b. One challenge in using Theorem 8.2 is choosing an appropriate improper integral for comparison to the given improper integral. In this case, we see that for large values of x, $\frac{1}{\sqrt[3]{x^2 - 0.5}} \approx \frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{2/3}}$, which suggests that we compare $\int_1^\infty \frac{1}{\sqrt[3]{x^2 - 0.5}}\,dx$ to $\int_1^\infty \frac{1}{x^{2/3}}\,dx$. In Example 2 we showed that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges if $p \le 1$, so we expect that $\int_1^\infty \frac{1}{\sqrt[3]{x^2 - 0.5}}\,dx$ also diverges. Therefore, we let $f(x) = \sqrt[3]{x^2 - 0.5}$ and $g(x) = x^{2/3}$ in statement (2) of Theorem 8.2; our goal is to show that $f(x) \ge g(x)$ on $[1, \infty)$. Note that $\sqrt[3]{x^2 - 0.5} < \sqrt[3]{x^2} = x^{2/3}$ on $[1, \infty)$, which implies that

$$\frac{1}{\sqrt[3]{x^2 - 0.5}} > \frac{1}{x^{2/3}} \text{ on } [1, \infty).$$

By statement (2) of Theorem 8.2, $\int_1^\infty \frac{1}{\sqrt[3]{x^2 - 0.5}}\,dx$ diverges.

Related Exercises 77, 79 ◀

We close this section with two important points. First, for improper integrals of the form $\int_a^\infty g(x)\,dx$, where g satisfies the conditions of Theorem 8.2, one need not find a function f that satisfies the theorem's conditions on the entire interval $[a, \infty)$. Rather, it suffices to find a function f such that

$$f(x) \ge g(x) \text{ on } [c, \infty), \text{ where } c > a, \quad \text{and} \quad \int_c^\infty f(x)\,dx \text{ converges.}$$

These two conditions are enough to conclude that $\int_a^\infty g(x)\,dx$ converges because $g(x)$ is bounded on $[a, c]$ and therefore $\int_a^c g(x)\,dx$ is finite. For example, even though e^{-x} is not greater than or equal to e^{-x^2} on $[0, \infty)$ (Figure 8.32), we can use the result of Example 7a to conclude that $\int_0^\infty e^{-x^2}\,dx$ (with lower limit of 0) converges—adding the finite number $\int_0^1 e^{-x^2}\,dx$ to the convergent integral $\int_1^\infty e^{-x^2}\,dx$ does not affect whether it converges. Similar principles can be applied to the other forms of improper integrals encountered in this section.

The second point is that we never discovered the value of the integral in Example 7a. In future chapters, we develop methods that provide accurate approximations to the value of a convergent integral, and in Section 16.3, we show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

SECTION 8.9 EXERCISES

Getting Started

1. What are the two general ways in which an improper integral may occur?

2. Evaluate $\int_2^\infty \frac{dx}{x^3}$ after writing the expression as a limit.

3. Rewrite $\int_2^\infty \frac{dx}{x^{1/5}}$ as a limit and then show that the integral diverges.

4. Evaluate $\int_0^1 \frac{dx}{x^{1/5}}$ after writing the integral as a limit.

5. Write $\lim\limits_{a\to-\infty} \int_a^0 f(x)\,dx + \lim\limits_{b\to\infty} \int_0^b f(x)\,dx$ as an improper integral.

6. For what values of p does $\int_1^\infty x^{-p}\,dx$ converge?

Practice Exercises

7–58. Improper integrals *Evaluate the following integrals or state that they diverge.*

