Contents

5.1 §5.5 Substitution Rule

We start the first integration technique using this example: $\int \cos x dx = \sin x + C$, then $\int \cos 2x dx =?$.

The Substitution Rule is a result of the Chain Rule. Recall that the chain rule is: if $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable, then $\boxed{F^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x)}$.

Remember the concept of a differential: $dx \approx \Delta x$ for independent variable x and $\boxed{dy = y'dx}$ since $y' = \frac{dy}{dx}$ (for a constant c, $d(cy) = cy'dx$) for dependent variable y. If $u = g(x)$ then u is a dependent function and so $du = u'dx = g'(x)dx$ since $\frac{du}{dx} = u' = g'(x)$. Or we can write $d[g(x)] = g'(x)dx$.

5.1.1 Substitution Rule for Indefinite Integrals

Theorem (Thm 5.6: Substitution Rule for Indefinite Integrals). If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\ g'(x)dx = \int f(g(x))\ d[g(x)] = \int f(u)\ du.$$

Remark. In practice, sometimes it is convenient to use $\boxed{g'(x)dx = d[g(x)]}$ without explicitly writing down the substitution $u = g(x)$.

Proof. (read the proof after class) Suppose that $F(x)$ is an antiderivative of $f(x)$, i.e.

$$F(x) + C = \int f(x)\ dx.$$

Let $u = g(x)$. Then, we have

$$\begin{aligned} &D_x(F(x)) = F'(x) = f(x) \text{ or equivalently} \\ &D_u(F(u)) = F'(u) = f(u) \text{ or equivalently} \\ &\int f(u)\ du = F(u) + C \end{aligned}$$

By the Chain Rule, we have

$$D_x(F(g(x))) = \underbrace{D_u(F(u))}_{=f(u)}\underbrace{D_x(u)}_{=\frac{du}{dx}} = f(u)D_x(u) = f(g(x))g'(x)$$

and so

$$\begin{aligned} \int f(g(x))g'(x)dx =& \int D_x(F(g(x)))\ dx = F(g(x)) + C \\ =& F(u) + C = \int f(u)\ du \end{aligned}$$

□

Some Examples in the Textbook:

Example 1.(perfect substitutions)

(a) $\int 2(2x+1)^3 \; dx$

Solution.

$\square$

(b) $\int 10e^{10x} \; dx$

Solution.

$\square$

Example 2.(Introducing a constant)

(a) $\int x^4(x^5+6)^9 \ dx$

Solution.

If $u = x^5 + 6$ then $du = 5x^4dx$ and so we have

$$\begin{aligned}\int x^4(\underbrace{x^5+6}_{=u})^9 \ dx &= \frac{1}{5}\int(\underbrace{x^5+6}_{=u})^9 \underbrace{(5x^4) \ dx}_{=du}\\ &= \frac{1}{5}\int u^9 \ du = \frac{1}{5}(\frac{u^10}{10}) + C = \frac{1}{50}(x^5+6)^{10} + C.\end{aligned}$$

Or we do the following without writing the substitution u explicitly.

$$\begin{aligned}\int x^4(x^5+6)^9 \ dx = \int (x^5+6)^9 d(\frac{x^5}{5}) &= \int (x^5+6)^9 d\left(\frac{x^5+6}{5}\right)\\ =&\frac{1}{5}\int (x^5+6)^9 d(x^5+6) = \frac{1}{50}(x^5+6)^{10}.\end{aligned}$$

$\square$

(b) $\int \cos^3 x \sin x \ dx$

Solution.

$\square$

Example 3.(more than one substitution)

$\int \dfrac{x}{\sqrt{x+1}} \ dx$

Solution.

□

Example 4.(integration of $\int f(ax)\ dx$)

(a) $\int e^{ax}\ dx$

Solution.

□

(b) $\int b^{x}\ dx, b > 0, b \neq 1$

Solution.

□

(c) $\int \dfrac{1}{\sqrt{a^2 - x^2}}\ dx, a > 0$

Solution.

□

More Examples:

Example 1. If $u = x^4 + 2$, $du = d(x^4+2) = u'dx = 4x^3dx$.

$$\begin{aligned}\int x^3\cos(x^4+2)\ dx &= \int \cos(x^4+2)\ \frac{4x^3}{4}dx\\ &= \frac{1}{4}\int \underbrace{\cos(x^4+2)}_{\cos u}\ \underbrace{4x^3dx}_{=du} = \frac{1}{4}\int \cos u\ du = \frac{1}{4}\sin u + C\\ &= \frac{1}{4}\sin(x^4+2) + C.\end{aligned}$$

Or

$$\boxed{\int x^3\cos(x^4+2)\ dx = \int \cos(x^4+2)\ d\left(\frac{x^4+2}{4}\right) = \frac{1}{4}\int \cos u\ du = \cdots\cdots} \qquad \square$$

