8.1.80
$$d(t) = \int_0^t v(y)\,dy = \int_0^t v_T\left(\frac{e^{ay}-1}{e^{ay}+1}\right)dy = v_T\int_0^t \frac{e^{ay}}{e^{ay}+1}\,dy + -v_T\int_0^t \frac{1}{e^{ay}+1}\,dy.$$

The first integral can be computed by letting $u = e^{ay}+1$ so that $du = ae^{ay}\,dy$. The first integral is then equal to $(v_T/a)\int_2^{e^{at}+1}\frac{1}{u}\,du = (v_T/a)(\ln(e^{at}+1)-\ln(2))$. The second integral is $-v_T\int_0^t \frac{1}{e^{ay}+1}\cdot\frac{e^{ay}}{e^{ay}}\,dy$. Again, let $u = e^{ay}+1$ and note that $du = ae^{ay}\,dy$. Substitution gives

$$\begin{aligned}-(v_T/a)\int_2^{e^{at}+1}\frac{1}{u(u-1)}\,du &= -(v_T/a)\int_2^{e^{at}+1}\left(\frac{1}{u-1}-\frac{1}{u}\right)dy \\ &= -(v_T/a)\left(\ln(u-1)-\ln(u)\right)\Big|_2^{e^{at}+1} = -(v_T/a)(at-\ln(e^{at}+1)-(0-\ln 2)).\end{aligned}$$

Adding the results of the two integrations gives

$$d(t) = (v_T/a)(\ln(e^{at}+1)-\ln(2)) + -(v_T/a)(at-\ln(e^{at}+1)-(0-\ln 2)) = (v_T/a)(2\ln(e^{at}+1)-at-\ln 4)$$

8.2 Integration by Parts

8.2.1 It is based on the product rule. In fact, the rule can be obtained by writing down the product rule, then integrating both sides and rearranging the terms in the result.

8.2.2 $u = x$ so $du = dx$, and $dv = \cos x\,dx$ so $v = \sin x$. Then the integration by parts formula gives

$$x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

8.2.3 $u = \ln x$ so $du = \frac{dx}{x}$, and $dv = x\,dx$ so $v = \frac{x^2}{2}$. Then the integration by parts formula gives

$$\frac{x^2\ln x}{2} - \int\frac{x^2}{2x}\,dx = \frac{x^2\ln x}{2} - \int\frac{x}{2}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C = \frac{x^2(2\ln x - 1)}{4} + C.$$

8.2.4 One can use integration by parts for definite integrals via the formula

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

8.2.5 Those for which the choice for dv is easily integrated and when the resulting new integral is no more difficult than the original.

8.2.6 It is generally a good idea to let dv be something easy to integrate. In this case, we would let $dv = e^{ax}\,dx$, leaving $u = x^n$. Note that differentiating x^n results in something simpler (lower degree,) while integrating it make it more complicated (higher degree). However, differentiating or integrating e^{ax} yields essentially the same thing (a constant times the function e^{ax}).

8.2.7 Let $u = \tan x + 2$, so that $du = \sec^2 x\,dx$. Then

$$\int \sec^2 x\ln(\tan x + 2)\,dx = \int \ln u\,du = u\ln u - u + C = (\tan x + 2)\ln(\tan x + 2) - \tan x + C.$$

8.2.8 Let $u = \sin x$ so that $du = \cos x\,dx$. Then

$$\int \cos x\ln(\sin x)\,dx = \int \ln u\,du = u\ln u - u + C = \sin x\ln(\sin x) - \sin x + C.$$

8.2.9 Let $u = x$ and $dv = \cos 5x\,dx$. Then $du = dx$ and $v = \dfrac{1}{5}\sin 5x$. Then

$$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \frac{1}{5}\int \sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C.$$

8.2.10 Let $u = x$ and $dv = \sin 2x\,dx$. Then $du = dx$ and $v = -\frac{1}{2}\cos(2x)$. So

$$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

8.2.11 Let $u = t$ and $dv = e^{6t}\,dt$. Then $du = dt$ and $v = \dfrac{1}{6}\cdot e^{6t}$. Then

$$\int te^{6t}\,dt = \frac{1}{6}te^{6t} - \frac{1}{6}\int e^{6t}\,dt = \frac{1}{6}te^{6t} - \frac{1}{36}e^{6t} + C.$$

