Homework 5 Solution

Tuesday, October 12, 2021 10:57 AM

Recall.

Improper Integral of Type I ($f(x)$ continuous)

(a) $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

(b) $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

(c) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$

which is convergent $\Longleftrightarrow$ both converges

Improper integrals of Type II

($f(x)$ is unbounded at a point)

(a) $f(x)$ is unbounded at b: $\lim_{x\to b^-} f(x) = +\infty$ or $-\infty$

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

(b) $f(x)$ is unbounded at a: $\lim_{x\to a^+} f(x) = +\infty / -\infty$

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

(c) $f(x)$ is unbounded at $\underline{c} \in (a, b)$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

Note. $\int_a^b f(x)dx$ converges $\Longleftrightarrow$ both converges

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5. (1 point) `Library/Wiley/setAnton_Section_7.8/Anton_7_8_Q10.pg`

Evaluate the integrals that converge, enter 'DNC' if integral Does Not Converge.

$$\int_{-\infty}^{2} \frac{3}{x^2+49}\,dx = \_\_\_\_\_\_\_\_$$

Solution: *(Instructor solution preview: show the student solution after due date.)*

Sol. $\displaystyle \int \frac{3}{x^2+49}\,dx = \frac{3}{49}\int \frac{dx}{1+\left(\frac{x}{7}\right)^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

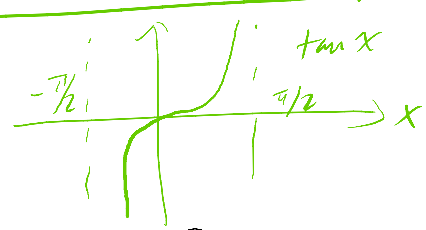
$u = \frac{x}{7}$, $dx = 7du$

$$\frac{3}{49}\cdot 7 \cdot \int \frac{du}{1+u^2}$$

$$= \frac{3}{7}\arctan\left(\frac{x}{7}\right) + C$$

$$\Rightarrow \boxed{\int_{-\infty}^{2} \frac{3}{x^2+49}dx = \lim_{t\to-\infty}\int_t^2 \frac{3}{x^2+49}\,dx}$$

$$= \frac{3}{7}\lim_{t\to-\infty} \arctan\left(\frac{x}{7}\right)\Big|_t^2$$



$$= \frac{3}{7}\left[\arctan\left(\frac{2}{7}\right) - \lim_{t\to-\infty}\tan^{-1}\left(\frac{t}{7}\right)\right]$$

$$= \frac{3}{7}\left[\arctan\left(\frac{2}{7}\right) + \frac{\pi}{2}\right]$$

6. (1 point) `Library/Wiley/setAnton_Section_7.8/Anton_7_8_Q18.pg`

Evaluate the integrals that converge, enter 'DNC' if integral Does Not Converge.

$\int_0^8 \frac{6}{\sqrt[3]{x}}\, dx =$ ________ $\frac{6}{\sqrt[3]{x}}$ undefined at 0

Solution: *(Instructor solution preview: show the student solution after due date.)*

sol. $\int \frac{6}{x^{\frac{1}{3}}}\, dx = 6\int x^{-\frac{1}{3}}\, dx = 6 \cdot \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$

$= 6 \cdot \frac{3}{2}\, x^{\frac{2}{3}} + C$

$= 9\, x^{\frac{2}{3}} + C$

The p-test for Improper Integrals of Unbounded Integrand

$$\boxed{\int_0^1 \frac{1}{x^p}\, dx = \begin{cases} \text{diverges, if } p \geq 1 \\ \text{converges, if } p < 1 \end{cases}}$$

$\Rightarrow \int_0^8 \frac{6}{\sqrt[3]{x}}\, dx = \lim_{t \to 0^+} \int_t^8 \frac{6}{\sqrt[3]{x}}\, dx$

$= 9 \lim_{t \to 0^+} x^{\frac{2}{3}} \Big|_t^8$

$= 9\left[8^{\frac{2}{3}} - \lim_{t \to 0^+} t^{\frac{2}{3}} \right] = 9 \cdot 8^{\frac{2}{3}}$

$= 9\left(8^{\frac{1}{3}}\right)^2 = 9 \times 4 = 36$

7. (1 point) `Library/Wiley/setAnton_Section_7.8/Anton_7_8_Q6.pg`

Evaluate the integrals that converge, enter 'DNC' if integral Does Not Converge.