Example 2.

$$\begin{aligned}\int \sec^2(3\theta)\ d\theta &= \frac{1}{3}\int \sec^2(3\theta)\ 3d\theta = \frac{1}{3}\int \sec^2 u\ du\\ &= \frac{1}{3}\tan u + C = \frac{1}{3}\tan(3\theta) + C \quad \text{(we let } u = 3\theta).\end{aligned}$$

Example 3.

$$\int \frac{1}{x}\ dx = \begin{cases} \int \frac{1}{x}\ dx = \ln x + C & \text{if } x > 0\\ \\ \int \frac{1}{x}\ dx = \int \frac{1}{-x}\ d(-x) = \ln(-x) + C & \text{if } x < 0\end{cases}$$

$$= \ln|x| + C$$

Example 4 Manipulate with the integrand before substitution.

$$\begin{aligned}\int \tan x\ dx &= \int \frac{\sin x}{\cos x}\ dx = \int \frac{-d(\cos x)}{\cos x} = -\int \frac{d(\cos x)}{\cos x}\\ &= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\cos x^{-1}| + C\\ &= \ln|\sec x| + C.\end{aligned}$$

Example 5. Manipulate with the integrand before substitution (try $\int \csc x\ dx$ as an exercise).

$$\begin{aligned}\int \sec x\ dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\ dx\\ &= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\ dx = \ln|\sec x + \tan x| + C.\end{aligned}$$

Example 6. $u = \tan^{-1} x \implies du = \frac{dx}{1+x^2}$

$$\int \frac{\tan^{-1} x}{x^2+1}\, dx = \int u\, du = \cdots\cdots$$

Example 7. Let $u = e^x$. Then $du = e^x dx$ and $e^{2x} = u^2$.

$$\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{du}{1+u^2} = \cdots\cdots$$

Example 8. $u = x^2$. Then $du = 2xdx$.

$$\int \frac{x}{1+x^4}\, dx = \frac{1}{2}\int \frac{du}{1+u^2} = \cdots\cdots$$

Example 9. $u = \tan\theta \implies du = \sec^2\theta d\theta$

$$\int \tan^2\theta \sec^2\theta\, d\theta = \int u^2\, du = \cdots\cdots$$

Example 10. $u = \frac{\pi}{x} \implies du = -\pi x^{-2}\, dx$

$$\int \frac{\cos(\frac{\pi}{x})}{x^2}\, dx = -\frac{1}{\pi}\int \cos(\frac{\pi}{x})(-\pi x^{-2}\, dx) = -\frac{1}{\pi}\int \cos u\, du = \cdots\cdots$$

Example 11. Find $\int \frac{dx}{a^2+x^2}$, $a \neq 0$.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 12. Find $\frac{dx}{\sqrt{a^2-x^2}}$, $a > 0$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{a}\frac{dx}{\sqrt{1-(\frac{x}{a})^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

$\square$

5.1.2 Substitution Rule for Definite Integrals

Theorem (Thm 5.7: The Substitution Rule for Definite Integrals). If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

$\square$

Remark 1. If we do not explicitly write down the substitution, we do not change the lower limit and the upper limit. For example,

$$\int_0^1 x^3(1-x^4)\ dx = \boxed{\frac{-1}{4}\int_0^1 (1-x^4)d(1-x^4)} = \left(\frac{-1}{4}\right)\frac{(1-x^4)^2}{2}\Big|_0^1.$$

Remark 2. We can avoid changing the lower limit and the upper limit. For example,

$$u = 1-x^4, \quad \int x^3(1-x^4)\ dx = \boxed{\frac{-u^2}{8}} \text{Thus, } \int_0^1 x^3(1-x^4)\ dx = \frac{-u^2}{8}\Big|_{x=0}^{x=1}.$$

$\square$

Proof. This proof amounts to finding the value of both the left hand side and the right hand side and showing that they are equal.

For the LHS: Let $F(x)$ be an antiderivative of $f(x)$. Then, by the Substitution Rule for Indefinite Integrals, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, and so by Part 2 of the FTC, we have

$$\int_a^b f(g(x))g'(x)\ dx = F(g(b)) - F(g(a))$$

For the RHS: Since $F(x)$ be an antiderivative of $f(x)$, $F(u)$ is an antiderivative for $f(u)$ and so by FTC Part 2, we have

$$\int_{g(a)}^{g(b)} f(u)\ du = F(g(b)) - F(g(a))$$

The result now follows. $\square$

Some examples:

Example. $\int_0^7 \sqrt{4+3x}\ dx$

Here we set $u = g(x) = 4 + 3x$ and compute as follows.

