Contents

Chapter 10, for the most part, is dedicated to the study of convergence and divergence of infinite series. Since this involves the study of the convergence and divergence of the sequence of **partial sums**. We begin with the study of sequences.

10.2 §10.2 Sequences

Definition. A **sequence** can be thought of as a list of numbers written in a definite order

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

The a_1 is called the first **term**, a_2 is called the second term, etc. and in general a_n is called the nth term.

The subscript n in a_n is called an **index**, and it indicates the order of terms in the sequence. The choice of a starting index is arbitrary, but sequences usually begin with $n = 0$ or $n = 1$.

We are dealing with infinite sequence so that each a_n has a successor a_{n+1}.

Examples.

A sequence may be defined with an explicit formula of the form $a_n = f(n)$, for $n = 1, 2, 3, \dots$. See (a), (b) and (c).

(a)

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \underbrace{\frac{n}{n+1}}_{n^{th} \text{ term}}, \ldots\right\}$$

(b) Although a sequence is a list of numbers, **the sequence does not have to start at** $n = 1$.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} = \left\{0, 1, \sqrt{2}, \sqrt{3} \ldots, \sqrt{n-3}, \ldots\right\}$$

(c)

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} = \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \underbrace{\cos\frac{n\pi}{6}}_{n^{th} \text{ term}}, \ldots\right\}$$

(d) Sometimes sequences are defined **recursively**. For example, $a_{n+1} = f(a_n)$.

Fibonacci sequence is one such example

$$f_1 = 1 \text{ and } f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2},\ n \geq 3$$

$$\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

To generate a term in this sequence, we need to know the two terms just before it. What number comes after 21 in the Fibonacci sequence?

Definition (**Limit** of a Sequence)**.** A sequence $\{a_n\}$ has a real number L as a limit, written as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

provided the limit exists. In this case, we say that $\{a_n\}$ **converges or is convergent.** Otherwise, we say that the sequence sequence **diverges** or is **divergent**.

Remark.(Rigorous definition of limit) $\lim_{n\to\infty} a_n = L \iff$ for every positive number ϵ, there is a corresponding N_ϵ such that $|L - a_n| < \epsilon$ for all $n \geq N_\epsilon$. □

Example. The sequence $\{a_n = (-1)^n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$ is divergent, since the terms of the sequence oscillate between 1 and -1. So the limit $\lim_{n\to\infty}(-1)^n$ does not exist.

The following theorem is obvious.

Theorem. If $\{a_n\}$ is a convergent sequence and $\lim\limits_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}.$$

Proof. $\lim\limits_{n\to\infty} a_n = L \iff$ the values of the sequence get closer and closer to L as n get larger and larger $\iff \lim_{n\to\infty} a_{n+1} = L$. $\square$

The following result allows us for example to use **L'Hospital's Rule** if helpful.

Theorem (Thm 10.1 Function Value Theorem) If one can associate a function $f(x)$ with the sequence $\{a_n\}$ in the following sense

$$a_n = f(n),$$

then

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L.$$

$\square$

Remark. The sequence above $a_n = f(n)$ is a function of n. It is not continuous.

Example. If $r > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \implies \lim_{n\to\infty} \frac{1}{n^r} = 0$$

since $\frac{1}{n^r} = f(n)$, where $f(x) = \frac{1}{x^r}$.

Example. Find

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

Solution. Note if we define $f(x) = \frac{\ln x}{x}$, then $f(n) = \frac{\ln n}{n}$ and so we need only find L where $\lim\limits_{x\to\infty} f(x) = L$ and use the Function Value Theorem. We do this using L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\therefore \quad \lim_{x\to\infty} \frac{\ln n}{n} = 0.$$

Recall the following three typical limits in Calculus I.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &=1\\ \lim_{x\to\infty} x^{\frac{1}{x}} &=1\\ \lim_{x\to\infty}(1+\frac{1}{x})^x &=e.\end{aligned}$$

Because of the correspondence between limits of sequences and limits of functions at infinity, we have the following properties that are analogous to those of functions in Chapter 2.

Limit Laws for Sequences: Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences and c be any constant. Then

$$\begin{aligned}&\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n\\ &\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} c = c\\ &\lim_{n\to\infty}(a_nb_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)\\ &\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0\\ &\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } p>0 \text{ and } a_n>0.\end{aligned}$$

□

Example 1. Determine whether the sequence is convergent or divergent. Find the limit if it is convergent.

a. $a_n = \dfrac{3n^3}{n^3+1}$

Solution. We set up a function to use the Function Value Theorem. Set $f(x) = \frac{3x^3}{x^3+1}$ and we can use L'hospital's Rule to find $\lim_{x\to\infty}\frac{3x^3}{x^3+1}$

$$\lim_{x\to\infty}\frac{3x^3}{x^3+1} = \lim_{x\to\infty}\frac{9x^2}{3x^2} = \lim_{x\to\infty}\frac{9}{3} = \lim_{x\to\infty} 3 = 3 \implies \lim_{x\to\infty}\frac{3n^3}{n^3+1} = \lim_{x\to\infty} a_n = 3.$$

□

b. $b_n = \left(\dfrac{n+5}{n}\right)^n$

Solution.

$\square$

c. $c_n = n^{1/n}$

Solution.

$\square$

Terminology for Sequences

Definition. • A sequence $\{a_n\}$ is called **increasing** if $a_k < a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 < a_2 < a_3 < \cdots$$

• A sequence $\{a_n\}$ is called **nondecreasing** if $a_k \leq a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 \leq a_2 \leq a_3 \leq \cdots$$

• A sequence $\{a_n\}$ is called **decreasing** if $a_k > a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 > a_2 > a_3 > \cdots$$

• A sequence $\{a_n\}$ is called **nonincreasing** if $a_k \geq a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 \geq a_2 \geq a_3 \geq \cdots$$

• A sequence $\{a_n\}$ is called **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction).

Example. Show that the sequence $\{\frac{n}{n^2+1}\}$ is decreasing.

Solution.

Method I. Set $f(x) = \frac{x}{x^2+1}$. It suffices to show that $f(x)$ is decreasing.

$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0 \text{ for all } x > 1.$$

Method II.

$$\begin{aligned}\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} &\iff n[(n+1)^2+1] > (n+1)(n^2+1)\\ &\iff n^3+2n^2+2n > n^3+n^2+n+1\\ &\iff n^2+n > 1\end{aligned}$$

The last inequality is true for all $n \geq 1$ and so the first inequality is true for all $n \geq 1$. Therefore, $\{a_n = \frac{n}{n^2+1}\}$ is decreasing.

Definition. • A sequence $\{a_n\}$ is **bounded above** if there is some number M such that

$$a_n \leq M \qquad \text{for all } n.$$

• A sequence $\{a_n\}$ is **bounded below** if there is some number m such that

$$m \leq a_n \qquad \text{for all } n.$$

• If the sequence $\{a_n\}$ is both bounded above and bounded below, then the sequence $\{a_n\}$ is said to be **bounded**.

Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

□

Example 84 (Reading) Suppose (show) that the sequence given recursively by

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{2}a_n + 2 \quad \text{for } n = 0, 1, 2, 3, \ldots$$

is monotonic and bounded. Find $\lim_{n\to\infty} a_n$.

