Chapter 8

Integration Techniques

8.1 Basic Approaches

8.1.1 Let $u = 4 - 7x$. Then $du = -7\,dx$ and we obtain $-\dfrac{1}{7}\displaystyle\int u^{-6}\,du$.

8.1.2 $\displaystyle\int (\sec x + 1)^2\,dx = \int (\sec^2 x + 2\sec x + 1)\,dx = \tan x + 2\ln|\sec x + \tan x| + x + C$.

8.1.3 $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

8.1.4 $\displaystyle\int \frac{4x^3 + x^2 + 4x + 2}{x^2+1}\,dx = \int \left(4x + 1 + \frac{1}{x^2+1}\right)dx = 2x^2 + x + \tan^{-1} x + C$.

8.1.5 Complete the square in the denominator to get $\displaystyle\int \frac{10}{(x-2)^2+1}\,dx$.

8.1.6

a. $\displaystyle\int \frac{2x+1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$.

b. $\ln(x^2+1) + \tan^{-1} x + C$.

8.1.7 Let $u = 3 - 5x$ so that $du = -5\,dx$. Substituting gives

$$-\frac{1}{5}\int u^{-4}\,du = \frac{1}{15}u^{-3} + C = \frac{1}{15(3-5x)^3} + C.$$

8.1.8 Let $u = 9x - 2$ so that $du = 9\,dx$. Substituting gives

$$\frac{1}{9}\int u^{-3}\,du = \frac{-1}{18}u^{-2} + C = \frac{-1}{18(9x-2)^2} + C.$$

8.1.9 Let $u = 2x - \pi/4$ so that $du = 2\,dx$. Substituting gives

$$\frac{1}{2}\int_{-\pi/4}^{\pi/2} \sin u\,du = \frac{1}{2}\left(-\cos u\right)\Big|_{-\pi/4}^{\pi/2} = \frac{1}{2}\left(0 + \sqrt{2}/2\right) = \frac{\sqrt{2}}{4}.$$

8.1.10 Let $u = 3 - 4x$ so that $du = -4\,dx$. Substituting gives

$$-\frac{1}{4}\int e^u\,du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{3-4x} + C.$$

8.1.11 Let $u = \ln(2x)$ so that $du = \dfrac{dx}{x}$. Substituting gives

$$\int u\,du = u^2/2 + C = \frac{1}{2}\ln^2 2x + C.$$

8.1.12 Let $u = 4 - x$ so that $du = -dx$. Substituting gives

$$-\int_9^4 u^{-1/2}\,du = \int_4^9 u^{-1/2}\,du = 2\sqrt{u}\Big|_4^9 = 6 - 4 = 2.$$

8.1.13 Let $u = e^x + 1$ so that $du = e^x\,dx$ Substituting gives

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + 1) + C.$$

8.1.14 Let $u = x^2 + 1$ so that $du = 2x\,dx$. Note that $0^2 + 1 = 1$ and $1^2 + 1 = 2$. Substituting gives

$$\frac{1}{2}\int_1^2 3^u\,du = \frac{1}{2\ln 3}3^u\Big|_1^2 = \frac{1}{2\ln 3}(9 - 3) = \frac{3}{\ln 3}.$$

8.1.15 Let $u = \ln x^2 = 2\ln x$. Then $du = \dfrac{2}{x}\,dx$. Substituting gives

$$\frac{1}{2}\int_0^4 u^2\,du = \left(u^3/6\right)\Big|_0^4 = 32/3.$$

8.1.16 Let $u = t^3$ so that $du = 3t^2\,dt$. Note that $0^3 = 0$ and $1^3 = 1$. Substituting gives

$$\frac{1}{3}\int_0^1 \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u\Big|_0^1 = \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}.$$

8.1.17 Let $u = s - 1$, so that $du = ds$ and $s = u + 1$. Note that $1 - 1 = 0$ and $2 - 1 = 1$, so the new limits of integration are 0 and 1. Substituting gives

$$\int_0^1 u^9(u+1)\,du = \int_0^1 (u^{10} + u^9)\,du = \left(\frac{u^{11}}{11} + \frac{u^{10}}{10}\right)\Big|_0^1 = \frac{1}{11} + \frac{1}{10} = \frac{21}{110}.$$