$$\begin{aligned}\int_0^7 \sqrt{4+3x}\ dx =& \frac{1}{3}\int_0^7 \sqrt{4+3x}\ d(4+3x) = \frac{1}{3}\int_{g(0)}^{g(7)} u^{\frac{1}{2}}\ du \\ =& \frac{1}{3}\int_4^{25} u^{\frac{1}{2}}\ du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=4}^{u=25} \\ =& \frac{2}{9}[125-8] = \frac{234}{9}.\end{aligned}$$

Example. $\int_0^1 xe^{-x^2}\ dx$

Here we set $u = g(x) = -x^2$ and compute as follows.

$$\begin{aligned}\int_0^1 xe^{-x^2}\ dx &= -\frac{1}{2}\int_0^1 e^{-x^2}\ (-2xdx) = -\frac{1}{2}\int_{g(0)}^{g(1)} e^u\ du \\ &= -\frac{1}{2}\int_0^{-1} e^u\ du = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}\left[\frac{1}{e}-1\right] \\ &= \frac{e-1}{2e}.\end{aligned}$$

Example. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\ dx$

Here we set $u = g(x) = \sin^{-1}x$ and compute as follows.

$$\begin{aligned}\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\ dx &= \int_0^1 \sin^{-1}x(\frac{1}{\sqrt{1-x^2}}\ dx) \\ &= \int_{g(0)}^{g(1)} u\ du = [\frac{u^2}{2}]_0^{\sin^{-1}1} \\ &= \frac{(\sin^{-1}1)^2}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.\end{aligned}$$

Example 5. Evaluate the following integrals.

(a) $\int_0^2 \dfrac{1}{(x+3)^3}\ dx$

Solution.

$\square$

(b) $\int_2^3 \dfrac{x^2}{x^3-7}\ dx$

Solution.

$\square$

(c) $\int_0^{\pi/2} \sin^4 x \cos x\ dx$

Solution.

$\square$

Example 6. Evaluate $\int_0^{\pi/2} \cos^2 x\ dx$

Solution.

$\square$

5.2 §5.4 Working with Integrals: More Theorems

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section, we discuss the role of symmetry in integrals, we use the slice-and-sum strategy to define the average value of a function, and we explore a theoretical result called the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions

Theorem (Integrals of Symmetric Functions)**.** Suppose that $f(x)$ is continuous on the symmetric interval $[-a, a]$. Then

(a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

(b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$.

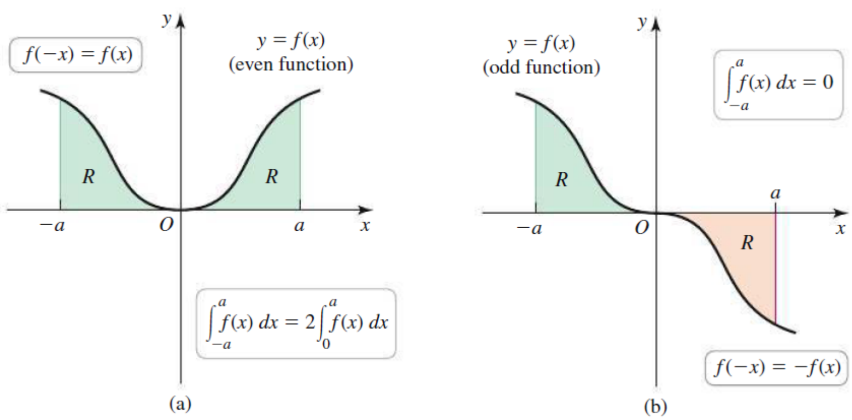

Figure 1: Integrals of Even and Odd Functions

Proof. We split the integral in two

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = -\int_0^{-a} f(x)\, dx + \int_0^a f(x)\, dx.$$

For the first part, we use the substitution $u = -x$, then

$$-\int_0^{-a} f(x)\, dx = -\int_0^a f(-u)\,(-du) = \int_0^a f(-u)\, du.$$

Therefore,

$$\int_{-a}^{a} f(x)\, dx = \int_0^a f(-v)\, dv + \int_0^a f(v)\, dv.$$

(a) If f is even, then $f(-v) = f(v)$, thus,

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(v)\ dv + \int_{0}^{a} f(v)\ dv = 2\int_{0}^{a} f(v)\ dv.$$

(b) If f is odd, then $f(-v) = -f(v)$, thus,

$$\int_{-a}^{a} f(x)\ dx = -\int_{0}^{a} f(v)\ dv + \int_{0}^{a} f(v)\ dv = 0.$$

□

Example. The cosine function is an even function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \cos x\ dx = 2\int_{0}^{\frac{\pi}{3}} \cos x\ dx = 2\sin x\Big|_{x=0}^{x=\frac{\pi}{3}} = 2[\frac{\sqrt{3}}{2}] = \sqrt{3}.$$

Example. The sine function is an odd function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \sin^5 x\ dx = 0.$$

Example Evaluate the following integrals using symmetry arguments.

a. $\displaystyle\int_{-2}^{2} (x^4 - 3x^3)\ dx$ **b.** $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 4\sin^3 x)\ dx$

Solution.

