8.5 Partial Fractions

8.5.1 Proper rational functions can be integrated using partial fraction decomposition.

8.5.2 Your answers may vary.

a. $x - 1$.

b. $(x-1)^3$.

c. $x^2 + x + 1$.

d. $(x^2 + x + 1)^2$

8.5.3

a. $\dfrac{A}{x-3}$.

b. $\dfrac{A_1}{x-4}, \dfrac{A_2}{(x-4)^2}, \dfrac{A_3}{(x-4)^3}$.

c $\dfrac{Ax+B}{x^2+2x+6}$.

8.5.4 The first step is to divide the numerator by the denominator via long division in order to write the quotient as the sum of a polynomial and a proper rational function. Thus we would write

$$\frac{x^2+2x-3}{x+1} = x + 1 - \frac{4}{x+1}.$$

8.5.5 $\dfrac{4x}{(x-4)(x-5)} = \dfrac{A}{x-4} + \dfrac{B}{x-5}$.

8.5.6 $\dfrac{4x+1}{(2x-1)(2x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1}$.

8.5.7 $\dfrac{x+3}{(x-5)^2} = \dfrac{A}{x-5} + \dfrac{B}{(x-5)^2}$.

8.5.8 $\dfrac{2}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

8.5.9 $\dfrac{4}{x(x+1)(x-1)(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{D}{x+2} + \dfrac{E}{x-2}$.

8.5.10 $\dfrac{20x}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1}$.

8.5.11 $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.

8.5.12 $\dfrac{2x^2+3}{(x^2-8x+16)(x^2+3x+4)} = \dfrac{2x^2+3}{(x-4)^2(x^2+3x+4)} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{Cx+D}{x^2+3x+4}$.

8.5.13 $\dfrac{x^4+12x^2}{(x-2)^2(x^2+x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+x+2} + \dfrac{Ex+F}{(x^2+x+2)^2}$.

8.5.14 $\dfrac{6x^4-4x^3+15x^2-5x+7}{(x-2)(2x^2+3)^2} = \dfrac{A}{x-2} + \dfrac{Bx+C}{2x^2+3} + \dfrac{Dx+E}{(2x^2+3)^2}$.

8.5.15 $\dfrac{x}{(x-2)^2(x+2)^2(x^2+4)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} + \dfrac{Ex+F}{x^2+4} + \dfrac{Gx+H}{(x^2+4)^2}$.

8.5.16 $\dfrac{x^2+2x+6}{x^3(x^2+x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+x+1} + \dfrac{Fx+G}{(x^2+x+1)^2}$.

8.5.17 $\dfrac{5x-7}{x^2-3x+2} = \dfrac{5x-7}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$. Thus, $A(x-2)+B(x-1) = 5x-7$. Equating coefficients gives $A+B=5$ and $-2A-B=-7$. Solving this system yields $A=2$, $B=3$. Thus,

$$\frac{5x-7}{x^2-3x+2} = \frac{2}{x-1} + \frac{3}{x-2}.$$

8.5.18 $\dfrac{11x-10}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$. Thus, $A(x-1)+Bx = 11x-10$. Equating coefficients gives $A+B=11$, $-A=-10$. Solving this system yields $A=10$, $B=1$. Thus,

$$\frac{11x-10}{x^2-x} = \frac{10}{x} + \frac{1}{x-1}.$$

8.5.19 $\dfrac{6}{x^2-2x-8} = \dfrac{6}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$. Thus, $6 = A(x+2)+B(x-4)$. Equating coefficients gives $A+B=0$ and $2A-4B=6$. Solving this system yields $A=1$ and $B=-1$. Thus,

$$\frac{6}{x^2-2x-8} = \frac{1}{x-4} - \frac{1}{x+2}.$$

8.5.20 $\dfrac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$. Thus, $x^2-4x+11 = A(x-1)(x+1) + B(x-3)(x+1) + C(x-3)(x-1)$. Letting $x=1$ gives $B=-2$, letting $x=-1$ gives $C=2$, and letting $x=3$ gives $A=1$. Thus,

$$\frac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \frac{1}{x-3} + \frac{-2}{x-1} + \frac{2}{x+1}.$$

