)**.** Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the **volume** of the resulting solid of revolution is is given by

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[f(x)]^2\, dx.$$

Example 3. Let R be the region bounded by the curve $f(x) = (x+1)^2$, the x-axis, and the lines $x = 0$ and $x = 2$ (Figure 6.30a). Find the volume of the solid of revolution obtained by revolving R about the x-axis.

Solution.

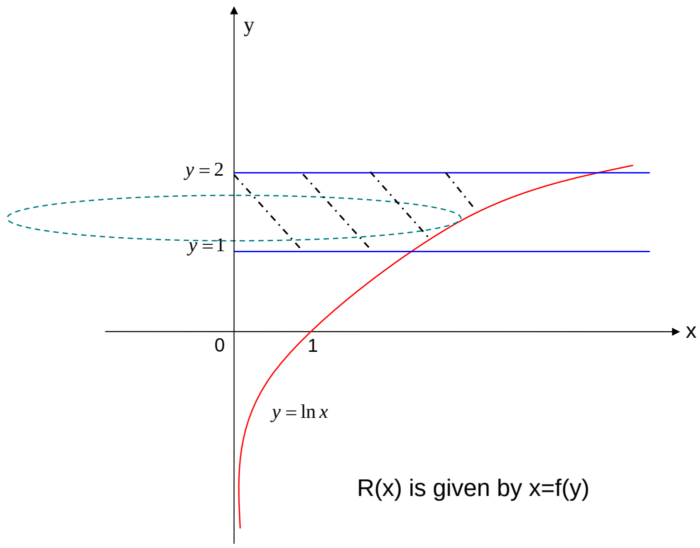
□

What we learned about revolving regions about the x-axis applies to revolving regions about the y-axis.

Theorem (Disk Method about the y-Axis)**.** Let $x = g(y)$ be continuous with $g(y) \geq 0$ on the interval $[c, d]$. If the region R bounded by the graph of g, the y-axis, and the lines $y = c$ and $y = d$ is revolved about the y-axis, the **volume** of the resulting solid of revolution is is given by

$$V = \int_c^d A(y)\ dy = \int_c^d \pi[g(y)]^2\ dy.$$

Example Find the volume of the solid obtained by rotating the region bounded by the curves $y = \ln x$, $y = 1$, $y = 2$ and $x = 0$ about the y-axis.



Solution. The graphs of all these curves are well known. It is clear from the diagram that we want to integrate wrt y. Also,

$$y = \ln x \iff x = e^y$$

and for each fixed y the radius of our disc is $x = e^y$ and so the cross sectional area is $A(y) = \pi(e^y)^2 = \pi e^2 y$.

$$\begin{aligned}
\text{Volume } &= \int_1^2 \pi e^{2y}\, dy \\
&= \frac{\pi}{2}\int_1^2 e^{2y}\, 2d(y) = \frac{\pi}{2}\int_1^2 e^{2y}\, d(2y) \\
&= \frac{\pi}{2}\left[e^{2y}\right]_1^2 = \frac{\pi}{2}(e^4 - e^2)
\end{aligned}$$

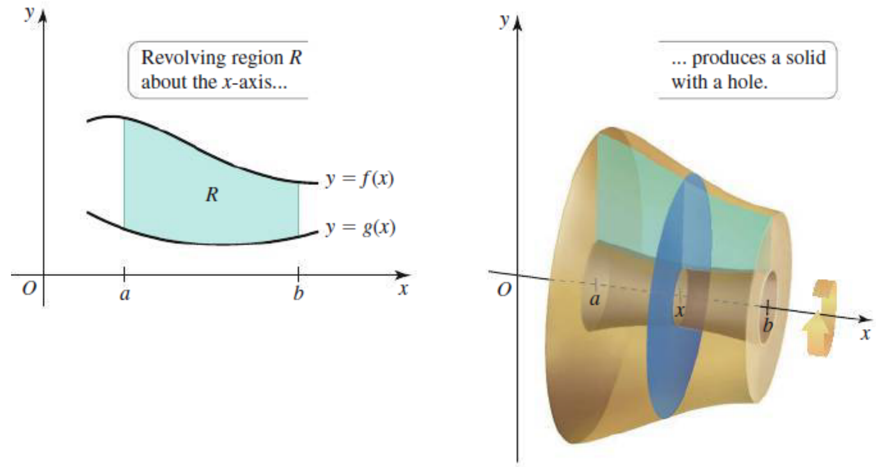
$\square$

Washer Method

The volume of a $\boxed{\textbf{washer}}$ of height Δx and inside radius r and outside radius R is

$$\text{Volume of Washer } = \pi R^2 \Delta x - \pi r^2 \Delta x = \pi(R^2 - r^2)\Delta x,$$

where R and r are functions of x. □



Theorem (Washer Method about the x-Axis)**.** Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

