## Contents



## 5.1 Antiderivatives (Indefinite Integrals)

### 5.1.1 Definition and Properties of Indefinite Integrals

In Calculus I, we spent considerable time considering the derivatives of a function and their applications and discussed, “the other direction,” antiderivatives. That is, given a function $f(x)$, we discussed functions $F(x)$ such that $F^{'}(x) = f(x)$. We first review this topic.

Given a function $y = f(x)$, a *differential equation* is one that incorporates $y$, $x$, and the derivatives of $y$. For instance, a simple differential equation is:

$$y^{'} = 2x.$$

Solving a differential equation amounts to finding a function $y$ that satisfies the given equation. Take a moment and consider that equation; can you find a function $y$ such that $y^{'} = 2x$?

Can you find another? And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. “Finding another” may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding “yet another” is not difficult; the function $y = x^2+123,456,789$ also has a derivative of $2x$. The differential equation $y^{'} = 2x$ has many solutions. This leads us to some definitions.

**Definition** (Antiderivatives and Indefinite Integrals)**.** Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F^{'}(x) = f(x)$.
The set of all antiderivatives of $f(x)$ is the **indefinite integral of** $f$, denoted by

$$\int f(x)\ dx.$$

Knowing one antiderivative of $f$ allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

**Theorem** (Antiderivative Forms)**.** Let $F(x)$ be any antiderivatives of $f(x)$ on an interval $I$. Then all the antiderivatives of $f$ on $I$ have the form

$$F(x) + C,$$

where $C$ is an arbitrary constant.

**Remark.** Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential $dx$. For now, $dx$ simply means that $x$ is the independent variable, or the variable of integration. The notation $\int f(x)\ dx$ represents all the antiderivatives of $f$.

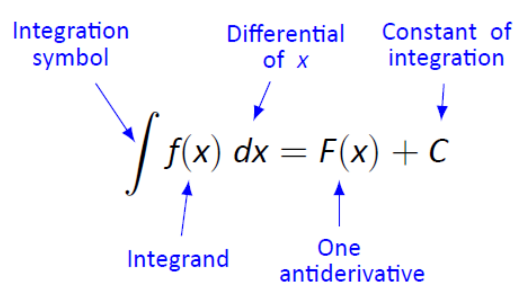



Figure 1: Understanding the indefinite integral notation.

Figure 1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an “elongated S,” representing “take the sum.” We will later see how *sums* and *antiderivatives* are related. The $\int$ symbol and the differential $dx$

are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and $dx$ are multiplied together; the $dx$ does not "just sit there."

**Example** Determine the following indefinite integrals

**a.** $\int 3x^3\,dx$ **b.** $\int \frac{1}{1+x^2}\,dx$ **c.** $\int \sin t\,dt$

**Solution.**

$\square$

**Theorem** (Power Rule for Indefinite Integrals)**.**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and $C$ is an arbitrary constant.

**Theorem** (Constant Multiple and Sum Rules)**.**

$$\textbf{Constant Multiple Rule: } \int cf(x)\,dx = c\int f(x)\,dx, \text{ for real numbers } c$$

$$\textbf{Sum Rule: } \int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

**Example** Determine the following indefinite integrals

**a.** $\int (3x^5 + 2 - 5\sqrt{x})\, dx$ **b.** $\int \left( \frac{4x^{19} - 5x^{-8}}{x^2} \right)\, dx$ **c.** $\int (z^2 + 1)(2z - 5)\, dz$

**Solution.**

$\square$

We restate a list of derivatives here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn ($D_x$ denotes $\frac{d}{dx}$).

$$D_x(x^n) = nx^{n-1} \implies \int x^n\ dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$D_x(e^x) = e^x \implies \int e^x\ dx = e^x + C$$

$$D_x(a^x) = a^x \ln a \implies \int a^x\ dx = \frac{a^x}{\ln a} + C$$

$$D_x(\ln|x|) = \frac{1}{x} \implies \int \frac{dx}{x} = \ln|x| + C$$

$$D_x(\sin x) = \cos x \implies \int \cos x\ dx = \sin x + C$$

$$D_x(\cos x) = -\sin x \implies \int \sin x\ dx = -\cos x + C$$

$$D_x(\tan x) = \sec^2 x \implies \int \sec^2 x\ dx = \tan x + C$$

$$D_x(\sec x) = \sec x \tan x \implies \int \sec x \tan x\ dx = \sec x + C$$

$$D_x(\cot x) = -\csc^2 x \implies \int \csc^2\ dx = -\cot x + C$$

$$D_x(\csc x) = -\csc x \cot x \implies \int \csc x \cot x\ dx = -\csc x + C$$

$$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \implies \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \implies \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C = -\sin^{-1} x + C$$

$$D_x(\tan^{-1} x) = \frac{1}{1+x^2} \implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$D_x(\cot^{-1} x) = \frac{-1}{1+x^2} \implies \int \frac{-dx}{1+x^2} = \cot^{-1} x + C = -\tan^{-1} x + C$$

$$D_x(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \implies \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

$$D_x(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} \implies \int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}|x| + C = -\sec^{-1}|x| + C$$

$$D_x(\ln|\sec x|) = \tan x \implies \int \tan x\ dx = \ln|\sec x| + C$$

$$D_x(\ln|\sin x|) = \cot x \implies \int \cot x\ dx = \ln|\sin x| + C$$

**Example** Determine the following indefinite integrals

**a.** $\int \sin(3x)\, dx$ **b.** $\int \sec ax \tan ax\, dx$, where $a \neq 0$ is a real number

**Solution.**

$\square$

**Example** Determine the following indefinite integrals

**a.** $\int \sec^2 3x\, dx$ **b.** $\int \cos \frac{x}{2}\, dx$

**Solution.**

$\square$

**Example** Determine the following indefinite integrals

**a.** $\int e^{ax}\,dx$ **b.** $\int \frac{1}{a^2+x^2}\,dx$

**Solution.**

$\square$

**Example** Determine the following indefinite integrals

**a.** $\int e^{-10t}\,dt$ **b.** $\int \frac{4}{\sqrt{9-x^2}}\,dx$ **c.** $\int \frac{1}{\sqrt{16x^2+1}}\,dx$

**Solution.**

$\square$

### 5.1.2 Initial Value Problems (reading)

An equation involving an unknown function and its derivatives is called a **differential equation**. The equation

$$\frac{dy}{dx} = f(x) \tag{5.1}$$

is a simple example of a differential equation. Solving this equation means finding a function $y$ with a derivative $f$. Therefore, the solutions of Equation 5.1 are the antiderivatives of $f$. If $F$ is one antiderivative of $f$, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point $(x_0, y_0)$ that is, $y(x_0) = y_0$. The problem of finding a function $y$ that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \tag{5.2}$$

with the additional condition

$$y(x_0) = y_0 \tag{5.3}$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function $y$ that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5.$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function $y$ that also satisfies the initial condition, we need to find $C$ such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in Figure 2.