8.2.12 Let $u = 2x$ and $dv = e^{3x}\,dx$. Then $du = 2\,dx$ and $v = \frac{e^{3x}}{3}$. Then

$$\int 2xe^{3x}\,dx = \frac{2xe^{3x}}{3} - \frac{2}{3}\int e^{3x}\,dx = \frac{2xe^{3x}}{3} - \frac{2e^{3x}}{9} + C.$$

8.2.13 Let $u = \ln 10x$ and $dv = x\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{x^2}{2}$. Then

$$\int x\ln 10x\,dx = \frac{x^2}{2}\ln 10x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln 10x - \frac{x^2}{4} + C = \frac{x^2}{4}(2\ln 10x - 1) + C.$$

8.2.14 Let $u = s$ and $dv = e^{-2s}\,ds$. Then $du = ds$ and $v = -\dfrac{1}{2}e^{-2s}$. Then

$$\int se^{-2s}\,ds = -\frac{1}{2}se^{-2s} + \frac{1}{2}\int e^{-2s}\,ds = -\frac{1}{2}se^{-2s} - \frac{1}{4}e^{-2s} + C.$$

8.2.15 Let $u = 2w + 4$ and $dv = \cos 2w\,dw$. Then $du = 2\,dw$ and $v = \dfrac{1}{2}\sin 2w$. The integration by parts formula gives

$$(w+2)\sin 2w - \int \sin 2w\,dw = (w+2)\sin 2w + \frac{1}{2}\cos 2w + C.$$

8.2.16 Let $u = \theta$ and $dv = \sec^2\theta\,d\theta$. Then $du = d\theta$ and $v = \tan\theta$. Then

$$\int \theta\sec^2\theta\,d\theta = \theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta + \ln|\cos\theta| + C.$$

8.2.17 Let $u = x$ and $dv = 3^x\,dx$. Then $du = dx$ and $v = \dfrac{3^x}{\ln 3}$. Then we have

$$\frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx = \frac{x3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + C = \frac{3^x}{\ln 3}\left(x - \frac{1}{\ln 3}\right) + C.$$

8.2.18 Let $u = \ln x$ and $dv = x^9\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{1}{10}x^{10}$. Then we have

$$\frac{1}{10}x^{10}\ln x - \frac{1}{10}\int x^9\,dx = \frac{1}{10}x^{10}\ln x - \frac{1}{100}x^{10} + C = \frac{1}{100}x^{10}(10\ln x - 1) + C.$$

8.2.19 Let $u = \ln x$ and $dv = x^{-10}\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = -\dfrac{1}{9}x^{-9}$. Then

$$\int \frac{\ln x}{x^{10}}\,dx = -\frac{1}{9x^9}\ln x + \frac{1}{9}\int x^{-10}\,dx = -\frac{1}{9x^9}\ln x + -\frac{1}{81x^9} + C.$$

8.2.20 Let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $v = x$. Then

$$\int \sin^{-1} x\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

The fact that $-\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = \sqrt{1-x^2} + C$ follows from the ordinary substitution $u = 1 - x^2$.

8.2.21

$$\int x\sin x\cos x\,dx = \frac{1}{2}\int x\cdot(2\sin x\cos x)\,dx = \frac{1}{2}\int x\sin 2x\,dx.$$

Now using the result of problem 10, we have

$$\int x\sin x\cos x\,dx = \frac{1}{2}\cdot\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right) + C = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C.$$

8.2.22 Let $u = e^x$ and $dv = e^x\sin e^x\,dx$. Then $du = e^x\,dx$ and $v = -\cos e^x$. Then we have

$$-e^x\cos e^x + \int e^x\cos e^x\,dx = -e^x\cos e^x + \sin e^x + C = \sin e^x - e^x\cos e^x + C.$$

8.2.23 Let $u = x^2$ and $dv = \sin 2x\,dx$. Then $du = 2x\,dx$ and $v = -\frac{1}{2}\cos 2x$. Then we have

$$\int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx.$$

Now we consider computing this last term $\displaystyle\int x\cos 2x\,dx$ as a new problem. Let $u = x$ and $dv = \cos 2x\,dx$. Then $du = dx$ and $v = \dfrac{1}{2}\sin 2x$. So