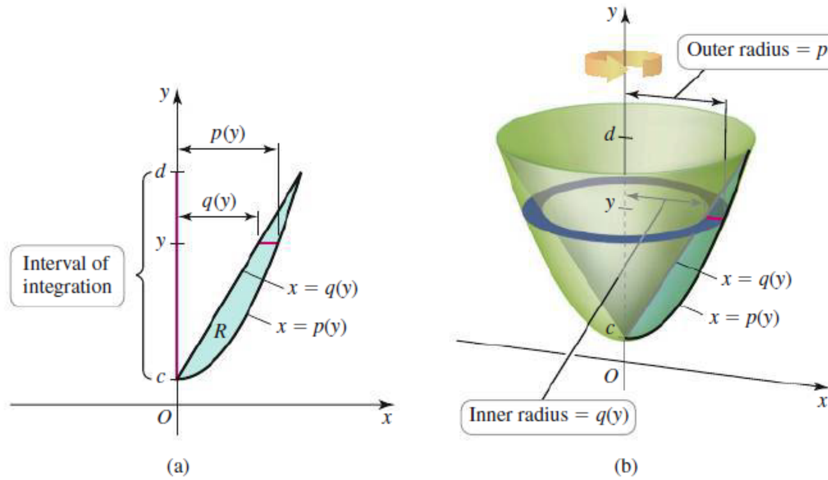
$$V = \int_a^b A(x)\, dx = \int_a^b \pi \left\{[f(x)]^2 - [g(x)]^2\right\}\, dx.$$

Example 4. The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results when R is revolved about the x-axis?

Solution.

□

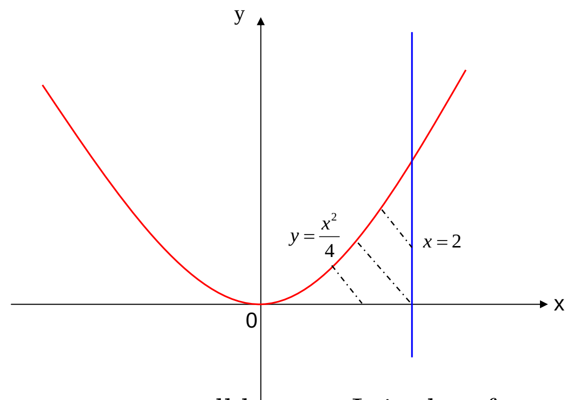
Again, what we learned about revolving regions about the x-axis applies to revolving regions about the y-axis.



Theorem (Washer Method about the y-Axis)**.** Let $p(y)$ and $q(y)$ be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d A(y)\ dy = \int_c^d \pi \left\{[p(y)]^2 - [q(y)]^2\right\}\ dy.$$

Example. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \frac{1}{4}x^2$, $x = 2$ and $y = 0$ about the y-axis. (What if the region is rotating about the x-axis?)



Solution. The graphs of all these curves are well known. It is clear from the diagram that we want to integrate wrt y. Also,

$$y = \frac{1}{4}x^2 \iff x = 2\sqrt{y}$$

and for each fixed y the inside radius $x = 2\sqrt{y}$, the outside radius is $x = 2$ and so the cross sectional area is $A(y) = \pi(2^2 - (2\sqrt{y})^2) = \pi(4 - 4y)$.

$$\text{Volume } = \int_0^1 \underbrace{\pi(4-4y)}_{=A(y)} \, dy = \pi\left[4y - 2y^2\right]_0^1 = 2\pi.$$

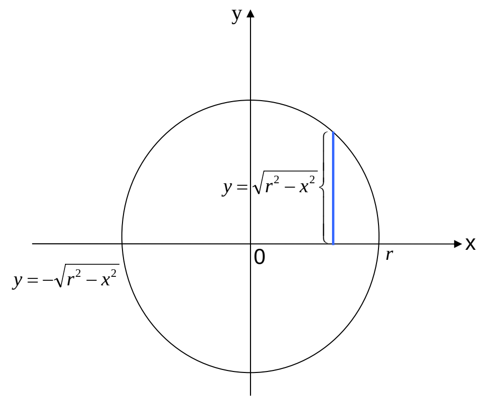
Example 5. Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution.

$\square$

More examples

Example. Find a formula for the volume of a sphere of radius r.



Solution. A sphere of radius r can be thought of as being the result of rotating the curve of a semi-circle $y = \sqrt{r^2 - x^2}$ about the x-axis.
For each fixed x between $x = -r$ and $x = r$, the cross section is a circle of radius $y = \sqrt{r^2 - x^2}$ and the area of this circle is given by

$$A(x) = \pi(\sqrt{r^2 - x^2})^2$$

and so the volume is

$$\begin{aligned}
\text{Volume} &= \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 \, dx \\
&= \pi \int_{-r}^{r} (r^2 - x^2) \, dx = \left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} \\
&= \pi\left[[r^3 - \frac{1}{3}r^3] - [r^2(-r) - \frac{1}{3}(-r)^3]\right] = \pi\left[2r^3 - \frac{2}{3}r^3\right] \\
&= \frac{4}{3}\pi r^3.
\end{aligned}$$

Remark. A slight modification of this Example allows us to find the volume of the cap of a sphere.

Example. Find a formula for the volume of the cap of a sphere of radius r and height h.