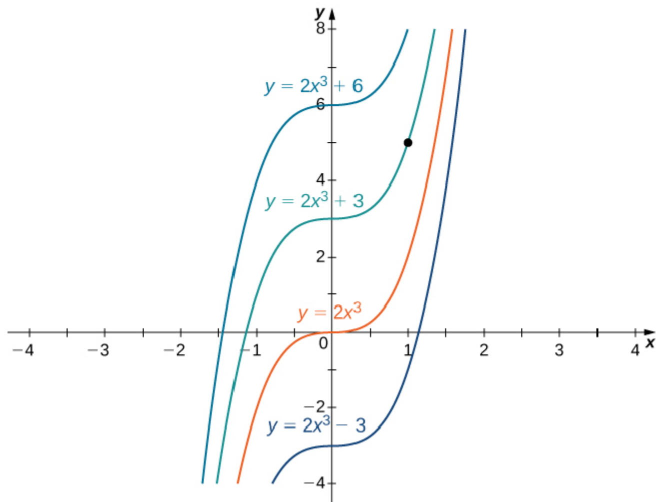



Figure 2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

**Example** Solve the initial value problem $f'(x) = x^2 - 2x$ with $f(1) = \frac{1}{3}$.

**Solution.**

$\square$

**Initial Value Problems for Velocity and Position** (reading)

Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem

$$s^{'}(t) = v(t), s(0) = s_0, \text{ where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v^{'}(t) = a(t), v(0) = v_0, \text{ where } v_0 \text{ is the (known) initial velocity.}$$

**Example** Runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.

**Solution.**

□

**Example** Neglecting air resistance, the motion of an object moving vertically near Earths surface is determined by the acceleration due to gravity, which is approximately $9.8m/s^2$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of $40m/s$ from the edge of a cliff that is $100m$ above a river.

**a.** Find the velocity $v(t)$ of the object, for $t \geq 0$.

**b.** Find the position $s(t)$ of the object, for $t \geq 0$.

**c.** Find the maximum height of the object above the river.

**d.** With what speed does the object strike the river?

**Solution.**

$\square$

### 5.1.3 Power Rule for Indefinite Integrals

Recall that the power rule for indefinite integrals is

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.}$$

Let's see some examples in this section.

**Example 1.** Find

$$\int \left(x^2 + \sqrt[3]{x} - \sqrt{x}\right) \, dx.$$

**Example 2.** Find

$$\int \left(x^{100} + \frac{\sqrt{x} - x^{3/2}}{x}\right) \, dx.$$

**Example 3.** Find

$$\int \left( x^{10} + \sqrt{x} + \frac{1}{\sqrt{x}} \right) \, dx.$$

**Example 4.** Find

$$\int \left( x^{-3} + \frac{x - x^2}{\sqrt{x}} \right) \, dx.$$

## 5.2 Definite Integrals

### 5.2.1 Review of $\sum$ notation and some formulas

Note: The definition of a definite integral involves the use of sigma ($\sum$) notation to describe a sum.

**Sigma Notation.** If $f$ is a function defined on a finite set $\{k, k+1, \ldots, m-1, m\}$ then

$$\sum_{i=k}^{m} f(i) = f(k) + f(k+1) + \cdots + f(m-1) + f(m).$$

□

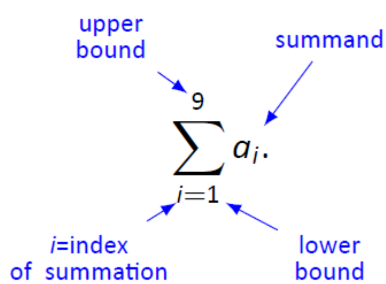



Figure 3: Understanding summation notation.

The upper case sigma represents the term "sum". The index of summation above is $i$; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

**Examples:**

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30,$$

$$\sum_{i=-1}^{7} i = -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = 27.$$

□

**Remark 1.** One should notice that in $\sum_{i=k}^{m} f(i)$ the $i$ is simply an **indexing** variable and so

$$\sum_{i=k}^{m} f(i) = \sum_{j=k}^{m} f(j).$$

**Remark 2.** One can do a change of variable in the sigma notation.

$$\begin{aligned}\sum_{i=3}^{6}(i-2)^2 =& (3-2)^2 + (4-2)^2 + (5-2)^2 + (6-2)^2 \\ =& 1^2 + 2^2 + 3^2 + 4^2 = \sum_{i=1}^{4} i^2\end{aligned}$$

$$\therefore \quad \sum_{i=3}^{6}(i-2)^2 = \sum_{j=1}^{4} j^2 \text{ where } j = i-2.$$

**Example.** Write $2^3 + \cdots + n^3$ in sigma notation starting with $j = 0$.

**Solution.**

$$2^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3 = \sum_{j=0}^{n-2}(j+2)^3, \text{ where } j = i-2. \qquad \square$$

**Some Basic Properties and Formulas.**

**1.** $\sum_{i=1}^{n} c = nc$

**2.** $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

**3.** $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

**4.** $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

**5.** $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

**6.** $\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

**Proof.** To establish Part **1**, we need only think of $c$ as the function $f(x) = c$ and then

$$\sum_{i=1}^{n} c = \sum_{i=1}^{n} f(i) = f(1) + \cdots + f(n) = c + \cdots + c = nc.$$