8.5.21 By long division, $\dfrac{2x^2+5x+6}{x^2+3x+2} = 2 + \dfrac{-x+2}{(x+1)(x+2)}$. We use a partial fraction decomposition on $\dfrac{-x+2}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Therefore $A(x+2)+B(x+1) = -x+2$, so $A+B=-1$ and $2A+B=2$. So $A=3$ and $B=-4$. Thus

$$\frac{2x^2+5x+6}{x^2+3x+2} = 2 + \frac{3}{x+1} - \frac{4}{x+2}.$$

8.5.22 By long division, we can write $\dfrac{x^4+2x^3+x}{x^2-1}$ as $x^2+2x+1+\dfrac{3x+1}{x^2-1} = (x+1)^2 + \dfrac{3x+1}{(x-1)(x+1)}$. We use a partial fraction decomposition on $\dfrac{3x+1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$. We have $A(x+1)+B(x-1) = 3x+1$, so $A+B=3$ and $A-B=1$. Therefore, $A=2$ and $B=1$. We have

$$\frac{x^4+2x^3+x}{x^2-1} = (x+1)^2 + \frac{2}{x-1} + \frac{1}{x+1}.$$

8.5.23 If we write $\dfrac{3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$, we have $3 = A(x+2)+B(x-1)$. Letting $x=-2$ yields $B=-1$ and letting $x=1$ yields $A=1$. Thus, the original integral is equal to

$$\int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx = \ln|x-1| - \ln|x+2| + C = \ln\left|\frac{x-1}{x+2}\right| + C.$$

8.5.24 If we write $\dfrac{8}{(x-2)(x+6)} = \dfrac{A}{x-2} + \dfrac{B}{x+6}$, we have $8 = A(x+6) + B(x-2)$. Letting $x = -6$ yields $B = -1$ and letting $x = 2$ yields $A = 1$. Thus the original integral is equal to

$$\int \left(\frac{1}{x-2} - \frac{1}{x+6} \right) dx = \ln|x-2| - \ln|x+6| + C.$$

8.5.25 If we write $\dfrac{6}{x^2-1} = \dfrac{6}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$, then we have $6 = A(x+1) + B(x-1)$. Letting $x = -1$ yields $B = -3$ and letting $x = 1$ yields $A = 3$. Thus, the original integral is equal to

$$\int \left(\frac{3}{x-1} - \frac{3}{x+1} \right) dx = 3\left(\ln|x-1| - \ln|x+1|\right) + C = 3\ln\left|\frac{x-1}{x+1}\right| + C.$$

8.5.26 If we write $\dfrac{1}{t^2-9} = \dfrac{A}{t-3} + \dfrac{B}{t+3}$, then we have $1 = A(t+3) + B(t-3)$. Letting $t = -3$ yields $B = -1/6$ and letting $t = 3$ yields $A = 1/6$. Thus the original integral is equal to

$$\int_0^1 \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) dt = \frac{1}{6}\left(\ln|t-3| - \ln|t+3|\right)\Big|_0^1 = -\frac{\ln 2}{6}.$$

8.5.27 $\dfrac{8x-5}{(x-1)(3x-2)} = \dfrac{A}{x-1} + \dfrac{B}{3x-2}$. Thus $8x - 5 = A(3x-2) + B(x-1) = (3A+B)x - 2A - B$. So $3A + B = 8$ and $-2A - B = -5$. Solving gives $A = 3$ and $B = -1$. We have

$$\int \frac{8x-5}{3x^2-5x+2}\, dx = \int \left(\frac{3}{x-1} - \frac{1}{3x-2} \right) dx = 3\ln|x-1| - \frac{1}{3}\ln|3x-2| + C.$$

8.5.28 $\dfrac{7x-2}{3x^2-2x} = \dfrac{7x-2}{x(3x-2)} = \dfrac{A}{x} + \dfrac{B}{3x-2}$. Then $7x - 2 = A(3x-2) + Bx = (3A+B)x - 2A$. So $A = 1$ and $B = 4$. We have

$$\int_1^2 \frac{7x-2}{3x^2-2x}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{4}{3x-2} \right) dx = \left(\ln x + \frac{4}{3}\ln(3x-2) \right)\Big|_1^2 = \ln 2 + \frac{4}{3}\ln 4 = \ln 2 + \frac{8}{3}\ln 2 = \frac{11\ln 2}{3}.$$

8.5.29 If we write $\dfrac{5x}{x^2-x-6} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$, then we have $5x = A(x+2) + B(x-3)$. Letting $x = -2$ yields $B = 2$ and letting $x = 3$ yields $A = 3$. Thus the original integral is equal to

$$\int_{-1}^2 \left(\frac{3}{x-3} + \frac{2}{x+2} \right) dx = \left(3\ln|x-3| + 2\ln|x+2|\right)\Big|_{-1}^2 = \ln(16) - \ln(64) = -\ln 4.$$