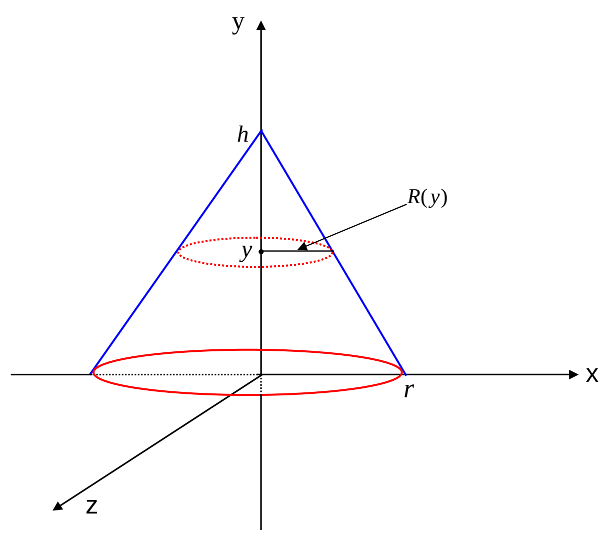
Solution.

$$\text{Volume} = \int_{r-h}^{r} \pi(\sqrt{r^2 - x^2})^2 \; dx = \int_{r-h}^{r} \pi(r^2 - x^2) \; dx$$
$$= \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{r-h}^{r} = \frac{\pi}{3}(3rh^2 - h^3).$$

□

Remark. These examples involved either the disc method or the washer method. However, recall that the general method of finding volume by using known cross sectional is more general than this.

Example (reading). Find a formula for the volume of a right circular cone whose base is a circle of radius r and those height is h.



Solution. We set up a coordinate system so that the base of the cone is on the xz-plan and the y-axis goes straight through the center. We integrate wrt y. By similar triangles, we see that as y varies from $y = 0$ to $y = h$, the radius $R(y)$ of the cross section circle at height y is found as follows

$$\frac{h}{r} = \frac{h-y}{R(y)} \implies R(y) = \frac{r(h-y)}{h}$$

The cross sectional area is given by

$$A(y) = \pi R(y)^2 = \pi \left(\frac{r(h-y)}{h}\right)^2$$

and

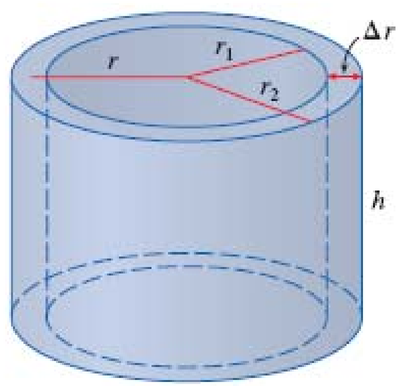
$$\begin{aligned} \text{Volume} &= \int_0^h \pi \left(\frac{r(h-y)}{h}\right)^2 \, dy = \frac{\pi r^2}{h^2} \int_0^h (y^2 - 2hy + h^2) \, dy \\ &= \frac{\pi r^2}{h^2} \left[\frac{y^3}{3} - hy^2 + h^2 y\right]_0^h = \frac{\pi r^2}{h^2} \left[\frac{h^3}{3} - h^3 + h^3\right] \\ &= \frac{\pi r^2 h}{3}. \end{aligned}$$

$\square$

6.4 §6.4 Volumes By Cylindrical Shells

Several volume problems that we have seen are those of finding the volume of rotation of a region in the xy-plane about an axis. Sometimes the solution is simplified by the Method of Shells. This method uses the difference between the volumes of two concentric cylinders.

Find the volume of the cylinder shell with (1) $r_1 =$ inside radius; (2) $r_2 =$ outside radius; (3) $h =$ height; (4) Can no top or bottom.



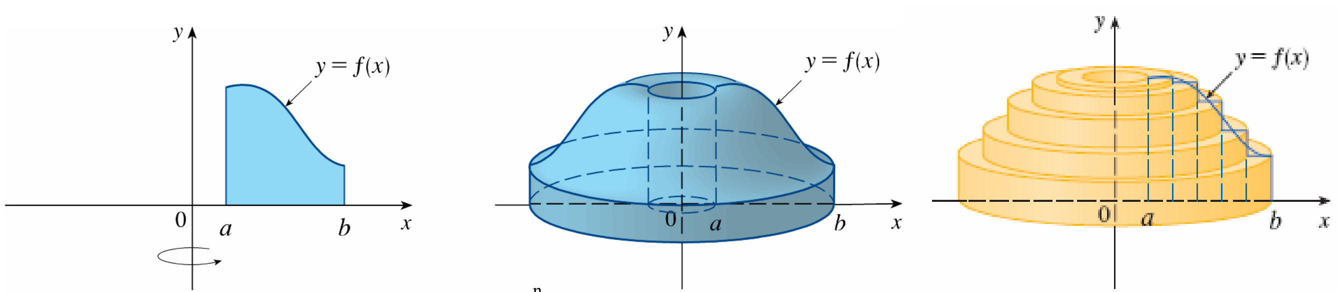
$$
\begin{aligned}
\text{Then, Volume of Shell } &= V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\
&= \pi h(r_2 + r_1)(r_2 - r_1) \text{ (sample point must be between } r_1 \text{ and } r_2) \\
&= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \qquad \left(r_1 \le \frac{r_2 + r_1}{2} \le r_2\right) \\
&= 2\pi r h \Delta r, \text{ where } r \text{ is the average radius} \\
&= [\text{circumference}][\text{height}][\text{thickness}].
\end{aligned}
$$