Theorem (The Squeeze Theorem). If $\{a_n\}$ and $\{c_n\}$ are two convergent sequences with the same limit, $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, and $a_n \le b_n \le c_n$ for $n > n_0$, then

$$\lim_{n\to\infty} b_n = L.$$

$\square$

Example 4. Find the limit of the sequence $a_n = \dfrac{\cos n}{n^2+1}$.

Solution.

$\square$

$\boxed{\textbf{Theorem.} \quad \text{If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0.}$

Remark. This Theorem does NOT say that if $\{|a_n|\}$ is convergent, then $\{a_n\}$ is convergent. $\square$

Example. Determine whether the following sequence is convergent or divergent.

$$a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}.$$

Solution. Notice

$$|a_n| = \left|\frac{(-1)^n n^3}{n^3+2n^2+1}\right| = \frac{n^3}{n^3+2n^2+1}$$

and we can use repeated applications of L'Hospital's Rule

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3}{n^3+2n^2+1} = \lim_{n\to\infty} \frac{3n^2}{3n^2+4n} = \lim_{n\to\infty} \frac{6n}{6n+4} = \lim_{n\to\infty} \frac{6}{6} = 1.$$

Therefore, $\{|a_n|\}$ is convergent. But

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n}((2n)^3}{(2n)^3+2(2n)^2+1} = 1,$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+1}((2n+1)^3}{(2n+1)^3+2(2n+1)^2+1}$$

$$= \lim_{n\to\infty} (-1)\frac{((2n+1)^3}{(2n+1)^3+2(2n+1)^2+1} = -1.$$

Since the even terms converge to 1 and the odd terms converge to -1, there is no limit. Therefore, $\{a_n\}$ is divergent. That is, **absolute convergence does not mean convergence**. □

Example. Determine wether the following sequence is convergent or divergent.

$$a_n = \frac{\sin 2n}{1+\sqrt{n}}.$$

Solution. We use The Squeeze Theorem to show the convergence of the absolute sequence $\{|a_n|\}$ and then use the above Theorem to get the convergence of $\{a_n\}$.

$$0 \le \left|\frac{\sin 2n}{1+\sqrt{n}}\right| \le \frac{1}{1+\sqrt{n}} \text{ and } \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$$

$$\therefore \quad \{b_n = \left|\frac{\sin 2n}{1+\sqrt{n}}\right|\} \text{ is convergence with limit} = 0 \text{ by Squeeze Thm.}$$

$$\therefore \quad \{a_n = \frac{\sin 2n}{1+\sqrt{n}}\} \text{ is convergent to 0.}$$

□

Theorem - Continuity and Limits: If $\lim\limits_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L) = f(\lim\limits_{n\to\infty} a_n)$.

Example. Determine whether the following sequence is convergent or divergent.

$$a_n = (2^{3n+1})^{\frac{1}{n}}.$$

Solution.

$$a_n = (2^{3n+1})^{\frac{1}{n}} = (2^{3n}2)^{\frac{1}{n}} = (2^{3n})^{\frac{1}{n}}(2)^{\frac{1}{n}} = 8(2)^{\frac{1}{n}}$$

Since 2^x is continuous at $x = 0$, we see that

$$\lim_{n\to\infty} a_n = 8(2)^{\lim_{n\to\infty}\frac{1}{n}} = 8 \cdot 2^0 = 8.$$

Example. Determine the limit of $\{a_n = \cos\frac{\pi}{n}\}$.

Solution. Recall that $f(x) = \cos x$ is continuous on the real line. Also, notice that

$$\lim_{n\to\infty} \frac{\pi}{n} = 0 = L \qquad \therefore \lim_{n\to\infty} \cos\frac{\pi}{n} = \lim_{n\to\infty} f(\frac{\pi}{n}) = f(L) = \cos 0 = 1.$$

□

Geometric Sequences

Theorem.(**Geometric Sequences**) The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ ($\{r^n\}$ is monotonic if $r > 0$ and oscillates if $r < 0$) and divergent for all other values of r. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Example 3. Determine the limits of the following sequences.

a. $a_n = 5(0.6)^n - \dfrac{1}{3^n}$ **b.** $b_n = \dfrac{2n^2 + n}{2^n(3n^2 - 4)}$

Solution.

□

Growth Rates of Sequences

The results in Chapter 4 (section 4.7 L'Hôpital's Rule) about the relative growth rates of functions are applied to sequences.

Theorem (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$;

$$\{\ln^q n\} << \{n^p\} << \{n^p \ln^r n\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

Example 6. Compare growth rates of sequences to determine whether the following sequences converge.

a. $\left\{\dfrac{\ln n^{10}}{0.00001n}\right\}$ **b.** $\left\{\dfrac{n^8 \ln n}{n^{8.001}}\right\}$ **c.** $\left\{\dfrac{n!}{10^n}\right\}$

Solution.

$\square$

10.3 §10.3 Infinite Series

An infinite series (or simply a series) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = \underbrace{\sum a_n}_{\text{short form}}.$$

We are trying to **add all the terms in a sequence** $\{a_n\}$. Naturally, when one is adding infinitely many numbers one ask what exactly does this mean.

$$1 + 2 + 3 + \cdots + n + \cdots > \text{ any finite number?}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots \text{ Is this equal to a finite number?}$$

These are the type of questions we will be asking. To answer these questions we need to develop some mathematical tools.

Definition (Infinite Series)**.** Given a series $\sum a_n$, we define the N^{th} **partial sum** to be

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N, \quad N = 1, 2, \ldots.$$

The series $\sum a_n$ is said to be **convergent** provided the *sequence of partial sums* $\{S_N\}$ converges and in this case, the value of the series is

$$\sum_{i=1}^{\infty} a_n = L \iff \lim_{N\to\infty} S_N = L.$$

The series is said to be **divergent** if it is not convergent, i.e. $\{S_N\}$ is divergent.

Theorem - Geometric Series Test. A Geometric series is one that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \text{ with } a \neq 0.$$

This series is convergent if $|r| < 1$ with

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ when } |r| < 1$$

and it is divergent if $|r| \geq 1$. (Notice that we must write the series in the form given in order to find the sum.)

Proof (Reading). Let S_N denote the N^{th} partial sum.

$$S_N = \sum_{n=1}^{N} ar^{n-1}$$

We need to study $\lim_{n\to\infty} S_N$. Certainly, if $r = 1$, then $S_N = \sum_{i=1}^{N} ar^{n-1} = \sum_{i=1}^{N} a = Na$ and $\lim_{n\to\infty} S_N = \pm\infty$. Now, we assume that $r \neq 1$ Then

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{N-1} \\
rS_N &= r\sum_{n=1}^{N} ar^{n-1} = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N
\end{aligned}$$

$$\begin{aligned}
S_N - rS_N &= (a + ar + ar^2 + ar^3 + \cdots + ar^{N-1}) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N) \\
&= (a - ar^N)
\end{aligned}$$

$$(1-r)S_N = (a - ar^N) \implies S_N = \frac{a - ar^N}{1-r} = \frac{a(1-r^N)}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

We are using the **Theorem.** The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Example 1. Evaluate the following geometric series or state that the series diverges.

a. $\sum_{k=0}^{\infty} 1.1^k$ **b.** $\sum_{k=0}^{\infty} e^{-k}$ **c.** $\sum_{k=2}^{\infty} 3(-0.75)^k$

Solution.

