➤ Recall that if $f(x) \geq 0$ on $[a, b]$ and the region bounded by the graph of f and the x-axis on $[a, b]$ is revolved about the x-axis, then the volume of the solid generated is $V = \int_a^b \pi f(x)^2\, dx$.

We integrate by parts with the following assignments.

$u = (\ln x)^2$	$dv = dx$
$du = \dfrac{2 \ln x}{x}\, dx$	$v = x$

The integration is carried out as follows, using the indefinite integral of $\ln x$ just given:

$$V = \int_1^e \pi (\ln x)^2\, dx \quad \text{Disk method}$$

$$= \pi \left(\underbrace{(\ln x)^2}_{u} \underbrace{x}_{v} \Big|_1^e - \int_1^e \underbrace{x}_{v} \underbrace{\frac{2 \ln x}{x}\, dx}_{du} \right) \quad \text{Integration by parts}$$

$$= \pi \left(x(\ln x)^2 \Big|_1^e - 2 \int_1^e \ln x\, dx \right) \quad \text{Simplify.}$$

$$= \pi \left(x(\ln x)^2 \Big|_1^e - 2(x \ln x - x) \Big|_1^e \right) \quad \int \ln x\, dx = x \ln x - x + C$$

$$= \pi(e(\ln e)^2 - 2e \ln e + 2e - 2). \quad \text{Evaluate and simplify.}$$

$$= \pi(e - 2) \approx 2.257 \quad \text{Simplify.}$$

Related Exercises 45, 47 ◄

QUICK CHECK 3 How many times do you need to integrate by parts to reduce $\int_1^e (\ln x)^6\, dx$ to an integral of $\ln x$? ◄

SECTION 8.2 EXERCISES

Getting Started

1. On which derivative rule is integration by parts based?
2. Use integration by parts to evaluate $\int x \cos x\, dx$ with $u = x$ and $dv = \cos x\, dx$.
3. Use integration by parts to evaluate $\int x \ln x\, dx$ with $u = \ln x$ and $dv = x\, dx$.
4. How is integration by parts used to evaluate a definite integral?
5. What type of integrand is a good candidate for integration by parts?
6. How would you choose dv when evaluating $\int x^n e^{ax}\, dx$ using integration by parts?

7–8. *Use a substitution to reduce the following integrals to* $\int \ln u\, du$. *Then evaluate the resulting integral using the formula for* $\int \ln x\, dx$.

7. $\int (\sec^2 x) \ln (\tan x + 2)\, dx$
8. $\int (\cos x) \ln (\sin x)\, dx$

Practice Exercises

9–40. Integration by parts *Evaluate the following integrals using integration by parts.*

9. $\int x \cos 5x\, dx$
10. $\int x \sin 2x\, dx$
11. $\int t e^{6t}\, dt$
12. $\int 2x e^{3x}\, dx$
13. $\int x \ln 10x\, dx$
14. $\int s e^{-2s}\, ds$
15. $\int (2w + 4) \cos 2w\, dw$
16. $\int \theta \sec^2 \theta\, d\theta$
17. $\int x\, 3^x\, dx$
18. $\int x^9 \ln x\, dx$
19. $\int \frac{\ln x}{x^{10}}\, dx$
20. $\int \sin^{-1} x\, dx$
21. $\int x \sin x \cos x\, dx$
22. $\int e^{2x} \sin e^x\, dx$
23. $\int x^2 \sin 2x\, dx$
24. $\int x^2 e^{4x}\, dx$
25. $\int t^2 e^{-t}\, dt$
26. $\int t^3 \sin t\, dt$
27. $\int e^x \cos x\, dx$
28. $\int e^{3x} \cos 2x\, dx$
29. $\int e^{-x} \sin 4x\, dx$
30. $\int e^{-2\theta} \sin 6\theta\, d\theta$
31. $\int e^{3x} \sin e^x\, dx$
32. $\int_0^1 x^2\, 2^x\, dx$
33. $\int_0^\pi x \sin x\, dx$
34. $\int_1^e \ln 2x\, dx$
35. $\int_0^{\pi/2} x \cos 2x\, dx$
36. $\int_0^{\ln 2} x e^x\, dx$
37. $\int_1^{e^2} x^2 \ln x\, dx$
38. $\int x^2 \ln^2 x\, dx$
39. $\int_0^1 \sin^{-1} y\, dy$
40. $\int e^{\sqrt{x}}\, dx$
41. Evaluate the integral in part (a) and then use this result to evaluate the integral in part (b).
    a. $\int \tan^{-1} x\, dx$ **b.** $\int x \tan^{-1} x^2\, dx$