8.1.18 Let $u = t - 3$ so that $du = dt$ and $t = u + 3$. Note that $3 - 3 = 0$ and $7 - 3 = 4$, so the new limits of integration are 0 and 4. Substituting gives

$$\int_0^4 (u-3)\sqrt{u}\,du = \int_0^4 (u^{3/2} - 3u^{1/2})\,du = \left(\frac{2}{5}u^{5/2} - 2u^{3/2}\right)\Big|_0^4 = \frac{64}{5} - 16 = -\frac{16}{5}.$$

8.1.19 Let $u = \ln w - 1$. Then $du = \dfrac{dw}{w}$ and $\ln w = u + 1$. Substituting gives

$$\int u^7(u+1)\,du = \int (u^8 + u^7)\,du = \frac{u^9}{9} + \frac{u^8}{8} + C = \frac{(\ln w - 1)^9}{9} + \frac{(\ln w - 1)^8}{8} + C.$$

8.1.20 Let $u = 1 + e^x$ so that $du = e^x\,dx$ and $e^x = u - 1$. Substituting gives

$$\int u^9(2-u)\,du = \int (2u^9 - u^{10})\,du = \frac{u^{10}}{5} - \frac{u^{11}}{11} + C = \frac{(1+e^x)^{10}}{5} - \frac{(1+e^x)^{11}}{11} + C$$
$$= (1+e^x)^{10}\left(\frac{11}{55} - \frac{5(1+e^x)}{55}\right) + C = \frac{(1+e^x)^{10}(6 - 5e^x)}{55} + C.$$

8.1.21 $\displaystyle\int \frac{x}{x^2+4}\,dx + 2\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + \tan^{-1}(x/2) + C.$

8.1.22 $\displaystyle\int \frac{\sin x + 1}{\cos x}\,dx = \int (\tan x + \sec x)\,dx = \ln|\sec x| + \ln|\sec x + \tan x| + C.$

8.1.23 Let $u = 3e^x + 4$ so that $du = 3e^x\,dx$. Substituting gives

$$\frac{1}{3}\int \csc u\,du = -\frac{1}{3}\ln|\csc u + \cot u| + C = -\frac{1}{3}\ln|\csc(3e^x+4) + \cot(3e^x+4)| + C.$$

8.1.24 $\displaystyle\int_4^9 x\,dx - \int_4^9 x^{-1}\,dx = \frac{x^2}{2}\Big|_4^9 - \ln x\Big|_4^9 = \frac{81}{2} - \frac{16}{2} - (\ln 9 - \ln 4) = \frac{65}{2} - \ln\left(\frac{9}{4}\right).$

8.1.25 We may rewrite the integrand as $\displaystyle\int_0^{\pi/4}\left(\frac{\sec\theta}{\sec\theta\csc\theta} + \frac{\csc\theta}{\sec\theta\csc\theta}\right)d\theta$. This can then be simplified as

$$\int_0^{\pi/4}(\sin\theta + \cos\theta)\,d\theta = (-\cos\theta + \sin\theta)\Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1-0) = 1.$$

8.1.26 $\displaystyle\int 4e^{-3x}\,dx + \int e^{-5x}\,dx = (-4/3)e^{-3x} + (-1/5)e^{-5x} + C.$

8.1.27

$$\int \frac{2}{\sqrt{1-x^2}}\,dx - 3\int \frac{x}{\sqrt{1-x^2}}\,dx = 2\sin^{-1}x - 3\int \frac{x}{\sqrt{1-x^2}}\,dx.$$

Let $u = 1 - x^2$ so that $du = -2x\,dx$. Substituting gives

$$2\sin^{-1}x + \frac{3}{2}\int u^{-1/2}\,du = 2\sin^{-1}x + 3\sqrt{u} + C = 2\sin^{-1}x + 3\sqrt{1-x^2} + C.$$

8.1.28

$$\int \frac{3x}{\sqrt{4-x^2}}\,dx + \int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{3x}{\sqrt{4-x^2}}\,dx + \sin^{-1}(x/2).$$

Let $u = 4 - x^2$ so that $du = -2x\,dx$. Substituting gives

$$-\frac{3}{2}\int u^{-1/2}\,du + \sin^{-1}(x/2) = -3\sqrt{u} + \sin^{-1}(x/2) + C = -3\sqrt{4-x^2} + \sin^{-1}(x/2) + C.$$

8.1.29 $\displaystyle\int_{\pi/4}^{\pi/2}\sqrt{1+\cot^2 x}\,dx = \int_{\pi/4}^{\pi/2}\csc x\,dx = -\ln|\csc x + \cot x|\Big|_{\pi/4}^{\pi/2} = -\ln|1+0| + \ln|\sqrt{2}+1| = \ln(\sqrt{2}+1).$