$\square$

Example 2. Write $1.0\overline{35} = 1.0353535\cdots$ as a geometric series and express its value as a fraction.

Solution.

$\square$

More examples.

Example (Reading). Determine if the geometric series

$$\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$$

is convergent or divergent and if convergent find its sum.

Solution. First, we write in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 10 \frac{10^{n-1}}{(-9)^{n-1}} = \sum_{n=1}^{\infty} (10) \left(\underbrace{\frac{-10}{9}}_{=r} \right)^{n-1}$$

so that we can apply the Geometric Series Test. $r = \frac{-10}{9}$ has absolute value greater than 1 and so the series is divergent. □

Example (Reading). Determine if the geometric series

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$$

is convergent or divergent and if convergent find its sum.

Solution. First, we write in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left[\frac{1}{4} \cdot \frac{(-3)^{n-1}}{4^{n-1}} \right] = \sum_{i=1}^{\infty} \left[\frac{1}{4} \left(-\frac{3}{4} \right)^{n-1} \right]$$

so that we can apply the Geometric Series Test. Since $r = \frac{-3}{4}$ has absolute value less than 1 this series converges and

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{a}{1-r} = \frac{\frac{1}{4}}{1 - \frac{-3}{4}} = \frac{1}{4} \frac{1}{\frac{4+3}{4}} = \frac{1}{7}.$$

□

Telescoping Series

With geometric series, we carried out the entire evaluation process by finding a formula for the sequence of partial sums and evaluating the limit of the sequence. Not many infinite series can be subjected to this sort of analysis. With another class of series, called **telescoping series**, it can also be done. Here are some examples.

Example 3. Evaluate the following series.

a. $\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}-\cos\frac{1}{k+1}\right)$ **b.** $\sum_{k=3}^{\infty}\frac{1}{(k-2)(k-1)}$

Solution.

$\square$

Theorem (Limit Laws for Series). Let $\sum a_n$ and $\sum b_n$ be two series.

(i) If $\sum a_n$ is convergent, then $\sum ca_n = c\sum a_n$;

(ii) If $\sum a_n$ and $\sum b_n$ are convergent, $\sum(a_n \pm b_n)$ is convergent and $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$.

(iii) If $\sum a_n$ is convergent and $\sum b_n$ is divergent $\Rightarrow \sum(a_n + b_n)$ is divergent.

(iv) If both $\sum(a_n + b_n)$ and $\sum a_n$ are convergent, then so is $\sum b_n$.

These follow immediately from the corresponding laws for sequence. □

Remark 1. If $\sum a_n$ is divergent, $\sum ca_n$ is divergent for any $c \neq 0$.

Remark 2. If M is a positive integer, then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=M}^{\infty} a_n$ either both converge or both diverge.

Remark 3. $\sum a_n$ and $\sum b_n$ are both divergent $\not\Rightarrow \sum(a_n + b_n)$ is divergent.

Example 4. Evaluate the following series.

a. $\sum\limits_{k=2}^{\infty}\left(\dfrac{1}{3^k} - \dfrac{2}{3^{k+2}}\right)$ **b.** $\sum\limits_{k=1}^{\infty}\left(5(\dfrac{2}{3})^k - \frac{7^{k-1}}{6^k}\right)$

Solution.

□

Example. Determine if the series

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n}$$

is convergent or divergent and if convergent find its sum.

Solution. We first recognize this as being the sum of two geometric series. One of which is convergent and the other is divergent.

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1+3^{n-1}}{2^{n-1}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}}}_{\text{convergent}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{3^{n-1}}{2^{n-1}}}_{\text{divergent}}$$

$$\therefore \qquad \sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n} \text{ is divergent.}$$

□

Theorem - Harmonic Series Diverges The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} \cdots + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \underbrace{\frac{1}{16} \cdots + \frac{1}{32}}_{>\frac{1}{2}} + \cdots$$

tells us that we are adding infinitely many terms all greater than $\frac{1}{2}$.

Example. Determine if the series

$$\sum_{n=1}^{\infty}\left[\frac{3}{5^n}+\frac{2}{n}\right]$$

is convergent or divergent and if convergent find its sum.

Solution. We first notice recognize this series as the sum of a convergent and a divergent series

$$\sum_{n=1}^{\infty}\left[\frac{3}{5^n}+\frac{2}{n}\right]=\underbrace{\sum_{n=1}^{\infty}\left[\frac{3}{5}\cdot\frac{1}{5^{n-1}}\right]}_{\text{convergent}}+2\underbrace{\sum_{n=1}^{\infty}\left[\frac{1}{n}\right]}_{\text{divergent}}$$

$$\therefore \qquad \sum_{n=1}^{\infty}\left[\frac{3}{5^n}+\frac{2}{n}\right] \text{ is divergent.}$$

$\square$

Example. Be careful. Recall the convergent telescoping series example done above.

$$\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}}_{\text{convergent}}=\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right]=\underbrace{\sum_{n=1}^{\infty}\left[\frac{1}{n}\right]}_{\text{divergent}}+\underbrace{\sum_{n=1}^{\infty}\left[\frac{-1}{n+1}\right]}_{\text{divergent}}$$

$\square$

Remark. This is what we mentioned before, the sum/difference of two divergent series may not be divergent.

10.4 §10.4 The Divergence and Integral Tests

The Divergence Test

Theorem - n^{th}- Term Test for Divergence If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.

- If $\lim\limits_{n\to\infty} a_n$ either does not exist or does not equal 0, then the series

$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Proof. We prove the contrapositive. That is, we assume that the series converges to value S and show that $\sum_{n=1}^{\infty} a_n$ exits and equals 0.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \Big[a_1 + \cdots + a_n - (a_1 + \cdots + a_{n-1})\Big] \\
&= \lim_{n\to\infty} \Big[\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\Big] = \lim_{n\to\infty} \Big[S_n - S_{n-1}\Big] \\
&= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.
\end{aligned}$$

Remark. This is a **test for divergence only** and

$$\lim_{n\to\infty} a_n = 0 \text{ does \textbf{not} imply that } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

As can be seen from the Harmonic Series:

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

□

Example. Show that

$$\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$$

is divergent but $\lim_{n\to\infty} \ln(1+\frac{1}{n}) = 0$

Solution. The function $\ln x$ is continuous at $x = 1$, and $\lim_{n\to\infty} 1 + \frac{1}{n} = 1$ so by Continuity and Limits, we have

$$\lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln(\lim_{n\to\infty}(1+\frac{1}{n}))) = \ln 1 = 0$$

Finally, we have a **telescoping divergent series** as can be seen from:

$$\begin{aligned}
\ln(1+\frac{1}{n}) &= \ln(\frac{n+1}{n}) = \ln(n+1) - \ln n \\
S_N &= \sum_{n=1}^{N} \ln(1+\frac{1}{n}) = \sum_{n=1}^{N} \Big[\ln(n+1) - \ln n\Big] \\
&= \Big[\ln(2) - \ln 1\Big] + \Big[\ln(3) - \ln 2\Big] + \Big[\ln(4) - \ln 3\Big] \\
&\quad + \cdots + \Big[\ln(N) - \ln(N-1)\Big] + \Big[\ln(N+1) - \ln N\Big] \\
&= \ln(N+1)
\end{aligned}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(N+1) = \infty \therefore \text{ The series } \textstyle\sum_{n=1}^{\infty} \ln(1+\frac{1}{n}) \text{ is divergent.}$$

□

Example 1. Determine whether the following series diverge or state that the Divergence Test is inconclusive

a. $\sum_{k=0}^{\infty} \frac{k}{k+1}$ **b.** $\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$ **c.** $\sum_{k=1}^{\infty} \frac{1}{k}$ **d.** $\sum_{k=1}^{\infty} \frac{1}{k^2}$

Solution.