7. $\int_3^\infty \frac{dx}{x^2}$

8. $\int_2^\infty \frac{dx}{x}$

9. $\int_2^\infty \frac{dx}{\sqrt{x}}$

10. $\int_0^\infty e^{-2x}\,dx$

11. $\int_0^\infty e^{-ax}\,dx,\ a > 0$

12. $\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x}}$

13. $\int_0^\infty \cos x\,dx$

14. $\int_{-\infty}^{-1} \frac{dx}{x^3}$

15. $\int_{-\infty}^\infty \frac{dx}{x^2 + 100}$

16. $\int_{-\infty}^\infty \frac{dx}{x^2 + a^2},\ a > 0$

17. $\displaystyle\int_7^{\infty} \frac{dx}{(x+1)^{1/3}}$

18. $\displaystyle\int_2^{\infty} \frac{dx}{(x+2)^2}$

19. $\displaystyle\int_1^{\infty} \frac{3x^2+1}{x^3+x}\, dx$

20. $\displaystyle\int_1^{\infty} 2^{-x}\, dx$

21. $\displaystyle\int_2^{\infty} \frac{\cos(\pi/x)}{x^2}\, dx$

22. $\displaystyle\int_{-\infty}^{-2} \frac{1}{z^2} \sin\frac{\pi}{z}\, dz$

23. $\displaystyle\int_0^{\infty} \frac{e^u}{e^{2u}+1}\, du$

24. $\displaystyle\int_{-\infty}^{a} \sqrt{e^x}\, dx,\ a > 0$

25. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3x}}{1+e^{6x}}\, dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\, dx$

27. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\, dx$

28. $\displaystyle\int_1^{\infty} \frac{\tan^{-1} s}{s^2+1}\, ds$

29. $\displaystyle\int_{-\infty}^{\infty} \frac{(\tan^{-1} t)^2}{t^2+1}\, dt$

30. $\displaystyle\int_{-\infty}^{0} e^x\, dx$

31. $\displaystyle\int_1^{\infty} \frac{dv}{v(v+1)}$

32. $\displaystyle\int_1^{\infty} \frac{dx}{x^2(x-1)}$

33. $\displaystyle\int_2^{\infty} \frac{dy}{y \ln y}$

34. $\displaystyle\int_{-\infty}^{-4/\pi} \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx$

35. $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

36. $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x \ln^p x},\ p > 1$

37. $\displaystyle\int_0^{8} \frac{dx}{\sqrt[3]{x}}$

38. $\displaystyle\int_1^{2} \frac{dx}{\sqrt{x-1}}$

39. $\displaystyle\int_0^{\pi/2} \tan\theta\, d\theta$

40. $\displaystyle\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}}$

41. $\displaystyle\int_0^{\pi/2} \sec x \tan x\, dx$

42. $\displaystyle\int_3^{4} \frac{dz}{(z-3)^{3/2}}$

43. $\displaystyle\int_0^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

44. $\displaystyle\int_0^{\ln 3} \frac{e^y}{(e^y-1)^{2/3}}\, dy$

45. $\displaystyle\int_0^{1} \frac{x^3}{x^4-1}\, dx$

46. $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt[3]{x-1}}$

47. $\displaystyle\int_0^{10} \frac{dx}{\sqrt[4]{10-x}}$

48. $\displaystyle\int_1^{11} \frac{dx}{(x-3)^{2/3}}$

49. $\displaystyle\int_0^{2} \frac{dx}{(x-1)^2}$

50. $\displaystyle\int_0^{9} \frac{dx}{(x-1)^{1/3}}$

51. $\displaystyle\int_{-2}^{2} \frac{dp}{\sqrt{4-p^2}}$

52. $\displaystyle\int_0^{\infty} xe^{-x}\, dx$

53. $\displaystyle\int_0^{1} \ln x\, dx$

54. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$

55. $\displaystyle\int_0^{\ln 2} \frac{e^x}{\sqrt{e^{2x}-1}}\, dx$

56. $\displaystyle\int_0^{1} \frac{dx}{x+\sqrt{x}}\, dx$

57. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\, dx$

58. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

59. Perpetual annuity Imagine that today you deposit $\$B$ in a savings account that earns interest at a rate of $p\%$ per year compounded continuously (see Section 7.2). The goal is to draw an income of $\$I$ per year from the account forever. The amount of money that must be deposited is $B = I\int_0^{\infty} e^{-rt}\, dt$, where $r = p/100$. Suppose you find an account that earns 12% interest annually and you wish to have an income from the account of \$5000 per year. How much must you deposit today?

60. Draining a pool Water is drained from a swimming pool at a rate given by $R(t) = 100e^{-0.05t}$ gal/hr. If the drain is left open indefinitely, how much water drains from the pool?

61. Bioavailability When a drug is given intravenously, the concentration of the drug in the blood is $C_i(t) = 250e^{-0.08t}$, for $t \geq 0$. When the same drug is given orally, the concentration of the drug in the blood is $C_o(t) = 200(e^{-0.08t} - e^{-1.8t})$, for $t \geq 0$. Compute the bioavailability of the drug.

