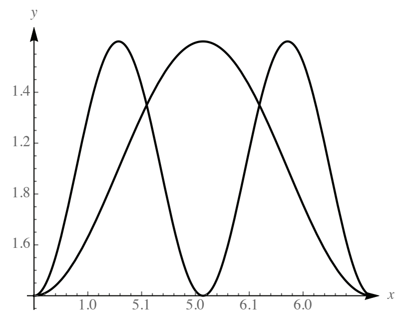
$\sec^{n-2} x \tan x-(n-2)\displaystyle\int(\sec^{n-2} x)(\sec^2 x-1)\,dx = \sec^{n-2} x\tan x-(n-2)\displaystyle\int \sec^n x\,dx+(n-2)\displaystyle\int \sec^{n-2} x\,dx.$

Combining like terms then gives $(n-1)\displaystyle\int \sec^n x\,dx = \sec^{n-2} x\tan x + (n-2)\displaystyle\int \sec^{n-2} x\,dx$, so as long as $n \neq 1$ we have

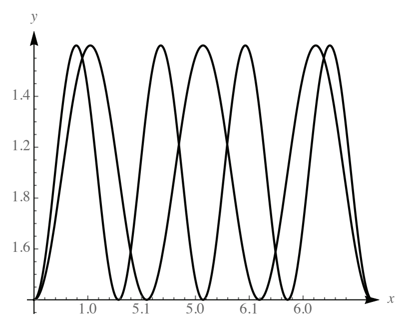
$$\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

8.3.75

a. $\displaystyle\int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 2x)\,dx = \frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\Big|_0^\pi = \frac{\pi}{2}.$
$\displaystyle\int_0^\pi \sin^2 2x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 4x)\,dx = \frac{1}{2}\left(x-\frac{\sin 4x}{4}\right)\Big|_0^\pi = \frac{\pi}{2}.$



b. $\displaystyle\int_0^\pi \sin^2 3x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 6x)\,dx = \frac{1}{2}\left(x-\frac{\sin 6x}{6}\right)\Big|_0^\pi = \frac{\pi}{2}.$
$\displaystyle\int_0^\pi \sin^2 4x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 8x)\,dx = \frac{1}{2}\left(x-\frac{\sin 8x}{8}\right)\Big|_0^\pi = \frac{\pi}{2}.$



c. $\displaystyle\int_0^\pi \sin^2 nx\,dx = \frac{1}{2}\int_0^\pi (1-\cos 2nx)\,dx = \frac{1}{2}\left(x-\frac{\sin 2nx}{2n}\right)\Big|_0^\pi = \frac{\pi}{2}.$

d. Yes. $\displaystyle\int_0^\pi \cos^2 nx\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2nx)\,dx = \frac{1}{2}\left(x+\frac{\sin 2nx}{2n}\right)\Big|_0^\pi = \frac{\pi}{2}.$

e. Claim: The corresponding integrals are all equal to $\frac{3\pi}{8}$. Proof:

$$\begin{aligned}\int_0^\pi \sin^4 nx\,dx &= \int_0^\pi \left(\frac{1-\cos 2nx}{2}\right)^2 dx\\ &= \int_0^\pi \frac{1-2\cos 2nx+\cos^2 2nx}{4}\,dx = \int_0^\pi \frac{1}{4}\,dx - \frac{1}{2}\int_0^\pi \cos 2nx\,dx + \frac{1}{4}\int_0^\pi \cos^2 2nx\,dx\\ &= \frac{\pi}{4} - \frac{1}{2}\left(\frac{\sin 2nx}{2n}\right)\Big|_0^\pi + \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{4} - 0 + \frac{\pi}{8} = \frac{3\pi}{8}.\end{aligned}$$

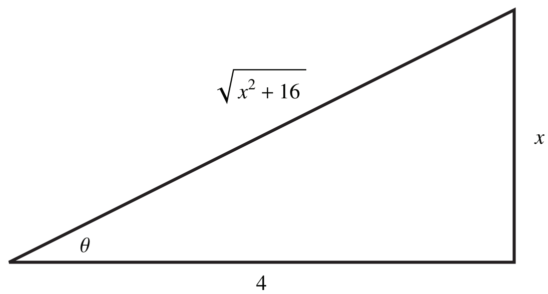
8.4 Trigonometric Substitutions

8.4.1 This would suggest $x = 3\sec\theta$, because then $\sqrt{x^2-9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta$, for $\theta \in [0, \pi/2)$.

8.4.2 This would suggest $x = 6\tan\theta$, because then $\sqrt{x^2+36} = 6\sqrt{\tan^2\theta+1} = 6\sqrt{\sec^2\theta} = 6\sec\theta$, for $|\theta| < \frac{\pi}{2}$.

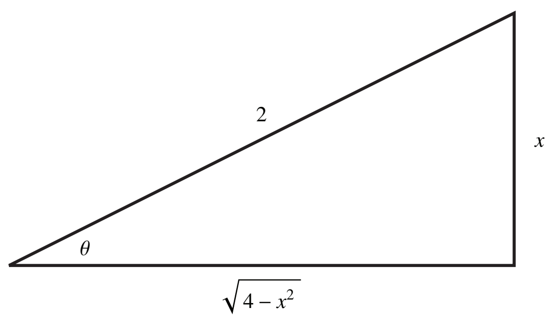
8.4.3 This would suggest $x = 10\sin\theta$, because then $\sqrt{100-x^2} = 10\sqrt{1-\sin^2\theta} = 10\sqrt{\cos^2\theta} = 10\cos\theta$, for $|\theta| \le \frac{\pi}{2}$.