8.5.30 If we write $\dfrac{21x^2}{x^3-x^2-12x} = \dfrac{21x^2}{x(x-4)(x+3)} = \dfrac{21x}{(x-4)(x+3)} = \dfrac{A}{x-4} + \dfrac{B}{x+3}$, then we have $21x = A(x+3) + B(x-4)$. Letting $x = 4$ yields $A = 12$. Letting $x = -3$ yields $B = 9$. Thus, the original integral is equal to

$$\int \left(\frac{12}{x-4} + \frac{9}{x+3} \right) dx = 12\ln|x-4| + 9\ln|x+3| + C.$$

8.5.31 Let $\dfrac{6x^2}{x^4-5x^2+4} = \dfrac{6x^2}{(x-2)(x+2)(x-1)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$. Then

$$6x^2 = A(x+2)(x-1)(x+1) + B(x-2)(x-1)(x+1) + C(x-2)(x+2)(x+1) + D(x-2)(x+2)(x-1).$$

Letting $x = 2$ gives $A = 2$, letting $x = -2$ gives $B = -2$, letting $x = 1$ gives $C = -1$, and letting $x = -1$ gives $D = 1$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x-2} - \frac{2}{x+2} - \frac{1}{x-1} + \frac{1}{x+1} \right) dx = \ln\left|\frac{(x-2)^2(x+1)}{(x+2)^2(x-1)}\right| + C.$$

8.5.32 Let $\dfrac{4x-2}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$. Thus,

$$4x - 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1).$$

Letting $x = 0$ gives $A = 2$, letting $x = 1$ gives $B = 1$, and letting $x = -1$ gives $C = -3$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1}\right) dx = \ln\left|\frac{x^2(x-1)}{(x+1)^3}\right| + C.$$

8.5.33 After performing long division, we have that the original integrand is equal to $3 + \dfrac{13x-12}{(x-1)(x-2)}$, and if we write $\dfrac{13x-12}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$, then $13x - 12 = A(x-2) + B(x-1)$. Letting $x = 1$ yields $A = -1$ and letting $x = 2$ yields $B = 14$. Thus the original integral is equal to

$$3x - \int \frac{1}{x-1}\, dx + 14\int \frac{1}{x-2}\, dx = 3x - \ln|x-1| + 14\ln|x-2| + C.$$

8.5.34 After performing long division, we have that the original integrand is equal to $2z-1+\dfrac{7z+1}{(z+3)(z-2)}$. If we write $\dfrac{7z+1}{(z+3)(z-2)} = \dfrac{A}{z+3} + \dfrac{B}{z-2}$, then $7z + 1 = A(z-2) + B(z+3)$. Letting $z = 2$ yields $B = 3$ and letting $z = -3$ yields $A = 4$. Thus, the original integral is equal to

$$\int \left(2z - 1 + \frac{4}{z+3} + \frac{3}{z-2}\right) dz = z^2 - z + 4\ln|z+3| + 3\ln|z-2| + C.$$

8.5.35 Let $\dfrac{x^2+12x-4}{x(x-2)(x+2)} = \dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x+2}$. Then $x^2+12x-4 = A(x-2)(x+2)+Bx(x+2)+Cx(x-2)$. Letting $x = 0$ gives $A = 1$, letting $x = 2$ gives $B = 3$, and letting $x = -2$ gives $C = -3$. Thus, the original integral is equal to $\displaystyle\int \left(\frac{1}{x} + \frac{3}{x-2} - \frac{3}{x+2}\right) dx = \ln\left|\frac{x(x-2)^3}{(x+2)^3}\right| + C.$

8.5.36 Let $\dfrac{z^2+20z-15}{z(z+5)(z-1)} = \dfrac{A}{z}+\dfrac{B}{z+5}+\dfrac{C}{z-1}$. Then $z^2+20z-15 = A(z+5)(z-1)+Bz(z-1)+Cz(z+5)$. Letting $z = 0$ gives $A = 3$, letting $z = -5$ gives $B = -3$, and letting $z = 1$ gives $C = 1$. Thus, the original integral is equal to $\displaystyle\int \left(\frac{3}{z} - \frac{3}{z+5} + \frac{1}{z-1}\right) dz = \ln\left|\frac{z^3(z-1)}{(z+5)^3}\right| + C.$

8.5.37 If we write $\dfrac{1}{x^4-10x^2+9} = \dfrac{1}{(x-1)(x+1)(x-3)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x-3} + \dfrac{D}{x+3}$ then $1 = A(x+1)(x-3)(x+3) + B(x-1)(x-3)(x+3) + C(x-1)(x+1)(x+3) + D(x-1)(x+1)(x-3)$. Letting $x = -1$ yields $B = 1/16$. Letting $x = 3$ yields $C = 1/48$. Letting $x = -3$ yields $D = -1/48$, and letting $x = 1$ yields $A = -1/16$. Thus the original integral is equal to

$$\begin{aligned}&\int \left(-\frac{1/16}{x-1} + \frac{1/16}{x+1} + \frac{1/48}{x-3} - \frac{1/48}{x+3}\right) dx\\ &= -\frac{1}{16}\ln|x-1| + \frac{1}{16}\ln|x+1| + \frac{1}{48}\ln|x-3| - \frac{1}{48}\ln|x+3| + C\\ &= \ln\left|\frac{(x+1)^3(x-3)}{(x-1)^3(x+3)}\right|^{1/48} + C.\end{aligned}$$