8.1.30 Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$. Substituting gives

$$\int_{\pi/4}^{\pi/3}\cot u\,du = \ln|\sin u|\Big|_{\pi/4}^{\pi/3} = \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2} = \frac{\ln 3 - \ln 2}{2} = \frac{1}{2}\ln\frac{3}{2}.$$

8.1.31 Note that by completing the square, we have $x^2 - 2x + 10 = (x^2 - 2x + 1) + 9 = (x-1)^2 + 9$. So

$$\int \frac{dx}{(x-1)^2 + 3^2} = \frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right) + C.$$

8.1.32 Note that by completing the square, we have $x^2 + 4x + 8 = (x+2)^2 + 4$. Thus we have

$$\int_0^2 \frac{(x+2)-2}{(x+2)^2+4}\,dx = \int_2^4 \frac{u}{u^2+4}\,du - \int_2^4 \frac{2}{u^2+4}\,du$$

where $u = x + 2$. This is equal to

$$\left(\frac{1}{2}\ln|u^2+4| - \tan^{-1}(u/2)\right)\Big|_2^4 = \frac{1}{2}\ln(5/2) + \frac{\pi}{4} - \tan^{-1}2.$$

8.1.33 We reduce the integrand by long division to $x+1+\dfrac{2x+1}{x^2+x+2}$. Then we have

$$\int\left(x+1+\frac{2x+1}{x^2+x+2}\right)dx=\frac{x^2}{2}+x+\int\frac{2x+1}{x^2+x+1}\,dx.$$

For the remaining integral, we let $u=x^2+x+1$ so that $du=(2x+1)\,dx$. The last integral is therefore equal to $\ln|u|+C=\ln|x^2+x+1|+C$. So our final result is

$$\frac{x^2}{2}+x+\ln|x^2+x+1|+C.=\frac{x^2}{2}+x+\ln(x^2+x+1)+C.$$

8.1.34 Note that long division gives $\dfrac{x^2+2}{x-1}=x+1+\dfrac{3}{x-1}$. Thus we have

$$\int_2^4\left(x+1+\frac{3}{x-1}\right)dx=\left(x^2/2+x+3\ln|x-1|\right)\Big|_2^4=8+4+3\ln 3-(2+2+0)=8+3\ln 3.$$

8.1.35 By long division, we can write the integrand as $t^2+t+\dfrac{1}{t^2+1}$. Then we have

$$\int_0^1\left(t^2+t+\frac{1}{t^2+1}\right)dt=\left(\frac{t^3}{3}+\frac{t^2}{2}+\tan^{-1}t\right)\Big|_0^1=\left(\frac{1}{3}+\frac{1}{2}+\frac{\pi}{4}\right)-0=\frac{3\pi+10}{12}.$$

8.1.36 Note that long division gives $\dfrac{t^3-2}{t+1}=t^2-t+1-\dfrac{3}{t+1}$. Our integral is therefore

$$\int\left(t^2-t+1-\frac{3}{t+1}\right)dt=t^3/3-t^2/2+t-3\ln|t+1|+C.$$

8.1.37 Note that $27-6\theta-\theta^2=-(\theta^2+6\theta+9-36)=-((\theta+3)^2-36)=36-(\theta+3)^2$. Thus our integral is

$$\int\frac{d\theta}{\sqrt{36-(\theta+3)^2}}=\sin^{-1}((\theta+3)/6)+C.$$

8.1.38 The integral can be written as $\displaystyle\int\frac{x}{(x^2+1)^2}\,dx$. Let $u=x^2+1$ so that $du=2x\,dx$. Substitution gives

$$\frac{1}{2}\int u^{-2}\,du=\frac{-1}{2u}+C=\frac{-1}{2(x^2+1)}+C.$$

8.1.39

$$\begin{aligned}\int\frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}\,d\theta&=\int\frac{1-\sin\theta}{1-\sin^2\theta}\,d\theta=\int\frac{1-\sin\theta}{\cos^2\theta}\,d\theta\\&=\int\sec^2\theta\,d\theta-\int\frac{\sin\theta}{\cos^2\theta}\,d\theta=\tan\theta-\int\frac{\sin\theta}{\cos^2}\theta\,d\theta.\end{aligned}$$