8.4.4



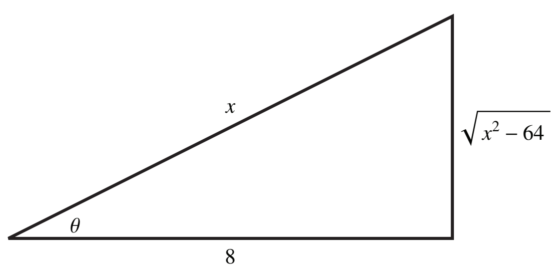
If $\tan\theta = \dfrac{x}{4}$, then $16\tan^2\theta = x^2$, so $16(\sec^2\theta - 1) = x^2$. Thus $\sec^2\theta = \dfrac{x^2+16}{16}$ and $\cos^2\theta = 1-\sin^2\theta = \dfrac{16}{x^2+16}$. Thus $\sin^2\theta = \dfrac{x^2}{16+x^2}$ and we have $\sin\theta = \dfrac{x}{\sqrt{16+x^2}}$, for $|\theta| < \dfrac{\pi}{2}$.

8.4.5



If $x = 2\sin\theta$ then $\dfrac{x^2}{4} = \sin^2\theta = \dfrac{1}{\csc^2\theta}$. Then $\cot^2\theta = \csc^2\theta - 1 = \dfrac{4}{x^2} - 1 = \dfrac{4-x^2}{x^2}$. So $\cot\theta = \dfrac{\sqrt{4-x^2}}{x}$ for $0 < |\theta| \le \dfrac{\pi}{2}$.

8.4.6



If $x = 8\sec\theta$, the $\tan^2\theta = \sec^2\theta - 1 = \dfrac{x^2}{64} - 1 = \dfrac{x^2-64}{64}$. Thus $\tan\theta = \dfrac{\sqrt{x^2-64}}{8}$.

8.4.7 Let $x = 5\sin\theta$, so that $dx = 5\cos\theta\,d\theta$. Note that $\sqrt{25-x^2} = 5\cos\theta$. Then $\displaystyle\int_0^{5/2} \frac{1}{\sqrt{25-x^2}}\,dx = \int_0^{\pi/6} \frac{5\cos\theta}{5\cos\theta}\,d\theta = \frac{\pi}{6}$.

Checking without using a trigonometric substitution:

$$\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \arcsin\left(\frac{x}{5}\right)\Big|_0^{5/2} = \left(\frac{\pi}{6} - 0\right) = \frac{\pi}{6}.$$

8.4.8 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\, d\theta$. Note that $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\int_0^{3/2} \frac{1}{(9-x^2)^{3/2}}\, dx = \int_0^{\pi/6} \frac{3\cos\theta}{27\cos^3\theta}\, d\theta = \frac{1}{9}\int_0^{\pi/6} \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta\,\Big|_0^{\pi/6} = \frac{1}{9}\left(\frac{1}{\sqrt{3}} - 0\right) = \frac{\sqrt{3}}{27}.$$

8.4.9 Let $x = 10\sin\theta$ so that $dx = 10\cos\theta\, d\theta$. Note that $\sqrt{100-x^2} = 10\cos\theta$. Then $\int_5^{5\sqrt{3}} \sqrt{100-x^2}\, dx =$ $100\int_{\pi/6}^{\pi/3} \cos^2\theta\, d\theta = 50\int_{\pi/6}^{\pi/3} (1+\cos 2\theta)\, d\theta = 50\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/6}^{\pi/3} = 50\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right) = \frac{25\pi}{3}.$

8.4.10 Let $x = 2\sin\theta$, so that $dx = 2\cos\theta\, d\theta$. Note that $\sqrt{4-x^2} = 2\cos\theta$. Thus, $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\, dx =$ $\int_0^{\pi/4} \frac{4\sin^2\theta\cdot 2\cos\theta}{2\cos\theta}\, d\theta = 4\int_0^{\pi/4} \sin^2\theta\, d\theta = 2\left(\frac{\pi}{4} - \int_0^{\pi/4}\cos 2\theta\, d\theta\right) = \frac{\pi}{2} - 2\left(\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{\pi}{2} - 1.$

8.4.11 Let $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/3} \sin^2\theta\, d\theta = \int_{\pi/6}^{\pi/3} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{\theta}{2}\,\Big|_{\pi/6}^{\pi/3} - \frac{\sin 2\theta}{4}\,\Big|_{\pi/6}^{\pi/3} = \frac{\pi}{12}.$$

8.4.12 Let $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/2} \cot^2\theta\, d\theta = \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\, d\theta = (-\cot\theta - \theta)\,\Big|_{\pi/6}^{\pi/2} = 0 - \pi/2 - (-\sqrt{3} - \pi/6) = \sqrt{3} - \frac{\pi}{3}.$$

8.4.13 Let $x = 4\sin\theta$ so that $dx = 4\cos\theta\, d\theta$. Note that $\sqrt{16-x^2} = 4\cos\theta$. Thus,

$$\int \frac{1}{\sqrt{16-x^2}}\, dx = \int \frac{4\cos\theta}{4\cos\theta}\, d\theta = \theta + C = \sin^{-1}\left(\frac{x}{4}\right) + C.$$

8.4.14 Let $t = 6\sin\theta$ so that $dt = 6\cos\theta\, d\theta$ and $\sqrt{36-t^2} = 6\cos\theta$. Then

$$\int \sqrt{36-t^2}\, dt = \int 36\cos^2\theta\, d\theta = 18\int (1+\cos 2\theta)\, d\theta = 18\left(\theta + \frac{\sin 2\theta}{2}\right) + C = 18\,(\theta + \sin\theta\cos\theta)$$
$$= 18\left(\sin^{-1}\left(\frac{t}{6}\right) + \frac{t}{6}\cdot\frac{\sqrt{36-t^2}}{6}\right) + C = 18\sin^{-1}\left(\frac{t}{6}\right) + \frac{t\sqrt{36-t^2}}{2} + C.$$

8.4.15 Let $x = 3\tan\theta$ so that $dx = 3\sec^2\theta\, d\theta$. Note that $\sqrt{x^2+9} = \sqrt{9(\tan^2\theta+1)} = 3\sec\theta$. Substituting gives

$$\int \frac{1}{9}\cot\theta\csc\theta\, d\theta = -\frac{1}{9}\csc\theta + C = -\frac{1}{9}\csc(\tan^{-1}(x/3)) + C = -\frac{\sqrt{x^2+9}}{9x} + C.$$