When we represent the height h as a function of r, we get

$$\text{Volume of Shell } = 2\pi r h(r)\ \Delta r.$$

Theorem. The volume of the solid in the following figure, obtained by rotating the region under the curve $y = f(x)$ (from a to b) about the y-axis, is

$$V = \int_a^b 2\pi x f(x)\,dx, \qquad \text{where } 0 \le a < b.$$



$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i) \Delta_x = \int_a^b 2\pi x f(x) dx$$

Example (reading). Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

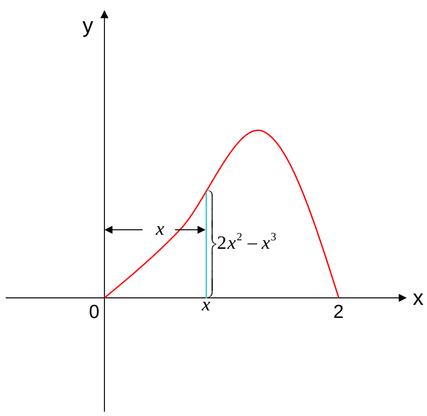
Solution. For the graph, we focus on the function $y = 2x^2 - x^3$:

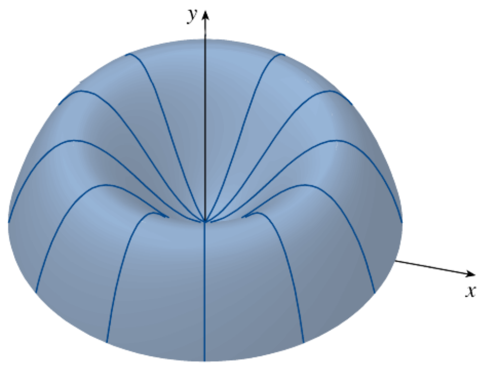
(1) Where is $y = 2x^2 - x^3$ neg. and pos.? $0 = 2x^2 - x^3 \implies 0 = x^2(2 - x)$;

(2) Where is $y = 2x^2 - x^3$ inc. and dec.? $0 = y' = x(4 - 3x) \implies x = 0, 4/3$;

(3) Where is $y = 2x^2 - x^3$ CU and CD? $y'' = 4 - 6x \implies$ CU if $x < 2/3$ and CD if $x > 2/3$.

With this information in hand we sketch the graph of $y = 2x^2 - x^3$.





Computer-generated picture of the solid

The intersection points with the x-axis are $(0, 0)$ and $(2, 0)$. We can see that the shell in the graph has radius x, circumference $2\pi x$ and height $f(x) = 2x^2 - x^3$. Thus, by the shell method, the volume is

$$
\begin{aligned}
V = &\int_0^2 2\pi x(2x^2 - x^3)\, dx = 2\pi \int_0^2 x(2x^2 - x^3)\, dx \\
=&2\pi \left[\frac{x^4}{2} - \frac{x^5}{5} \right]_0^2 = 2\pi(8 - 32/5) = \frac{16}{5}\pi.
\end{aligned}
$$

□

Example. Find the volume of the solid obtained when the region bounded by the curves $y = 3 + 2x - x^2$ and $x + y = 3$ is rotated about the y-axis.

Solution. As we shall see the method of shells is the best way to attack this problem.

The graph of $y = 3 - x$ is well known and so we focus on $y = 3 + 2x - x^2$.

(1) Where is $y = 3+2x-x^2$ neg. and pos.? $0 = -3-2x+x^2 \implies 0 = (x-3)(x+1)$;

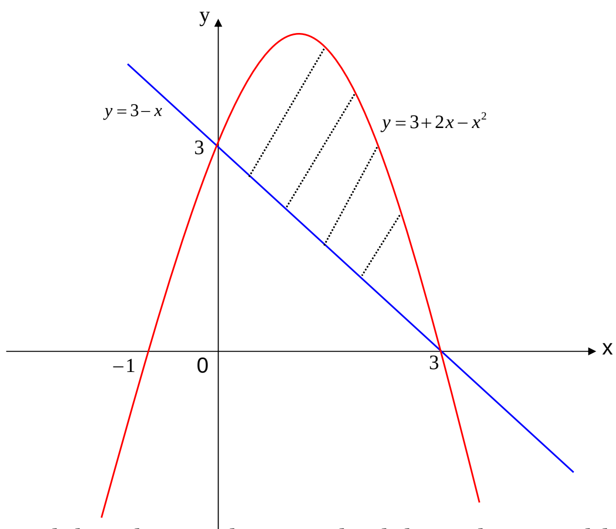
(2) Where is $y = 3 + 2x - x^2$ inc. and dec.? $0 = y' = 2 - 2x \implies x = 1$;

(3) Where is $y = 3 + 2x - x^2$ CU and CD? $y'' = -2 \implies$ CD everywhere.