Part **2** is the distributive law and Part **3** follows by simply writing out the terms and then using the commutative law and the associative law to collect terms

For Part **4**, set $S = \sum_{i=1}^{n} i$, write it out forwards and backward and add to get:

$$\begin{aligned} S =& 1 + 2 + 3 + \cdots + (n-1) + n \\ S =& n + (n-1) + \cdots + 3 + 2 + 1 \\ \text{Add} \quad 2S =& (n+1) + (n+1) + \cdots + (n+1) = n(n+1) \\ \text{Therefore} \quad S = \sum_{i=1}^{n} i =& \frac{n(n+1)}{2}. \end{aligned}$$

(*Read the following proof if you are interested.*)

We prove **5** and omit the proof of **6** which is similar. Set $S = \sum_{i=1}^{n} i^2$. Observe first that

$$\begin{aligned} &\sum_{i=1}^{n} [(1+i)^3 - i^3] \\ &\quad = (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) \\ &\quad = (n+1)^3 - 1 = n^3 + 3n^2 + 3n \end{aligned}$$

and on the other hand

$$\begin{aligned} \sum_{i=1}^{n} [(1+i)^3 - i^3] &= \sum_{i=1}^{n} [1 + 3i + 3i^2 + i^3 - i^3] \\ &= \sum_{i=1}^{n} [1 + 3i + 3i^2] \\ &= \sum_{i=1}^{n} 1 + 3\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} i^2 \\ &= n + 3\frac{n(n+1)}{2} + 3S = \frac{3}{2}n^2 + \frac{5}{2}n + 3S. \end{aligned}$$

Therefore,

$$n^3 + 3n^2 + 3n = \frac{3}{2}n^2 + \frac{5}{2}n + 3S$$

and so we have

$$\begin{aligned} S =& \frac{1}{3}[n^3 + 3n^2 + 3n - (\frac{3}{2}n^2 + \frac{5}{2}n)] \\ =& \frac{1}{3}[n^3 + \frac{3}{2}n^2 + \frac{1}{2}n] = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

$\square$

### 5.2.2 Riemann Sums

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is what we will do here.

Consider the region given in Figure 4, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region?

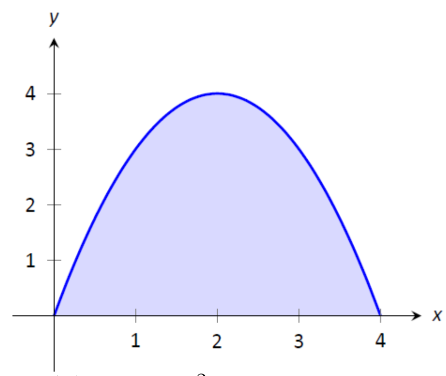



Figure 4: A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use 4 rectangles with an equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**.

The **Left Hand Rule**(LHR) says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 5 (a), the rectangles drawn have height determined by the Left Hand Rule (LHR); the heights are $f(0), f(1), f(2), f(3)$.

The **Right Hand Rule** (RHR) says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In Figure 5 (b), the rectangles drawn are drawn using $f(1), f(2), f(3), f(4)$ as their heights.

The **Midpoint Rule** (MPR) says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. Figure 5 (c), the rectangles drawn were made using the Midpoint Rule (MPR), with heights of $f(0.5), f(1.5), f(2.5), f(3.5)$.

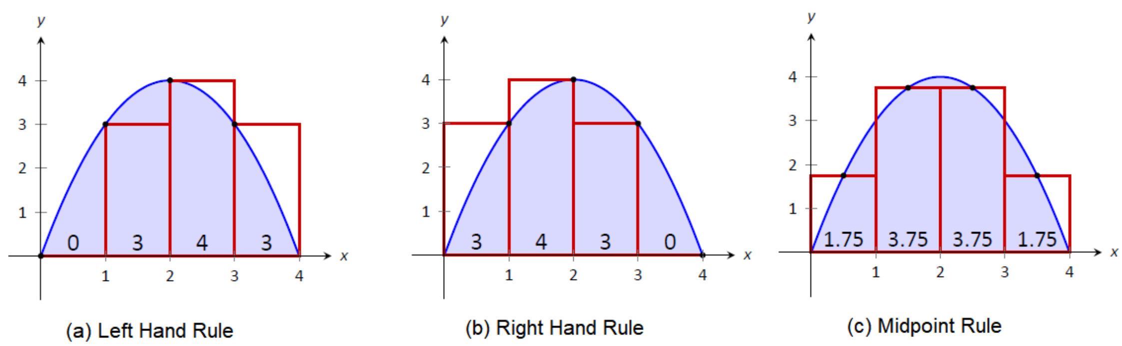



Figure 5: Approximating the area of the shaded region under $f(x) = 4x - x^2$ using the Left Hand Rule, Left Hand Rule, and Midpoint Rule, using 4 equally spaced subintervals

We now calculate the three approximations. We break the interval $[0, 4]$ into four subintervals as before.

**(1)** In Figure 5(a) we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule.

Note how in the first subinterval, $[0, 1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height$\times$ width) for our Left Hand Rule approximation:

$$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 =$$
$$0 + 3 + 4 + 3 = 10.$$

**(2)** Figure 5(b) shows 4 rectangles drawn under $f$ using the Right Hand Rule; note how the $[3, 4]$ subinterval has a rectangle of height 0.

In this example, these rectangle seem to be the mirror image of those found in part (a) of the Figure. This is because of the symmetry of our shaded region. Our approximation gives the same answer as before, though calculated a different way:

$$\begin{aligned} f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = \\ 3 + 4 + 3 + 0 = 10. \end{aligned}$$

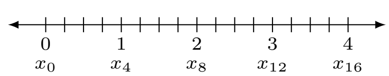
**(3)** Figure 5(c) shows 4 rectangles drawn under $f$ using the Midpoint Rule.