8.4.16 Let $x = 5\tan\theta$ so that $dx = 5\sec^2\theta\, d\theta$. Note that $25 + x^2 = 25\sec^2\theta$. Thus, $\int \frac{x^2}{(25+x^2)^2}\, dx =$ $\int \frac{25\tan^2\theta\cdot 5\sec^2\theta}{25^2\sec^4\theta}\, d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1-\cos 2\theta)\; d\theta = \frac{1}{10}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C =$ $\frac{1}{10}(\tan\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left(\tan^{-1}\left(\frac{x}{5}\right) - \frac{5x}{25+x^2}\right) + C.$

8.4.17 Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Note that $x^2+4 = 4\tan^2\theta + 4 = 4(\tan^2\theta+1) = 4\sec^2\theta$. Then

$$\int_0^1 \frac{x^2}{x^2+4}\,dx = \int_0^{\pi/4} \frac{4\tan^2\theta}{4\sec^2\theta} 2\sec^2\theta\,d\theta = 2\int_0^{\pi/4} \tan^2\theta\,d\theta$$
$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta = 2(\tan\theta - \theta)\Big|_0^{\pi/4} = 2\left(1 - \frac{\pi}{4}\right) = 2 - \frac{\pi}{2}.$$

8.4.18 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$. Note that $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. Substituting gives

$$\int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{x^2+1}} + C.$$

8.4.19 Let $x = 9\sec\theta$ with $\theta \in (0, \pi/2)$. Then $dx = 9\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-81} = 9\tan\theta$. Then $\displaystyle\int \frac{1}{\sqrt{x^2-81}}\,dx = \int \frac{9\sec\theta\tan\theta}{9\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{9} + \frac{\sqrt{x^2-81}}{9}\right| + C$. Note that because $x > 9$, the absolute value signs are unnecessary, and the final result can be written as $\ln(\sqrt{x^2-81}+x) + C$.

8.4.20 Let $x = 7\sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = 7\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-49} = 7\tan\theta$. Then $\displaystyle\int \frac{1}{\sqrt{x^2-49}}\,dx = \int \frac{7\sec\theta\tan\theta}{7\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{7} + \frac{\sqrt{x^2-49}}{7}\right| + C$. (Note also that the absolute value signs can be omitted because $x > 7$, and if we replace $-\ln(7) + C$ by a different arbitrary constant D, we can write the result as $\ln(x + \sqrt{x^2-49}) + D$.

8.4.21 Let $x = 8\sin\theta$ so that $dx = 8\cos\theta\,d\theta$ and $\sqrt{64-x^2} = 8\cos\theta$. Then,

$$\int \sqrt{64-x^2}\,dx = \int 64\cos^2\theta\,d\theta = 32\int (1+\cos 2\theta)\,d\theta = 32\theta + 16\sin 2\theta + C$$
$$= 32\theta + 32\sin\theta\cos\theta + C = 32\sin^{-1}\left(\frac{x}{8}\right) + \frac{x\sqrt{64-x^2}}{2} + C.$$

8.4.22 Let $t = 3\sin\theta$, so that $dt = 3\cos\theta\,d\theta$. Note that $\sqrt{9-t^2} = \sqrt{9(\cos^2\theta)} = 3\cos\theta$. Substituting gives

$$\int \frac{1}{9}\csc^2\theta\,d\theta = -\frac{1}{9}\cot\theta + C = -\frac{\sqrt{9-t^2}}{9t} + C.$$

8.4.23 Let $x = 5\sin\theta$ so that $dx = 5\cos\theta\,d\theta$. Note that $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$. Substituting gives

$$\int \frac{5\cos\theta}{125\cos^3\theta}\,d\theta = \frac{1}{25}\int \sec^2\theta\,d\theta = \frac{1}{25}\tan\theta + C = \frac{x}{25\sqrt{25-x^2}} + C.$$

8.4.24 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Note that $\sqrt{9-x^2} = 3\cos\theta$. Thus, $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta\cdot 3\cos\theta}{9\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int \csc^2 - 1\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(x/3) + C$.

8.4.25 Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\int \frac{\sqrt{9-x^2}}{x}\,dx = \int \frac{3\cos\theta\cdot 3\cos\theta}{3\sin\theta}\,d\theta = 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$
$$= 3\left(\int \csc\theta\,d\theta - \int \sin\theta\,d\theta\right)\,d\theta = 3\left(-\ln|\csc\theta + \cot\theta| + \cos\theta\right)$$
$$= -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + \sqrt{9-x^2} + C.$$

8.4.26 Let $x = \sec\theta$ so that $dx = \sec\theta\tan\theta\,d\theta$. Note that $\sqrt{x^2-1} = \sqrt{\tan^2\theta} = \tan\theta$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\,d\theta = (\tan\theta - \theta)\Big|_{\pi/4}^{\pi/3} = \sqrt{3} - \pi/3 - (1-\pi/4) = \sqrt{3} - 1 - \frac{\pi}{12}.$$

8.4.27 Let $x = \frac{1}{3}\tan\theta$ so that $dx = \frac{1}{3}\sec^2\theta\,d\theta$. Note that $\sqrt{9x^2+1} = \sec\theta$. Thus $\int_0^{1/3} \frac{1}{(9x^2+1)^{3/2}}\,dx =$ $\int_0^{\pi/4} \frac{\frac{1}{3}\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/4}\cos\theta\,d\theta = \frac{1}{3}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{6}$.