□

The Integral Test

In this subsection we relate the convergence of series with the convergence of improper integrals. The p-Test for improper integrals is useful here.

Recall: The p-Test for Improper Integrals.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \le 1 \\ \text{converges, if } p > 1 \end{cases}$$

Theorem (Integral Test). If the tail of a sequence $\{a_n\}$ can be given by the values of a function $f(x)$ which is continuous, positive, and decreasing on $[a, \infty)$ then

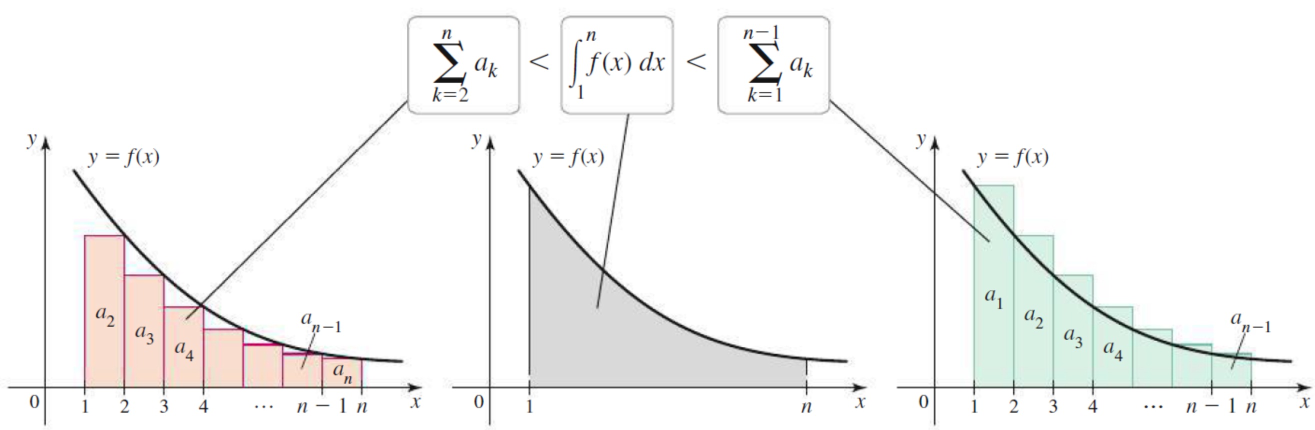
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_a^\infty f(x)\,dx \text{ converges.}$$

That is:

(i) If $\sum_{n=1}^\infty a_n$ is convergent , then $\int_a^\infty f(x)\,dx$ is convergent.

(ii) If $\int_a^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ is convergent.

$$\sum_{k=2}^{n} a_k < \int_1^n f(x)\,dx < \sum_{k=1}^{n-1} a_k.$$



We explain why this is true using the following Example.

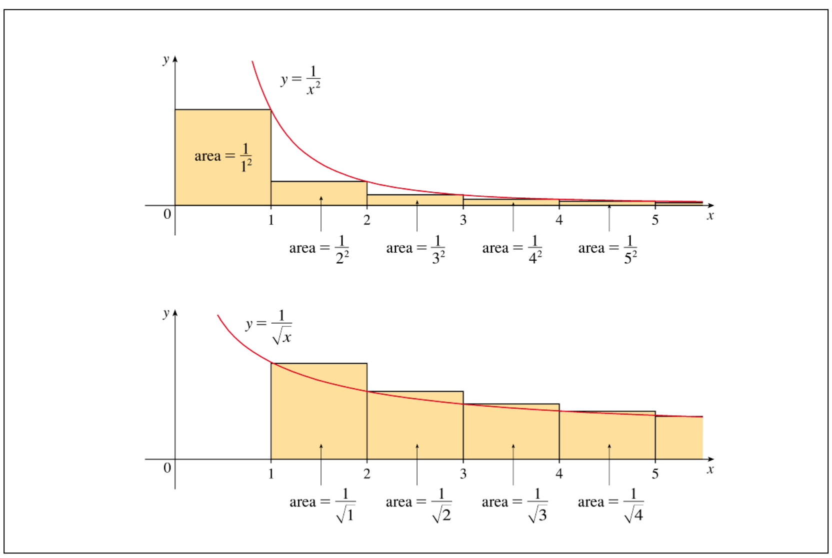
Example. Let us consider two series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and the corresponding functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{\sqrt{x}}$. Both of $f(x)$ and $g(x)$ are continuous, positive, and decreasing on $[1, \infty)$. Moreover, by the p-Test for Improper Integrals we have

$$\int_1^{\infty} \frac{1}{x^2}\, dx \text{ is convergent, } (p = 2 > 1) \text{ while}$$
$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx \text{ is divergent } (p = \tfrac{1}{2} < 1).$$

Therefore, by the Integral Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent, but}$$
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is divergent.}$$

To see what is going on here, we graph $y = f(x)$ and $y = g(x)$ and interpret the sequence on the graph of its corresponding function.



Comparing series with integrals

Example 2. (Applying the Integral Test) Determine whether the following series converge.

a. $\sum\limits_{k=1}^{\infty} \frac{k}{k^2+1}$ **b.** $\sum\limits_{k=3}^{\infty} \frac{1}{\sqrt{2k-5}}$ **c.** $\sum\limits_{n=1}^{\infty} \frac{1}{k^2+4}$

Solution.

$\square$

In general, the Integral Test combined with The p-Test for Improper Integrals give us:

Theorem (p **-Test for Series).** The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

$\square$

Example 3. (Using the p-series test) Determine whether the following series converge or diverge.

a. $\sum\limits_{k=1}^{\infty} k^{-3}$ **b.** $\sum\limits_{k=3}^{\infty} \frac{1}{\sqrt[4]{k^3}}$ **c.** $\sum\limits_{k=4}^{\infty} \frac{1}{(k-1)^2}$

Solution.

□

Example. Determine whether the series

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$$

is convergent or divergent.

Solution.

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \quad \text{and} \quad p = \frac{3}{2} \implies \text{convergence.}$$

□

Example. (Reading) Determine the values of p for which the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$$

is converges.