This gives an approximation of $\int_0^4 (4x - x^2)\, dx$ as:

$$\begin{aligned} f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = \\ 1.75 + 3.75 + 3.75 + 1.75 = 11. \end{aligned}$$

Our three methods provide two approximations: 10 and 11.

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. We will approximate area using 16 equally spaced subintervals and the Right Hand Rule. Before doing so, it will pay to do some careful preparation.




The figure shows a number line of $[0, 4]$ divided, or *partitioned*, into 16 equally spaced subintervals. We denote 0 as $x_0$; we have marked the values of $x_4$, $x_8$, $x_{12}$ and $x_{16}$. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_9 = 2.25$, in general, we want a method of determining the value of $x_i$ without consulting the figure. Consider:

$$x_i = \underbrace{x_0}_{\textit{starting value}} + i \overbrace{\Delta x}_{\textit{subinterval size}}, i = 1, \ldots, 16.$$

So $x_9 = x_0 + 9(4/16) = 2.25$.

If we had partitioned $[0,4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute $x_{32}$ as

$$x_{32} = x_0 + 32(4/100) = 1.28.$$

(That was far faster than creating a sketch first.)

Given any subdivision of $[0,4]$, the first subinterval is $[x_0, x_1]$; the second is $[x_1, x_2]$; the $i$ th subinterval is $[x_{i-1}, x_i]$.

When using the Left Hand Rule, the height of the $i$ th rectangle will be $f(x_{i-1})$.

When using the Right Hand Rule, the height of the $i$ th rectangle will be $f(x_i)$.

When using the Midpoint Rule, the height of the $i$ th rectangle will be $f\left(\frac{x_{i-1}+x_i}{2}\right)$.

Thus the approximation with 16 equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

**Left Hand Rule:** $\sum_{i=1}^{16} f(x_{i-1})\Delta x = \sum_{i=1}^{16} f\Big(x_0 + (i-1)\Delta x\Big)\Delta x$

**Right Hand Rule:** $\sum_{i=1}^{16} f(x_i)\Delta x = \sum_{i=1}^{16} f\Big(x_0 + i\Delta x\Big)\Delta x$

**Midpoint Rule:** $\sum_{i=1}^{16} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = \sum_{i=1}^{16} f\Big(x_0 + (i-\frac{1}{2})\Delta x\Big)\Delta x$

**Example.** Approximate the area of the region under $\int_0^4 (4x - x^2)\, dx$ on $[0,4]$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

**Solution.** Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the area as

$$\sum_{i=1}^{16} f(x_i)\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Note that

$$x_i = 0 + i\Delta x$$

Using the summation formulas, we have the approximation

$$\begin{aligned}
\sum_{i=1}^{16} f(x_i)\Delta x &= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \big(4i\Delta x - (i\Delta x)^2\big)\Delta x \\
&= \sum_{i=1}^{16} (4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (5.4) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3 \frac{16(17)(33)}{6} \qquad (\Delta x = 0.25) \\
&= 10.625
\end{aligned}$$

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 6 the function and the 16 rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

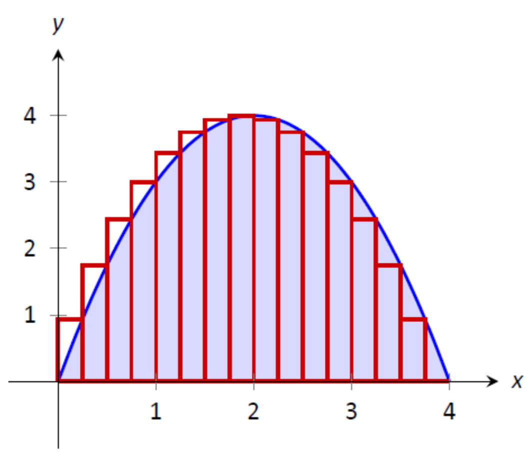



Figure 6: Approximation with the Right Hand Rule and 16 evenly spaced subintervals.

Notice Equation (5.4); by changing the 16's to 1,000's (and appropriately changing the value of $\Delta x$), we can use that equation to sum up 1000 rectangles!

We do so here, skipping from the original summand to the equivalent of Equation

(5.4) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\begin{aligned}\sum_{i=1}^{1000} f(x_i)\Delta x &= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3 \sum_{i=1}^{1000} i^2 \\ &= (4\Delta x^2)\frac{1000 \cdot 1001}{2} - \Delta x^3 \frac{1000(1001)(2001)}{6} \\ &= 10.666656\end{aligned}$$

Using many, many rectangles, we have a likely good approximation.

Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as $\Delta x$, and
2. each rectangle's height is determined by evaluating $f$ at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating $f$ at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that do not have the same size. We refer to the length of the $i$ th subinterval as $\Delta x_i$. Also, one could determine each rectangle's height by evaluating $f$ at *any* point $c_i$ in the $i$ th subinterval. Thus the height of the $i$ th subinterval would be $f(c_i)$, and the area of the $i$ th rectangle would be $f(c_i)\Delta x_i$. These ideas are formally defined below.

**Definition** (Partition)**.** A **partition** of a closed interval $[a, b]$ is a set of numbers $x_0$, $x_1$, $x_2$, $\ldots$ $x_n$ where

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

The length of the $i$ th subinterval, $[x_{i-1}, x_i]$, is $\Delta x_i = x_i - x_{i-1}$. If $[a, b]$ is partitioned into subintervals of equal length, we let $\Delta x$ represent the length of each subinterval. The **size of the partition**, denoted $||\Delta x||$, is the length of the largest subinterval of the partition.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

**Definition** (Riemann Sum)**.** Let $f$ be defined on a closed interval $[a, b]$, let $P$ :

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

be a partition of $[a, b]$ and let $x_i^*$ denote any value in the $i$ th subinterval.

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a **Riemann sum** of $f$ on $[a, b]$.

**Remark.** "Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working on an example, let's summarize some of what we have learned in a convenient way.

**Riemann Sum Concepts:** Consider $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, the approximation of the area of the region under $f(x) \geq 0$ on $[a, b]$.