8.4.28 Let $z = 6\tan\theta$ so that $dz = 6\sec^2\theta\,d\theta$. Note that $z^2+36 = 36\sec^2\theta$. Thus

$$\begin{aligned}\int_0^6 \frac{z^2}{(z^2+36)^2}\,dz &= \int_0^{\pi/4} \frac{36\tan^2\theta\cdot 6\sec^2\theta}{36^2\sec^4\theta}\,d\theta = \frac{1}{6}\int_0^{\pi/4}\frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\ &= \frac{1}{6}\int_0^{\pi/4}\sin^2\theta\,d\theta = \frac{1}{12}\int_0^{\pi/4}(1-\cos 2\theta)\,d\theta \\ &= \frac{1}{12}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{1}{12}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi-2}{48}\end{aligned}$$

8.4.29 Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $4 + x^2 = 4(\sec^2\theta)$. Then

$$\begin{aligned}\int \frac{dx}{(4+x^2)^2} &= \int \frac{2\sec^2\theta}{2^4\sec^4\theta}\,d\theta = \frac{1}{8}\int\cos^2\theta\,d\theta = \frac{1}{16}\int(1+\cos 2\theta)\,d\theta \\ &= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{16}(\theta + \sin\theta\cos\theta) + C = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C.\end{aligned}$$

8.4.30 Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$. Substituting gives

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &\int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int \sin^3\theta\cos^2\theta\,d\theta \\ &= \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta = -\int(u^2-u^4)\,du \\ &= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = -\left(\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right) + C \\ &= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C = -\frac{1}{15}(1-x^2)^{3/2}(3x^2+2) + C.\end{aligned}$$

Along the way we made the substitution $u = \cos x$.

8.4.31 Let $x = 4\sin\theta$ so that $dx = 4\cos\theta\,d\theta$. Note that $\sqrt{16-x^2} = 4\cos\theta$. Then $\int \frac{x^2}{\sqrt{16-x^2}}\,dx =$ $\int \frac{16\sin^2\theta\cdot 4\cos\theta}{4\cos\theta}\,d\theta = 16\int\sin^2\theta\,d\theta = 8\int(1-\cos 2\theta)\,d\theta = 8\left(\theta - \frac{\sin 2\theta}{2}\right) + C = 8\theta - 8\sin\theta\cos\theta + C =$ $8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$.

8.4.32 Let $x = 6\sec\theta$ with $\theta \in (0, \pi/2)$. Then $dx = 6\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2-36} = 6\tan\theta$. Then

$$\int \frac{1}{(x^2-36)^{3/2}}\,dx = \int \frac{6\sec\theta\tan\theta}{6^3\tan^3\theta}\,d\theta = \frac{1}{36}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{36}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Then we have

$$\frac{1}{36}\int u^{-2}\,du = -\frac{1}{36u} + C = -\frac{1}{36\sin\theta} + C = -\frac{x}{36\sqrt{x^2-36}} + C.$$

8.4.33 Let $x = 3\sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = 3\sec\theta\tan\theta$ and $\sqrt{x^2-9} = 3\tan\theta$. Thus we have $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\sec\theta\tan\theta\cdot 3\tan\theta}{3\sec\theta}\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int \sec^2\theta - 1\,d\theta = 3(\tan\theta - \theta) + C =$ $\displaystyle\sqrt{x^2-9} - 3\tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right) + C = \sqrt{x^2-9} - 3\sec^{-1}(x/3) + C.$

8.4.34 Let $x = \sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$. Thus,

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int 1 + \cos 2\theta\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$
$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\tan^{-1}\sqrt{x^2-1} + \frac{\sqrt{x^2-1}}{x^2}\right) + C.$$

8.4.35 Let $x = \sec\theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \tan\theta$. Then

$$\int \frac{1}{x(x^2-1)^{3/2}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan^3\theta}\,d\theta = \int \cot^2\theta\,d\theta$$
$$= \int \csc^2\theta - 1\,d\theta = -\cot\theta - \theta + C = -\frac{1}{\sqrt{x^2-1}} - \sec^{-1}x + C.$$

8.4.36 Let $x = 8\sec\theta$ so that $dx = 8\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-64} = 8\tan\theta$. Then $\displaystyle\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}} =$ $\displaystyle\int_{\pi/4}^{\pi/3} \frac{8\sec\theta\tan\theta}{8\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_{\pi/4}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1) = \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right).$

8.4.37 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$. Substituting gives

$$\int_{\pi/6}^{\pi/4} \cot\theta\csc\theta\,d\theta = (-\csc\theta)\Big|_{\pi/6}^{\pi/4} = -(\sqrt{2} - 2) = 2 - \sqrt{2}.$$

8.4.38 Let $x = 2\sin\theta$, so that $dx = 2\cos\theta\,d\theta$. Note that when $x = 1$ we have $\theta = \dfrac{\pi}{6}$ and when $x = \sqrt{2}$ we have $\theta = \dfrac{\pi}{4}$. Also $\sqrt{4-x^2} = 2\sqrt{\cos^2\theta} = 2\cos\theta$. Substituting gives

$$\frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\,d\theta = -\frac{1}{4}(\cot\theta)\Big|_{\pi/6}^{\pi/4} = -\frac{1}{4}\left(1 - \sqrt{3}\right) = \frac{\sqrt{3}-1}{4}.$$

8.4.39 Let $x = 10\sin\theta$ so that $dx = 10\cos\theta\,d\theta$. Note that $\sqrt{100-x^2} = 10\cos\theta$. Thus,

$$\int \frac{x^2}{(100-x^2)^{3/2}}\,dx = \int \frac{100\sin^2\theta\cdot 10\cos\theta}{1000\cos^3\theta}\,d\theta = \int \tan^2\theta\,d\theta$$
$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C = \frac{x}{\sqrt{100-x^2}} - \sin^{-1}(x/10) + C.$$

8.4.40 Let $y = 5\sec\theta$ so that $dy = 5\sec\theta\tan\theta\,d\theta$. Then $\sqrt{y^2-25} = 5\tan\theta$. Then, $\displaystyle\int_{10/\sqrt{3}}^{10} \frac{1}{\sqrt{y^2-25}}\,dy =$ $\displaystyle\int_{\pi/6}^{\pi/3} \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_{\pi/6}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(\sqrt{3}) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right).$

8.4.41 Let $x = \dfrac{\tan\theta}{2}$ so that $dx = \dfrac{\sec^2\theta}{2}\,d\theta$ and $\sqrt{1+4x^2} = \sec\theta$. Then

$$\int \frac{1}{(1+4x^2)^{3/2}}\,dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{\sin\theta}{2} + C = \frac{x}{\sqrt{1+4x^2}} + C.$$