The intersection points of $y = 3 + 2x - x^2$ and $x + y = 3$ are $(0, 3)$ and $(3, 0)$ because

$$3 - x = 3 + 2x - x^2 \implies 0 = 3x - x^2 \implies x = 0 \text{ or } 3$$

and so the intersection points are $(0, 3)$ and $(3, 0)$. With this information in hand, we can sketch the required region.



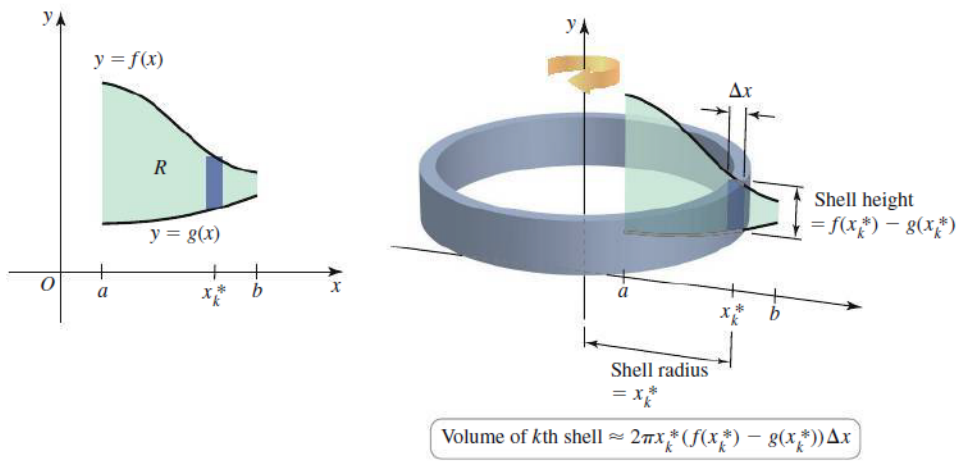
This volume can be found by the washer method but this would require that we integrate wrt y and represent the volume as the sum of two integrals. However, the shell method allows us to find the volume with a single integration.

$$\begin{aligned}\text{Volume } &= \int_0^3 2\pi x[(3+2x-x^2)-(3-x)]\,dx = \int_0^3 2\pi x[3x-x^2]\,dx \\ &= \int_0^3 2\pi x[3x-x^2]\,dx = 2\pi\int_0^3 (3x^2-x^3)\,dx \\ &= 2\pi\left[x^3-\frac{x^4}{4}\right]_0^3 = 2\pi\left[27-\frac{81}{4}\right] = 2\pi\left[\frac{108-81}{4}\right] = \pi\left[\frac{27}{2}\right].\end{aligned}$$

□

Q. Find the volume of the solid obtained when the region is rotated about the x-axis.

We can generalize this method to the case that the region R is bounded by two curves, $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ on $[a, b]$.



Theorem (Volumn by the Shell Method)**.** Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis, is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\, dx, \qquad \text{where } \ 0 \leq a < b.$$

Remark 1. If $g(x) = 0$, the theorem is reduced to the previous result.

Remark 2. We can derive the result when a region R is revolved about the x-axis.

Remark 3. We can derive the result when a region R is revolved about other lines.

Example 1. (A sine bowl) Let R be the region bounded by the graph of $f(x) = sin(x^2)$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$ (Figure 6.44). Find the volume of the solid generated when R is revolved about the y-axis.

Solution.

$\square$

Example 2. Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and the line $y = 2$. Find the volume of the solid generated when R is revolved about the x-axis.

Solution.

$\square$

Example 3. A cylindrical hole with radius r is drilled symmetrically through the center of a sphere with radius a, where $0 \leq r \leq a$. What is the volume of the remaining material?

Solution.

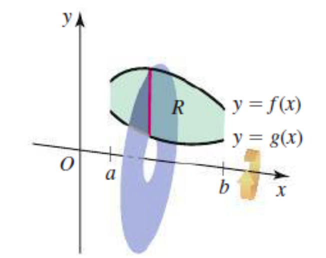
$\square$

Summary:

- The disk method is just a special case of the washer method. For **solids of revolution**, either method can be used. In practice, one method usually produces an integral that is easier to evaluate than the other method.
- The method of cylindrical shells is a second way of computing volumes of **solids of revolution**. It frequently leads to simpler computations than does the method of cross sections (disc method or washer method).

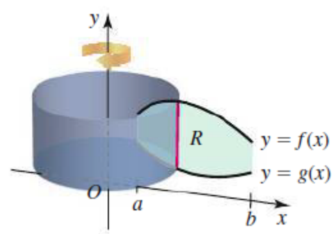
SUMMARY **Disk/Washer and Shell Methods**

Integration with respect to x



Disk/washer method about the x-axis
Disks/washers are *perpendicular* to the x-axis.

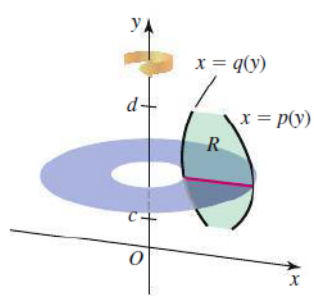
$$\int_a^b \pi(f(x)^2 - g(x)^2)\,dx$$



Shell method about the y-axis
Shells are *parallel* to the y-axis.