Let $u=\cos\theta$ so that $du=-\sin\theta\,d\theta$. Substituting gives

$$\tan\theta+\int u^{-2}\,du=\tan\theta-\frac{1}{u}+C=\tan\theta-\sec\theta+C.$$

8.1.40 $\displaystyle\int\frac{1-x}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\,dx=\int\frac{(1-x)(1+\sqrt{x})}{1-x}\,dx=\int(1+\sqrt{x})\,dx=x+2x^{3/2}/3+C.$

8.1.41 $\displaystyle\int\frac{1}{\sec x-1}\cdot\frac{\sec x+1}{\sec x+1}\,dx=\int\frac{\sec x+1}{\sec^2 x-1}\,dx=\int\frac{\sec x+1}{\tan^2 x}\,dx=\int\frac{\sec x}{\tan^2 x}\,dx+\int\cot^2 x\,dx=$ $\displaystyle\int\cot x\csc x\,dx+\int\cot^2 x\,dx=-\csc x+\int(\csc^2 x-1)\,dx=-\csc x-\cot x-x+C.$

8.1.42 $\displaystyle\int \frac{1}{1-\csc\theta}\cdot\frac{1+\csc\theta}{1+\csc\theta}\,d\theta = \int \frac{1+\csc\theta}{1-\csc^2\theta}\,d\theta = \int \frac{1+\csc\theta}{-\cot^2\theta}\,d\theta = \int \frac{-\csc\theta}{\cot^2\theta}\,d\theta - \int \frac{1}{\cot^2\theta}\,d\theta =$ $\displaystyle\int -\tan\theta\sec\theta\,d\theta - \int \tan^2\theta\,d\theta = -\sec\theta - \int(\sec^2\theta-1)\,d\theta = -\sec\theta-\tan\theta+\theta+C.$

8.1.43 Let $u = 1+\sinh 3x$. Then $du = 3\cosh 3x\,dx$. Substituting gives

$$\frac{1}{3}\int\frac{1}{u}\,du = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|1+\sinh x|+C.$$

8.1.44 Let $u = x^2+1$. Then $du = 2x\,dx$. Note that $0^2+1=1$ and $\sqrt{3}^2+1=4$, and $6x^3\,dx = 3x^2 2x\,dx = 3(u-1)\,du$. Substituting gives

$$3\int_1^4\frac{u-1}{\sqrt{u}}\,du = 3\int_1^4(u^{1/2}-u^{-1/2})\,du = 3\left(\frac{2}{3}u^{3/2}-2u^{1/2}\right)\Big|_1^4 = 3\left(\frac{16}{3}-4-\frac{2}{3}+2\right) = 8.$$

8.1.45 We rewrite the integral by multiplying the numerator and denominator of the integrand by e^x. We have

$$\int\frac{e^x}{e^x-2e^{-x}}\,dx = \int\frac{e^x}{e^x-2e^{-x}}\cdot\frac{e^x}{e^x}\,dx = \int\frac{e^{2x}}{e^{2x}-2}\,dx.$$

Now let $u = e^{2x}-2$ so that $du = 2e^{2x}\,dx$. Substituting gives

$$\frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|e^{2x}-2|+C.$$

8.1.46 $\displaystyle\int\frac{e^{2z}}{e^{2z}-4e^{-z}}\cdot\frac{e^z}{e^z}\,dz = \int\frac{e^{3z}}{e^{3z}-4}\,dz$. Let $u = e^{3z}-4$ so that $du = 3e^{3z}\,dz$. Substituting gives

$$\frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|e^{3z}-4|+C.$$

8.1.47 $\displaystyle\int\frac{dx}{x^{-1}+1} = \int\frac{1}{x^{-1}+1}\cdot\frac{x}{x}\,dx = \int\frac{x}{x+1}\,dx$. Using long division, we have $\dfrac{x}{x+1} = 1-\dfrac{1}{x+1}$. Then

$$\int\left(1-\frac{1}{x+1}\right)dx = x-\ln|x+1|+C.$$

8.1.48 $\displaystyle\int\frac{dy}{y^{-1}+y^{-3}} = \int\frac{1}{y^{-1}+y^{-3}}\cdot\frac{y^3}{y^3}\,dy = \int\frac{y^3}{y^2+1}\,dy$. Using long division, we have

$$\frac{y^3}{y^2+1} = y-\frac{y}{y^2+1}.$$

Thus our integral is equal to

$$\int\left(y-\frac{y}{y^2+1}\right)dy = \frac{y^2}{2}-\int\frac{y}{y^2+1}\,dy.$$

