On the volume of zonotopes

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Abstract

We study properties of a projection of an hypercube $I^n = [0,1]^n$ onto a k-dimensional subspace. This projection can be seen as a zonotope in $\mathbb{R}^k$, which is the Minkowski sum of linear segments. Studying particular transformations and small perturbations of a projection, we find necessary conditions for it to have the maximal volume. Finally, we give a new approach to find the optimal upper bound on the volume of a projection onto a two-dimensional subspace.

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1 Introduction

We let the *Minkoski sum* of two sets A, B be the set $A + B = \{a + b : a \in A, b \in B\}$ and we denote by $[\mathbf{v}, \mathbf{w}] = \{\lambda\mathbf{v} + (1-\lambda)\mathbf{w} : \lambda \in [0,1]\}$ the *linear segment* between two vectors $\mathbf{v}, \mathbf{w}$. We define a *zonotope* in $\mathbb{R}^k$ as the Minkowski sum of several line segments.

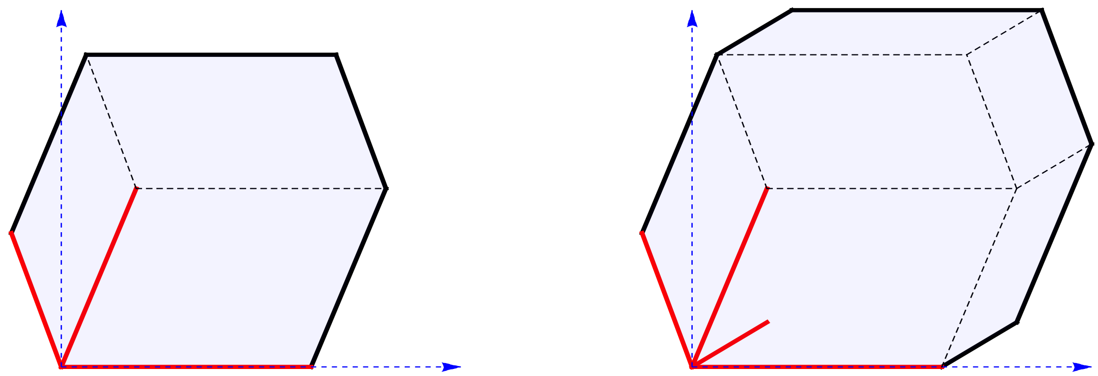

Figure 1: Example of two *zonotopes* in $\mathbb{R}^2$, generated by 3 and 4 line segments (in red).

We study properties of the volume of such convex sets under a certain constraint. Define the *n-dimensional unit cube* $I^n = \sum_{i=1}^n [\mathbf{0}, \mathbf{e}_i]$, where $\{\mathbf{e}_i\}_{i\in[n]}$ is the standard basis of $\mathbb{R}^n$, and we consider the zonotopes given by a I^n projection onto k-dimensional subspace H^k ($\mathbf{P} : \mathbb{R}^n \mapsto H^k$):

$$Z = I^n | H^k = \sum_{i=1}^n [\mathbf{0}, \mathbf{P}\mathbf{e}_i]. \tag{1.1}$$

We prove in Section 2 that every zonotope is an affine transformation of a projection of an unit cube, but we refer the reader to C. Zong's book [8, Chapter 2] for a deeper review of hypercubes projections.

Our goal is to obtain some necessary conditions for maximizers of

$$\mathrm{Vol}(Z), \qquad \text{for } Z = I^n | H^k. \tag{1.2}$$

Some bounds on this volume are known. Since $I^n | H^k \supset I^n \bigcap H^k$ for all subspaces H^k, we get a tight lower bound from the following theorem

Theorem 1.1 (Vaaler, 1979, [7]). *Let $C^n = [-1/2, 1/2]^n$, for every k-dimensional subspace H^k of $\mathbb{R}^n$, we have*

$$\mathrm{Vol}(C^n \bigcap H^k) \geq 1,$$

where equality is attained if and only if H^k is spanned by k axes of $\mathbb{R}^n$.

Therefore, we have that $\mathrm{Vol}(I^n \mid H^k) \geq 1$. This result was proved independently by Chackerian and Filliman, without using Valeer's theorem, in [1] and in the same paper two upper bounds were also proven.

Theorem 1.2 (Chakerian and Filliman, [1]).

$$\mathrm{Vol}(I^n \mid H^k) \leq \frac{\omega_{k-1}^k}{\omega_k^{k-1}} \left(\frac{n}{k}\right)^{k/2} \tag{1.3}$$

Theorem 1.3 (Chakerian and Filliman, [1])**.**

$$\mathrm{Vol}(I^n \mid H^k) \leq \sqrt{\frac{n!}{(n-k)!k!}} \tag{1.4}$$

For a fixed k and $n \to \infty$ this two bounds behaves similarly, although (1.3) is asymptotically tight, while it is known that inequality (1.4) is most efficient for a relatively small n and equality is attained only for $k = 1, n-1$.