8.4.42 Let $x = \frac{1}{3}\sec\theta$, where $\theta \in (0, \pi/2)$. Then $dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$. Note that $\sqrt{9x^2-1} = \tan\theta$. Then

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = \int \frac{\frac{1}{3}\sec\theta\tan\theta}{\frac{1}{9}\sec^2\theta\tan\theta}\,d\theta = 3\int \cos\theta\,d\theta = 3\sin\theta + C = \frac{\sqrt{9x^2-1}}{x} + C.$$

8.4.43 Let $x = 4\tan\theta$ so that $dx = 4\sec^2\theta\,d\theta$. Note that $\sqrt{x^2+16} = 4\sec\theta$. Thus, $\int_0^{4/\sqrt{3}} \frac{1}{\sqrt{x^2+16}}\,dx = \int_0^{\pi/6} \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int_0^{\pi/6} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/6} = \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln 1 = \ln\frac{3}{\sqrt{3}} = \ln 3 - \frac{1}{2}\ln 3 = \frac{1}{2}\ln 3$.

8.4.44 Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{16+4x^2} = 4\sec\theta$. Then

$$\int \frac{1}{\sqrt{16+4x^2}}\,dx = \int \frac{2\sec^2\theta}{4\sec\theta}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \frac{1}{2}\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

8.4.45 Let $x = 9\sin\theta$ so that $dx = 9\cos\theta\,d\theta$. Note that $81 - x^2 = 81\cos^2\theta$. Thus,

$$\begin{aligned}\int \frac{x^3}{(81-x^2)^2}\,dx &= \int \frac{9^3\sin^3\theta \cdot 9\cos\theta}{9^4\cos^4\theta}\,d\theta = \int \tan^3\theta\,d\theta \\ &= \int (\tan\theta)(\sec^2\theta - 1)\,d\theta = \int \sec^2\theta\tan\theta\,d\theta - \int \tan\theta\,d\theta \\ &= \frac{\sec^2\theta}{2} + \ln|\cos\theta| + C = \frac{81}{2(81-x^2)} + \ln\left|\frac{\sqrt{81-x^2}}{9}\right| + C.\end{aligned}$$

This can be written as $\frac{81}{2(81-x^2)} + \ln\sqrt{81-x^2} + C$.

8.4.46 Let $x = (1/\sqrt{2})\sin\theta$ so that $dx = (1/\sqrt{2})\cos\theta\,d\theta$ and $\sqrt{1-2x^2} = \cos\theta$. Then

$$\int \frac{1}{\sqrt{1-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{\cos\theta}{\cos\theta}\,d\theta = \frac{1}{\sqrt{2}}\cdot\theta + C = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}x) + C.$$

8.4.47 Let $x = 2\sec\theta$ so that $dx = 2\sec\theta\tan\theta\,d\theta$ and $x^2 - 4 = 4\tan^2\theta$. Thus, $\int_{4/\sqrt{3}}^{4} \frac{1}{x^2(x^2-4)}\,dx = \int_{\pi/6}^{\pi/3} \frac{2\sec\theta\tan\theta}{4\sec^2\theta\cdot 4\tan^2\theta}\,d\theta = \frac{1}{8}\int_{\pi/6}^{\pi/3}\frac{\cos^2\theta}{\sin\theta}\,d\theta = \frac{1}{8}\int_{\pi/6}^{\pi/3}\frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \frac{1}{8}\int_{\pi/6}^{\pi/3}\csc\theta - \sin\theta\,d\theta = \frac{1}{8}\left(-\ln|\csc\theta+\cot\theta| + \cos\theta\right)\Big|_{\pi/6}^{\pi/3} = \frac{1}{8}\left(-\ln(\sqrt{3}(2-\sqrt{3})) + \frac{1-\sqrt{3}}{2}\right) = \frac{1}{16}\left(1 - \sqrt{3} - \ln(21 - 12\sqrt{3})\right)$.

8.4.48 Let $x = \frac{3}{2}\sin\theta$, so that $dx = \frac{3}{2}\cos\theta\,d\theta$. Note that $\sqrt{9-4x^2} = 3\cos\theta$. Thus $\int\sqrt{9-4x^2}\,dx = \frac{3}{2}\int\cos\theta\cdot 3\cos\theta\,d\theta = \frac{9}{2}\int\cos^2\theta\,d\theta = \frac{9}{4}\int 1+\cos 2\theta\,d\theta = \frac{9}{4}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{9\theta}{4} + \frac{9\sin\theta\cos\theta}{4} + C = \frac{9\sin^{-1}(2x/3)}{4} + \frac{x\sqrt{9-4x^2}}{2} + C$.

8.4.49 Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$. Note that $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$. Substituting gives $\int_0^{\pi/6}\sec^3\theta\,d\theta$. Recall from section 8.2 number 48 that $\int\sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$. Thus the original integral is equal to $\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/6} = \frac{1\cdot 2\cdot 1}{2\cdot\sqrt{3}\cdot\sqrt{3}} + \frac{1}{2}\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{\ln 3}{4}$.

8.4.50 Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. Then $\int (36-9x^2)^{-3/2}\,dx = \frac{1}{27}\int \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{1}{27}\int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \frac{1}{108}\int \sec^2\theta\,d\theta = \frac{1}{108}\tan\theta + C = \frac{x}{108\sqrt{4-x^2}} + C.$

8.4.51 Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Note that $\sqrt{4+x^2} = 2\sec\theta$. Then $\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2\theta\cdot 2\sec^2\theta}{2\sec\theta}\,d\theta = 4\int \tan^2\theta\sec\theta\,d\theta = 4\int(\sec^2\theta - 1)\sec\theta\,d\theta = 4\left(\int \sec^3\theta\;d\theta - \int \sec\theta\,d\theta\right) = 4\left(\frac{1}{2}\left(\sec\theta\tan\theta + \int \sec\theta\,d\theta\right) - \int \sec\theta\,d\theta\right) = 2\sec\theta\tan\theta - 2\int \sec\theta\,d\theta = 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| + C = \frac{x\sqrt{4+x^2}}{2} - 2\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$. This can be written as $\frac{x\sqrt{4+x^2}}{2} - 2\ln(x + \sqrt{4+x^2}) + C$.