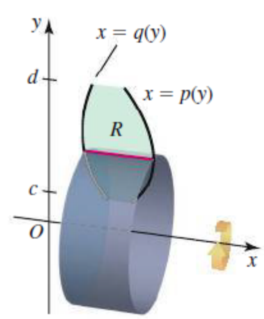
$$\int_a^b 2\pi x(f(x) - g(x))\,dx$$

Integration with respect to y



Disk/washer method about the y-axis
Disks/washers are *perpendicular* to the y-axis.

$$\int_c^d \pi(p(y)^2 - q(y)^2)\,dy$$

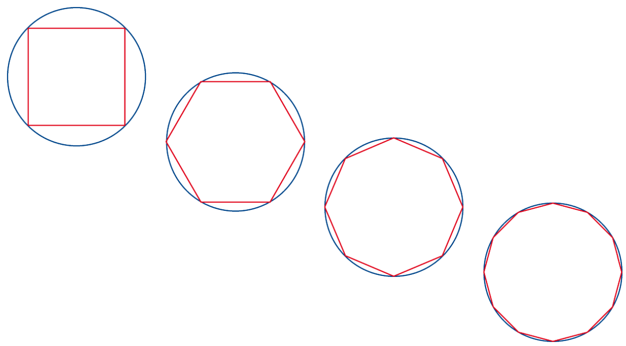


Shell method about the x-axis
Shells are *parallel* to the x-axis.

$$\int_c^d 2\pi y(p(y) - q(y))\,dy$$

6.5 §6.5 The length of Curves

As in all applications of definite integrals we have seen, the key is to obtain an appropriate Riemann Sum approximation. We treat the case of a function $y = f(x)$ having a continuous derivative $f'(x)$ on the interval $[a, b]$.



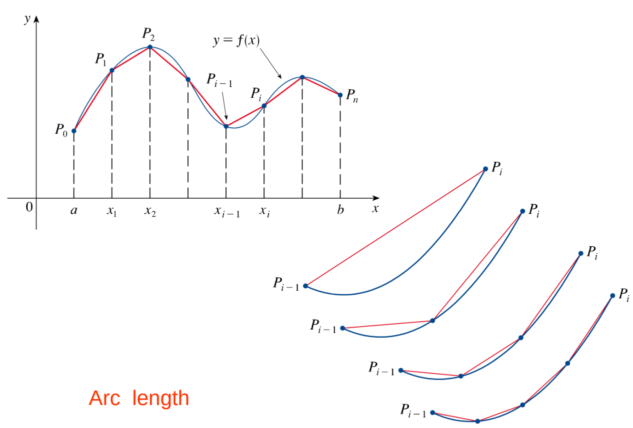
Circumference is the limit of lengths of inscribed polygons

Derivation:

Let $y = f(x)$ be a function with continuous derivative $f'(x)$ on the interval $[a, b]$. To find the Riemann Sum we use the following outline.

1. Partition the interval $[a, b]$ into n equal parts of length $\Delta x = \frac{b-a}{n}$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$



2. Form a polygonal path to approximate the graph of the curve $y = f(x)$ by connecting the points by straight line segments.

$$\begin{aligned} P_0 &= (x_0, f(x_0)) \\ P_1 &= (x_1, f(x_1)) \\ &\vdots \\ P_{n-1} &= (x_{n-1}, f(x_{n-1})) \\ P_n &= (x_n, f(x_n)) \end{aligned}$$

3. Find an expression for the distance $|P_{i-1}P_i|$ from P_{i-1} to P_i.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}. \end{aligned}$$

4. Apply the Mean Value Theorem for Derivatives to f on the interval $[x_{i-1}, x_i]$ to get

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x$$

for some x_i^* between x_{i-1} and x_i. Thus we have

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\ &= \sqrt{1 + (f'(x_i^*))^2}\Delta x. \end{aligned}$$

5. We now have

$$\text{Arc Length } = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \lim_{n\to\infty} \sum_{i-1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x.$$

Since $f'(x)$ is assumed to be continuous, $\sqrt{1 + [f'(x)]^2}$ is continuous and this definite integral has a finite value, i.e. $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ exists.

$\square$

Theorem (The Arc Length Formula)**.** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.}$$

Remark. If the curve is given by the equation $x = g(y), c \leq y \leq d$ and $g'(y)$ is continuous, then the length of the curve is given by

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2} dy.}$$

Example 1. Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$ (Figure 6.56).

Solution.

□

Example 2. Find the length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$.

Solution.

□

Example 3. Confirm that the circumference of a circle of radius r is $2\pi r$.

Solution.

$\square$

Example 4. Find the length of the curve $f(x) = x^2$ on the interval $[0, 2]$.

Solution.

$\square$

Example 5.[Arc length for $x = g(y)$] Find the length of the curve $y = f(x) = x^{2/3}$ between $x = 0$ and $x = 8$ (Figure 6.58).

Solution.