To compute this last integral, let $u = y^2+1$ so that $du = 2y\,dy$. We have

$$\frac{y^2}{2}-\frac{1}{2}\int\frac{1}{u}\,du = \frac{y^2}{2}-\frac{1}{2}\ln|u|+C = \frac{y^2}{2}-\frac{1}{2}\ln|y^2+1|+C.$$

8.1.49 Let $u = 9+\sqrt{t+1}$. Then $du = \dfrac{dt}{2\sqrt{t+1}}$, or $dt = 2(u-9)\,du$. Our integral is

$$\begin{aligned}\int\sqrt{9+\sqrt{t+1}}\,dt &= \int\sqrt{u}(2(u-9))\,du = 2\int(u^{3/2}-9u^{1/2})\,du = 2\left(\frac{2}{5}u^{5/2}-6u^{3/2}\right)+C\\ &= \frac{4}{5}u^{3/2}(u-15)+C = \frac{4}{5}(9+\sqrt{t+1})^{3/2}(9+\sqrt{t+1}-15)+C\\ &= \frac{4}{5}(9+\sqrt{t+1})^{3/2}(\sqrt{t+1}-6)+C.\end{aligned}$$

8.1.50 Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$, or $dx = 2u\,du$. Substituting gives

$$\int_2^3 \frac{2u}{1-u}\,du = -\int_2^3 \frac{2u}{u-1}\,du.$$

By long division, $\dfrac{2u}{u-1} = 2 + \dfrac{2}{u-1}$. Thus we have

$$-\int_2^3 \left(2 + \frac{2}{u-1}\right)du = (-2u - 2\ln|u-1|)\Big|_2^3 = -6 - 2\ln 2 - (-4-0) = -2 - 2\ln 2.$$

8.1.51 $\displaystyle\int_{-1}^0 \frac{x}{x^2+2x+2}\,dx = \int_{-1}^0 \frac{x}{(x+1)^2+1}\,dx$. Let $u = x+1$ so that $du = dx$. Substituting gives

$$\begin{aligned}\int_0^1 \frac{u-1}{u^2+1}\,du &= \int_0^1 \frac{u}{u^2+1}\,du - \int_0^1 \frac{1}{u^2+1}\,du\\ &= \left((1/2)\ln(u^2+1) - \tan^{-1}(u)\right)\Big|_0^1 = (1/2)\ln 2 - \frac{\pi}{4} = \frac{1}{4}(\ln 4 - \pi).\end{aligned}$$

8.1.52 $\displaystyle\int_{\pi/6}^{\pi/2} (\csc y)\,dy = -\ln|\csc y + \cot y|\Big|_{\pi/6}^{\pi/2} = -\ln|1+0| + \ln|2+\sqrt{3}| = \ln(2+\sqrt{3}).$

8.1.53 Let $u = e^x + 1$ so that $du = e^x\,dx$. Substituting gives

$$\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln|\sec(e^x+1) + \tan(e^x+1)| + C.$$

8.1.54 Let $u = 1 + \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Substituting gives

$$\begin{aligned}2\int_1^2 \sqrt{u}(u-1)\,du &= 2\int_1^2 (u^{3/2} - u^{1/2})\,du = 2\left(2u^{5/2}/5 - 2u^{3/2}/3\right)\Big|_1^2\\ &= 2(8\sqrt{2}/5 - 4\sqrt{2}/3 - (2/5 - 2/3)) = \frac{8}{15}\left(1+\sqrt{2}\right).\end{aligned}$$

8.1.55 Using the identity $\sin 2x = 2\sin x\cos x$, we have $2\displaystyle\int \sin^2 x\cos x\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. We have $2\displaystyle\int u^2\,du = 2u^3/3 + C = 2(\sin^3 x)/3 + C$.