8.4.52 Let $x = \frac{1}{2}\sec\theta$ where $\theta \in [0,\pi/2)$. Then $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$ and $\sqrt{4x^2-1} = \tan\theta$. Thus, $\int \frac{\sqrt{4x^2-1}}{x^2}\,dx = \int \frac{\tan\theta\cdot\frac{1}{2}\sec\theta\tan\theta}{\frac{1}{4}\cdot\sec^2\theta}\,d\theta = 2\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = 2\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = 2\int \sec\theta - \cos\theta\,d\theta = 2\left(\ln|\sec\theta + \tan\theta| - \sin\theta\right) + C = 2\ln(2x + \sqrt{4x^2-1}) - \frac{\sqrt{4x^2-1}}{x} + C.$

8.4.53 Let $x = \frac{5}{3}\sec\theta$ where $\theta \in [0,\pi/2)$. Then $dx = \frac{5}{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{9x^2-25} = 5\tan\theta$. Thus, $\int \frac{\sqrt{9x^2-25}}{x^3}\,dx = \int \frac{5\tan\theta\cdot\frac{5}{3}\cdot\sec\theta\tan\theta}{\frac{125}{27}\sec^3\theta}\,d\theta = \frac{9}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{9}{5}\int \frac{\sec^2\theta - 1}{\sec^2\theta}\,d\theta = \frac{9}{5}\int 1 - \cos^2\theta\,d\theta = \frac{9}{5}\int \sin^2\theta\,d\theta = \frac{9}{10}\int (1 - \cos 2\theta)\,d\theta = \frac{9\theta}{10} - \frac{9\sin 2\theta}{20} + C = \frac{9\theta}{10} - \frac{9\sin\theta\cos\theta}{10} = \frac{9\cos^{-1}(5/3x)}{10} - \frac{\sqrt{9x^2-25}}{2x^2} + C.$

8.4.54 Let $y = \tan\theta$ so that $dy = \sec^2\theta\,d\theta$. Note that $1 + y^2 = \sec^2\theta$. Thus,

$$\begin{aligned}
\int \frac{y^4}{1+y^2}\,dy &= \int \frac{\tan^4\theta\sec^2\theta}{\sec^2\theta}\,d\theta = \int \tan^4\theta\,d\theta \\
&= \int (\tan^2\theta)(\sec^2\theta - 1)\,d\theta = \int \tan^2\theta\sec^2\theta - \tan^2\theta\,d\theta \\
&= \int \tan^2\theta\sec^2\theta + 1 - \sec^2\theta\,d\theta = \int \tan^2\theta\sec^2\theta\,d\theta + \theta - \tan\theta.
\end{aligned}$$

Now recall that $y = \tan\theta$, so we have $\int y^2\,dy + \theta - \tan\theta = \frac{y^3}{3} + \theta - \tan\theta + C = \frac{y^3}{3} + \tan^{-1}(y) - y + C.$

8.4.55 Let $x = 10\sec\theta$ where $\theta \in (0,\pi/2)$. Then $dx = 10\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-100} = 10\tan\theta$. Thus,

$$\begin{aligned}
\int \frac{1}{x^3\sqrt{x^2-100}}\,dx &= \int \frac{10\sec\theta\tan\theta}{10^3\sec^3\theta\cdot 10\tan\theta}\,d\theta = \frac{1}{1000}\int \cos^2\theta\,d\theta \\
&= \frac{1}{2000}\int (1 + \cos 2\theta)\,d\theta = \frac{\theta}{2000} + \frac{\sin 2\theta}{4000} + C = \frac{\theta}{2000} + \frac{\sin\theta\cos\theta}{2000} + C \\
&= \frac{\sec^{-1}(x/10)}{2000} + \frac{\sqrt{x^2-100}}{200x^2} + C.
\end{aligned}$$

8.4.56 Let $x = 4\sec\theta$ where $\theta \in (0,\pi/2)$. Then $dx = 4\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-16} = 4\tan\theta$. Then

$$\begin{aligned}
\int \frac{x^3}{(x^2-16)^{3/2}}\,dx &= \int \frac{4^3\sec^3\theta\cdot 4\sec\theta\tan\theta}{64\tan^3\theta}\,d\theta = 4\int \frac{\sec^4\theta}{\tan^2\theta}\,d\theta \\
&= 4\int \frac{(\sec^2\theta)(1 + \tan^2\theta)}{\tan^2\theta}\,d\theta.
\end{aligned}$$

Let $u = \tan\theta$ so that $du = \sec^2\theta\, d\theta$. Then we have

$$4\int \frac{1+u^2}{u^2}\,du = 4\int u^{-2}+1\,du = -\frac{4}{u} + 4u + C$$
$$= -4\cot\theta + 4\tan\theta + C = -\frac{16}{\sqrt{x^2-16}} + \sqrt{x^2-16} + C.$$

8.4.57

a. False. In fact, we would have $\csc\theta = \dfrac{\sqrt{x^2+16}}{x}$.

b. True. Almost every number in the interval $[1, 2]$ is not in the domain of $\sqrt{1-x^2}$, so this integral isn't defined.

c. False. It does represent a finite real number, because $\sqrt{x^2-1}$ is continuous on the interval $[1, 2]$.

d. False. It can be so evaluated. The integral is equivalent to $\displaystyle\int \frac{1}{(x+2)^2+5}\,dx$, and this can be evaluated by the substitution $x + 2 = \sqrt{5}\tan\theta$.

8.4.58 Let A be the area of the ellipse. Using symmetry, we have $\dfrac{A}{4} = \displaystyle\int_0^b \frac{b}{a}\sqrt{a^2-x^2}\;dx$. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$. Substituting yields

$$\frac{b}{a}\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta = ab\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{ab}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{ab}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{\pi ab}{4}.$$

So the total area of the ellipse is $A = \pi ab$.