Solution. Set $u = \ln(\ln x)$ and $f(u) = u^{-p}$. Then $f(x) > 0$ provided $u > 0$ which is true when $x \geq 3$. Moreover,

$$f'(u) = (-p)u^{-p-1} < 0 \text{ when } x \geq 3.$$

Thus, $f(u)$ is continuous, positive and decreasing when $x \geq 3$. Further, $du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$.

$$\int_3^{\infty} \frac{1}{x \ln x [\ln(\ln x)]^p}\, dx = \int_{\ln(\ln 3)}^{\infty} \frac{1}{u^p}\, du \text{ convergent} \iff p > 1$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p} \text{ is convergent} \iff p > 1$$

□

Estimating the Value of Infinite Series

This sub-section is left for your own reading.

The Integral Test is powerful in its own right, but it comes with an added bonus. It can be used to estimate the value of a convergent series with positive terms. We define the **remainder** to be the error in approximating a convergent series by the sum of its first n terms; that is,

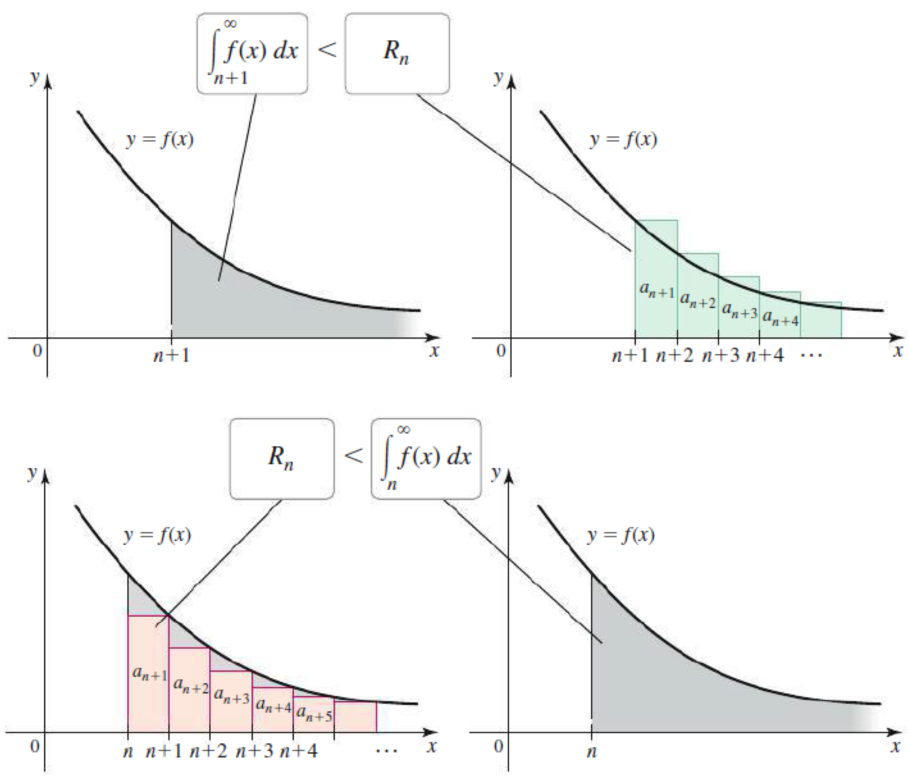
$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Theorem (Thm 10.13 Remainder Estimate for the Integral Test). Suppose that $f(k) = a_k$, where f is **continuous, positive and decreasing** for $x \geq n$ and $S = \sum_{k=1}^{\infty} a_k$ is convergent. Then, if $S_n = \sum\limits_{k=1}^{n} a_k$ is used to approximate S we find the error $R_n = S - S_n$ in this estimate is remainder R_n and we have

$$\int_{n+1}^{\infty} f(x)\ dx < S - S_n < \int_{n}^{\infty} f(x)\ dx$$

$$\Longrightarrow\ S_n + \int_{n+1}^{\infty} f(x)\ dx < S < S_n + \int_{n}^{\infty} f(x)\ dx.$$

□



Example 4. (Approximating a p-series)

a. How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

b. Find an approximation to the series using 50 terms of the series.

Solution.

$\square$

10.5 §10.5 Comparison Tests

As in the case of convergence and divergence of improper integrals there is a Comparison Test for convergence and divergence of series.

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series of positive terms, i.e. $a_n > 0$ and $b_n > 0$. Then

(i) $\sum b_n$ convergent and $a_n \leq b_n \implies \sum a_n$ convergent, and

(ii) $\sum b_n$ divergent and $a_n \geq b_n \implies \sum a_n$ divergent.

Proof. (Reading) Let

$$S_N = \sum_{n=1}^{N} a_n, \qquad T_N = \sum_{n=1}^{N} b_n, \qquad T = \sum_{n=1}^{\infty} b_n$$

(i) Since $a_n > 0$ and $b_n > 0$, the sequence $\{S_N\}$ and $\{T_N\}$ are increasing sequences.

$$\begin{aligned} a_n \leq b_n &\implies S_N \leq T_N \leq T \\ &\implies \{S_N\} \text{ is monotonic and bounded by } T \\ &\implies \{S_N\} \text{ converges } \implies \sum a_n \text{ converges.} \end{aligned}$$

(ii) Since $a_n > 0$ and $b_n > 0$, the sequence $\{S_N\}$ and $\{T_N\}$ are increasing sequences. Now, $\{T_N\}$ is increasing and divergent and hence $\{T_N\}$ goes to infinity as N goes to infinity.

$$a_n \geq b_n \implies S_N \geq T_N \to \infty \implies S_N \to \infty \implies \sum a_n \text{ is divergent.}$$

□

Remark. To Apply the Comparison Test, we need the convergence or divergence of some series to compare with. Here are two examples that we have seen.

- $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
- $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

The problem in using the Comparison Test is to come up with a known series to compare with. This to a large extent requires experience. **Do the problems!**

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$$

is convergent or divergent.

Solution. We use the Comparison Test.

$$\underbrace{\frac{3n+2}{n(n+1)}}_{>0} = \frac{3+2/n}{n+1} > \frac{3}{n+1} > \underbrace{\frac{3}{2n}}_{>0}$$

Now, the Harmonic Series is divergent and so we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent} \implies \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent}$$
$$\implies \sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)} \text{ divergent.}$$

$\square$

Example 1. (Using the Comparison Test) Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$ **b.** $\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$

Solution.

$\square$

Remark. Sometimes it is difficult to get the inequality to work for the Comparison Test and it is easier to use the Limit Comparison Test. For example, $\sum\limits_{n=1}^{\infty} \frac{n^3}{2n^4+1}$

Theorem - The Limit Comparison Test. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms, i.e. $a_n > 0$ and $b_n > 0$.