8.5.38 If we write $\dfrac{2}{x^2-4x-32} = \dfrac{A}{x-8} + \dfrac{B}{x+4}$, then we have $2 = A(x+4) + B(x-8)$. Letting $x = -4$ yields $B = -1/6$ and letting $x = 8$ yields $A = 1/6$. Thus, the original integral is equal to

$$\int_0^5 \left(\frac{1/6}{x-8} - \frac{1/6}{x+4}\right) dx = \frac{1}{6}\left(\ln|x-8| - \ln|x+4|\right)\Big|_0^5 = \frac{1}{6}(\ln(1/3) - \ln 2) = -\frac{\ln 6}{6}.$$

8.5.39 If we write $\frac{81}{x^3-9x^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}$, then $81 = Ax(x-9)+B(x-9)+C(x^2)$. Letting $x=0$ yields $B=-9$. Letting $x=9$ yields $C=1$. If we let $x=10$, then we have $81 = 10A+B+100C = 10A-9+100$, so $A=-1$. Thus, the original integral is equal to $\int \left(-\frac{1}{x}-\frac{9}{x^2}+\frac{1}{x-9}\right)\,dx = \ln\left|\frac{(x-9)}{x}\right| + \frac{9}{x}+C.$

8.5.40 If we write $\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$, then we have $16x^2 = A(x+2)^2+B(x-6)(x+2)+C(x-6)$. Letting $x=6$ yields $A=9$. Letting $x=-2$ yields $C=-8$. Letting $x=0$ gives $0=36-12B+48$, so $B=7$. Thus the original integral is equal to

$$\int \left(\frac{9}{x-6}+\frac{7}{x+2}-\frac{8}{(x+2)^2}\right)\,dx = \ln\left|(x+2)^7(x-6)^9\right| + \frac{8}{x+2}+C.$$

8.5.41 If we write $\frac{x}{(x+3)^2} = \frac{A}{x+3}+\frac{B}{(x+3)^2}$, then we have $x = A(x+3)+B$. Letting $x=-3$ yields $B=-3$, and then letting $x=-2$ yields $A=1$. Thus the original integral is equal to

$$\int_{-1}^{1} \left(\frac{1}{x+3}-\frac{3}{(x+3)^2}\right)\,dx = \left(\ln|x+3|+\frac{3}{x+3}\right)\Big|_{-1}^{1} = \ln 4+\frac{3}{4}-\left(\ln 2+\frac{3}{2}\right) = \ln 2-\frac{3}{4}.$$

8.5.42 If we write $\frac{1}{x^3-2x^2-4x+8} = \frac{1}{(x+2)(x-2)^2} = \frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$, then $1 = A(x-2)^2 + B(x+2)(x-2)+C(x+2)$. Letting $x=2$ yields $C=1/4$. Letting $x=-2$ yields $A=1/16$. Letting $x=3$ yields $1=\frac{1}{16}+5B+5\cdot\frac{1}{4}$, so $B=-\frac{1}{16}$. Thus, the original integral is equal to

$$\int \left(\frac{1/16}{x+2}-\frac{1/16}{x-2}+\frac{1/4}{(x-2)^2}\right)\,dx = \frac{1}{16}\left(\ln|x+2|-\ln|x-2|\right)-\frac{1}{4(x-2)}+C.$$

8.5.43 If we write $\frac{2}{x^3+x^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$, then $2 = Ax(x+1)+B(x+1)+Cx^2$. Letting $x=0$ yields $B=2$, and letting $x=-1$ yields $C=2$. Then letting $x=1$ yields $A=-2$. So the original integral is equal to

$$\int \left(-\frac{2}{x}+\frac{2}{x^2}+\frac{2}{x+1}\right)\,dx = 2\left(\ln|x+1|-\ln|x|\right)-\frac{2}{x}+C.$$