$\int_0^{+\infty} xe^{-x^2}\, dx =$ ________

Solution: *(Instructor solution preview: show the student solution after due date.)*

$\int_0 \quad xe^{-x} \quad dx =$ ____________

Solution: (*Instructor solution preview: show the student solution after due date.*)

Sol. $\int x e^{-x^2} dx \;\underset{du = -2x\,dx}{\overset{u = -x^2}{=\!=}}\; \frac{-1}{2}\int e^u \, du$

$= \frac{-1}{2} e^{-x^2} + C$

$\Rightarrow \int_0^{+\infty} x e^{-x^2} dx = \lim_{t\to\infty} \int_0^t x e^{-x^2} dx$

$= -\frac{1}{2} \lim_{t\to\infty} e^{-x^2} \Big|_0^t$

$= -\frac{1}{2} \left[\lim_{t\to\infty} e^{-t^2} - 1 \right]$ $\qquad e^{-t^2} = \frac{1}{e^{t^2}}$

$= -\frac{1}{2}(0 - 1) = \frac{1}{2}$

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8. **(1 point)** Library/UCSB/Stewart5_7_8/Stewart5_7_8_32.pg

Consider the integral

$$\int_0^1 \frac{-4}{\sqrt{1-x^2}} dx$$

$\frac{-4}{\sqrt{1-x^2}}$ undefined at 1

Sol. $\int \frac{-4}{\sqrt{1-x^2}} dx = -4 \arcsin(x) + C$

$\Rightarrow \int_0^1 \frac{-4}{\sqrt{1-x^2}} dx = \lim_{t\to 1^-} \int_0^t \frac{-4\,dx}{\sqrt{1-x^2}}$

$$= -4\left[\lim_{t\to 1^-} \arcsin x \Big|_0^t\right]$$

$$= -4\left[\lim_{t\to 1^-} \arcsin t - 0\right]$$

$$= -4\left(\frac{\pi}{2} - 0\right) = -2\pi$$

9. **(1 point)** `Library/Michigan/Chap7Sec7/Q15.pg`

Calculate the integral, if it converges. If it diverges, enter **diverges** for your answer.

$\int_{-4}^{4} \frac{1}{v}\, dv =$ ______________ $\frac{1}{v}$ discontinous at 0

Solution: *(Instructor solution preview: show the student solution after due date.)*

$$\int_{-4}^{4} \frac{1}{v}\, dv \neq \ln|v| \Big|_{-4}^{4} \quad \times$$

$$\int_{-4}^{4} \frac{1}{v} dv = \int_{-4}^{0} \frac{1}{v} dv + \int_{0}^{4} \frac{1}{v} dv$$

both diverge by the p-test

$$\Rightarrow \int_{-4}^{4} \frac{1}{v} dv \text{ diverges}$$

10. **(1 point)** `Library/Michigan/Chap7Sec7/Q19.pg`

Calculate the integral, if it converges. If it diverges, enter **diverges** for your answer.

$\int_0^4 \frac{1}{u^2-16}\, du =$ ______________

Solution: *(Instructor solution preview: show the student solution after due date.)*

Sol. $\displaystyle \int \frac{du}{u^2-16} = \int \frac{du}{(u-4)(u+4)}$

$$\boxed{\frac{1}{u^2-16} = \frac{A}{u-4} + \frac{B}{u+4} = \frac{A(u+4) + B(u-4)}{u^2-16}}$$

$$= \frac{1}{8}\int \frac{du}{u-4} - \frac{1}{8}\int \frac{1}{u+4}\,du$$

$$= \frac{A(u+4)+B(u-4)}{u^2-16}$$

$\Rightarrow$ let $u=4$ $\left\{ 1 = 8A \right.$

let $u=-4$ $\left. 1 = -8B \right.$

$\Rightarrow A = \frac{1}{8}, \quad B = \frac{-1}{8}$

$$= \frac{1}{8}\ln|u-4| - \frac{1}{8}\ln|u+4| + C$$

$$\Rightarrow \int_0^4 \frac{1}{u^2-16}\,du = \frac{1}{8}\int_0^4 \frac{du}{u-4} - \frac{1}{8}\int_0^4 \frac{du}{u+4}$$

$$= \frac{1}{8}\lim_{t\to 4^-}\int_0^t \frac{du}{u-4} - \frac{1}{8}\ln|u+4|\Big|_0^4$$