1. If C is a finite positive number $(0 < C < \infty)$ and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$$
then either both series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, converge or both series diverge.
2. If $C = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
3. If $C = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof (Case 1). (Reading) Let m and M be such that $m < C < M$ and since $\frac{a_n}{b_n}$ is close to C we have
$$m < C < M \implies m < \frac{a_n}{b_n} < M \implies mb_n < a_n < Mb_n$$
for all large values of n, i.e. there exists some K such that $mb_n < a_n < Mb_n$ for all $n \geq K$. Also, by the Comparison Test we have
$$\begin{aligned}
\sum_{n=1}^{\infty} b_n \text{ convergent} &\implies \sum_{n=1}^{\infty} Mb_n \text{ convergent} \implies \sum_{n=1}^{\infty} a_n \text{ convergent}\\
\sum_{n=1}^{\infty} b_n \text{ divergent} &\implies \sum_{n=1}^{\infty} mb_n \text{ divergent} \implies \sum_{n=1}^{\infty} a_n \text{ divergent}
\end{aligned}$$
$\square$

Example. Determine whether the series
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n+3}$$
is convergent or divergent.

Solution. We looking at such a series ask yourself what is the dominant part of the term. In this case, the 2 doesn't do much and the 3 doesn't do much and so we think to compare with the divergent Harmonic Series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \text{ by L'H}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2n+3} \text{ is divergent since } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is.}$$

□

Example. Determine wether the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

is convergent or divergent.

Solution. By the p -Test $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent and for $n \geq 3$

$$\frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\left(\frac{4}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right) < \frac{1}{n}\frac{2}{n} \cdot 1 \cdots 1 = \frac{2}{n^2}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ is convergent.}$$

□

Example. Determine whether the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\frac{1}{n}}}$$

is convergent or divergent.

Solution. Because

$$\lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ by L'H}$$

$$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})} = e^{\lim_{n\to\infty} \ln(n^{\frac{1}{n}})} = e^0 = 1$$

seems to be telling us that maybe we should ignore $\frac{1}{n^{\frac{1}{n}}}$ and apply the limit Comparison Test using the divergent Harmonic Series $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n \cdot n^{\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n \cdot n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\frac{1}{n}}}$$

diverges by the limit Comparison Test. □

Example 2. (Using the Limit Comparison Test) Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$ **b.** $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

Solution.

□

10.6 §10.6 Alternating Series Test

For the most part we have been testing series of positive terms for convergence or divergence but now we force the series terms to alternate between positive and negative. It turns out that this test is easy to apply from the point of view that we do not have to compute an improper integral nor do we have to come up with some series to compare it with.

Definition. If $\{a_n\}$ is a sequence of **positive** terms, then the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots$$

is called an **alternate series**. □

Theorem - Alternating Series Test. The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges provided

(i) $a_n \geq a_{n+1} > 0$ for all n (nonincreasing), and

(ii) $\lim\limits_{n\to\infty} a_n = 0$.

(Since we need only test the **Tail** for convergence or divergence, the question is from some point on do the terms **alternate in sign**?; is the corresponding sequence decreasing or nonincreasing and does the limit of this sequence equal 0?.)

Proof (Reading). We first consider the subsequence of the partial sums corresponding to even N, i.e. $N = 2K$. We show $a_1 > S_{2K+2} > S_{2K}$.

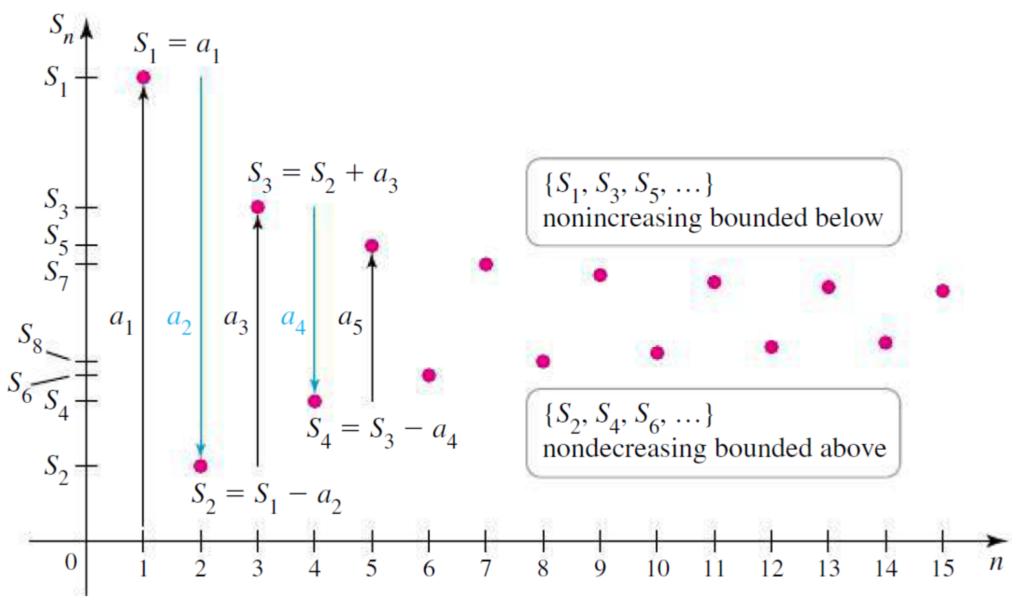
$$S_{2K} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} - \cdots - \underbrace{(a_{2K-2} - a_{2K-1})}_{\geq 0} - \underbrace{a_{2K}}_{>0} < a_1$$

$$S_{2K+2} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \cdots - \underbrace{(a_{2K-2} - a_{2K-1})} - \underbrace{(a_{2K} - a_{2K+1})}_{\geq 0} - \underbrace{a_{2K+2}}_{>0}$$

$$= S_{2K} + \underbrace{(a_{2K+1} - a_{2K+2})}_{\geq 0}$$

$\therefore \quad a_1 > S_{2K+2} > S_{2K}$ and $\{S_{2K}\}$ is bounded monotonic increasing.

$\therefore \quad \{S_{2K}\}$ has a limit. $\quad \lim_{K\to\infty} S_{2K} = S$



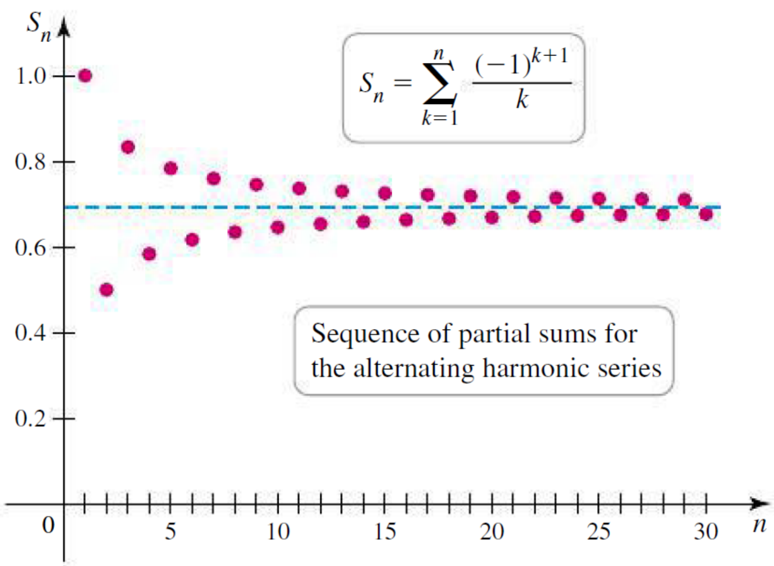
Now,

$$\lim_{K\to\infty} S_{2K+1} = \lim_{K\to\infty} [S_{2K} + a_{2K+1}] = \lim_{K\to\infty} [S_{2K}] + \lim_{K\to\infty} [a_{2K+1}] = S + 0 = S$$

What all this means is that both the odd partial sums and the even partial sums get closer and closer to S as N goes to infinity. Therefore. all the partial sums get closer and closer to S as N goes to infinity and S is the limit of the partial sums. $\square$

Recall that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem (Alternating Harmonic Series)**.** The alternating Harmonic Series $\sum_{n+1}^{\infty} \frac{(-1)^{n=1}}{n}$ converges.