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

Combining these results we have

$$\int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

8.2.24 Let $u = x^2$ and $dv = e^{4x}\,dx$. Then $du = 2x\,dx$ and $v = \dfrac{e^{4x}}{4}$. Then we have

$$\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx.$$

Now we consider computing this last integral $\displaystyle\int xe^{4x}\,dx$ as a new problem. Let $u = x$ and $dv = e^{4x}\,dx$. Then $du = dx$ and $v = \dfrac{e^{4x}}{4}$. Then we have

$$\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C.$$

Combining these results gives

$$\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C = e^{4x}\left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) + C.$$

8.2.25 Let $u = t^2$ and $dv = e^{-t}\,dt$. Then $du = 2t\,dt$ and $v = -e^{-t}$. We have

$$\int t^2 e^{-t}\,dt = -t^2 e^{-t} + 2\int te^{-t}\,dt.$$

To compute this last integral, we let $u = t$ and $dv = e^{-t}\,dt$. Then

$$\int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C.$$

Putting these results together, we obtain

$$\int t^2 e^{-t}\,dt = -t^2 e^{-t} + 2(-te^{-t} - e^{-t}) + C = -e^{-t}(t^2 + 2t + 2) + C.$$

8.2.26 Let $u = t^3$ and $dv = \sin t\,dt$. Then $du = 3t^2\,dt$ and $v = -\cos t$. Then

$$\int t^3 \sin t\,dt = -t^3 \cos t + 3\int t^2 \cos t.$$

We then consider the last integral. We now let $u = t^2$ and $dv = \cos t\,dt$. Then $du = 2t\,dt$ and $v = \sin t$. We have

$$3\int t^2 \cos t\,dt = 3t^2 \sin t - 6\int t \sin t\,dt.$$

For the last integral, we let $u = t$ and $dv = \sin t\,dt$. Then $du = dt$ and $v = \cos t$. We have

$$-6\int t \sin t\,dt = 6t\cos t - 6\int \cos t = 6t\cos t - 6\sin t + C.$$

Putting these results together, we have

$$\int t^3 \sin t\,dt = -t^3 \cos t + 3t^2 \sin t + 6t\cos t - 6\sin t + C.$$

8.2.27 Let $u = \cos x$ and $dv = e^x\,dx$. Then $du = -\sin x\,dx$ and $v = e^x$. We have

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin x$ and $dv = e^x\,dx$. Then $du = \cos x\,dx$ and $v = e^x$. Thus,

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

Putting these results together gives

$$\begin{aligned}
\int e^x \cos x\,dx &= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx \\
2\int e^x \cos x\,dx &= e^x(\cos x + \sin x) + C \\
\int e^x \cos x\,dx &= \frac{e^x}{2}(\cos x + \sin x) + C.
\end{aligned}$$

8.2.28 Let $u = \cos 2x$ and $dv = e^{3x}\,dx$. Then $du = -2\sin 2x\,dx$ and $v = \frac{1}{3}e^{3x}$. We have

$$\int e^{3x} \cos 2x\,dx = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x} \sin 2x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin 2x$ and $dv = e^{3x}\,dx$. Then $du = 2\cos 2x\,dx$ and $v = \frac{1}{3}e^{3x}$. Thus,

$$\int e^{3x}\sin 2x\,dx = \frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\,dx.$$

Putting these results together gives

$$\begin{aligned}
\int e^{3x}\cos 2x\,dx &= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}\int e^{3x}\cos 2x\,dx \\
\frac{13}{9}\int e^{3x}\cos 2x\,dx &= \frac{1}{3}e^{3x}\left(\cos 2x + \frac{2}{3}\sin 2x\right) + C \\
\int e^{3x}\cos 2x\,dx &= \frac{3}{13}e^{3x}\left(\cos 2x + \frac{2}{3}\sin 2x\right) + C.
\end{aligned}$$

