The original integral now becomes

$$\int \frac{dx}{\sqrt{x^2 + 4}} = \int \frac{2 \cosh t}{2 \cosh t}\, dt = \int dt = t + C.$$

Because $x = 2 \sinh t$, we have $t = \sinh^{-1} \frac{x}{2}$, which, by Theorem 7.7, leads to the result found in Solution 2.

This example shows that some integrals may be evaluated by more than one method. With practice, you will learn to identify the best method for a given integral.

Related Exercises 44, 49

➤ Recall that to complete the square with $x^2 + bx + c$, you add and subtract $(b/2)^2$ to and from the expression, and then factor to form a perfect square. You could also make the single substitution $x + 2 = 3 \sec \theta$ in Example 6.

EXAMPLE 6 **A secant substitution** Evaluate $\int_1^4 \frac{\sqrt{x^2 + 4x - 5}}{x + 2}\, dx.$

SOLUTION This example illustrates a useful preliminary step first encountered in Section 8.1. The integrand does not contain any of the patterns in Table 8.4 that suggest a trigonometric substitution. Completing the square does, however, lead to one of those patterns. Noting that $x^2 + 4x - 5 = (x + 2)^2 - 9$, we change variables with $u = x + 2$ and write the integral as

$$\int_1^4 \frac{\sqrt{x^2 + 4x - 5}}{x + 2}\, dx = \int_1^4 \frac{\sqrt{(x + 2)^2 - 9}}{x + 2}\, dx \quad \text{Complete the square.}$$

$$= \int_3^6 \frac{\sqrt{u^2 - 9}}{u}\, du. \quad u = x + 2,\ du = dx \text{ Change limits of integration.}$$

➤ The substitution $u = 3 \sec \theta$ can be rewritten as $\theta = \sec^{-1} (u/3)$. Because $u \ge 3$ in the integral $\int_3^6 \frac{\sqrt{u^2 - 9}}{u}\, du$, we have $0 \le \theta < \frac{\pi}{2}$.

This new integral calls for the secant substitution $u = 3 \sec \theta$ (where $0 \le \theta < \pi/2$), which implies that $du = 3 \sec \theta \tan \theta \, d\theta$ and $\sqrt{u^2 - 9} = 3 \tan \theta$. We also change the limits of integration: When $u = 3$, $\theta = 0$, and when $u = 6$, $\theta = \pi/3$. The complete integration can now be done:

$$\begin{aligned}
\int_1^4 \frac{\sqrt{x^2 + 4x - 5}}{x + 2}\, dx &= \int_3^6 \frac{\sqrt{u^2 - 9}}{u}\, du && u = x + 2,\ du = dx\\
&= \int_0^{\pi/3} \frac{3 \tan \theta}{3 \sec \theta}\, 3 \sec \theta \tan \theta \, d\theta && u = 3 \sec \theta,\ du = 3 \sec \theta \tan \theta \, d\theta\\
&= 3 \int_0^{\pi/3} \tan^2 \theta \, d\theta && \text{Simplify.}\\
&= 3 \int_0^{\pi/3} (\sec^2 \theta - 1)\, d\theta && \tan^2 \theta = \sec^2 \theta - 1\\
&= 3\, (\tan \theta - \theta) \Big|_0^{\pi/3} && \text{Evaluate the integral.}\\
&= 3\sqrt{3} - \pi. && \text{Simplify.}
\end{aligned}$$

Related Exercises 40, 60

SECTION 8.4 EXERCISES

Getting Started

1. What change of variables is suggested by an integral containing $\sqrt{x^2 - 9}$?
2. What change of variables is suggested by an integral containing $\sqrt{x^2 + 36}$?
3. What change of variables is suggested by an integral containing $\sqrt{100 - x^2}$?
4. If $x = 4 \tan \theta$, express $\sin \theta$ in terms of x.
5. Using the trigonometric substitution $x = 2 \sin \theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, express $\cot \theta$ in terms of x.
6. Using the trigonometric substitution $x = 8 \sec \theta$, $x \ge 8$ and $0 < \theta \le \frac{\pi}{2}$, express $\tan \theta$ in terms of x.

Practice Exercises

7–56. Trigonometric substitutions *Evaluate the following integrals using trigonometric substitution.*

7. $\int_0^{5/2} \frac{dx}{\sqrt{25 - x^2}}$ (*Hint:* Check your answer without using trigonometric substitution.)