Proof.

$\square$

Example 1. Determine whether the following series converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution.

$\square$

b.$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$

Solution.

□

c. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

Solution. We want to apply the Alternating Series Test to show convergence. The question is from some point on: do the terms alternate in sign; is the corresponding sequence decreasing; and does the limit of this sequence equal 0.

Certainly, the terms in the sequence $\{(-1)^{n-1}\frac{\ln n}{n}\}_{n=2}^{\infty}$ alternate in sign.

$$D_x(\frac{\ln x}{x}) = \frac{1-\ln x}{x^2} < 0 \text{ for } x > e \implies \frac{\ln n}{n} \text{ is decr. for } n > 3, \text{ and}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x} = 0 \implies \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \text{ is convergent.}$$

□

Example. Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n+4)}$$

for convergence or divergence.

Solution. Since we have an alternating series we need only show

(i) $\frac{1}{\ln(n+4)} \geq \frac{1}{\ln(n+5)} > 0$ for all n, and

(ii) $\lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$.

For (i), we first see that since $n+5 > 1$, $\ln(n+5) > 0$ and hence $\frac{1}{\ln(n+5)} > 0$.

$$\frac{1}{\ln(n+4)} \geq \frac{1}{\ln(n+5)} \iff \underbrace{\ln(n+5) \geq \ln(n+4)}_{\text{true since ln is an incr. func.}}$$

(ii) Since $\lim_{n\to\infty} \ln(n+4) = \infty$ we know that $\lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$. Therefore, by the Alternating Series Test

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n+4)} \text{ converges.}$$

□

Example. Test the series

$$\sum_{n=1}^{\infty}(-1)^n \cos\frac{\pi}{n}$$

for convergence or divergence.

Solution. It is very important to be sure that the Test that you want to apply actually applies. The Alternating Series Test is a Test for Convergence only. For $n \geq 3$, the sequence $\{(-1)^n \cos\frac{\pi}{n}\}_{n=3}^{\infty}$ does alternate sign but the Alternating Series Test does not apply because

$$\lim_{n\to\infty}\cos\frac{\pi}{n} = \cos\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \cos 0 = 1.$$

$\Big[$Recall - n^{th}**- Term Test for Divergence.**

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ is divergent.}\Big]$$

Now,

$$\begin{aligned}\lim_{n\to\infty}[(-1)^n\cos\frac{\pi}{n}] \neq 0 &\iff \lim_{n\to\infty}\left|(-1)^n\cos\frac{\pi}{n}\right| \neq 0 \\ &\iff \underbrace{\lim_{n\to\infty}\cos\frac{\pi}{n}}_{=1} \neq 0 \therefore \quad \sum_{n=1}^{\infty}(-1)^n\cos\frac{\pi}{n} \text{ is divergent.}\end{aligned}$$

□

Example. Test the series

$$\sum_{n=1}^{\infty}\left(-\frac{n}{5}\right)^n$$

for convergence or divergence.

Solution. This series diverges by the n-Term Test, since $\lim_{n\to\infty}\left(-\frac{n}{5}\right)^n \neq 0$ because

$$\lim_{n\to\infty}\left|-\frac{n}{5}\right|^n = \lim_{n\to\infty}\left(\frac{n}{5}\right)^n = \lim_{n\to\infty}\frac{n^n}{5^n} = \infty$$

□

Remainders in Alternating Series (Reading)

Theorem - Alternate Series Estimate of Remainder If $S = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent alternating series with

(i) $a_n \geq a_{n+1} > 0$ for all n, and

(ii) $\lim_{n\to\infty} a_n = 0$,

and $S_N = \sum_{n=1}^{N}(-1)^{n+1}a_n$, then the absolute remainder $|R_N| = |S - S_N|$ is bounded by a_{N+1}, ie.

$$|R_N| = |S - S_N| \leq a_{N+1}.$$

In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Proof. This result can be obtained by rereading the proof of the Alternating Series Test but we omit the proof.

Example. Find the N so that $|R_N| = |S - S_N| < .01$ where

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n}.$$

Solution. Set $a_n = \dfrac{n}{e^n}$. It is easy to show that the alternating series is convergent. Then, since $a_6 = .01287$ and $a_7 = .006383$ we see that $N = 7$ is required for the accuracy desired. $\square$

Absolute and Conditional Convergence

Notice that our tests often test the convergence of a series with positive terms. If our series $\sum a_n$ does not have positive terms, then we can use these tests to study the series $\sum |a_n|$.

Definition (Absolute and Conditional Convergence)**.** (1) A series $\sum a_n$ is said to be **absolutely convergent** provided $\sum |a_n|$ is convergent.

(2) A series $\sum a_n$ is said to be **conditionally convergent** provided $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Example. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent by the Alternating Series Test because the signs alternate and

(i) $\frac{1}{n} \geq \frac{1}{n+1} > 0$ for all $n = 1, 2, \cdots$, and

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$

BUT the series of absolute values terms is divergent because it is the Harmonic Series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$

So $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ is conditionally convergent. □

Theorem. (i) Absolute convergence $\Longrightarrow$ convergence, BUT

(ii) Convergence does not imply absolute convergence.

Proof.(read) Part (ii) is justified by the preceding Example. Part (i) follows by applying the Comparison Test.

$$0 \leq a_n + |a_n| \leq 2|a_n| \text{ and } \sum |a_n| \text{ convergent } \Longrightarrow \sum (a_n + |a_n|) \text{ convergent.}$$

Now,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| \text{ is convergent.}$$

□

Example 3. (Absolute and conditional convergence) Determine whether the following series diverge, converge absolutely, or converge conditionally.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ **b.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^3}}$ **c.** $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ **d.** $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$$

for convergence

Solution. We test the series $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ for absolute convergence by comparing it with a geometric series.

$$0 < |a_n| = \left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \text{ is a convergent geometric series since } r = \frac{1}{4}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{\sin 4n}{4^n} \text{ is absulutely convergent and hence convergent.}$$

□

10.7 §10.7 The Ratio and Root Tests

The **Ratio Test** and **Root Test** are very useful in determining whether a given series is absolutely convergent.

