EXAMPLE 5 **Multiple techniques needed** Find the area of the surface of revolution that results when the curve $f(x) = e^x$ on $[0, \ln 2]$ is revolved about the x-axis.

SOLUTION Recall from Section 6.6 that the integral $\int_a^b 2\pi\, f(x)\sqrt{1 + f'(x)^2}\, dx$ gives the area of the surface generated when $y = f(x)$ on $[a, b]$ is revolved about the x-axis. Because $f'(x) = e^x$, the area of the surface is

$$\int_0^{\ln 2} 2\pi e^x \sqrt{1 + (e^x)^2}\, dx.$$

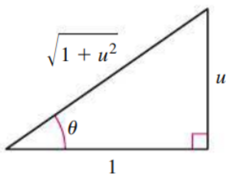
The presence of e^x and its derivative in the integrand suggests the substitution $u = e^x$:

$$\int_0^{\ln 2} 2\pi e^x \sqrt{1 + (e^x)^2}\, dx = 2\pi \int_1^2 \sqrt{1 + u^2}\, du \qquad u = e^x;\ du = e^x\, dx \quad x = 0 \Rightarrow u = 1;\ x = \ln 2 \Rightarrow u = 2$$

The new integrand contains $1 + u^2$, so we try the trigonometric substitution $u = \tan\theta$. Setting the definite integral aside for the moment, we first focus on evaluating the indefinite integral:

$$\begin{aligned} 2\pi \int \sqrt{1 + u^2}\, du &= 2\pi \int \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta && \text{Let } u = \tan\theta;\ du = \sec^2\theta\, d\theta. \\ &= 2\pi \int \sec^3\theta\, d\theta. && 1 + \tan^2\theta = \sec^2\theta \end{aligned}$$

To complete the solution, we rely on the secant reduction formula from Section 8.3 and return to the variable u:



$u = \tan\theta$

$$\begin{aligned} 2\pi \int \sec^3\theta\, d\theta &= 2\pi\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right) && \text{Secant reduction formula} \\ &= \pi(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C && \text{Evaluate } \textstyle\int \sec\theta\, d\theta. \\ &= \pi\left(u\sqrt{1 + u^2} + \ln\left|u + \sqrt{1 + u^2}\right|\right) + C. && \tan\theta = u;\ \sec\theta = \sqrt{1 + u^2} \end{aligned}$$

Evaluating the antiderivative using limits of integration from the integral in u gives the area of the surface:

$$\begin{aligned} \text{Area} &= \pi\left(u\sqrt{1 + u^2} + \ln\left(u + \sqrt{1 + u^2}\right)\right)\Big|_1^2 \\ &= \pi\left(2\sqrt{5} - \sqrt{2} + \ln\frac{2 + \sqrt{5}}{1 + \sqrt{2}}\right) \approx 11.37. \end{aligned}$$

Related Exercise 87 ◄

SECTION 8.6 EXERCISES

Getting Started

1–6. Choosing an integration strategy *Identify a technique of integration for evaluating the following integrals. If necessary, explain how to first simplify the integrand before applying the suggested technique of integration. You do not need to evaluate the integrals.*