$\square$

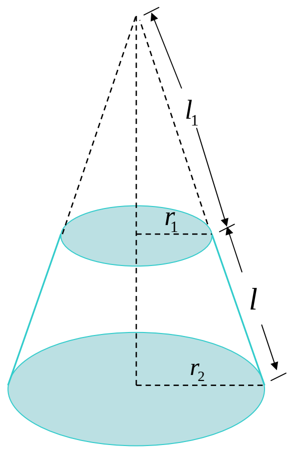
Example 6.[Arc length for $x = g(y)$] Find the length of the curve $y = f(x) = \ln(x + \sqrt{x^2 - 1})$ on the interval $[1, \sqrt{2}]$ (Figure 6.59).

Solution.

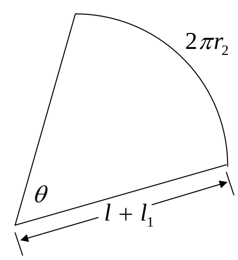
$\square$

6.6 §6.6 Surface Area

In this section, we find the surface area of a **surface of revolution** when the curve $y = f(x)$, $a \le x \le b$, is rotated about the x-axis. We approximate such a surface with frustums of a cone. Visualize a right circular cone of slant height $\ell_1 + \ell$ having base radius r_2 and cut the top off of the cone by removing a smaller cone of slant height ℓ_1. The remaining geometric object is a frustum.



Frustum from a cone



Surface area of the cone

We first find a suitable representation of the surface area of a frustum in terms of slant height ℓ and average radius $\frac{r_1+r_2}{2}$.

1. First observe that the surface area of a frustum is the difference of the surface areas of two cones.

2. Take our larger cone and cut it straight up the side from the base to the point making a cut of length $\ell_1 + \ell$, and lay it flat to obtain a sector of a circle whose radius is $\ell_1 + \ell$ the slant height of the cone. Observe that the circular perimeter is $2\pi r_2$, the perimeter of the base circle of our cone.

3. Observe that the area of the cone equals the area of this sector and use a geometric ratio to express the area of this sector of a circle in terms of the slant height $\ell_1 + \ell$ and the radius r_2.

Therefore,

$$\begin{aligned}
\text{Area of Cone } &= \text{Area of Sector of Circle}\\
&= \text{Fraction of Circle} \times \text{Area of Circle}\\
&= (\frac{\theta}{2\pi})(\pi(\ell_1+\ell)^2) = \frac{1}{2}(\ell_1+\ell)^2\theta\\
&= \frac{1}{2}(\ell_1+\ell)^2\big(2\pi\frac{r_2}{\ell_1+\ell}\big) \left\{ \frac{\theta}{2\pi} = \frac{2\pi r_2}{2\pi(\ell+\ell_1)} \qquad \text{Geo. Ratio} \right.\\
&= \pi(\ell_1+\ell)r_2\\
&= \pi \times \text{Slant Height of Cone} \times \text{Radius of Base circle}
\end{aligned}$$

4. Use similar triangles to express the surface area of a frustum in terms of slant height ℓ and average radius $\frac{r_1+r_2}{2}$.

$$\begin{aligned}
\frac{\ell_1}{r_1} = \frac{\ell_1+\ell}{r_2} &\implies \ell_1 r_2 = (\ell_1+\ell)r_1\\
&\implies (r_2-r_1)\ell_1 = r_1\ell
\end{aligned}$$

$$\begin{aligned}
\text{Area of Frustum } &= \text{Area of Big Cone} - \text{Area o Small Cone}\\
&= \pi(\ell+\ell_1)r_2 - \pi\ell_1 r_1\\
&= \pi r_1\ell + \pi\ell r_2 = \pi\ell(r_1+r_2)\\
&= \pi\ell\big(2\frac{r_1+r_2}{2}\big) = 2\pi\ell\Big(\underbrace{\frac{r_1+r_2}{2}}_{\text{ave. radius}}\Big)\\
&= 2\pi\ell r = 2\times\pi\times\text{slant ht.}\times\text{ave. radius}
\end{aligned}$$

It is now time to bring in the partition of our interval $[a, b]$ and revise the formula

$$\boxed{\text{Area of Frustum } = 2\pi r\ell}$$

using some of the same ideas that were used in arriving at our formula for arc length.

1. Partition the interval $[a, b]$ into n equal parts -

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

with each part having length $\Delta x = \frac{b-a}{n}$.

2. Form a polygonal path approximate the graph of the curve $y = f(x)$ by connecting the points by straight line segments.

$$\begin{aligned}
P_0 &= (x_0, f(x_0)) \\
P_1 &= (x_1, f(x_1)) \\
&\vdots \\
P_{n-1} &= (x_{n-1}, f(x_{n-1}) \\
P_n &= (x_n, f(x_n))
\end{aligned}$$

Each line segment $P_{i-1}P_i$ generates a frustum when rotated.