8.5.44 If we write $\frac{2}{t^3(t+1)} = \frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t+1}$ then we have $2 = At^2(t+1)+Bt(t+1)+C(t+1)+Dt^3$. Letting $t=0$ yields $C=2$, and letting $t=-1$ reveals that $D=-2$. Now if we let $t=1$ we have that $2=2A+2B+4-2$, so $A=-B$. Letting $t=2$ yields the equation $12=12A+6B=12A-6A=6A$, so $A=2$ and $B=-2$. So the original integral is equal to

$$\int_1^2 \left(\frac{2}{t}-\frac{2}{t^2}+\frac{2}{t^3}-\frac{2}{t+1}\right)\,dt = \left(2\left(\ln|t|-\ln|t+1|\right)+\frac{2}{t}-\frac{1}{t^2}\right)\Big|_1^2 = \ln(16/9)-1/4.$$

8.5.45 If we write $\frac{x-5}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$, then we have $x-5 = Ax(x+1)+B(x+1)+Cx^2$. Letting $x=0$ yields $B=-5$, and letting $x=-1$ yields $C=-6$. Then letting $x=1$ yields $-4=2A-10-6$, so $A=6$. The original integral is thus equal to

$$\int \left(\frac{6}{x}-\frac{5}{x^2}-\frac{6}{x+1}\right)\,dx = 6\left(\ln|x|-\ln|x+1|\right)+\frac{5}{x}+C.$$

8.5.46 Let $\frac{x^2}{(x-2)^3} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$. Then $x^2 = A(x-2)^2+B(x-2)+C$. Letting $x=2$ gives $C=4$. Letting $x=0$ gives $0=4A-2B+4$, and letting $x=1$ gives $1=A-B+4$. Solving the system of two linear equations results in $A=1$ and $B=4$. The original integral is therefore equal to

$$\int \left(\frac{1}{x-2}+\frac{4}{(x-2)^2}+\frac{4}{(x-2)^3}\right)\,dx = \ln|x-2|-\frac{4}{x-2}-\frac{2}{(x-2)^2}+C.$$

8.5.47 By long division, we can write the integrand as $x + \frac{2x}{(x-5)^2}$. We write

$$\frac{2x}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2},$$

so that $A(x-5) + B = 2x$. Then $A = 2$ and $B = 10$. We have

$$\int \frac{x^3 - 10x^2 + 27x}{x^2 - 10x + 25}\,dx = \int \left(x + \frac{2}{x-5} + \frac{10}{(x-5)^2}\right)dx = \frac{x^2}{2} + 2\ln|x-5| - \frac{10}{x-5} + C.$$

8.5.48 By long division, we can write the integrand as $1 + \frac{2x^2 - x + 2}{x(x-1)^2}$. We write

$$\frac{2x^2 - x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

So

$$2x^2 - x + 2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Letting $x = 0$ gives $A = 2$. Letting $x = 1$ gives $C = 3$. Letting $x = 2$ then gives $8 = 2 + 2B + 6$, so $B = 0$. We have

$$\int \frac{x^3 + 2}{x^3 - 2x^2 + x}\,dx = \int \left(1 + \frac{2}{x} + \frac{3}{(x-1)^2}\right)dx = x + 2\ln|x| - \frac{3}{x-1} + C.$$

8.5.49 Let $\frac{x^2 - 4}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$. Then $x^2 - 4 = A(x-1)^2 + Bx(x-1) + Cx$. Letting $x = 0$ gives $A = -4$. Letting $x = 1$ gives $C = -3$. Letting $x = 2$ gives $0 = -4 + 2B - 6$, so $B = 5$. The given integral is thus equal to

$$\int \left(-\frac{4}{x} + \frac{5}{x-1} - \frac{3}{(x-1)^2}\right)dx = \ln\left|\frac{(x-1)^5}{x^4}\right| + \frac{3}{x-1} + C.$$

8.5.50 Let $\frac{8(x^2+4)}{x(x^2+8)} = \frac{Ax + B}{x^2 + 8} + \frac{C}{x}$. Then $8(x^2+4) = Ax^2 + Bx + C(x^2+8)$. Letting $x = 0$ gives $C = 4$. Letting $x = 1$ gives $40 = A + B + 36$, so $A + B = 4$. Letting $x = -1$ gives $A - B = 4$, so $A = 4$ and $B = 0$. The original integral is thus equal to

$$\int \left(\frac{4x}{x^2+8} + \frac{4}{x}\right)dx = 2\ln(x^2+8) + 4\ln|x| + C = \ln((x^2+8)^2x^4) + C.$$

8.5.51 Let $\frac{x^2 + x + 2}{(x+1)(x^2+1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+1}$. Then $x^2 + x + 2 = (Ax + B)(x+1) + C(x^2+1)$. Letting $x = -1$ gives $C = 1$. Letting $x = 0$ gives $2 = B + 1$, so $B = 1$. Letting $x = 1$ gives $4 = 2A + 2 + 2$, so $A = 0$. The original integral is therefore equal to

$$\int \left(\frac{1}{x^2+1} + \frac{1}{x+1}\right)dx = \tan^{-1}(x) + \ln|x+1| + C.$$