□

Mean Value Theorem for Integrals

Recall that the average of numbers $y_1, \ldots, y_n$ is simply the sum divided by the number of values n

$$\text{Average value of } y = \frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{\sum_{i=1}^n y_i}{n}.$$

Now suppose that we have a continuous function of time t, say temperature $T(t)$, what would it mean to say the average temperature over the period of time, say from $t = a$ to $t = b$. One would probably accept an approximation obtained as follows.

(i) Partitioning the time period into n intervals of equal length

$$\Delta t = \frac{b-a}{n}$$

with endpoints

$$t_0 = a < \cdots < t_i = a + i\frac{b-a}{n} < \cdots < t_n = a + n\frac{b-a}{n} = b.$$

(ii) Selecting a sample temperature in each time interval $T(t_i^*)$ with

$$t_{i-1} \le t_i^* \le t_i.$$

(iii) Compute an approximation

$$\text{Average Temperature } \approx \frac{T(t_1^*) + \cdots + T(t_n^*)}{n} = \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{1}{n}.$$

Since

$$\frac{1}{n} = \frac{\Delta t}{b-a},$$

we find

$$\begin{aligned}
\text{Average Temperature } &\approx \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{1}{n} \\
&= \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{\Delta t}{b-a} \\
&= \frac{T(t_1^*)\Delta t + \cdots + T(t_n^*)\Delta t}{b-a}.
\end{aligned}$$

Notice that the preceding procedure conforms to the definition of the Definite integral. Thus if the function $T(t)$ for the temperature is known, we could get an accurate value for the average temperature.

$$\begin{aligned}\text{Average Temperature} &= \lim_{n\to\infty}\left(\frac{T(t_1^*)\Delta t+\cdots+T(t_n^*)\Delta t}{b-a}\right)\\ &= \frac{\lim_{n\to\infty}\sum_{i=1}^n T(t_i^*)\Delta t}{b-a}\\ &= \frac{1}{b-a}\int_a^b T(t)\ dt.\end{aligned}$$

Definition (Average Value of a Function f)**.** If f is a continuous function on the interval $[a,b]$, then the average value of f on $[a,b]$ is defined to be

$$\boxed{\bar{f}\ or\ f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

□

Recall - The Mean Value Theorem for Derivatives. If g is continuous on $[a,b]$ and differentiable on (a,b), then there is some c in $[a,b]$ such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}.$$

Theorem (**Thm 5.5: The Mean Value Theorem for Integrals**)**.** If f is a continuous function on the interval $[a,b]$ and f_{ave} is the average value of f on $[a,b]$, then there is some c in $[a,b]$ such that

$$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

Proof. By the FTC,

$$g(x) = \int_a^x f(t)\ dt \implies g'(x) = f(x)$$

and by the Mean Value Theorem for Derivatives, there is some c in $[a,b]$ such that

$$f(c) = g'(c) = \frac{g(b)-g(a)}{b-a} = \frac{\int_a^b f(t)\ dt - \int_a^a f(t)\ dt)}{b-a} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

□

Example 1 Let $f(x) = \sqrt{x}$.

(a) Find f_{ave} the average value of f on the interval $[0, 4]$.

(b) Find c between 0 and 4 such that $f(c) = f_{ave}$.

(c) Sketch the graph of $y = f(x)$ and a rectangle that has the same area as the area under the curve.

Solution.

(a)
$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\ dx = \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\ dx = \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}.$$

(b) $f(c) = f_{ave} \iff \sqrt{c} = \frac{4}{3} \iff c = \frac{16}{9} \approx 1.75.$

(c) **Hint:** $\int_a^b f(x)dx = (b-a)f(c)$. □

Example 2 Let $f(x) = 2 + 6x - 3x^2$. Find the number b such that f_{ave} the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Solution.

$$3 = f_{ave} = \frac{1}{b-0}\int_0^b (2 + 6x - 3x^2)\ dx = \frac{1}{b}\Big[2x + 3x^2 - x^3\Big]_0^b$$
$$\iff 3b = 2b + 3b^2 - b^3 \iff b = 3b^2 - b^3 \iff 1 = 3b - b^2$$
$$\iff b^2 - 3b + 1 = 0 \iff b = \frac{3 \pm \sqrt{(-3)^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

We may select b to be either of the values given by $\frac{3\pm\sqrt{5}}{2}$. □

Example 3 Find the point(s) on the interval $(0, 1)$ at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.

Solution.

□