42–47. Volumes of solids *Find the volume of the solid that is generated when the given region is revolved as described.*

42. The region bounded by $f(x) = \ln x$, $y = 1$, and the coordinate axes is revolved about the x-axis.

43. The region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-axis.

44. The region bounded by $f(x) = \sin x$ and the x-axis on $[0, \pi]$ is revolved about the y-axis.

45. The region bounded by $g(x) = \sqrt{\ln x}$ and the x-axis on $[1, e]$ is revolved about the x-axis.

46. The region bounded by $f(x) = e^{-x}$ and the x-axis on $[0, \ln 2]$ is revolved about the line $x = \ln 2$.

47. The region bounded by $f(x) = x \ln x$ and the x-axis on $[1, e^2]$ is revolved about the x-axis.

48. Integral of $\sec^3 x$ Use integration by parts to show that

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx.$$

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\int uv'dx = \left(\int u\,dx\right)\left(\int v'dx\right)$.

b. $\int uv'dx = uv - \int vu'dx$.

c. $\int v\,du = uv - \int u\,dv$.

50–53. Reduction formulas *Use integration by parts to derive the following reduction formulas.*

50. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$, for $a \neq 0$

51. $\int x^n \cos ax\,dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx$, for $a \neq 0$

52. $\int x^n \sin ax\,dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx$, for $a \neq 0$

53. $\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$

54–57. Applying reduction formulas *Use the reduction formulas in Exercises 50–53 to evaluate the following integrals.*

54. $\int x^2 e^{3x}\,dx$

55. $\int x^2 \cos 5x\,dx$

56. $\int x^3 \sin x\,dx$

57. $\int_1^e \ln^3 x\,dx$

58. Two methods Evaluate $\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$ in the following two ways.

a. Use integration by parts. **b.** Use substitution.

59. Two methods

a. Evaluate $\int \frac{x}{\sqrt{x+1}}\,dx$ using integration by parts.

b. Evaluate $\int \frac{x}{\sqrt{x+1}}\,dx$ using substitution.

c. Verify that your answers to parts (a) and (b) are consistent.

60. Two methods

a. Evaluate $\int x \ln x^2\,dx$ using the substitution $u = x^2$ and evaluating $\int \ln u\,du$.

b. Evaluate $\int x \ln x^2\,dx$ using integration by parts.

c. Verify that your answers to parts (a) and (b) are consistent.

61. Logarithm base *b* Prove that $\int \log_b x\,dx = \frac{1}{\ln b}(x\ln x - x) + C$.

62. Two integration methods Evaluate $\int \sin x \cos x\,dx$ using integration by parts. Then evaluate the integral using a substitution. Reconcile your answers.

63. Combining two integration methods Evaluate $\int \cos\sqrt{x}\,dx$ using a substitution followed by integration by parts.

64. Combining two integration methods Evaluate $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx$ using a substitution followed by integration by parts.

65. An identity Show that if f has continuous derivatives on $[a, b]$ and $f'(a) = f'(b) = 0$, then

$$\int_a^b x f''(x)\,dx = f(a) - f(b).$$

66. Integrating derivatives Use integration by parts to show that if f' is continuous on $[a, b]$, then

$$\int_a^b f(x)f'(x)\,dx = \frac{1}{2}(f(b)^2 - f(a)^2).$$

67. Function defined as an integral Find the arc length of the function $f(x) = \int_e^x \sqrt{\ln^2 t - 1}\,dt$ on $[e, e^3]$.

68. Log integrals Use integration by parts to show that for $m \neq -1$,

$$\int x^m \ln x\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right) + C$$

and for $m = -1$,

$$\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x + C.$$

69. Comparing volumes Let R be the region bounded by $y = \sin x$ and the x-axis on the interval $[0, \pi]$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the volume of the solid generated when R is revolved about the y-axis?

70. A useful integral

a. Use integration by parts to show that if f' is continuous, then

$$\int x f'(x)\,dx = xf(x) - \int f(x)\,dx.$$

b. Use part (a) to evaluate $\int xe^{3x}\,dx$.

71. Solid of revolution Find the volume of the solid generated when the region bounded by $y = \cos x$ and the x-axis on the interval $[0, \pi/2]$ is revolved about the y-axis.

Explorations and Challenges

72. Between the sine and inverse sine Find the area of the region bounded by the curves $y = \sin x$ and $y = \sin^{-1} x$ on the interval $[0, 1/2]$.

73. Two useful exponential integrals Use integration by parts to derive the following formulas for real numbers a and b.

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C$$