8.4.59

a. Recall that the area of a circular sector subtended by an angle θ is given by $\dfrac{\theta r^2}{2}$. So the area of the cap is this area minus the area of the isosceles triangle with two sides of length r and angle between them θ. So

$$A_{\text{cap}} = A_{\text{sector}} - A_{\text{triangle}} = \frac{\theta r^2}{2} - \frac{r^2\sin\theta}{2} = \frac{r^2}{2}(\theta - \sin\theta).$$

b. For a cap we have $0 \le \theta \le \pi$ so $0 \le \theta/2 \le \pi/2$. By symmetry, $\dfrac{A_{\text{cap}}}{2} = \displaystyle\int_{r\cos\theta/2}^{r}\sqrt{r^2-x^2}\,dx$. Let $x = r\cos\alpha/2$ so that $dx = -\frac{r}{2}\sin\alpha/2\,d\alpha$. Then we have

$$\frac{A_{\text{cap}}}{2} = \int_\theta^0 r\sin(\alpha/2)\cdot -\frac{r}{2}\sin(\alpha/2)\,d\alpha$$
$$= \frac{r^2}{2}\int_0^\theta \sin^2(\alpha/2)\,d\alpha = \frac{r^2}{4}\int_0^\theta (1-\cos\alpha)\,d\alpha = \frac{r^2}{4}(\alpha - \sin\alpha)\Big|_0^\theta = \frac{r^2}{4}(\theta - \sin\theta).$$

Thus $A_{\text{cap}} = \dfrac{r^2}{2}(\theta - \sin\theta)$.

8.4.60 Note that the given integral can be written $\displaystyle\int \frac{1}{(x-3)^2+25}\,dx = \int \frac{1}{u^2+25}\,du$ with $u = x-3$. Now let $u = 5\tan\theta$ so that $du = 5\sec^2\theta\,d\theta$ and $u^2 + 25 = 25\sec^2\theta$. Thus we have

$$\int \frac{5\sec^2\theta}{25\sec^2\theta}\,d\theta = \frac{\theta}{5} + C = \frac{\tan^{-1}((x-3)/5)}{5} + C.$$

8.4.61 $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du$ where $u = x+1$. Then let $u = 2\sin\theta$ so that $du = 2\cos\theta\,d\theta$. We have $\displaystyle\int \frac{2\cos\theta}{2\cos\theta}\,d\theta = \theta + C = \sin^{-1}\left(\frac{x+1}{2}\right) + C$.

8.4.62 Note that the given integral can be written $\displaystyle\frac{1}{2}\int \frac{1}{(u-3)^2+9}\,du = \frac{1}{2}\int \frac{1}{w^2+9}\,dw$ where $w = u-3$. Now let $w = 3\tan\theta$ so that $dw = 3\sec^2\theta\,d\theta$ and $w^2+9 = 9\sec^2\theta$. Thus we have

$$\frac{1}{2}\int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{\theta}{6} + C = \frac{\tan^{-1}((u-3)/3)}{6} + C.$$

8.4.63 Note that the given integral can be written $\displaystyle\int \frac{1}{(x+3)^2+9}\,dx = \int \frac{1}{u^2+9}\,du$ where $u = x+3$. Now let $u = 3\tan\theta$ so that $du = 3\sec^2\theta\,d\theta$ and $u^2+9 = 9\sec^2\theta$. Thus we have

$$\int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{\theta}{3} + C = \frac{\tan^{-1}((x+3)/3)}{3} + C.$$

8.4.64 Note that the given integral can be written as $\displaystyle\int \frac{(x-1)^2}{\sqrt{(x-1)^2+9}}\,dx = \int \frac{u^2}{\sqrt{u^2+9}}\,du$ where $u = x-1$. Now let $u = 3\tan\theta$ so that $du = 3\sec^2\theta\,d\theta$ and $u^2+9 = 9\sec^2\theta$. Thus we have

$$\begin{aligned}
\int \frac{9\tan^2\theta\cdot 3\sec^2\theta}{3\sec\theta}\,d\theta = 9\int \tan^2\theta\sec\theta\,d\theta &= 9\int (\sec^2\theta - 1)(\sec\theta)\,d\theta \\
&= 9\int (\sec^3\theta - \sec\theta)\,d\theta = 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta - \int \sec\theta\,d\theta\right) \\
&= \frac{9}{2}\left(\sec\theta\tan\theta - \int \sec\theta\,d\theta\right) \\
&= \frac{9}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right) + C \\
&= \frac{9}{2}\left(\frac{(x-1)\sqrt{(x-1)^2+9}}{9} - \ln\left|\frac{\sqrt{(x-1)^2+9}}{3} + \frac{x-1}{3}\right|\right) + C.
\end{aligned}$$

This can be written as $\displaystyle\frac{x-1}{2}\sqrt{x^2-2x+10} - \frac{9}{2}\ln(x-1+\sqrt{x^2-2x+10}) + C$. Note that in the middle of this derivation we used the reduction formula for $\displaystyle\int \sec^3\theta\,d\theta$ given in the previous section.

8.4.65 $\displaystyle\int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8x^2-8x+11}\,dx = \int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8(x-1/2)^2+9}\,dx$. Let $u = x - 1/2$, so that our integral becomes $\displaystyle\int_0^{3/2\sqrt{2}} \frac{1}{8u^2+9}\,du$. Now let $u = \frac{3}{\sqrt{8}}\tan\theta$ so that $du = \frac{3}{\sqrt{8}}\sec^2\theta\,d\theta$. Substituting gives

$$\int_0^{\pi/4} \frac{\frac{3}{\sqrt{8}}\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{6\sqrt{2}}\int d\theta = \frac{1}{6\sqrt{2}}\theta\,\Big|_0^{\pi/4} = \frac{\pi\sqrt{2}}{48}.$$

8.4.66 $\displaystyle\int_1^4 \frac{1}{t^2-2t+10}\,dt = \int_1^4 \frac{1}{(t-1)^2+9}\,dt$. Let $3\tan\theta = t-1$, so that $dt = 3\sec^2\theta\,d\theta$. Note that $(t-1)^2+9 = 9\sec^2\theta$. Then we have $\displaystyle\int_0^{\pi/4} \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta = \frac{1}{3}\theta\,\Big|_0^{\pi/4} = \frac{\pi}{12}$.

