Contents

In this brief chapter, we revisit the exponential and logarithmic functions.

7.1 §7.1 Logarithmic and Exponential Functions Revisited

Exponents

Here we go over the definition of x^y when x and y are arbitrary real numbers, with $x > 0$.

For any real number x and any positive integer $n = 1, 2, 3, \dots$ one defines

$$x^n = \overbrace{x \cdot x \cdot \dots \cdot x}^{n \text{ times}}$$

and, if $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}.$$

One defines $x^0 = 1$ for any $x \neq 0$.

To define $x^{p/q}$ for a general fraction $\frac{p}{q}$ one must assume that the number x is positive. One then defines

$$x^{p/q} = \sqrt[q]{x^p}.$$

It is shown in precalculus texts that the **power functions** satisfy the following properties:

$$\boxed{(xy)^a = x^a y^a, \quad x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad \left(x^a\right)^b = x^{ab}}$$

provided a and b are fractions. And these properties still hold if a and b are real numbers (not necessarily fractions.) We won't go through the proofs here.

Now instead of considering x^a as a function of x we can pick a positive number a and consider the **exponential function**

$$\boxed{f(x) = a^x.}$$

This function is defined for all real numbers x (as long as the base a is positive.).

The exponential function

$$\boxed{f(x) = e^x}$$

with base e (Euler's constant) is so prevalent in the sciences that it is often referred to as the **exponential function** or **the natural exponential function**. Recall that

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

and

$$\frac{de^x}{dx} = e^x.$$

Logarithmic Functions

The **logarithm function**, written $\log_a x$, is the inverse of the exponential function a^x. The function

$$f(x) = log_a x,$$

is called the **logarithmic function with base** a.

Theorem (Properties of Logs)**.** The following properties can be derived from the exponential function.

1. $\log_a(a^x) = x$
2. $a^{\log_a x} = x, \quad x > 0$
3. $\log_a(x^p) = p\log_a x, \quad x > 0$
4. $\log_a(A) = \dfrac{\log_b(A)}{\log_b(a)}$
5. $\log_a(B) + \log_a(C) = \log_a(BC)$
6. $\log_a(B) - \log_a(C) = \log_a(\frac{B}{C})$

The logarithm with base e is called the **Natural Logarithm**, and is written

$$\ln x = \log_e x.$$

Thus we have

$$\boxed{e^{\ln x} = x \qquad \ln e^x = x}$$

where the second formula holds for all real numbers x but the first one only makes sense for $x > 0$.

For any positive number a we have $a = e^{\ln a}$, and also

$$a^x = e^{x \ln a}.$$

By the chain rule you then get

$$\boxed{\frac{da^x}{dx} = a^x \ln a.}$$

Derivatives of Logarithms Since the natural logarithm is the inverse function of $f(x) = e^x$ we can find its derivative by implicit differentiation. Here is the computation (which you are supposed to be able to do yourself).

The function $f(x) = \log_a x$ satisfies

$$a^{f(x)} = x$$

Differentiate both sides, and use the chain rule on the left,

$$(\ln a) a^{f(x)} f'(x) = 1.$$

Then solve for $f'(x)$ to get

$$f'(x) = \frac{1}{(\ln a) a^{f(x)}}.$$

Finally we remember that $a^{f(x)} = x$ which gives us the derivative of a^x

$$\frac{da^x}{dx} = \frac{1}{x \ln a}.$$

In particular, the natural logarithm has a very simple derivative, namely, since $\ln e = 1$ we have

$$\boxed{\frac{d \ln x}{dx} = \frac{1}{x}.}$$

Therefore,

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0.}$$

Theorem. More Properties of Logs and Exponentials:

- $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}, \quad x \neq 0$
- $\dfrac{d}{dx}(\ln|u(x)|) = \dfrac{u^{'}(x)}{u(x)}, \quad u(x) \neq 0$
- $\int \frac{1}{x}\ dx = \ln|x| + C$
- $\frac{d}{dx}(e^{u(x)}) = e^{u(x)}u^{'}(x)$
- $\int e^x\ dx = e^x + C$

Example 1. Evaluate $\int_0^4 \dfrac{x}{x^2+9}\ dx$.

Solution.

$\square$

Example 2. Evaluate $\int \dfrac{e^x}{1+e^x}\ dx$.

Solution.

$\square$

Theorem (Derivatives and Integrals with Other Bases)**.** Let $b > 0$ and $b \neq 1$. Then

$$\frac{d}{dx}(\log_b |u(x)|) = \frac{1}{\ln b}\frac{u'(x)}{u(x)}, \quad u(x) \neq 0$$

$$\frac{d}{dx}(b^{u(x)}) = (\ln b)b^{u(x)}u'(x)$$

$$\boxed{\int b^x \ dx = \frac{1}{\ln b}b^x + C.}$$

Example 3. Evaluate the following integrals.

a. $\int x3^{x^2} \ dx$.

Solution.

□

b. $\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}} \ dx$.

Solution.

□

Theorem (General Power Rule)**.** For any real number p,

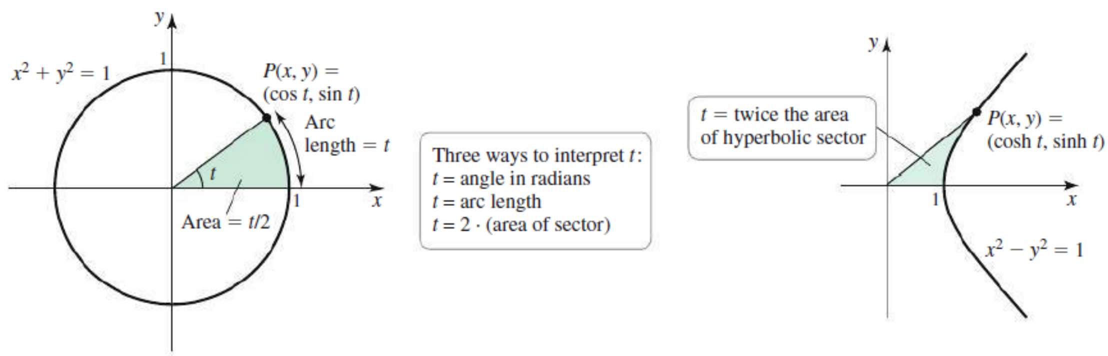
$$\boxed{\frac{d}{dx}(x^p) = px^{p-1} \quad \textit{and} \quad \frac{d}{dx}(u^p(x)) = pu^{p-1}(x)u'(x)}$$

Example 4 - Logarithmic Differentiation. Evaluate the derivative of $f(x) = x^{2x}$.

Solution.

7.3 §7.3 Hyperbolic Functions

In this section, we introduce a new family of functions called the hyperbolic functions, which are closely related to both trigonometric functions and exponential functions. Hyperbolic functions find widespread use in applied problems in fluid dynamics, projectile motion, architecture, and electrical engineering, to name just a few areas. Hyperbolic functions are also important in the development of many theoretical results in mathematics.



Definition.

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant

$$sechx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$cschx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Please read the textbook about the Calculus of these Hyperbolic functions if you are interested.