8.2.29 Let $u = \sin 4x$ and $dv = e^{-x}\,dx$. Then $du = 4\cos 4x\,dx$ and $v = -e^{-x}$. We have

$$\int e^{-x}\sin 4x\,dx = -e^{-x}\sin 4x + 4\int e^{-x}\cos 4x\,dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 4x$ and $dv = e^{-x}\,dx$. Then $du = -4\sin 4x\,dx$ and $v = -e^{-x}$. Thus,

$$\int e^{-x}\cos 4x\,dx = -e^{-x}\cos 4x - 4\int e^{-x}\sin 4x\,dx.$$

Putting these results together gives

$$\begin{aligned}
\int e^{-x}\sin 4x\,dx &= -e^{-x}\sin 4x - 4e^{-x}\cos 4x - 16\int e^{-x}\sin 4x\,dx \\
17\int e^{-x}\sin 4x\,dx &= -e^{-x}\sin 4x - 4e^{-x}\cos 4x + C \\
\int e^{-x}\sin 4x\,dx &= -\frac{e^{-x}}{17}\left(\sin 4x + 4\cos 4x\right) + C.
\end{aligned}$$

8.2.30 Let $u = \sin 6\theta$ and $dv = e^{-2\theta}\,d\theta$. Then $du = 6\cos 6\theta\,d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. We have

$$\int e^{-2\theta}\sin 6\theta\,d\theta = -\frac{1}{2}e^{-2\theta}\sin 6\theta + 3\int e^{-2\theta}\cos 6\theta\,d\theta.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 6\theta$ and $dv = e^{-2\theta}\,d\theta$. Then $du = -6\sin 6\theta\,d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. Thus,

$$\int e^{-2\theta}\cos 6\theta\,d\theta = -\frac{1}{2}e^{-2\theta}\cos 6\theta - 3\int e^{-2\theta}\sin 6\theta\,d\theta.$$

Putting these results together gives

$$\begin{aligned}
\int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{1}{2}e^{-2\theta}\sin 6\theta - \frac{3}{2}e^{-2\theta}\cos 6\theta - 9\int e^{-2\theta}\sin 6\theta\,d\theta \\
10\int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{1}{2}e^{-2\theta}\left(\sin 6\theta + 3\cos 6\theta\right) + C \\
\int e^{-2\theta}\sin 6\theta\,d\theta &= -\frac{e^{-2\theta}}{20}\left(\sin 6\theta + 3\cos 6\theta\right) + C.
\end{aligned}$$

8.2.31 Let $u = e^{2x}$ and $dv = e^x \sin e^x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = -\cos e^x$. Then we have

$$\int e^{3x} \sin e^x\,dx = -e^{2x}\cos e^x + 2\int e^{2x}\cos e^x\,dx.$$

To compute the last integral, we let $u = e^x$ and $dv = e^x \cos e^x\,dx$. Then $du = e^x\,dx$ and $v = \sin e^x$. Then the last integral is

$$2\int e^{2x}\cos e^x\,dx = 2e^x \sin e^x - 2\int e^x \sin e^x\,dx = 2e^x\sin e^x + 2\cos e^x + C.$$

Combining these results gives a final answer of

$$\int e^{3x}\sin e^x\,dx = -e^{2x}\cos e^x + 2e^x \sin e^x + 2\cos e^x + C.$$

8.2.32 Let $u = x^2$ and $dv = 2^x\,dx$, so that $du = 2x\,dx$ and $v = \dfrac{2^x}{\ln 2}$. Then we have

$$\left.\frac{x^2 2^x}{\ln 2}\right|_0^1 - \frac{2}{\ln 2}\int_0^1 x2^x\,dx = \frac{2}{\ln 2} - \frac{2}{\ln 2}\int_0^1 x2^x\,dx.$$

Now we let $u = x$ and $dv = 2^x\,dx$, so that $du = dx$ and $v = \dfrac{2^x}{\ln 2}$. The given integral is then equal to

$$\begin{aligned}\frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\left.\frac{x2^x}{\ln 2}\right|_0^1 - \frac{1}{\ln 2}\int_0^1 2^x\,dx\right) &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln 2}\left(\left.\frac{2^x}{\ln 2}\right|_0^1\right)\right)\\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \left(\frac{2}{\ln^2 2} - \frac{1}{\ln^2 2}\right)\right).\end{aligned}$$