8.5.52 Let $\frac{x^2 + 3x + 2}{x(x^2 + 2x + 2)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x}$. Then $x^2 + 3x + 2 = Ax^2 + Bx + C(x^2 + 2x + 2)$. Letting $x = 0$ gives $C = 1$. Letting $x = 1$ gives $6 = A + B + 5$, so $A + B = 1$. Letting $x = -1$ gives $0 = A - B + 1$. Solving the system of linear equations gives $A = 0$ and $B = 1$. The original integral is therefore equal to

$$\begin{aligned}\int \left(\frac{1}{x^2 + 2x + 2} + \frac{1}{x}\right)dx &= \int \left(\frac{1}{(x^2 + 2x + 1) + 1} + \frac{1}{x}\right)dx \\ &= \int \left(\frac{1}{(x+1)^2 + 1} + \frac{1}{x}\right)dx = \tan^{-1}(x+1) + \ln|x| + C.\end{aligned}$$

8.5.53 Let $\frac{2x^2+5x+5}{(x+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+2x+2}+\frac{C}{x+1}$. Then $2x^2+5x+5 = (Ax+B)(x+1)+C(x^2+2x+2)$. Letting $x=-1$ gives $C=2$. Letting $x=0$ gives $5=B+4$, so $B=1$. Letting $x=1$ gives $12=2A+2+2(5)$, so $A=0$. The original integral is therefore equal to

$$\int\left(\frac{1}{x^2+2x+2}+\frac{2}{x+1}\right)dx = \int\left(\frac{1}{(x+1)^2+1}+\frac{2}{x+1}\right)dx = \tan^{-1}(x+1)+\ln((x+1)^2)+C.$$

8.5.54 If we write $\frac{z+1}{z(z^2+4)} = \frac{A}{z}+\frac{Bz+C}{z^2+4}$, then we have that $z+1 = A(z^2+4)+(Bz+C)z$. Letting $z=0$ yields $A=1/4$, and we have $z+1=(1/4+B)z^2+Cz+1$, so equating coefficients gives $B=-1/4$ and $C=1$. So the original integral is equal to $\int\left(\frac{1}{4z}-\frac{z}{4(z^2+4)}+\frac{1}{z^2+4}\right)dz$. The middle term can be handled via the substitution $u=z^2+4$, and the last term is recognizable as the derivative of $\frac{1}{2}\tan^{-1}(z/2)$. Thus the original integral is equal to

$$\frac{1}{4}\ln|z|-\frac{1}{8}\ln(z^2+4)+\frac{1}{2}\tan^{-1}\frac{z}{2}+C.$$

8.5.55 If we write $\frac{20x}{(x-1)(x^2+4x+5)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+4x+5}$, then $20x = A(x^2+4x+5)+(Bx+C)(x-1)$. Letting $x=1$ yields $A=2$. Letting $x=0$ yields $0=10-C$, so $C=10$. Letting $x=2$ yields $40=34+2B+10$, so $B=-2$. The original integral is thus

$$\begin{aligned}\int\left(\frac{2}{x-1}-\frac{2x-10}{x^2+4x+5}\right)dx &= \int\left(\frac{2}{x-1}-\frac{(2x+4)-14}{x^2+4x+5}\right)dx\\ &= \int\left(\frac{2}{x-1}-\frac{(2x+4)}{x^2+4x+5}+\frac{14}{(x+2)^2+1}\right)dx\\ &= \ln\left|\frac{(x-1)^2}{x^2+4x+5}\right|+14\tan^{-1}(x+2)+C.\end{aligned}$$

8.5.56 Note that this rational function is already in decomposition form, so any attempt to decompose it further will be futile. Instead we write the given integral as the sum $\int\frac{2x}{x^2+4}\,dx+\int\frac{1}{x^2+4}\,dx$. For the first integral, let $u=x^2+4$ so that $du=2x\,dx$. It is then equal to $\int\frac{1}{u}\,du=\ln|x^2+4|+C$. The second integral can be written as $\frac{1}{4}\int\frac{1}{(x/2)^2+1}\,dx=\frac{1}{2}\tan^{-1}\frac{x}{2}+D$. So the original integral is equal to

$$\ln|x^2+4|+\frac{1}{2}\tan^{-1}\frac{x}{2}+E.$$

8.5.57 $\frac{x^3+5x}{(x^2+3)^2} = \frac{Ax+B}{x^2+3}+\frac{Cx+D}{(x^2+3)^2}$. Then

$$x^3+5x=(Ax+B)(x^2+3)+Cx+D=Ax^3+Bx^2+(3A+C)x+3B+D.$$

Then $A=1$, $B=0$, $3A+C=5$, and $3B+D=0$. So $C=2$ and $D=0$. We have

$$\int\frac{x^3+5x}{(x^2+3)^2}\,dx=\int\left(\frac{x}{x^2+3}+\frac{2x}{(x^2+3)^2}\right)dx=\frac{1}{2}\ln|x^2+3|-\frac{1}{x^2+3}+C.$$