1. When the $n$ subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The $i$ th term of an equally spaced partition is $x_i = a + i\Delta x$. (Thus $x_0 = a$ and $x_n = b$.)
3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\Big(x_0 + (i-1)\Delta x\Big)\Delta x$.
4. The Right Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\Big(x_0 + i\Delta x\Big)\Delta x$.
5. The Midpoint Rule summation is:

$$\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x = \sum_{i=1}^{n} f\Big(x_0 + (i - \frac{1}{2})\Delta x\Big)\Delta x.$$

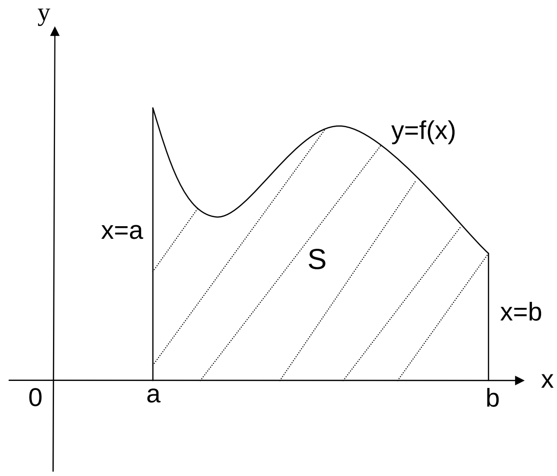
**Example** Evaluate the left, right, and midpoint Riemann sums for $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals. Make a conjecture about the exact area of the region under the curve.

**Solution.**

$\square$

### 5.2.3 Definite Integrals

Recall **The Area Problem.** Assume that $y = f(x) \geq 0$ on interval $[a, b]$. Find the area bounded by the curves $x = a$, $x = b$, $y = 0$ and $y = f(x)$. This problem is commonly referred to as "to find the area under the curve".




**Basic Idea of the Solution.** We assume that

• $f(x)$ is continuous on the interval $[a, b]$, (This is not necessary but it is convenient since it allows us to use **subintervals of the same length** and use any choice for **sample points**.)

• $x$ denotes the independent variable and $f$ denotes the function.

**Example.** Let $y = f(x) = 4x - x^2$ be defined on $[0, 4]$. Find the area under the curve. This is the example in the last sub section.

**I - Approximate solution.** An approximate solution is obtained by using a **Riemann sum** as now outlined.

**Step 1.** The interval $[a, b]$ (in the example $a = 0, b = 4$) is partitioned into $n$ **subintervals of equal length**, where $n$ is a finite number, for example, $n = 4, 10, 50$. Now, the length of the interval is $\Delta x = \frac{b-a}{n}$. The end points of our intervals are

$x_i = a + i(\frac{b-a}{n}) = a + i\Delta x.$

$$a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$$

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

In the example, if we choose $n = 4$, then we have 4 intervals of equal length of 1:

$$[0, 1], [1, 2], [2, 3], [3, 4].$$

**Step 2.** We assume that over a given subinterval $[x_i, x_{i+1}]$ the function is a constant with value $f(x_i^*)$ for some **sample point** $x_i^* \in [x_{i-1}, x_i]$, where $x_i^*$ can be chosen to be left endpoint $x_{i-1}$, right endpoint $x_i$ or midpoint $(x_{i-1} + x_i)/2$.

**Step 3.** Now

$$f(x_i^*)\Delta x = \begin{cases} \text{the area of a rectangle over } [x_{i-1}, x_i] \text{ with height } f(x_i^*) \\ \text{ OR} \\ \text{distance traveled over time } [x_{i-1}, x_i] \text{ when speed is } f(x_i^*) \end{cases}$$

**Step 4.** We now have an approximation to the solution that we seek given in the form of a **Riemann Sum**.

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \cong \begin{cases} \text{area as a sum of area of a rectangles} \\ \text{ OR} \\ \text{distance as a sum of distance over time } [x_{i-1}, x_i] \end{cases}$$

In the example, if we choose right endpoints as sample points

$$R_4 = \sum_{i=1}^{4} f(x_i) \cdot 1 = f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 3 + 4 + 3 + 0 = 10;$$

If we choose left endpoints as sample points

$$L_4 = \sum_{i=1}^{4} f(x_{i-1}) \cdot 1 = f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = 0 + 3 + 4 + 3 = 10;$$

If we choose midpoints as sample points (let $c_i = \frac{x_{i-1}+x_i}{2}$)

$$M_4 = \sum_{i=1}^{4} f(c_i)\cdot 1 = f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 = 1.75 + 3.75 + 3.75 + 1.75 = 11.$$

**II - Exact Solution.** To pass from an approximation to the exact value, the idea is that as we select smaller and **smaller subintervals**, the approximations become better and better.

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \begin{cases} \text{area in the area problem} \\ \text{OR} \\ \text{distance in the distance problem} \end{cases}$$

In the example, we split the interval $[0, 4]$ into $n$ subintervals

$$\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{8}{n}\right], \cdots, \left[4 - \frac{4}{n}, 4\right].$$

If we choose right endpoints $\frac{4i}{n}, i = 1, \ldots, n$ as samples points, then we have Riemann Sum

$$\begin{aligned} R_n =& \sum_{i=1}^{n} f(\frac{4i}{n})\frac{4}{n} = \sum_{i=1}^{n} \left(4\frac{4i}{n} - (\frac{4i}{n})^2\right)\frac{4}{n} = 4\left(\frac{4}{n}\right)^2 \sum_{i=1}^{n} i - \left(\frac{4}{n}\right)^3 \sum_{i=1}^{n} i^2 \\ =& \frac{64}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{64}{n}\frac{(n+1)}{2} - \frac{64}{n^2}\frac{(n+1)(2n+1)}{6} \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{n\to\infty} R_n =& \frac{64}{2}\lim_{n\to\infty}\frac{(n+1)}{n} - \frac{64}{6}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2} \\ =& 32 - \frac{64}{6}\cdot 2 = 32 - \frac{64}{3} = \frac{32}{3}. \end{aligned}$$

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

**Definition 5.1** (Definite integral). Let $f(x)$ be **a function** defined for $a \leq x \leq b$.

1. Partition $[a, b]$ into $n$ subintervals of width $\Delta x_i, i = 1, \ldots, n$.
2. Let $x_0 = a$, $x_1 = x_0 + \Delta x_1$, $\ldots$, $x_i = x_{i-1} + \Delta x_i$ for $i = 1, \ldots, n$, thus the subintervals are $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $x_n = b$.
3. Choose sample points $x_i^*$ between $x_{i-1}$ and $x_i$.