This can be written as

$$\frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln^2 2}\right) = \frac{2\ln^2 2}{\ln^3 2} - \frac{2(2\ln 2 - 1)}{\ln^3 2} = \frac{2(\ln 2 - 1)^2}{\ln^3 2}.$$

8.2.33 Let $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. Then

$$\int_0^\pi x\sin x\,dx = -x\cos x\,\Big|_0^\pi + \int_0^\pi \cos x\,dx = \pi - 0 + \sin x\,\Big|_0^\pi = \pi - 0 + 0 - 0 = \pi.$$

8.2.34 First note that $\displaystyle\int_1^e \ln 2x\,dx = \int_1^e \ln 2\,dx + \int_1^e \ln x\,dx = \ln 2(e-1) + \int_1^e \ln x\,dx.$

Let $u = \ln x$ and $dv = dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = x$. Then

$$\int_1^e \ln x\,dx = x\ln x\,\Big|_1^e - \int_1^e dx = e - (e-1) = 1.$$

Thus $\displaystyle\int_1^e \ln 2x\,dx = \ln 2(e-1) + 1.$

8.2.35 Let $u = x$ and $dv = \cos 2x\,dx$. Then $du = dx$ and $v = \dfrac{1}{2}\sin 2x$. Then

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x\,\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = 0 - \left(\frac{1}{2}\cdot\frac{(-\cos 2x)}{2}\right)\Big|_0^{\pi/2} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

8.2.36 Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Then

$$\int_0^{\ln 2} xe^x\,dx = xe^x\Big|_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx = 2\ln 2 - (e^x)\Big|_0^{\ln 2} = 2\ln 2 - (2-1) = 2\ln 2 - 1.$$

8.2.37 Let $u = \ln x$ and $dv = x^2\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. Then

$$\int_1^{e^2} x^2\ln x\,dx = \frac{1}{3}x^3\ln x\Big|_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx = \frac{2}{3}e^6 - \frac{1}{9}x^3\Big|_1^{e^2} = \frac{2}{3}e^6 - \frac{1}{9}\left(e^6 - 1\right) = \frac{5}{9}e^6 + \frac{1}{9}.$$

8.2.38 Let $u = \ln^2 x$ and $dv = x^2\,dx$. Then $du = \frac{2\ln x}{x}$ and $v = \frac{x^3}{3}$. We have

$$\int x^2\ln^2 x\,dx = \frac{1}{3}x^3\ln^2 x - \frac{2}{3}\int x^2\ln x\,dx.$$

To compute this last integral, let $u = \ln x$ and $dv = x^2\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. Then

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C.$$

Thus, $\displaystyle\int x^2\ln^2 x\,dx = \frac{x^3}{3}\ln^2 x - \frac{2x^3}{27}(3\ln x - 1) + C = \frac{x^3}{27}(9\ln^2 x - 6\ln x + 2) + C.$

8.2.39 By problem 20, $\displaystyle\int \sin^{-1} y\,dy = y\sin^{-1} y + \sqrt{1-y^2}$. Thus,

$$\int_0^1 \sin^{-1} y\,dy = \left(y\sin^{-1} y + \sqrt{1-y^2}\right)\Big|_0^1 = \left(\frac{\pi}{2} + 0\right) - (0+1) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}.$$

8.2.40 Let $u = \sqrt{x}$ and $dv = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. Then $du = \frac{dx}{2\sqrt{x}}$ and $v = 2e^{\sqrt{x}}$. Then

$$\int e^{\sqrt{x}}\,dx = \int \frac{\sqrt{x}e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

8.2.41

a. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x$. Then

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$$

b. Let $u = x^2$. Then $du = 2x\,dx$. Then

$$\int x\tan^{-1} x^2\,dx = \int \frac{1}{2}\tan^{-1} u\,du = \frac{1}{2}u\tan^{-1} u - \frac{1}{4}\ln(1+u^2) + C = \frac{1}{2}x^2\tan^{-1} x^2 - \frac{1}{4}\ln(1+x^4) + C.$$