62. Electronic chips Suppose the probability that a particular computer chip fails after a hours of operation is $0.00005\int_a^{\infty} e^{-0.00005t}\, dt$.

a. Find the probability that the computer chip fails after 15,000 hr of operation.

b. Of the chips that are still operating after 15,000 hr, what fraction of these will operate for at least another 15,000 hr?

c. Evaluate $0.00005\int_0^{\infty} e^{-0.00005t}\, dt$ and interpret its meaning.

63. Average lifetime The average time until a computer chip fails (see Exercise 62) is $0.00005\int_0^{\infty} te^{-0.00005t}\, dt$. Find this value.

64. Maximum distance An object moves on a line with velocity $v(t) = \dfrac{10}{(t+1)^2}$ mi/hr, for $t \geq 0$, where t is measured in hours. What is the maximum distance the object can travel?

65–76. Volumes *Find the volume of the described solid of revolution or state that it does not exist.*

65. The region bounded by $f(x) = x^{-2}$ and the x-axis on the interval $[1, \infty)$ is revolved about the x-axis.

66. The region bounded by $f(x) = (x^2+1)^{-1/2}$ and the x-axis on the interval $[2, \infty)$ is revolved about the x-axis.

67. The region bounded by $f(x) = \sqrt{\dfrac{x+1}{x^3}}$ and the x-axis on the interval $[1, \infty)$ is revolved about the x-axis.

68. The region bounded by $f(x) = (x+1)^{-3}$ and the x-axis on the interval $[0, \infty)$ is revolved about the y-axis.

69. The region bounded by $f(x) = \dfrac{1}{\sqrt{x}\ln x}$ and the x-axis on the interval $[2, \infty)$ is revolved about the x-axis.

70. The region bounded by $f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the x-axis on the interval $[0, \infty)$ is revolved about the x-axis.

71. The region bounded by $f(x) = (x-1)^{-1/4}$ and the x-axis on the interval $(1, 2]$ is revolved about the x-axis.

72. The region bounded by $f(x) = (x+1)^{-3/2}$ and the x-axis on the interval $(-1, 1]$ is revolved about the line $y = -1$.

73. The region bounded by $f(x) = \tan x$ and the x-axis on the interval $[0, \pi/2)$ is revolved about the x-axis.

74. The region bounded by $f(x) = -\ln x$ and the x-axis on the interval $(0, 1]$ is revolved about the x-axis.

75. The region bounded by $f(x) = (4-x)^{-1/3}$ and the x-axis on the interval $[0, 4)$ is revolved about the y-axis.

76. The region bounded by $f(x) = (x^2-1)^{-1/4}$ and the x-axis on the interval $(1, 2]$ is revolved about the y-axis.

77–86. Comparison Test *Determine whether the following integrals converge or diverge.*

77. $\int_1^\infty \frac{dx}{x^3+1}$

78. $\int_0^\infty \frac{dx}{e^x+x+1}$

79. $\int_3^\infty \frac{dx}{\ln x}$ (*Hint:* $\ln x \le x$.)

80. $\int_2^\infty \frac{x^3}{x^4-x-1}\,dx$

81. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

82. $\int_1^\infty \frac{1}{e^x(1+x^2)}\,dx$

83. $\int_1^\infty \frac{2+\cos x}{\sqrt{x}}\,dx$

84. $\int_1^\infty \frac{2+\cos x}{x^2}\,dx$

85. $\int_0^1 \frac{dx}{\sqrt{x^{1/3}+x}}$

86. $\int_0^1 \frac{\sin x+1}{x^5}\,dx$

87. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. If f is continuous and $0 < f(x) < g(x)$ on the interval $[0, \infty)$, and $\int_0^\infty g(x)\,dx = M < \infty$, then $\int_0^\infty f(x)\,dx$ exists.
   b. If $\lim_{x\to\infty} f(x) = 1$, then $\int_0^\infty f(x)\,dx$ exists.
   c. If $\int_0^1 x^{-p}\,dx$ exists, then $\int_0^1 x^{-q}\,dx$ exists, where $q > p$.
   d. If $\int_1^\infty x^{-p}\,dx$ exists, then $\int_1^\infty x^{-q}\,dx$ exists, where $q > p$.
   e. $\int_1^\infty \frac{dx}{x^{3p+2}}$ exists, for $p > -\frac{1}{3}$.

88. **Incorrect calculation**
   a. What is wrong with this calculation?
   $$\int_{-1}^1 \frac{dx}{x} = \ln|x|\Big|_{-1}^1 = \ln 1 - \ln 1 = 0$$
   b. Evaluate $\int_{-1}^1 \frac{dx}{x}$ or show that the integral does not exist.