8.5.58 After performing long division, we have that the original integrand is equal to $x-\frac{9x^2-1}{x(x^2+9)}$. If we write $\frac{9x^2-1}{x(x^2+9)}=\frac{A}{x}+\frac{Bx+C}{x^2+9}$, we have $9x^2-1=A(x^2+9)+(Bx+C)x$. Letting $x=0$ yields $A=-1/9$, and letting $x = 1$ yields $8 = 10(-1/9) + B + C$. Letting $x = -1$ yields $8 = 10(-1/9) + B - C$. Thus $C = 0$ and therefore $B = 82/9$. Therefore the original integral is equal to

$$\frac{x^2}{2} + \frac{1}{9}\ln|x| - \frac{82}{9}\int \frac{x}{x^2+9}\,dx = \frac{x^2}{2} + \frac{1}{9}\ln|x| - \frac{41}{9}\ln|x^2+9| + C.$$

8.5.59 $\dfrac{x^3+6x^2+12x+6}{(x^2+6x+10)^2} = \dfrac{Ax+B}{x^2+6x+10} + \dfrac{Cx+D}{(x^2+6x+10)^2}$. Then

$x^3+6x^2+12x+6 = (Ax+B)(x^2+6x+10)+Cx+D = Ax^3+(B+6A)x^2+(10A+6B+C)x+10B+D.$

Then $A = 1$, $B = 0$, $C = 2$, and $D = 6$. The given integral is then

$$\int \left(\frac{x}{x^2+6x+10} + \frac{2x+6}{(x^2+6x+10)^2}\right)dx = \int \frac{x}{(x+3)^2+1}\,dx - \frac{1}{x^2+6x+10} + C.$$

Note that

$$\int \frac{x}{(x+3)^2+1}\,dx = \int\left(\frac{x+3}{(x+3)^2+1} - \frac{3}{(x+3)^2+1}\right)dx = \frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) + C.$$

So the given integral is equal to

$$\frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) - \frac{1}{x^2+6x+10} + C.$$

8.5.60 If we write $\dfrac{1}{(y^2+1)(y^2+2)} = \dfrac{Ay+B}{y^2+1} + \dfrac{Cy+D}{y^2+2}$ then

$$1 = (Ay+B)(y^2+2) + (Cy+D)(y^2+1) = (A+C)y^3 + (B+D)y^2 + (2A+C)y + 2B + D.$$

Equating coefficients gives us the equations $A + C = 0$, $B + D = 0$, $2A + C = 0$, and $2B + D = 1$. Solving this system yields $A = C = 0$, $B = 1$ and $D = -1$. Thus the original integral is equal to

$$\int\left(\frac{1}{y^2+1} - \frac{1}{y^2+2}\right)dy = \tan^{-1} y - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C.$$

8.5.61 If we write $\dfrac{2}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$, then

$$2 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

Letting $x = 0$ yields $A = 2$. Expanding the right-hand side yields $2 = (2+B)x^4 + Cx^3 + (4+B+D)x^2 + (C+E)x + 2$. Equating coefficients gives us the equations $2+B = 0$, $C = 0$, $4+B+D = 0$, and $C+E = 0$, from which we can deduce that $B = -2$, $C = 0$, $D = -2$ and $E = 0$. The original integral is thus equal to

$$\int\left(\frac{2}{x} - \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}\right)dx = 2\ln|x| - \ln|x^2+1| + \frac{1}{x^2+1} + C.$$

8.5.62 If we write $\dfrac{1}{(x+1)(x^2+2x+2)^2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2} + \dfrac{Dx+E}{(x^2+2x+2)^2}$, then

$$1 = A(x^2+2x+2)^2 + (Bx+C)(x+1)(x^2+2x+2) + (Dx+E)(x+1).$$

Letting $x = -1$ yields $A = 1$. Then expanding the polynomial on the right-hand side gives

$$1 = (1+B)x^4 + (4+3B+C)x^3 + (8+4B+3C+D)x^2 + (8+2B+4C+D+E)x + (4+E+2C).$$

Equating coefficients and then solving for the unknowns yields $B = -1$, $C = -1$, $D = -1$, and $E = -1$. The original integral is thus equal to

$$\int\left(\frac{1}{x+1} - \frac{x+1}{x^2+2x+2} - \frac{x+1}{(x^2+2x+2)^2}\right)dx = \ln|x+1| - \frac{1}{2}\ln|x^2+2x+2| + \frac{1}{2(x^2+2x+2)} + C.$$