8.2.42 The volume is given by $V = \displaystyle\int_0^1 2\pi ye^y\,dy$. Let $u = y$ and $dv = e^y\,dy$. Then $du = dy$ and $v = e^y$. Then

$$V = 2\pi\left(ye^y\right)\Big|_0^1 - 2\pi\int_0^1 e^y\,dy = 2\pi\,(e - 0) - 2\pi(e^y)\Big|_0^1 = 2\pi e - 2\pi(e-1) = 2\pi.$$

8.2.43 Using shells, we have $\dfrac{V}{2\pi} = \displaystyle\int_0^{\ln 2} xe^{-x}\,dx$. Let $u = x$ and $dv = e^{-x}\,dx$, so that $du = dx$ and $v = -e^{-x}$. Then

$$\frac{V}{2\pi} = -xe^{-x}\bigg|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx = -\frac{1}{2}\ln 2 - e^{-x}\bigg|_0^{\ln 2} = -\frac{\ln 2}{2} - \left(\frac{1}{2} - 1\right) = \frac{1}{2}\left(1 - \ln 2\right).$$

Thus $V = \pi(1 - \ln 2)$.

8.2.44 Using shells, we have $\dfrac{V}{2\pi} = \displaystyle\int_0^{\pi} x\sin x\,dx$. Let $u = x$ and $dv = \sin x\,dx$, so that $du = dx$ and $v = -\cos x$. Then

$$\frac{V}{2\pi} = -x\cos x\bigg|_0^{\pi} + \int_0^{\pi}\cos x\,dx = \pi + \sin x\bigg|_0^{\pi} = \pi.$$

Thus $V = 2\pi^2$.

8.2.45 We have $V = \displaystyle\int_1^e \pi\ln x\,dx = \pi(x\ln x - x)\bigg|_1^e = \pi(e - e) - \pi(0 - 1) = \pi$.

8.2.46 Using shells, we have $\dfrac{V}{2\pi} = \displaystyle\int_0^{\ln 2}(\ln 2 - x)e^{-x}\,dx = \ln 2\int_0^{\ln 2} e^{-x}\,dx - \int_0^{\ln 2} xe^{-x}\,dx$.

In the course of solving problem 43, we deduced that $\displaystyle\int_0^{\ln 2} xe^{-x}\,dx = \frac{1 - \ln 2}{2}$. Thus,

$$\frac{V}{2\pi} = \ln 2\left(-e^{-x}\right)\bigg|_0^{\ln 2} - \frac{1 - \ln 2}{2} = \ln 2\left(-\frac{1}{2} + 1\right) - \frac{1 - \ln 2}{2} = \ln 2 - \frac{1}{2}.$$

Thus, $V = 2\pi\left(\ln 2 - \frac{1}{2}\right) = \pi(\ln 4 - 1)$.

8.2.47 Using disks, we have $\dfrac{V}{\pi} = \displaystyle\int_1^{e^2} x^2\ln^2 x\,dx$. By problem 38, we have $\displaystyle\int x^2\ln^2 x\,dx = \frac{1}{3}x^3\ln^2 x - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C$. Thus,

$$\frac{V}{\pi} = \left(\frac{1}{3}x^3\ln^2 x - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3\right)\bigg|_1^{e^2} = \left(\frac{4}{3}e^6 - \frac{4}{9}e^6 + \frac{2}{27}e^6\right) - \left(\frac{2}{27}\right) = \frac{26}{27}e^6 - \frac{2}{27}.$$

Thus, $V = \dfrac{\pi}{27}\left(26e^6 - 2\right)$.

8.2.48 Let $u = \sec x$ and $dv = \sec^2 x\,dx$, so that $du = \sec x\tan x\,dx$ and $v = \tan x$. Then $\displaystyle\int \sec^3 x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int \sec x\left(\sec^2 x - 1\right)\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$.

Thus $2\displaystyle\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx$, so

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx.$$

8.2.49

a. False. For example, suppose $u = x$ and $dv = x\,dx$. Then $\displaystyle\int uv'\,dx = \int x^2\,dx = \frac{x^3}{3} + C$, but $\displaystyle\int u\,dx\int v'\,dx = \left(\int x\,dx\right)^2 = \left(\frac{x^2}{2} + C\right)^2$.

b. True. This is one way to write the integration by parts formula.

c. True. This is the integration by parts formula with the roles of u and v reversed.