8. $\displaystyle\int_0^{3/2} \frac{dx}{(9 - x^2)^{3/2}}$

9. $\displaystyle\int_5^{5\sqrt{3}} \sqrt{100 - x^2}\, dx$

10. $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}\, dx$

11. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}\, dx$

12. $\displaystyle\int_{1/2}^{1} \frac{\sqrt{1 - x^2}}{x^2}\, dx$

13. $\displaystyle\int \frac{dx}{\sqrt{16 - x^2}}$

14. $\displaystyle\int \sqrt{36 - t^2}\, dt$

15. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 9}}$

16. $\displaystyle\int \frac{x^2}{(25 + x^2)^2}\, dx$

17. $\displaystyle\int_0^2 \frac{x^2}{x^2 + 4}\, dx$

18. $\displaystyle\int \frac{dx}{(1 + x^2)^{3/2}}$

19. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 81}}, x > 9$

20. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 49}}, x > 7$

21. $\displaystyle\int \sqrt{64 - x^2}\, dx$

22. $\displaystyle\int \frac{dt}{t^2\sqrt{9 - t^2}}$

23. $\displaystyle\int \frac{dx}{(25 - x^2)^{3/2}}$

24. $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$

25. $\displaystyle\int \frac{\sqrt{9 - x^2}}{x}\, dx$

26. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2 - 1}}{x}\, dx$

27. $\displaystyle\int_0^{1/3} \frac{dx}{(9x^2 + 1)^{3/2}}$

28. $\displaystyle\int_0^6 \frac{z^2}{(z^2 + 36)^2}\, dz$

29. $\displaystyle\int \frac{dx}{(4 + x^2)^2}$

30. $\displaystyle\int x^3\sqrt{1 - x^2}\, dx$

31. $\displaystyle\int \frac{x^2}{\sqrt{16 - x^2}}\, dx$

32. $\displaystyle\int \frac{dx}{(x^2 - 36)^{3/2}}, x > 6$

33. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x}\, dx, x > 3$

34. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2 - 1}}, x > 1$

35. $\displaystyle\int \frac{dx}{x(x^2 - 1)^{3/2}}, x > 1$

36. $\displaystyle\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2 - 64}}$

37. $\displaystyle\int_{1/\sqrt{3}}^{1} \frac{dx}{x^2\sqrt{1 + x^2}}$

38. $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4 - x^2}}$

39. $\displaystyle\int \frac{x^2}{(100 - x^2)^{3/2}}\, dx$

40. $\displaystyle\int_{10/\sqrt{3}}^{10} \frac{dy}{\sqrt{y^2 - 25}}$

41. $\displaystyle\int \frac{dx}{(1 + 4x^2)^{3/2}}$

42. $\displaystyle\int \frac{dx}{x^2\sqrt{9x^2 - 1}}, x > \frac{1}{3}$

43. $\displaystyle\int_0^{4/\sqrt{3}} \frac{dx}{\sqrt{x^2 + 16}}$

44. $\displaystyle\int \frac{dx}{\sqrt{16 + 4x^2}}$

45. $\displaystyle\int \frac{x^3}{(81 - x^2)^2}\, dx$

46. $\displaystyle\int \frac{dx}{\sqrt{1 - 2x^2}}$

47. $\displaystyle\int_{4/\sqrt{3}}^{4} \frac{dx}{x^2(x^2 - 4)}$

48. $\displaystyle\int \sqrt{9 - 4x^2}\, dx$

49. $\displaystyle\int_0^{1/\sqrt{3}} \sqrt{x^2 + 1}\, dx$

50. $\displaystyle\int (36 - 9x^2)^{-3/2}\, dx$

51. $\displaystyle\int \frac{x^2}{\sqrt{4 + x^2}}\, dx$

52. $\displaystyle\int \frac{\sqrt{4x^2 - 1}}{x^2}\, dx, x > \frac{1}{2}$

53. $\displaystyle\int \frac{\sqrt{9x^2 - 25}}{x^3}\, dx, x > \frac{5}{3}$

54. $\displaystyle\int \frac{y^4}{1 + y^2}\, dy$

55. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2 - 100}}, x > 10$

56. $\displaystyle\int \frac{x^3}{(x^2 - 16)^{3/2}}\, dx, x < -4$

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

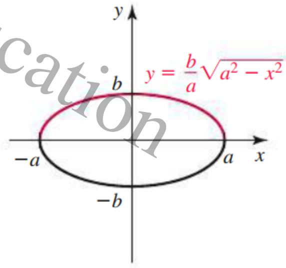
a. If $x = 4\tan\theta$, then $\csc\theta = 4/x$.

b. The integral $\displaystyle\int_1^2 \sqrt{1 - x^2}\, dx$ does not have a finite real value.

c. The integral $\displaystyle\int_1^2 \sqrt{x^2 - 1}\, dx$ does not have a finite real value.

d. The integral $\displaystyle\int \frac{dx}{x^2 + 4x + 9}$ cannot be evaluated using a trigonometric substitution.

58. **Area of an ellipse** The upper half of the ellipse centered at the origin with axes of length $2a$ and $2b$ is described by $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (see figure). Find the area of the ellipse in terms of a and b.

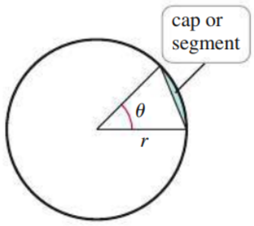


59. **Area of a segment of a circle** Use two approaches to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ (see figure) is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta).$$

a. Find the area using geometry (no calculus).

b. Find the area using calculus.



60–69. Completing the square *Evaluate the following integrals.*

60. $\displaystyle\int \frac{dx}{x^2 - 6x + 34}$

61. $\displaystyle\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

62. $\displaystyle\int \frac{du}{2u^2 - 12u + 36}$

63. $\displaystyle\int \frac{dx}{x^2 + 6x + 18}$

64. $\displaystyle\int \frac{x^2 - 2x + 1}{\sqrt{x^2 - 2x + 10}}\, dx$

65. $\displaystyle\int_{1/2}^{(\sqrt{2}+3)/(2\sqrt{2})} \frac{dx}{8x^2 - 8x + 11}$