Maximizers of (1.2) are known only for $k = 1, 2, n-2, n-1$ and for $n = 6, k = 3$. If $k = 1$, the projection of I^n of maximal volume is onto a main diagonal and, in this case, $\mathrm{Vol}(I^n|H^1) = \sqrt{n}$. For $k = 2$, the maximal projection is a regular $2n$-gon with edge length $\sqrt{2/n}$ and its volume is $\cot(\frac{\pi}{2n})$. This result can be easily verified by the *isoperimetric inequality* for $2n$-gons in $\mathbb{R}^2$, although we prove it with a completely different approach in Section 5. The cases $n-2, n-1$ are a consequence of

Theorem 1.4 (McMullen, 1984, [5])**.** *If H^k and H^{n-k} are orthogonal complementary subspaces of $\mathbb{R}^n$, then*

$$\mathrm{Vol}(I^n|H^k) = \mathrm{Vol}(I^n|H^{n-k}). \tag{1.5}$$

Our approach differs from those used by previous authors since we work with transformations of the zonotopes to derive necessary conditions for the maximizers of (1.2). In particular, we give in Section 2 an equivalent property for the zonotopes to be cubes projections, then, in Section 3, we describe how to modify a given projection to obtain another one. More specifically, in Section 4, we focus on small perturbations around a maximizer and we obtain some explicit necessary conditions for a projection to be a local maxima of (1.2). Finally, we give a new proof for the shape of the optimal projection onto a two-dimensional subspace and, thanks to the methods we use, our proof does not only point out the maximizer but shows that we can continuously transform any given projection into one of maximal volume, while increasing the volume in a monotonic way.

2 Definitions and preliminaries

Let n be a positive integer, we define by $[n]$ the set $\{1, \ldots, n\}$. Given a vector $\mathbf{x}$ in $\mathbb{R}^n$, we denote through $\mathbf{x}[i]$ its i-th coordinate in $\{\mathbf{e}_i\}_{i=1}^n$, the standard basis of $\mathbb{R}^n$, and for a matrix $M \in \mathrm{Mat}_{k\times k}(\mathbb{R})$ we use $M[i,j]$ for its (i,j)-th entry. Let I^n be the n-dimensional unit cube $\{\mathbf{x} \in \mathbb{R}^n : 0 \leq \mathbf{x}[i] \leq 1,\ \forall i \in [n]\}$. Throughout the paper, for $k \in [n-1]$, we will use H^k to denote a k-dimensional subspace of $\mathbb{R}^n$ and $\mathbb{R}^k \subseteq \mathbb{R}^n$ will be the subspace of vectors whose only non-zero coordinates are the first k (i.e $\mathbb{R}^k = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}[i] = 0,\ \forall i = k+1, \ldots, n\}$). Furthermore, we use $\langle \cdot, \cdot \rangle$ for the standard *inner product* of two vectors and $|\cdot|$ for the induced *norm*. Since $\mathbb{R}^n$ coincides with it dual space, we have that the *tensor product* correspond to the *outer product* and, given two vectors of same dimension $\mathbf{v}$ and $\mathbf{w}$, we can identify both the matrix and the operator with $(\mathbf{v} \otimes \mathbf{w})$, whose matrix components in the standard basis are given by $(\mathbf{v} \otimes \mathbf{w})[i,j] = \mathbf{v}[i]\mathbf{w}[j]$.

Definition 2.1. Let $S = \{\mathbf{v}_1, \ \ldots \ , \mathbf{v}_n\} \subset \mathbb{R}^k$ be a set of vectors. It defines

- A matrix $M^S = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n] \in \mathrm{Mat}_{k\times n}(\mathbb{R})$, whose columns are the vectors of S in label order, i.e $M^S[i,j] = \mathbf{v}_j[i]$ for $(i,j) \in [k] \times [n]$.
  We use M_i^S to identify the matrix i-th row, $i \in [k]$.

- A *zonotope* $Z_S := \{\sum_{i=1}^{n} [\mathbf{0}, \mathbf{v}_i]\}$ (the *Minkowski sum* of the line segments ending at $\mathbf{v}_i$). We call the $\mathbf{v}_i$-s its *generating vectors* and M^S its *generating matrix.*

- An operator A_S. Defined as follows

$$A_S = \left(\sum_{\mathbf{v} \in S} \mathbf{v} \otimes \mathbf{v} \right). \tag{2.1}$$

  We denote by A_S both the operator and the corresponding matrix in the standard basis.