1. $\int 4x \sin 5x\, dx$

2. $\int (1 + \tan x)\sec^2 x\, dx$

3. $\int \frac{x^3}{\sqrt{64 - x^2}}\, dx$

4. $\int \frac{\tan^2 x + 1}{\tan x}\, dx$

5. $\int \frac{5x^2 + 18x + 20}{(2x + 3)(x^2 + 4x + 8)}\, dx$

6. $\int \frac{\cos^5 x \sin^4 x}{1 - \sin^2 x}\, dx$

Practice Exercises

7–84. *Evaluate the following integrals.*

7. $\int_0^{\pi/2} \frac{\sin\theta}{1 + \cos^2\theta}\, d\theta$

8. $\int \cos^2 10x\, dx$

9. $\int_4^6 \frac{dx}{\sqrt{8x - x^2}}$

10. $\int \sin^9 x \cos^3 x\, dx$

11. $\int_0^{\pi/4} (\sec x - \cos x)^2\, dx$

12. $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$

13. $\int \frac{dx}{e^x\sqrt{1 - e^{2x}}}$

14. $\int \frac{x^{-2} + x^{-3}}{x^{-1} + 16x^{-3}}\, dx$

15. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$

16. $\int \frac{dx}{x^4 - 1}$

17. $\int_1^2 w^3 e^{w^2}\, dw$

18. $\int_5^6 x(x-5)^{10}\, dx$

19. $\int_0^{\pi/2} \sin^7 x\, dx$

20. $\int_1^3 \frac{dt}{\sqrt{t}(t+1)}$

21. $\int x^9 \ln 3x\, dx$

22. $\int \frac{dx}{(x-a)(x-b)}, a \neq b$

23. $\int \frac{\sin x}{\cos^2 x + \cos x}\, dx$

24. $\int \frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w}\, dw$

25. $\int \frac{dx}{x\sqrt{1-x^2}}$

26. $\int_{1/e}^1 \frac{dx}{x(\ln^2 x + 2\ln x + 2)}$

27. $\int \sin^4 \frac{x}{2}\, dx$

28. $\int \frac{3x^2 + 2x + 3}{x^4 + 2x^2 + 1}\, dx$

29. $\int \frac{2\cos x + \cot x}{1 + \sin x}\, dx$

30. $\int_{5/2}^{5\sqrt{3}/2} \frac{dv}{v^2\sqrt{25 - v^2}}$

31. $\int \sqrt{36 - 9x^2}\, dx$

32. $\int \frac{dx}{\sqrt{36x^2 - 25}}, x > 5/6$

33. $\int \frac{e^x}{a^2 + e^{2x}}\, dx, a \neq 0$

34. $\int_0^{\pi/9} \frac{\sin 3x}{\cos 3x + 1}\, dx$

35. $\int_0^{\pi/4} (\tan^2\theta + \tan\theta + 1)\sec^2\theta\, d\theta$

36. $\int x\, 10^x\, dx$

37. $\int_0^{\pi/6} \frac{dx}{1 - \sin 2x}$

38. $\int_{\pi/6}^{\pi/2} \cos x \ln(\sin x)\, dx$

39. $\int \sin x \ln(\sin x)\, dx$

40. $\int \sin 2x \ln(\sin x)\, dx$

41. $\int \cot^{3/2} x \csc^4 x\, dx$

42. $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

43. $\int \frac{x^9}{\sqrt{1 - x^{20}}}\, dx$

44. $\int \frac{dx}{x^3 - x^2}$

45. $\int_0^{\ln 2} \frac{1}{(1+e^x)^2}\, dx$

46. $\int \frac{dx}{e^{2x} + 1}$

47. $\int \frac{2x^3 + x^2 - 2x - 4}{x^2 - x - 2}\, dx$

48. $\int \frac{\sqrt{16 - x^2}}{x^2}\, dx$

49. $\int \tan^3 x \sec^9 x\, dx$

50. $\int \tan^7 x \sec^4 x\, dx$

51. $\int_0^{\pi/3} \tan x \sec^{7/4} x\, dx$

52. $\int t^2 e^{3t}\, dt$

53. $\int e^x \cot^3 e^x\, dx$

54. $\int \frac{2x^2 + 3x + 26}{(x-2)(x^2+16)}\, dx$

55. $\int \frac{3x^2 + 3x + 1}{x^3 + x}\, dx$

56. $\int_{\pi}^{3\pi/2} \sin 2x\, e^{\sin^2 x}\, dx$

57. $\int \sin\sqrt{x}\, dx$

58. $\int w^2 \tan^{-1} w\, dw$

59. $\int \frac{dx}{x^4 + x^2}$

60. $\int_0^{\pi/2} e^{-3x}\cos x\, dx$

61. $\int_0^{\sqrt{2}/2} e^{\sin^{-1} x}\, dx$

62. $\int_0^{\pi/2} \sqrt{1 + \cos\theta}\, d\theta$

63. $\int x^a \ln x\, dx, a \neq -1$

64. $\int \frac{\ln ax}{x}\, dx, a \neq 0$

65. $\int_0^{1/6} \frac{dx}{\sqrt{1 - 9x^2}}$

66. $\int \frac{x}{\sqrt{1 - 9x^2}}\, dx$

67. $\int \frac{x^2}{\sqrt{1 - 9x^2}}\, dx$

68. $\int \frac{e^x}{e^{2x} + 2e^x + 17}\, dx$

69. $\int \frac{dx}{1 - x^2 + \sqrt{1 - x^2}}$

70. $\int \ln(x^2 + a^2)\, dx, a \neq 0$

71. $\int \frac{1 - \cos x}{1 + \cos x}\, dx$

72. $\int x^2 \sinh x\, dx$

73. $\int_9^{16} \sqrt{1 + \sqrt{x}}\, dx$

74. $\int \frac{e^{3x}}{e^x - 1}\, dx$

75. $\int_1^3 \frac{\tan^{-1}\sqrt{x}}{x^{1/2} + x^{3/2}}\, dx$

76. $\int \frac{x}{x^2 + 6x + 18}\, dx$

77. $\int \cos^{-1} x\, dx$

78. $\int (\cos^{-1} x)^2\, dx$

79. $\int \frac{\sin^{-1} x}{x^2}\, dx$

80. $\int_{-2}^{-1} \sqrt{-4x - x^2}\, dx$

81. $\int \frac{x^4 + 2x^3 + 5x^2 + 2x + 1}{x^5 + 2x^3 + x}\, dx$

82. $\int \frac{dx}{1 + \tan x}$

83. $\int e^x \sin^{998}(e^x)\cos^3(e^x)\, dx$

84. $\int \frac{\tan\theta + \tan^3\theta}{(1 + \tan\theta)^{50}}\, d\theta$

85. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. More than one integration method can be used to evaluate $\int \frac{dx}{1 - x^2}$.
   b. Using the substitution $u = \sqrt[3]{x}$ in $\int \sin\sqrt[3]{x}\, dx$ leads to $\int 3u^2 \sin u\, du$.
   c. The most efficient way to evaluate $\int \tan 3x \sec^2 3x\, dx$ is to first rewrite the integrand in terms of $\sin 3x$ and $\cos 3x$.
   d. Using the substitution $u = \tan x$ in $\int \frac{\tan^2 x}{\tan x - 1}\, dx$ leads to $\int \frac{u^2}{u - 1}\, du$.

86. **Area** Find the area of the region bounded by the curves $y = \frac{x}{x^2 - 2x + 2}$, $y = \frac{2}{x^2 - 2x + 2}$, and $x = 0$.

87. **Surface area** Find the area of the surface generated when the curve $f(x) = \sin x$ on $[0, \pi/2]$ is revolved about the x-axis.