Then the definite integral of $f(x)$ over the interval $[a, b]$ is the limit of the following Riemann sum provided that this limit exists. If it does exist, we say that $f$ is **integrable** on $[a, b]$. This is denoted by

$$\boxed{\int_a^b f(x)dx = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,}$$

where $\Delta = max\{\Delta x_1, \ldots, \Delta x_n\}$ with $\Delta x_i = x_i - x_{i-1}$.

**Remark 1.** In the above, if $f(x)$ is **continuous** then the subintervals can be **equally spaced**, then $\Delta x = \Delta x_i = \frac{b-a}{n}, i = 1, \ldots, n$. And the definition of $\int_a^b f(x)dx$ for a continuous function $f(x)$ is

$$\boxed{\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.}$$

**Remark 2.** The sample point $x_i^*$ is **any point** in the $i$th interval $i = 1, 2, \ldots, n$. Especially, $x_i^*$ can be chosen to be

(1) left endpoints $x_i^* = x_{i-1} = a + (i-1)\frac{b-a}{n}$.

(2) right endpoints $x_i^* = x_i = a + (i)\frac{b-a}{n}$.

(3) midpoints.

$$\begin{aligned} x_i^* = \bar{x}_i = x_{i-1} + \frac{1}{2}(x_i - x_{i-1}) &= \frac{x_{i-1} + x_i}{2} \\ = a + (i-1)\frac{b-a}{n} + \frac{1}{2}\frac{b-a}{n} &= a + (2i-1)\frac{b-a}{2n}. \end{aligned}$$

□

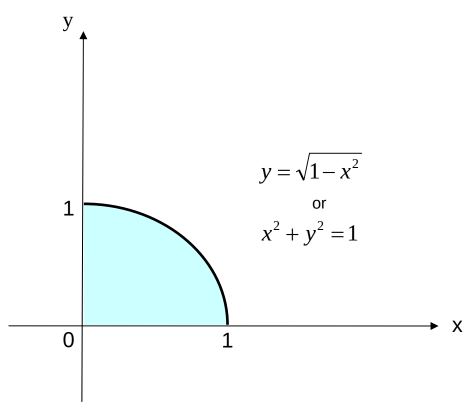
**Evaluating Definite Integrals**

Most of the functions encountered in this text are integrable (see Exercise 83 for an exception). In fact, if $f$ is continuous on $[a, b]$ or if $f$ is bounded on $[a, b]$ with a finite number of discontinuities, then $f$ is integrable on $[a, b]$. The proof of this result goes beyond the scope of this text.

**Theorem (Integrable Functions).** If $f$ is **continuous** on $[a, b]$, or if $f$ is bounded and has only a finite number of jump discontinuities (or **piecewise continuous**), then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists. □

**Example: Evaluating definite integrals using geometry** Evaluate $\int_0^1 \sqrt{1-x^2}\, dx$ by interpreting it as an area.

**Solution.** Set $f(x) = \sqrt{1-x^2}$. If we sketch the graph of $y = f(x)$ over the interval $[0, 1]$,




we see that we have a curve corresponding to the quarter of a circle of radius 1 and so the area of the region is $A = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$.

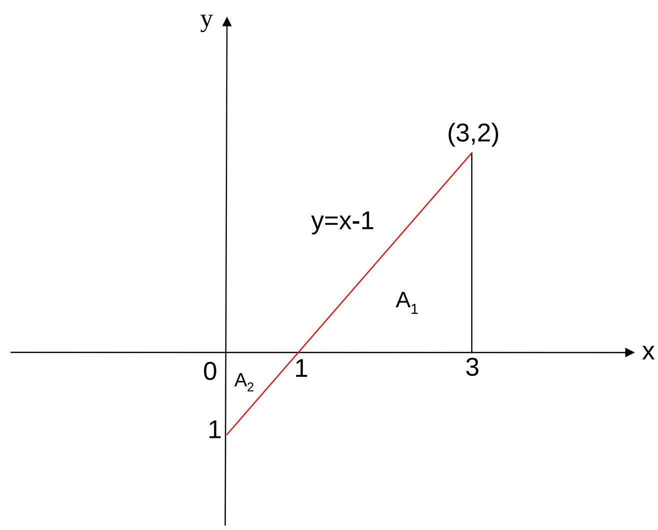
$$\int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi}{4}.$$

□

**Remark.** The discussion so far has been for non-negative valued functions. If $f(x)$ can take both positive and negative values, then $\int_a^b f(x)\,dx$ is equal to the area up minus the area down. See the following example.

**Example: Area up minus area down** Evaluate $\int_0^3 (x-1)\,dx$ by interpreting it as an area.

**Solution.** Set $f(x) = x - 1$. If we sketch the graph of $y = f(x)$ over the interval $[0,3]$ we see that we have straight line forming the hypotenuse of two triangles: one under the interval $[0,1]$ and the other over $[1,3]$.