8.1.56 Using the double angle identity $\cos 2x = 2\cos^2 x - 1$, we have

$$\int_0^{\pi/2} \sqrt{2\cos^2 x}\,dx = \int_0^{\pi/2} \sqrt{2}\cos x\,dx = \sqrt{2}\sin x\Big|_0^{\pi/2} = \sqrt{2}(1-0) = \sqrt{2}.$$

8.1.57 Rewrite the integral as $\int \frac{1}{\sqrt{x}} \cdot \frac{1}{1+(\sqrt{x})^2}\,dx$ and let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, and substituting gives

$$2\int \frac{1}{1+u^2}\,du = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$$

8.1.58 Let $u = \sqrt{p}$ so that $u^2 = p$ and $2u\,du = dp$. Substituting gives

$$\begin{aligned}\int_0^1 \frac{2u}{4-u}\,du &= 2\int_0^1 \left(-1 - \frac{4}{u-4}\right)du\\ &= 2\left(-u - 4\ln|u-4|\right)\Big|_0^1 = 2(-1 - 4\ln 3 - (0 - 4\ln 4)) = 2(\ln(256/81) - 1).\end{aligned}$$

8.1.59 Note that $x^2+6x+13=(x^2+6x+9)+4=(x+3)^2+4$. Also note that we can write the numerator $x-2=x+3-5=\frac{1}{2}(2x+6)-5$. We have

$$\int \frac{\frac{1}{2}(2x+6)}{x^2+6x+13}\,dx - \int \frac{5}{(x+3)^2+4}\,dx.$$

For the first integral, let $u=x^2+6x+13$ so that $du=(2x+6)\,dx$. We have (for just the first integral)

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C=\frac{1}{2}\ln(x^2+6x+13)+C.$$

The second integrand has antiderivative equal to $\frac{5}{2}\tan^{-1}((x+3)/2)$, so the original integral is equal to

$$\frac{1}{2}\ln(x^2+6x+13)-\frac{5}{2}\tan^{-1}((x+3)/2)+C.$$

8.1.60 $3\int_0^{\pi/4}\sqrt{1+\sin 2x}\cdot\frac{\sqrt{1-\sin 2x}}{\sqrt{1-\sin 2x}}\,dx = 3\int_0^{\pi/4}\frac{\cos 2x}{\sqrt{1-\sin 2x}}\,dx$. Let $u=1-\sin 2x$ so that $du=-2\cos 2x\,dx$. Substituting gives

$$\frac{-3}{2}\int_1^0 u^{-1/2}\,du = 3\sqrt{u}\Big|_0^1 = 3.$$

8.1.61 Let $u=e^x$ so that $du=e^x\,dx$. Substituting gives

$$\int\frac{1}{u^2+2u+1}\,du=\int(u+1)^{-2}\,du=-\frac{1}{u+1}+C=-\frac{1}{e^x+1}+C.$$

8.1.62 By long division, the integrand can be written as $-x^3-x+1+\frac{4x+2}{x^2+x+1}$. Then

$$\int\left(-x^3-x+1+\frac{2(2x+1)}{x^2+x+1}\right)=-\frac{x^4}{4}-\frac{x^2}{2}+x+2\ln|x^2+x+1|+C.$$

8.1.63 The denominator factors as $(x+1)^2$.

$$\int_1^3\frac{2}{(x+1)^2}\,dx=-2\left(\frac{1}{x+1}\right)\Big|_1^3=-2\left(\frac{1}{4}-\frac{1}{2}\right)=\frac{1}{2}.$$

8.1.64 The denominator factors as $(s+1)^3$.

$$\int_0^2\frac{2}{(s+1)^3}\,ds=-\,(s+1)^{-2}\Big|_0^2=-\left(\frac{1}{9}-1\right)=\frac{8}{9}.$$

8.1.65

a. False. This seem to use the untrue "identity" that $\frac{a}{b+c}=\frac{a}{b}+\frac{a}{c}$.

b. False. The degree of the numerator is already less than the degree of the denominator, so long division won't help.

c. False. This is false because $\frac{d}{dx}\ln|\sin x+1|+C\neq\frac{1}{\sin x+1}$. The substitution $u=\sin x+1$ can't be carried out because $du=\cos x\,dx$ can't be accounted for.

d. False. In fact, $\int e^{-x}\,dx=-e^{-x}+C\neq\ln e^x+C$.

8.1.66 $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$ so that $du = \cos x\,dx$. We then have

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C.$$

8.1.67

$$\int \csc x\,dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx.$$

Let $u = \csc x + \cot x$ so that $du = -\csc^2 x - \csc x \cot x\,dx$. Substituting then yields

$$-\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\csc x + \cot x| + C.$$

8.1.68

a. If $u = \cot x$, then $du = -\csc^2 x\,dx$. Substituting gives $-\int u\,du = -\frac{u^2}{2} + C = -\frac{\cot^2 x}{2} + C$.

b. If $u = \csc x$, then $du = -\csc x \cot x\,dx$. Substituting gives $-\int u\,du = -\frac{u^2}{2} + C = -\frac{\csc^2 x}{2} + C$.

c. The seemingly different answers are the same, since $-(\cot^2 x)/2$ and $-(\csc^2 x)/2$ differ by a constant. In fact, $-\frac{\cot^2 x}{2} - \left(-\left(\frac{\csc^2 x}{2}\right)\right) = \frac{1}{2}$.