8.2.50 Let $u = x^n$ and $dv = e^{ax}\,dx$. Then $du = nx^{n-1}\,dx$ and $v = \dfrac{e^{ax}}{a}$. Then

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx.$$

8.2.51 Let $u = x^n$ and $dv = \cos ax\,dx$. Then $du = nx^{n-1}\,dx$ and $v = \dfrac{\sin ax}{a}$. Then

$$\int x^n \cos ax\,dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx.$$

8.2.52 Let $u = x^n$ and $dv = \sin ax\,dx$. Then $du = nx^{n-1}\,dx$ and $v = -\dfrac{\cos ax}{a}$. Then

$$\int x^n \sin ax\,dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx.$$

8.2.53 Let $u = \ln^n x$ and $dv = dx$. Then $du = \dfrac{n\ln^{n-1}(x)}{x}\,dx$ and $v = x$. Then

$$\int \ln^n(x)\,dx = x\ln^n x - n\int \ln^{n-1}(x)\,dx.$$

8.2.54

$$\begin{aligned}\int x^2 e^{3x}\,dx &= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}\,dx\\ &= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx\right)\\ &= \frac{1}{3}\left(x^2e^{3x} - \frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x}\right) + C\\ &= \frac{e^{3x}}{3}\left(x^2 - \frac{2}{3}x + \frac{2}{9}\right) + C.\end{aligned}$$

8.2.55

$$\begin{aligned}\int x^2\cos 5x\,dx &= \frac{x^2\sin 5x}{5} - \frac{2}{5}\int x\sin 5x\,dx\\ &= \frac{x^2\sin 5x}{5} - \frac{2}{5}\left(-\frac{x\cos 5x}{5} + \frac{1}{5}\int \cos 5x\,dx\right)\\ &= \frac{1}{5}\left(x^2\sin 5x + \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x\right) + C.\end{aligned}$$

8.2.56

$$\begin{aligned}\int x^3\sin x\,dx &= -x^3\cos x + 3\int x^2\cos x\,dx\\ &= -x^3\cos x + 3\left(x^2\sin x - 2\int x\sin x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x - 6\left(-x\cos x + \int \cos x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

8.2.57

$$\begin{aligned}\int_1^e \ln^3 x\,dx &= x\ln^3 x\Big|_1^e - 3\int_1^e \ln^2 x\,dx\\ &= e - 3\left(x\ln^2 x\right)\Big|_1^e + 6\int_1^e \ln x\,dx\\ &= e - 3e + 6(x\ln x - x)\Big|_1^e\\ &= e - 3e + 6((e-e)-(0-1)) = 6-2e.\end{aligned}$$

8.2.58

a. Let $u = \ln\cos x$ and $dv = \sin x\,dx$. Then $du = -\tan x\,dx$ and $v = -\cos x$. We have

$$\begin{aligned}\int_0^{\pi/3} \sin x\ln(\cos x)\,dx &= -\cos x\ln\cos x\Big|_0^{\pi/3} - \int_0^{\pi/3}\sin x\,dx = -\frac{1}{2}\ln\frac{1}{2} + \left(\cos x\Big|_0^{\pi/3}\right)\\ &= -\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} - 1 = \frac{1}{2}\ln 2 - \frac{1}{2} - \frac{\ln 2 - 1}{2}.\end{aligned}$$

b. Let $u = \cos x$. Then $du = -\sin x\,dx$. Substituting gives

$$-\int_1^{1/2} \ln u\,du = (u\ln u - u)\Big|_{1/2}^1 = (0-1) - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right) = \frac{\ln 2 - 1}{2}.$$

8.2.59

a. Let $u = x$ and $dv = \dfrac{dx}{\sqrt{x+1}}$. Then $du = dx$ and $v = 2\sqrt{x+1}$. Then

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x\sqrt{x+1} - \int 2\sqrt{x+1}\,dx. = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2} + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

b. Let $u = x+1$. Then $du = dx$ and $x = u-1$. Substituting gives

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int (u^{1/2} - u^{-1/2})\,du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

c. The answer to b can be written as the answer to a:

$$\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C = \frac{2}{3}(x+1)^{1/2}(x+1-3) + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