3. Apply the Mean Value Theorem for Derivatives to $f(x_i) - f(x_{i-1})$ to get a formula for the slant height

$$\begin{aligned}
\text{Slant Height} =& |P_{i-1}P_i| \\
&= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\
&= \sqrt{1 + (f'(x_i^*))^2}\Delta x
\end{aligned}$$

We now have that the surface area of a surface of revolution of the curve $y = f(x) \geq 0$, $a \leq x \leq b$, with continuous derivative is given by

$$\begin{aligned}
\text{Surface Area } &= \lim_{n\to\infty} \sum_{i=1}^{n} \text{Surface area of the } i^{th} \text{ Frustum} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\Delta x \\
&= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx = \int_a^b 2\pi y\sqrt{1 + (y')^2}\ dx
\end{aligned}$$

Theorem (Area of a Surface of Revolution). Suppose the function f is nonnegative and has a continues derivative. Then the **surface area** of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about the x-axis is

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx = \int_a^b 2\pi y\ ds,}$$

where $ds = \sqrt{1+(f'(x))^2}\ dx$.

Remark 1. Let's see how we have the second formula. Note that if we define an arc length function by

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\ dt$$

then by the Fundamental Theorem of Calculus Part I, $ds = \sqrt{1+(f'(x))^2}\ dx$ and so

$$\boxed{\text{Surface Area } = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx = \int_a^b 2\pi y\ ds.}$$

Remark 2. If the curve is given by the equation $x = g(y), c \le y \le d$, then the formula for the surface area (rotating the curve about the y-axis) is given by

$$\boxed{S = \int_c^d 2\pi x\sqrt{1+[g'(y)]^2}dy = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}dy.}$$

Example 1. The graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of the surface generated (Figure 6.65)?

Solution.

$\square$

Example 2. Find the surface area of a sphere of radius r.

Solution.

$\square$

Remark. Then we can find the surface area of a **spherical zone**.

Example 3. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1, 2]$ about the x-axis (Figure 6.67). Find the surface area.

Solution.

$\square$

Example 4. Consider the function $y = \ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$. Find the area of the surface generated when the part of the curve between the points $(\frac{5}{4}, 0)$ and $(\frac{17}{8}, \ln 2)$ is revolved about the y-axis (Figure 6.68).

Solution.

$\square$

More examples

Example. Find the surface area of the surface obtained when

$$y = 1 - x^2, \qquad 0 \le x \le 1$$

is rotated about the y-axis.

Solution. First, we find x as a function of y and the value of the radical $\sqrt{1+(\frac{dx}{dy})^2}$.

$$y = 1 - x^2 \implies x = f(y) = \sqrt{1-y} \text{ since } x \ge 0$$

$$\begin{aligned}\sqrt{1+(\frac{dx}{dy})^2} &= \sqrt{1+(-\frac{1}{2}\frac{1}{\sqrt{1-y}})^2}\\ &= \sqrt{1+\frac{1}{4(1-y)}}\end{aligned}$$

since $\frac{dx}{dy}$ is non-positive $x = f(y)$ is defined on the interval $0 \le y \le 1$ and

$$\begin{aligned}f(y)\sqrt{1+(f'(y))^2} &= \sqrt{1-y}\sqrt{1+(\frac{dx}{dy})^2}\\ &= \sqrt{1-y}\sqrt{1+\frac{1}{4(1-y)}}\\ &= \sqrt{1-y+\frac{1}{4}} = \sqrt{\frac{5}{4}-y} = \frac{1}{2}\sqrt{5-y}\end{aligned}$$

$$\begin{aligned}\text{Surface Area} &= \int_0^1 2\pi f(y)\sqrt{1+(f'(y))^2}\,dy\\ &= \int_0^1 2\pi\frac{1}{2}\sqrt{5-y}\,dy = \Big[-\frac{2\pi}{3}(5-y)^{\frac{3}{2}}\Big]_0^1\\ &= \frac{2\pi}{3}\Big((5)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\Big).\end{aligned}$$

□

Example. Show that surface area of **Gabriel's Horn** is infinite while its volume is finite. Gabriel's Horn is defined to be that surface obtained when

$$y = \frac{1}{x}, \qquad 1 \leq x$$

is rotated about the x-axis.

Solution. First, we find the value of the radical $\sqrt{1 + (y')^2}$.

$$y = x^{-1} \implies y' = -x^{-2} \implies \sqrt{1 + (y')^2} = \sqrt{1 + (-x^{-2})^2} = \sqrt{1 + x^{-4}}$$

Now, we want to show that

$$\int_1^\infty 2\pi x^{-1}\sqrt{1 + x^{-4}}\ dx \text{ is divergent}$$

$$\text{or equivalently that } \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\ dx \text{ is divergent.}$$

We do this by the Comparison Test. Notice that

$$x^{-1}\sqrt{1 + x^{-4}} > \frac{1}{x}$$

Now,

$$\int_1^\infty \frac{dx}{x} \text{ is divergent (p-Test with } p = 1)$$

$$\therefore \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\ dx \text{ is divergent (Comparison Test)}$$

Finally,

$$\begin{aligned}\text{Volume of Gabriel's Horn} &= \int_1^\infty \pi(\frac{1}{x})^2\ dx = \lim_{t\to\infty} \int_1^t \pi(\frac{1}{x})^2\ dx \\ &= \lim_{t\to\infty} \Big[-\pi x^{-1} \Big]_1^t = \lim_{t\to\infty} \Big[-\pi t^{-1} + \pi \Big]_1^t = \pi.\end{aligned}$$