8.1.69

a. If $u = \tan x$ then $du = \sec^2 x\,dx$. Substituting gives $\int u\,du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$.

b. If $u = \sec x$, then $du = \sec x \tan x\,dx$. Substituting gives $\int u\,du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C$.

c. The seemingly different answers are the same, since $\frac{\tan^2 x}{2}$ and $\frac{\sec^2 x}{2}$ differ by a constant. In fact, $\frac{\tan^2 x}{2} - \frac{\sec^2 x}{2} = -\frac{1}{2}$.

8.1.70

a. Note that long division gives $\frac{x+2}{x+4} = 1 - \frac{2}{x+4}$. Thus our integral is equal to

$$\int \left(1 - \frac{2}{x+4}\right)\,dx = x - 2\ln|x+4| + C.$$

b. Let $u = x + 4$. Then $du = dx$ and $x + 2 = u - 2$. Substituting gives

$$\int \frac{u-2}{u}\,du = \int \left(1 - \frac{2}{u}\right)\,du = u - 2\ln|u| + C = x + 2 - 2\ln|x+2| + C = x - 2\ln|x+2| + 2 + C.$$

c. In part b, $2 + C$ is a constant, so we can replace $2 + C$ by C to obtain the answer to part a.

8.1.71

a. Let $u = x + 1$ so that $du = dx$. Note that $x = u - 1$, so that $x^2 = (u-1)^2$. Substituting gives

$$\int \frac{u^2 - 2u + 1}{u}\,du = \int (u - 2 + (1/u))\,du = u^2/2 - 2u + \ln|u| + C = (x+1)^2/2 - 2(x+1) + \ln|x+1| + C.$$

b. By long division, $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$. Thus,

$$\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx = x^2/2 - x + \ln|x+1| + C.$$

c. The seemingly different answers are the same, because they differ by a constant. In fact,

$$\frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| - \left(\frac{x^2}{2} - x + \ln|x+1|\right) = -\frac{3}{2}.$$

8.1.72 The curves intersect when $x^3 = 8x$, so at $x = 0$ and $x = \pm\sqrt{8}$. By symmetry, we have $A = 2\int_0^{\sqrt{8}} \dfrac{8x - x^3}{x^2+1}\,dx$. Using long division, we can write $\dfrac{8x - x^3}{x^2+1} = -x + \dfrac{9x}{x^2+1}$. Thus,

$$\begin{aligned} A &= 2\int_0^{\sqrt{8}} (-x + \frac{9x}{x^2+1})\,dx = 2\int_0^{\sqrt{8}} (-x)\,dx + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx \\ &= -2\,x^2/2\,\Big|_0^{\sqrt{8}} + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx = -8 + 2\int_0^{\sqrt{8}} \frac{9x}{x^2+1}\,dx. \end{aligned}$$

To compute this last integral, let $u = x^2 + 1$ so that $du = 2x\,dx$. Then we have

$$A = -8 + 9\int_1^9 \frac{1}{u}\,du = -8 + 9\,\ln u\,\Big|_1^9 = -8 + 9\ln 9 \approx 11.775.$$

8.1.73

$$A = \int_2^4 \frac{x^2 - 1}{x^3 - 3x}\,dx.$$

Let $u = x^3 - 3x$ so that $du = 3x^2 - 3\,dx$. Substituting gives

$$A = \frac{1}{3}\int_2^{52} \frac{1}{u}\,du = \frac{1}{3}\,\ln u\,\Big|_2^{52} = \frac{1}{3}(\ln 52 - \ln 2) = \frac{\ln 26}{3}.$$

8.1.74

$$\begin{aligned} V &= 2\pi\int_0^3 \frac{x}{x+2}\,dx = 2\pi\int_0^3 (1 - \frac{2}{x+2})\,dx \\ &= 2\pi\,(x - 2\ln(x+2))\,\Big|_0^3 = 2\pi(3 - 2\ln 5 - (0 - 2\ln 2)) = 2\pi(3 - 2\ln(2/5)). \end{aligned}$$