89. **Area between curves** Let R be the region bounded by the graphs of $y = e^{-ax}$ and $y = e^{-bx}$, for $x \ge 0$, where $a > b > 0$. Find the area of R in terms of a and b.

90. **Area between curves** Let R be the region bounded by the graphs of $y = x^{-p}$ and $y = x^{-q}$, for $x \ge 1$, where $q > p > 1$. Find the area of R in terms of p and q.

T 91. **Regions bounded by exponentials** Let $a > 0$ and let R be the region bounded by the graph of $y = e^{-ax}$ and the x-axis on the interval $[b, \infty)$.
   a. Find $A(a, b)$, the area of R as a function of a and b.
   b. Find the relationship $b = g(a)$ such that $A(a, b) = 2$.
   c. What is the minimum value of b (call it b^*) such that when $b > b^*$, $A(a, b) = 2$ for some value of $a > 0$?

Explorations and Challenges

92–93. Improper integrals with infinite intervals and unbounded integrands *For a real number a, suppose* $\lim_{x\to a^+} f(x) = -\infty$ *or* $\lim_{x\to a^+} f(x) = \infty$. *In these cases, the integral* $\int_a^\infty f(x)\,dx$ *is improper for two reasons:* ∞ *appears in the upper limit and f is unbounded at* $x = a$. *It can be shown that* $\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$, *for any* $c > a$. *Use this result to evaluate the following improper integrals.*

92. $\int_0^\infty x^{-x}(\ln x + 1)\,dx$

93. $\int_1^\infty \frac{dx}{x\sqrt{x-1}}$

94. **The family $f(x) = \frac{1}{x^p}$ revisited** Consider the family of functions $f(x) = \frac{1}{x^p}$, where p is a real number. For what values of p does the integral $\int_0^1 f(x)\,dx$ exist? What is its value?

T **95–98. Numerical methods** *Use numerical methods or a calculator to approximate the following integrals as closely as possible. The exact value of each integral is given.*

95. $\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx = -\frac{\pi \ln 2}{2}$

96. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

97. $\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$

98. $\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

99. **Decaying oscillations** Let $a > 0$ and b be real numbers. Use integration to confirm the following identities. (See Exercise 73 of Section 8.2)
   a. $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$
   b. $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

100. **The Eiffel Tower property** Let R be the region between the curves $y = e^{-cx}$ and $y = -e^{-cx}$ on the interval $[a, \infty)$, where $a \ge 0$ and $c > 0$. The center of mass of R is located at $(\bar{x}, 0)$, where $\bar{x} = \frac{\int_a^\infty xe^{-cx}\,dx}{\int_a^\infty e^{-cx}\,dx}$. (The profile of the Eiffel Tower is modeled by the two exponential curves; see the Guided Project *The exponential Eiffel Tower*.)
   a. For $a = 0$ and $c = 2$, sketch the curves that define R and find the center of mass of R. Indicate the location of the center of mass.
   b. With $a = 0$ and $c = 2$, find equations of the lines tangent to the curves at the points corresponding to $x = 0$.
   c. Show that the tangent lines intersect at the center of mass.
   d. Show that this same property holds for any $a \ge 0$ and any $c > 0$; that is, the tangent lines to the curves $y = \pm e^{-cx}$ at $x = a$ intersect at the center of mass of R.

   (*Source:* P. Weidman and I. Pinelis, *Comptes Rendu, Mechanique,* 332, 571–584, 2004)

101. **Many methods needed** Show that $\int_0^\infty \frac{\sqrt{x}\ln x}{(1+x)^2}\,dx = \pi$ in the following steps.
   a. Integrate by parts with $u = \sqrt{x}\ln x$.
   b. Change variables by letting $y = 1/x$.
   c. Show that $\int_0^1 \frac{\ln x}{\sqrt{x}(1+x)}\,dx = -\int_1^\infty \frac{\ln x}{\sqrt{x}(1+x)}\,dx$ (and that both integrals converge). Conclude that $\int_0^\infty \frac{\ln x}{\sqrt{x}(1+x)}\,dx = 0$.
   d. Evaluate the remaining integral using the change of variables $z = \sqrt{x}$.

   (*Source: Mathematics Magazine* 59, 1, Feb 1986)