Theorem - The Ratio Test (for Absolute Convergence) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

(i)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent and hence convergent.}$$

(ii)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \implies \sum_{n=1}^{\infty} a_n \text{is divergent.}$$

(iii)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \implies \sum_{n=1}^{\infty} a_n \text{ the test fails, i.e. no conclusion.}$$

Proof (Reading). (i) Assume $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ Then we may select an r such that $L < r < 1$ and there is some there is some M such that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < r < 1 &\implies \frac{|a_{n+1}|}{|a_n|} < r \text{ for all } n \geq M \\
&\implies |a_{n+1}| < r|a_n| \text{ for all } n \geq M \\
&\implies |a_{M+1}| < r|a_M|, |a_{M+2}| < r|a_{M+1}| < r^2|a_M|, \ldots \\
&\implies |a_{M+k}| < r^k|a_M|.
\end{aligned}$$

Now, we compare the series $\sum_{n=1}^{\infty} |a_{M+n}|$ to the geometric series

$$\sum_{n=1}^{\infty} |a_M|r^n \text{ convergent geometric series.}$$

$\therefore$, $\sum_{n=1}^{\infty} |a_{M+n}|$ is convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then there is some r with $L > r > 1$ and some m such that

$$|a_{n+1}| > r|a_n| \text{ for all } n \geq m$$

Continue to follow the proof of (ii) to finish the proof of (ii).

(iii) The series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ both satisfy $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ but one is divergent and the other is convergent. $\square$

Example 1. Use the Ratio Test to determine whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ **b.** $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ **c.** $\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n}(n^2+4)$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

for absolutely convergence, conditionally convergent or divergence.

Solution. Since the series has positive terms, it is either absolutely convergent or divergent. We use the Limit Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \frac{100^n}{n!} = \lim_{n\to\infty} \frac{(n+1)}{100} = \infty > 1 \\
\therefore \quad & \sum_{n=1}^{\infty} \frac{n!}{100^n} \text{ diverges.}
\end{aligned}$$

□

The **Root Test** is similar to the Ratio Test and is well suited to series whose general term involves powers.

Theorem - Root Test (for Absolute Convergence) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

(i)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}$$

(and hence convergent).

(ii)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \text{ or } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty \implies \sum_{n=1}^{\infty} a_n \text{is divergent.}$$

(iii)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \implies \sum_{n=1}^{\infty} a_n \text{ the test fails, i.e. no conclusion.}$$

Proof. The proof is similar to the Ratio Test and we omit it. □

Example 2. Use the Root Test to determine whether the following series converge.

a. $\sum_{k=1}^{\infty}\left(\frac{3-4k^2}{7k^2+6}\right)^k$ **b.** $\sum_{k=1}^{\infty}\frac{(-2)^k}{k^{10}}$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$$

for absolutely convergence, conditionally convergent or divergence.

Solution. We test the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ for absolute convergence or divergence by The Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2}{n}\right)^n}=\lim_{n\to\infty}\frac{2}{n}=0<1$$

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}\text{ is absolutely convergent and hence convergent.}$$

Example. Test the series

$$\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n$$

for absolutely convergence, conditionally convergence, or divergence.

Solution. We test the series $\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n$ for absolute convergence or divergence by The Root Test.

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^n\right|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n}\\
&= \lim_{n\to\infty}\frac{2n}{n+1} = 2 > 1
\end{aligned}$$

$$\therefore \quad \sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n \text{ is divergent by part (ii) of the Root Test.}$$

10.8 §10.8 Choosing a Convergence Test

We have discussed several different tests for convergence and/or divergence. We provide you with a summary of the Tests that we have studied. Your author suggests a strategy to help you but nothing is as helpful as experience. Do the problems!

Strategy. Classify the series according to form.

1. Does the series have the form of a p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{convergent if } p > 1 \\ \text{Divergent if } p \leq 1. \end{cases}$$
2. Does the series have the form of a geometric series
$$\sum_{n=1}^{\infty} ar^n \begin{cases} \text{convergent if } |r| > 1 \\ \text{Divergent if } |r| \leq 1. \end{cases}$$
3. Does the series have the form 'similar' to a p-series or a geometric series? If so, think of using the Comparison Test or Limit Comparison Test.
4. Does a quick application of the n^{th}-Term Test do the trick?
5. Do we have an Alternating Series?
6. If the series involves a factorial then a Ratio Test may help?
7. If a_n has the form b_n^n, think of the Root Test.
8. Does a_n have the form $a_n = f(n)$ for some continuous, positive,decreasing function with $\int_1^\infty f(x)\, dx$ easily computed? If so think of the Integral Test.

□

Summary of Series Tests

Test	Series	Convergent	Divergent	Comment
n^{th}–Term	$S=\sum_{n=0}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Geometric	$S=\sum_{n=0}^{\infty} ar^n$	$\|r\|<1$	$\|r\|\geq 1$	$S=\frac{a}{1-r}$
Telescoping	$\sum_{n=1}^{\infty}(b_n-b_{n+1})$	$\lim_{n\to\infty} b_n = L$		$S=b_1-L$
p-Series	$\sum_{n=1}^{\infty}\frac{1}{n^p}$	$p>1$	$p\leq 1$	
Alternating	$S=\sum_{n=1}^{\infty}(-1)^n a_n$	$0<a_{n+1}\leq a_n$ $\lim_{n\to\infty} a_n = 0$		Remainder $\|R_N\|\leq a_{N+1}$
Integ. f cont. pos. & decr.	$S=\sum_{n=1}^{\infty} a_n$ $a_n=f(n)\geq 0$	$\int_1^{\infty} f(x)dx$ Convergent	$\int_1^{\infty} f(x)dx$ Divergent	$\int_{N+1}^{\infty} f(x)dx < R_N$ $< \int_N^{\infty} f(x)dx$
Root	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\sqrt[n]{\|a_n\|}<1$	$\lim_{n\to\infty}\sqrt[n]{\|a_n\|}>1$	
Ratio	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\frac{\|a_{n+1}\|}{\|a_n\|}<1$	$\lim_{n\to\infty}\frac{\|a_{n+1}\|}{\|a_n\|}>1$	
Comparison	$S=\sum_{n=0}^{\infty} a_n$	$0<a_n\leq b_n$ $\sum_{n=0}^{\infty} b_n$ Conv.	$0<b_n\leq a_n$ $\sum_{n=0}^{\infty} b_n$ Div.	
Lim. Comp. $(a_n, b_n>0)$	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\frac{a_n}{b_n}=$ $L\geq 0(L<\infty)$ $\sum_{n=0}^{\infty} b_n$ Conv.	$\lim_{n\to\infty}\frac{a_n}{b_n}=$ $L>0(or\ \infty)$ $\sum_{n=0}^{\infty} b_n$ Div.	

$\square$

Test the series in the textbook for absolute convergence, conditional convergence, or divergence.

Example 1. $\sum_{k=1}^{\infty} \frac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}$.

Solution.

$\square$

Example 2. $\sum_{k=1}^{\infty} \left(1 - \frac{10}{k}\right)^k$

Solution.

$\square$

Example 3. $\sum_{k=4}^{\infty} \frac{1}{\sqrt[4]{k^2 - 6k + 9}}$

Solution.

$\square$

Example 4. $\sum_{k=1}^{\infty} k^2 e^{-2k}$

Solution.

$\square$

Example 5. $\sum_{k=2}^{\infty} \sqrt[3]{\frac{k^2-1}{k^4+4}}$

Solution.

$\square$