8.2.60

a. Let $u = x^2$, so that $du = 2x\,dx$. Then

$$\int x\ln x^2\,dx = \frac{1}{2}\int \ln u\,du = \frac{1}{2}\left(u\ln u - u\right) + C = \frac{1}{2}\left(x^2\ln(x^2) - x^2\right) + C.$$

b. Let $u = \ln x$ and $dv = x\,dx$. Then $du = \dfrac{1}{x}\,dx$ and $v = \dfrac{x^2}{2}$. Then

$$\int x\ln x^2\,dx = 2\int x\ln x\,dx = 2\left(\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right) = x^2\ln x - \frac{x^2}{2} + C.$$

c. The answer to the first part is $\dfrac{1}{2}\left(x^2\ln(x^2) - x^2\right) + C = x^2\ln(x) - \dfrac{x^2}{2} + C.$ which is the answer to the second part.

8.2.61 Using the change of base formula, we have $\int \log_b x\,dx = \int \frac{\ln x}{\ln b}\,dx = \frac{1}{\ln b}\,(x\ln x - x) + C.$

8.2.62 By parts: Let $u = \sin x$ and $dv = \cos x\,dx$, so that $du = \cos x\,dx$ and $v = \sin x$. Then

$$\int \sin x\cos x\,dx = \sin^2 x - \int \sin x\cos x\,dx,$$

so $\int \sin x\cos x\,dx = \frac{\sin^2 x}{2} + C.$

By substitution: Let $u = \sin x$, so that $du = \cos x\,dx$. Then we have $\int \sin x\cos x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$. The two answers are the same.

8.2.63 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $2\int \frac{\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}}\,dx = 2\int z\cos z\,dz$. Now let $u = z$ and $dv = \cos z\,dz$. then $du = dz$ and $v = \sin z$. Then by Integration by Parts, we have

$$2\int z\cos z\,dz = 2\left(z\sin z - \int \sin z\,dz\right) = 2z\sin z + 2\cos z + C.$$

Thus, the original given integral is equal to $2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + C$.

8.2.64 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx = 2\int_0^{\pi/2} z\sin z\,dz$. Now let $u = z$ and $dv = \sin z\,dz$. Then $du = dz$ and $v = -\cos z$. Then by Integration by Parts, we have

$$2\int_0^{\pi/2} z\sin z\,dz = 2(-z\cos z)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} \cos z = 0 + 2\sin z\Big|_0^{\pi/2} = 2.$$

8.2.65 Let $u = x$ and $dv = f''(x)\,dx$. Then $du = dx$ and $v = f'(x)$. We have

$$\int_a^b xf''(x)\,dx = xf'(x)\Big|_a^b - \int_a^b f'(x)\,dx = (0-0) - f(x)\Big|_a^b = -(f(b) - f(a)) = f(a) - f(b).$$

8.2.66 Let $u = f(x)$ and $dv = f'(x)\,dx$. Then $du = f'(x)\,dx$ and $v = f(x)$. We have

$$\int_a^b f(x)f'(x)\,dx = f(x)^2\Big|_a^b - \int_a^b f(x)f'(x)\,dx.$$

Thus, $2\int_a^b f(x)f'(x)\,dx = f(x)^2\Big|_a^b$, so $\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\,f(x)^2\Big|_a^b = \frac{1}{2}\left(f(b)^2 - f(a)^2\right).$

8.2.67 By the Fundamental Theorem, $f'(x) = \sqrt{\ln^2 x - 1}$. So the arc length is $\int_e^{e^3} \sqrt{1 + (f'(x))^2}\,dx = \int_e^{e^3} \ln x\,dx = (x\ln x - x)\Big|_e^{e^3} = 3e^3 - e^3 - (e - e) = 2e^3.$

8.2.68 Suppose $m \neq -1$ and let $u = \ln x$ and $dv = x^m\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^{m+1}}{m+1}$. Then $\int x^m\ln x\,dx = \frac{x^{m+1}}{m+1}\ln x - \frac{1}{m+1}\int x^m\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right) + C.$

For the case $m = -1$ we are computing $\int \frac{1}{x}\ln x\,dx$, so letting $u = \ln x$ so that $du = \frac{1}{x}\,dx$ yields $\int u\,du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C.$