$V = u-4$

$$\int_{-4}^{0} \frac{dv}{v}$$

diverges by p-test

$$= \frac{1}{8}\left[\lim_{t\to 4^-}\ln|t-4| - \ln 4\right] - \frac{1}{8}\ln|u+4|\Big|_0^4$$

$|t-4| \to 0^+$ as $t \to 4^-$

$$= \frac{1}{8}(-\infty - \ln 4) - \frac{1}{8}\ln|u+4|\Big|_0^4$$

so $\int_0^4 \frac{1}{u^2-16}\,du$ diverges

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11. **(1 point)** Library/WHFreeman/Rogawski_Calculus_Early_Transcendentals_Second_ _Integration/7.6_Improper_Integrals/7.6.25.pg

Determine if the improper integral converges and, if so, evaluate it.

$\int_0^\infty \frac{dx}{7+x} =$ __________

Write F if the integral doesn't converge.

$$\int_0^\infty \frac{dx}{7+x} \quad \underline{\underline{u = 7+x}} \quad \int_7^\infty \frac{du}{u}$$ diverges by the p-test

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12. **(1 point)** Library/Utah/Quantitative_Analysis/set12_Definite_Integrals_Techniques_of_Integration/s1p12.pg

12. **(1 point)** Library/Utah/Quantitative_Analysis/set12_Definite_Integrals_Techniques_of_Integration/s1p12.pg

Find what value of c does

$\int_7^\infty \frac{c}{x^3} dx = 1$?

Answer: ______________________

Correct Answers:

$$\int \frac{1}{x^3} dx = \int x^{-3} dx$$

$$= \frac{x^{-2}}{-2} + C$$

$$= -\frac{1}{2} \cdot \frac{1}{x^2}$$

$$\Rightarrow \int_7^\infty \frac{c}{x^3} dx$$

$$= \lim_{t \to \infty} \int_7^t \frac{c}{x^3} dx = \frac{-c}{2} \lim_{t \to \infty} \frac{1}{x^2} \Big|_7^t$$

$$= \frac{-c}{2} \left[\lim_{t \to \infty} \frac{1}{t^2} - \frac{1}{49} \right]$$

$$= \frac{-c}{2} \left(0 - \frac{1}{49} \right) = \frac{c}{98} = 1$$

$$\Rightarrow c = 98$$

3. **(1 point)** Library/UMN/calculusStewartET/s_7_1_34.pg

First make a substitution and then use integration by parts to evaluate the integral

$$\int 4x^3 e^{-x^2} dx.$$

Answer: ______________________

Correct Answers:

- (-2)*e^(-x^2)*(x^2+1)+C

$$u = -x^2$$

4. **(1 point)** `Library/UVA-Stew5e/setUVA-Stew5e-C05S05-Substitution/5-5-41.pg`

Evaluate the indefinite integral.

$$\int \frac{5x+3}{x^2+1}dx$$

$$\int \frac{5x\,dx}{x^2+1} + \int \frac{3}{x^2+1}\,dx$$

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2. **(1 point)** `Library/Wiley/setAnton_Section_7.1/Anton_7_1_Q23.pg`

Evaluate the integral by any method.

$\int \frac{e^{-x}}{9-e^{-2x}}\,dx =$ __________ $+C$

Solution: (*Instructor solution preview: show the student solution*

SOLUTION

For $u = e^{-x}$ we have $-du = e^{-x}dx$ hence using partial fractions

$$\int \frac{e^{-x}}{9-e^{-2x}}\,dx = -\int \frac{1}{9-u^2}\,du = \frac{1}{6}\ln\left(\left|\frac{u-3}{u+3}\right|\right) + C =$$

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$(9-u)(9+u)$