Therefore,

$$\int_0^3 (x-1)\,dx = \underbrace{\int_0^1 (x-1)\,dx}_{=-\text{Area down}} + \underbrace{\int_1^3 (x-1)\,dx}_{=\text{Area up}} = -\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{3}{2}$$

**Note.** The point is that if $x \in [0,1]$, then the function $f(x)$ is negative and if $x \in [1,3]$, then the function $f(x)$ is positive and so the Riemann sum splits into two parts

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \underbrace{\sum_{i=1}^{k} f(x_i^*)\Delta x}_{\leq 0 \text{ over } [0,1]} + \underbrace{\sum_{i=k+1}^{n} f(x_i^*)\Delta x}_{\geq 0 \text{ over } [1,3]}$$

**Example: Evaluating definite integrals using geometry**

**a.** $\int_2^4 (2x+3)\,dx$ **b.** $\int_1^6 (2x-6)\,dx$ **c.** $\int_3^4 \sqrt{1-(x-3)^2}\,dx$

**Solution.**

$\square$

## Properties of Definite Integrals

**Theorem** (Properties of the Definite Integral)**.** Suppose $f(x)$ and $g(x)$ are both integrable.

(0) $\int_a^a f(x)\ dx = 0$ (since $\Delta x = 0$)

(1) $\int_a^b c\ dx = c(b-a)$, where $c$ is any constant;

(2) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$, where $a < b$, because $\Delta x$ changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$ ;

(3) $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)dx \pm \int_a^b g(x)\ dx$;

(4) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, where $c$ is any constant.

(5) $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$, where $c$ is any constant ($c$ need not be between $a$ and $b$.);

(6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$;

(7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$;

(8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

(9) The function $|f(x)|$ is also integrable on $[a, b]$, and $\int_a^b |f(x)|\ dx$ is the sum of the areas of the regions bounded by the graph $f$ and the $x$-axis on $[a, b]$. □

**Hint of proof:** Write an integral in terms of the limit of a sum. See details in the textbook.

**Theorem.** If $f$ is continuous on $[a, b]$, then

$$\boxed{\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx.}$$

**Example.** Properties of integrals Assume that $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. Evaluate the following integrals, if possible.

**a.** $\int_0^7 2f(x)\,dx =$ **b.** $\int_5^0 f(x)\,dx =$ **c.** $\int_5^7 f(x)\,dx =$ **d.** $\int_7^0 6f(x)\,dx =$

**e.** $\int_0^7 |f(x)|\,dx =$

**Solution.**

$\square$

## 5.3 Fundamental Theorem of Calculus

The results in this section simplify the computation of a definite integral greatly. We begin by defining **functions in terms of definite integrals**. This may seem to be a strange way to define a function, but these functions are common place in applications such as in physics, chemistry and statistics.

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\, dx$ is the "area under $f$ " on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\, dt$. It computes the area under $f$ on $[a, x]$ as illustrated in Figure 7. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\, dt = 0$. If $f(t) > 0$, then $F(x) > 0$ when $x > a$ (consider the figure for a visual understanding). $F(x)$ is called an Area Function in the textbook.

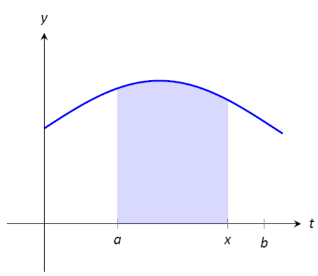



Figure 7: The area of the shaded region is $F(x) = \int_a^x f(t)\, dt$.

We can also apply calculus ideas to $F(x)$; in particular, we can compute its derivative. While this may seem like an innocuous thing to do, it has far–reaching implications, as demonstrated by the fact that the result is given as an important theorem.

**Theorem (The Fundamental Theorem of Calculus - Part 1).** If $f(x)$ is continuous on the interval $[a, b]$, then the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on $(a, b)$ and

$$F'(x) = \frac{d\, F(x)}{dx} = \frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

$\square$

**Remark.** The FTC(Fundamental Theorem of Calculus) Part 1 requires that the lower limit is a constant and the upper limit is the variable with respect to which we are computing the derivative.

There are times when these conditions are not met and yet we can apply FTC in conjunction with the Chain Rule. This requires **rewriting the integral** using the Properties of a Definite Integral (5.2.3) (see Example 2 - 3 in what follows). □

*Proof.* Assume that $a < x < b$. Use Properties 2 and 5 from the list of Properties of the Definite Integrals to get

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\ dx - \int_a^x f(t)\ dt}{h} \\
&= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\ dt + \int_x^a f(t)\ dt}{h} = \lim_{h\to 0} \frac{\int_x^a f(t)\ dt + \int_a^{x+h} f(t)\ dt}{h} \\
&= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h}.
\end{aligned}$$

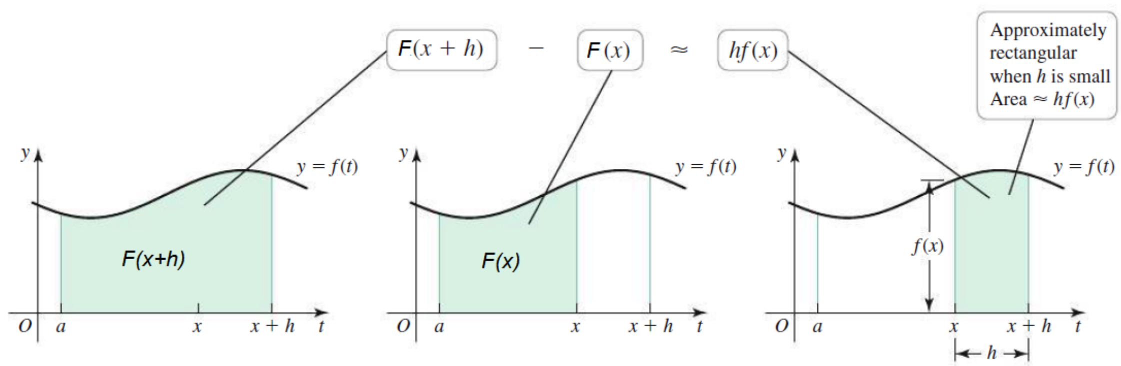



Figure 8: Key idea of the proof of the Fundamental Theorem of Calculus

Let $m = f(x + h_m)$ be the **absolute minimum** of $f(x)$ on the interval between $x$ and $x + h$ so that $h_m$ between 0 and $h$ and $M = f(x + h_M)$ be the **absolute maximum** of $f(x)$ on the interval between $x$ and $x + h$ with $h_M$ between 0 and $h$. Then using Property 8 of Properties of Definite Integrals (5.2.3) we have

$\boxed{mh \leq \int_x^{x+h} f(t)\ dt \leq Mh}$ and so we have

$$
\begin{aligned}
f(x) = \underbrace{\lim_{h\to 0} f(x+h_m)}_{h_m\to 0 \text{ as } h\to 0} &= \lim_{h\to 0} m = \lim_{h\to 0} \frac{mh}{h} \\
&\leq F'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h} \\
&\leq \lim_{h\to 0} \frac{Mh}{h} = \lim_{h\to 0} M = \underbrace{\lim_{h\to 0} f(x+h_M)}_{h_M\to 0 \text{ as } h\to 0} = f(x)
\end{aligned}
$$

Thus, $F'(x) = f(x)$ on $(a,b)$.