8.1.75

a. $V = \pi\displaystyle\int_0^2 (x^2+1)\,dx = \pi\,\left(x^3/3 + x\right)\Big|_0^2 = \pi(8/3 + 2) = \frac{14\pi}{3}.$

b. $V = 2\pi\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx$. Let $u = x^2 + 1$ so that $du = 2x\,dx$. Substituting gives

$$\pi\int_1^5 u^{1/2}\,du = \frac{2\pi}{3}\left(u^{3/2}\right)\Big|_1^5 = \frac{2\pi}{3}(5\sqrt{5} - 1).$$

8.1.76

a. Note that $x - x^2 = -(x^2 - x + 1/4 - 1/4) = -((x-1/2)^2 - 1/4) = 1/4 - (x-1/2)^2$. We can write our integral as

$$\int \frac{dx}{\sqrt{1/4-(x-1/2)^2}} = 2\int \frac{dx}{\sqrt{1-(2x-1)^2}}.$$

Let $u = 2x-1$. Then $du = 2\,dx$. Substituting gives

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(2x-1) + C.$$

b. We can write our integral as $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$. Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Substituting gives $2\displaystyle\int \frac{du}{\sqrt{1-u^2}} = 2\sin^{-1} u + C = 2\sin^{-1}\sqrt{x} + C$.

c. By parts a and b, it follows that both $\sin^{-1}(2x-1)$ and $2\sin^{-1}(\sqrt{x})$ are antiderivatives of $\dfrac{1}{\sqrt{x-x^2}}$. Therefore, $2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1) = C$ for some constant C. To determine C, we let $x = 0$, giving $2\sin^{-1}(0) - \sin^{-1}(-1) = C$. Thus $0 - \left(-\dfrac{\pi}{2}\right) = C$, so $C = \dfrac{\pi}{2}$.

8.1.77

$$\begin{aligned} A &= 2\pi\int_0^1 \sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi\int_0^1 \sqrt{x+5/4}\,dx \\ &= 2\pi\left((2/3)(x+5/4)^{3/2}\right)\Big|_0^1 = \frac{4\pi}{3}(27/8 - 5\sqrt{5}/8) = \frac{9\pi}{2} - \frac{5\sqrt{5}\pi}{6}. \end{aligned}$$

8.1.78 $A = 2\pi\displaystyle\int_0^{\ln 2} (e^x + e^{-x}/4)\sqrt{1+(e^x - e^{-x}/4)^2}\,dx$. Note that

$$1 + (e^x - e^{-x}/4)^2 = 1 + e^{2x} - 1/2 + e^{-2x}/16 = e^{2x} + 1/2 + e^{-2x}/16 = (e^x + e^{-x}/4)^2.$$

Thus we have

$$\begin{aligned} 2\pi\int_0^{\ln 2} (e^x + e^{-x}/4)^2\,dx &= 2\pi\int_0^{\ln 2} (e^{2x} + 1/2 + e^{-2x}/16)\,dx = 2\pi\left(e^{2x}/2 + x/2 - e^{-2x}/32\right)\Big|_0^{\ln 2} \\ &= 2\pi(2 + (\ln 2)/2 - 1/128 - (1/2 - 1/32)) = \pi\left(\frac{195}{64} + \ln 2\right). \end{aligned}$$

8.1.79 $L = \displaystyle\int_0^1 \sqrt{1+\frac{25x^{1/2}}{16}}\,dx$. Let $u^2 = 1 + \dfrac{25x^{1/2}}{16}$. Then $2u\,du = \dfrac{25}{32\sqrt{x}}\,dx$. Note that $\sqrt{x} = \dfrac{16}{25}(u^2-1)$, and that $dx = \dfrac{64\sqrt{x}}{25}u\,du = \dfrac{1024}{625}(u^3-u)\,du$. Substituting gives

$$\begin{aligned} L &= \int_1^{\sqrt{41/16}} \frac{1024}{625}(u^4-u^2)\,du = \frac{1024}{625}\left(u^5/5 - u^3/3\right)\Big|_1^{\sqrt{41/16}} \\ &= \frac{1024}{625}((\sqrt{41/16})^5/5 - (\sqrt{41/16})^3/3 - (1/5-1/3)) = \frac{1024}{625}\left(\frac{2}{15} + \frac{1763\sqrt{41}}{15360}\right) \\ &= \frac{2048 + 1763\sqrt{41}}{9375} \approx 1.423. \end{aligned}$$