8.5.63 $\frac{9x^2+x+21}{(3x^2+7)^2} = \frac{Ax+B}{3x^2+7} + \frac{Cx+D}{(3x^2+7)^2}$. Then

$$9x^2+x+21 = (Ax+B)(3x^2+7)+Cx+D = 3Ax^3+3Bx^2+(7A+C)x+7B+D.$$

So $A=0$, $B=3$, $C=1$, and $D=0$. We have

$$\begin{aligned}\int \frac{9x^2+x+21}{(3x^2+7)^2}\,dx &= \int \left(\frac{3}{3x^2+7} + \frac{x}{(3x^2+7)^2}\right)dx = \int \frac{1}{x^2+(7/3)}\,dx + \int \frac{x}{(3x^2+7)^2}\,dx \\ &= \sqrt{\frac{3}{7}}\tan^{-1}\left(\sqrt{\frac{3}{7}}x\right) - \frac{1}{6(3x^2+7)} + C.\end{aligned}$$

8.5.64 $\frac{9x^5+6x^3}{(3x^2+1)^3} = \frac{Ax+B}{3x^2+1} + \frac{Cx+D}{(3x^2+1)^2} + \frac{Ex+F}{(3x^2+1)^3}$. Then

$$9x^5+6x^3 = (Ax+B)(3x^2+1)^2+(Cx+D)(3x^2+1)+Ex+F.$$

This can be written as

$$9x^5+6x^3 = 9Ax^5+9Bx^4+(6A+3C)x^3+(6B+3D)x^2+(A+C+E)x+B+D+F.$$

Therefore, $A=1$, $B=C=D=0$, $E=-1$, and $F=0$. We have

$$\int \frac{9x^5+6x^3}{(3x^2+1)^3}\,dx = \int\left(\frac{x}{3x^2+1} - \frac{x}{(3x^2+1)^3}\right)dx = \frac{1}{6}\ln(3x^2+1) + \frac{1}{12(3x^2+1)^2} + C.$$

8.5.65

a. False. Because the given integrand is improper, the first step would be to use long division to write the integrand as the sum of a polynomial and a proper rational function.

b. False. This is easy to evaluate via the substitution $u = 3x^2+x$.

c. False. The discriminant of the denominator is $b^2-4ac = 169-168 = 1 > 0$, so the denominator factors into linear factors of the real numbers.

d. True. The discriminant of the denominator is $b^2-4ac = 169-172 = -3 < 0$, so the given quadratic expression is irreducible.

8.5.66 We are seeking $\int_0^1 \frac{x-x^2}{(x+1)(x^2+1)}\,dx$. We can write the integrand as $\frac{A}{x+1} + \frac{Bx+C}{x^2+1}$, and then

$$x-x^2 = A(x^2+1)+(Bx+C)(x+1) = (A+B)x^2+(C+B)x+A+C.$$

Thus $A+B=-1$, $C+B=1$, and $A+C=0$. Solving gives $A=-1$, $B=0$, and $C=1$. Thus we have

$$\int_0^1 \frac{x-x^2}{(x+1)(x^2+1)}\,dx = \int_0^1\left(-\frac{1}{x+1} + \frac{1}{x^2+1}\right)dx = \left(-\ln|x+1|+\tan^{-1}x\right)\Big|_0^1 = \frac{\pi}{4} - \ln 2.$$

8.5.67 We are seeking $\int_{-2}^2 \frac{10}{x^2-2x-24}\,dx = \int_{-2}^2 \frac{10}{(x-6)(x+4)}\,dx$. If we write

$$\frac{10}{x^2-2x-24} = \frac{10}{(x-6)(x+4)} = \frac{A}{x-6} + \frac{B}{x+4},$$

then $10 = A(x+4)+B(x-6)$. Letting $x=-4$ gives $B=-1$ and letting $x=6$ gives $A=1$. Thus the area in question is given by

$$-\int_{-2}^2\left(-\frac{1}{x+4} + \frac{1}{x-6}\right)dx = \left(\ln(x+4) - \ln|x-6|\right)\Big|_{-2}^2 = \ln 6 - \ln 4 - (\ln 2 - \ln 8) = \ln 6.$$