Recall that differentiability implies continuity and so $F(x)$ is continuous on the open interval $(a,b)$. Thus, we need only show that $F(x)$ is right continuous at $a$ and left continuous at $b$. We show right continuity at $a$ only.

First observe that $\mathbf{F(a) = 0}$. Let $m = f(c_m)$ with $c_m$ in $[a,x]$ be the absolute minimum of $f(x)$ on $[a,x]$ and $M = f(c_M)$ with $c_M$ in $[a,x]$ be the absolute maximum of $f(x)$ on $[a,x]$.

$$
\begin{aligned}
f(a)(0) &= \lim_{x\to a^+} f(c_m)(x-a) \\
&\leq \lim_{x\to a^+} F(x) = \lim_{x\to a^+} \int_a^x f(t)\ dt \\
&\leq \lim_{x\to a^+} f(c_M)(x-a) = f(a)(0)
\end{aligned}
$$

Thus, $F(a) = \lim_{x\to a} F(x)$. □

**Example 1.** Notation: $D_x = \frac{d}{dx}$.

$$D_x \left( \int_{-1}^{x} \sqrt[3]{t^3+t-1}\ dt \right) = \frac{d}{dx} \int_{-1}^{x} \sqrt[3]{t^3+t-1}\ dt = \sqrt[3]{x^3+x-1}.$$

□

**Example 2.** Find **(a)** $D_x(\int_1^x \sin^2 t\, dt)$ **(b)** $D_x(\int_x^5 \sqrt{t^2+1}\, dt)$ **(c)** $D_x(\int_0^{x^2} \cos t^2\, dt)$

**Solution.**

$\square$

**Example 3.** Find $D_x(\int_{\cos x}^{5x} \cos(u^2)\ du)$.

**Solution.**

$$\begin{aligned}
D_x\left(\int_{\cos x}^{5x} \cos(u^2)\ du\right) &= -\underbrace{D_x\left(\int_0^{\cos x} \cos(u^2)\ du\right)}_{v=\cos x} + \underbrace{D_x\left(\int_0^{5x} \cos(u^2)\ du\right)}_{w=5x} \\
&= -D_x\left(\int_0^{v} \cos(u^2) du\right) + D_x\left(\int_0^{w} \cos(u^2)\ du\right) \\
&= -D_v\left(\int_0^{v} \cos(u^2) du\right)(D_x(v)) + D_w\left(\int_0^{w} \cos(u^2)\ du\right)(D_x(w)) \\
&= -(\cos v^2) D_x(\cos x) + \cos(w^2) D_x(5x) \\
&= -(\cos(\cos^2 x))(-\sin x) + (\cos(5x)^2)(5).
\end{aligned}$$

$\square$.

Using the properties of definite integrals, we know

$$\int_a^b f(t)\ dt = \int_a^c f(t)\ dt + \int_c^b f(t)\ dt = -\int_c^a f(t)\ dt + \int_c^b f(t)\ dt$$
$$= -F(a) + F(b) = F(b) - F(a).$$

We now see how indefinite integrals and definite integrals are related: we can **evaluate a definite integral using antiderivatives!** This is the second part of the Fundamental Theorem of Calculus.

**Theorem (The Fundamental Theorem of Calculus - Part 2).** If $f(x)$ is continuous on the interval $[a, b]$ then

$$\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\Big|_a^b,$$

where $F(x)$ is any antiderivative of $f$, i.e. $F'(x) = f(x)$. □

*Proof.* Let $g(x) = \int_a^x f(t)\ dt$. Then

$$g(b) - g(a) = \int_a^b f(x)\ dx - \int_a^a f(x)\ dx = \int_a^b f(t)\ dt - 0 = \int_a^b f(t)\ dt.$$

If $F(x)$ is any other antiderivative of $f(x)$ then $F'(x) = f(x) = g'(x)$ and so $F(x)$ and $g(x)$ differ by a constant, i.e. $F(x) = g(x) + C$ for some constant $C$. Thus,

$$F(b) - F(a) = g(b) - g(a) = \int_a^b f(x)\ dx.$$

□

**Remark 1.** Notice the notation $\Big|$, it is customary and convenient to denote the difference $F(b) - F(a)$ by $F(x)\Big|_a^b$

**Remark 2.** We must be careful. FTC does not apply to functions $f(x)$ which are not continuous, such as

$$\int_{-1}^1 \frac{1}{x^4}\ dx.$$

because the integrand is not continuous on $[-1, 1]$. Actually, one can show that this integral does not exist (In Calculus II you will study improper internal and see that this is such an integral).

**Remark 3.** When working with the absolute value function, we must find where the integrand is positive and where it is negative. That is, we must write an absolute value function as a piece-wise function.

**Example** Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2.

**a.** $\displaystyle\int_0^{10} (60x - 6x^2)\, dx$ **b.** $\displaystyle\int_0^{2\pi} 3\sin x\, dx$ **c.** $\displaystyle\int_{1/16}^{1/4} \frac{\sqrt{t}-1}{t}\, dt$

**Solution.**

□

In summary, in this section we have learned the FTC (summary of Theorem 5.3 and Theorem 5.3):

**Theorem** (The Fundamental Theorem of Calculus)**.** Suppose $f$ is continuous on $[a, b]$. Then

1. If $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$. It can be written as $\boxed{\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)}$.

2. $\int_a^b f(x)dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$, that is, $F' = f$. It can also be written as $\boxed{\displaystyle\int_a^b F'(x)dx = F(b) - F(a)}$.