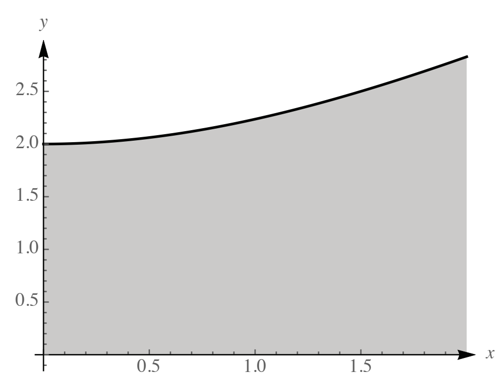
8.4.67 Note that the given integral can be written as $\int \frac{(x-4)^2}{(25-(x-4)^2)^{3/2}}\,dx$. Let $u = x-4$, and note that we have $\int \frac{u^2}{(25-u^2)^{3/2}}\,du$. Now let $u = 5\sin\theta$ so that $du = 5\cos\theta\,d\theta$, and note that $\sqrt{25-u^2} = 5\cos\theta$. Thus we have

$$\int \frac{25\sin^2\theta\cdot 5\cos\theta}{5^3\cos^3\theta}\,d\theta = \int \tan^2\theta\,d\theta = \int \sec^2\theta - 1\,d\theta = \tan\theta - \theta + C = \frac{x-4}{\sqrt{25-(x-4)^2}} - \sin^{-1}\left(\frac{x-4}{5}\right) + C.$$

8.4.68 $\int \frac{1}{\sqrt{(x-1)(3-x)}}\,dx = \int \frac{1}{\sqrt{1-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C = \sin^{-1}(x-2) + C.$

8.4.69 $\int_{2+\sqrt{2}}^{4} \frac{1}{\sqrt{(x-1)(x-3)}}\,dx = \int_{2+\sqrt{2}}^{4} \frac{1}{\sqrt{(x-2)^2-1}}\,dx = \int_{\sqrt{2}}^{2} \frac{1}{\sqrt{u^2-1}}\,du$, where $u = x-2$. Now let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\,d\theta$. Then $\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta)\Big|_{\pi/4}^{\pi/3} =$ $\ln(2+\sqrt{3}) - \ln(\sqrt{2}+1) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) = \ln((2+\sqrt{3})(\sqrt{2}-1)).$

8.4.70



The area is given by $\int_0^2 \sqrt{4+x^2}\,dx$. Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$. Then we have

$$\int_0^{\pi/4} 2\sec^2\theta\cdot 2\sec\theta\,d\theta = 4\int_0^{\pi/4} \sec^3\theta\,d\theta = 2\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4} = 2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).$$

8.4.85

a. Because $t \in [0, \pi]$ so that $sint \geq 0$, we have

$$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(1+\cos t)(\cos a - \cos t)}}\, dt = \int_a^b \sin t \sqrt{\frac{1}{g(1+\cos t)(\cos a - \cos t)}}\, dt.$$

Let $u = \cos t$ so that $du = -\sin t\, dt$. Then the given integral is equal to

$$-\frac{1}{\sqrt{g}} \int_{\cos a}^{\cos b} \sqrt{\frac{1}{(1+u)(\cos a - u)}}\, du.$$

Now we complete the square:

$$\begin{aligned}(1+u)(\cos a - u) &= \cos a + (\cos a - 1)u - u^2 \\ &= -\left(u^2 - (\cos a - 1)u + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2\right) + \cos a \\ &= \cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2 = \left(\frac{\cos a + 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2.\end{aligned}$$

Thus, setting $v = u - \dfrac{\cos a - 1}{2}$ we have that the original integral is equal to
$-\dfrac{1}{\sqrt{g}} \displaystyle\int_{(\cos a+1)/2}^{\cos b - \frac{\cos a - 1}{2}} \frac{1}{\sqrt{k^2 - v^2}}\, dv$ where $k = \dfrac{(\cos a + 1)}{2}$.
Now, $\displaystyle\int \frac{1}{\sqrt{k^2 - v^2}}\, dv = \int \frac{k\cos\theta}{k\cos\theta}\, d\theta = \theta + C = \sin^{-1}(v/k) + C$ where $v = k\sin\theta$.
Therefore, the original integral is equal to

$$\begin{aligned}-\frac{1}{\sqrt{g}} \sin^{-1}\left(\frac{2v}{\cos a + 1}\right)\Bigg|_{(\cos a+1)/2}^{\cos b - (\cos a - 1)/2} &= \frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a + 1}{\cos a + 1}\right) - \sin^{-1}\left(\frac{2\cos b - \cos a + 1}{\cos a + 1}\right)\right) \\ &= \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2\cos b - \cos a + 1}{\cos a + 1}\right)\right).\end{aligned}$$

b. Letting $b = \pi$, we have that the integral is equal to

$$\begin{aligned}&\frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-2 - \cos a + 1}{\cos a + 1}\right)\right) \\ &= \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \sin^{-1}(-1)\right) = \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt{g}}.\end{aligned}$$

8.4.86 Let $x = 2\tan^{-1} u$ so that $u = \tan(x/2)$ and $\sec^2(x/2) = 1 + \tan^2(x/2) = 1 + u^2$. Also, $\cos^2(x/2) = 1/(1+u^2)$. By the double angle identity,

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \cos^2(x/2) - u^2\cos^2(x/2) = (1-u^2)\cos^2(x/2) = \frac{1-u^2}{1+u^2}.$$

Also, $\sin x = 2\sin(x/2)\cos(x/2) = 2\tan(x/2)\cos^2(x/2) = \dfrac{2u}{1+u^2}$. Now
$\displaystyle\int \frac{1}{1+\sin x + \cos x}\, dx = 2\int \frac{1}{1+u^2} \cdot \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}}\, du = 2\int \frac{1}{1+u^2+2u+1-u^2}\, du = 2\int \frac{1}{2+2u}\, du =$
$\displaystyle\int \frac{1}{1+u}\, du = \ln|1+u| + C = \ln|1+\tan(x/2)| + C.$