- A subspace of $\mathbb{R}^n$, that we call H_S^k, given by the row space of M^S

- A projection operator $\mathbf{P}_S$ from R^n to H_S^k, we use $\mathbf{P}_S$ to denote both the operator and the corresponding matrix in the standard basis.

Definition 2.2. A set of vectors $S = \{\mathbf{v}_1, \ \dots \ , \mathbf{v}_n\} \subset \mathbb{R}^k$ is called an (n, k)-*frame* if $\operatorname{Span}\{S\} = \mathbb{R}^k$

Definition 2.3. Let $S = \{\mathbf{v}_1, \ \dots \ , \mathbf{v}_n\} \subset \mathbb{R}^k$.

- We call S an (n, k)-*uframe* if there exist an orthonormal basis $\{\mathbf{f}_1, \ \dots \ , \mathbf{f}_n\}$ of $\mathbb{R}^n$ such that for all $\mathbf{v}_i \in S$ we have $\mathbf{v}_i = \mathbf{P}(\mathbf{f}_i)$, where $\mathbf{P}$ denotes the orthogonal projection from $\mathbb{R}^n$ to $\mathbb{R}^k$.

- We say that S give a *unit decomposition* if it satisfies

$$A_S|_{\mathbb{R}^k} = \sum_{i=1}^{n} (\mathbf{v}_i \otimes \mathbf{v}_i)|_{\mathbb{R}^k} = \mathrm{Id}_k, \tag{2.2}$$

  where Id_k is the identity operator in $\mathbb{R}^k$.

Lemma 2.4. *For any set of vectors $S \subseteq \mathbb{R}^k$, the operator A_S is self-adjoint and semi-positive definite. Furthermore, if S is an (n, k)-frame, A_S is positive definite.*

Proof.
By the one to one correspondence of linear operators and their matrix in any base, it is enough to prove the assumption for the matrix A_S. By the geometrical definition of the tensor product $\mathbf{v} \otimes \mathbf{v}$, as the operator given by the composition of the projection onto the linear span of $\mathbf{v}$ and a scaling by $|\mathbf{v}|^2$, we have that $(\mathbf{v} \otimes \mathbf{v})\mathbf{x} = \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v}$. Consequently, we get

$$\begin{aligned} {A_s}^t &= \sum_{\mathbf{v} \in S} (\mathbf{v} \otimes \mathbf{v})^t = \sum_{\mathbf{v} \in S} \mathbf{v} \otimes \mathbf{v} = A_S, \\ \langle \mathbf{x}, A_S \mathbf{x} \rangle &= \sum_{\mathbf{v} \in S} \langle \mathbf{x}, (\mathbf{v} \otimes \mathbf{v})\mathbf{x} \rangle = \sum_{\mathbf{v} \in S} \langle \mathbf{x}, \mathbf{v} \rangle^2 \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^k, \end{aligned}$$

thus, we prove that the operator A_S is self-adjoint and semi-positive definite.
Let us now suppose that $\operatorname{Span}\{S\} = \mathbb{R}^k$, this means that for every vector $\mathbf{x} \in \mathbb{R}^k$, we can write it as a linear combination of $\{\mathbf{v}_1, \ \dots \ , \mathbf{v}_n\}$. In particular, if $\mathbf{x} \neq \mathbf{0}$, there exists at least one $\mathbf{v}$ such that $\langle \mathbf{x}, \mathbf{v} \rangle \neq 0$. Using this observation, the last inequality becomes strict, and thus A_S is positive definite. □

By *diagonalization theorem*, we have, as a direct consequence of the previous Lemma, that A_S can be expressed as $A_S = Q^t D Q$, where Q is an orthogonal matrix whose columns are A_S's eigenvectors and D is diagonal and its entries are the corresponding eigenvalues. Furthermore, if Z_S is non-degenerate (i.e $\mathrm{Span}\{S\} = \mathbb{R}^k$), all eigenvalues are non-zero, as A_S is positive definite. We can therefore define a new operator for S.

Definition 2.5. Let $\mathrm{Span}\{S\} = \mathbb{R}^k$, we define the operator B_S as follows

$$B_S = A_S^{-1/2}. \tag{2.3}$$

We will denote through B_S both the operator and the corresponding matrix in the standard basis.

Looking at the structure of the matrix A_S, we see that

$$A_S[l,j] = \sum_{i=1}^{n} (\mathbf{v}_i \otimes \mathbf{v}_i)[l,j] = \sum_{i=1}^{n} (\mathbf{v}_i[l]\mathbf{v}_i[j]) = \langle M_l^S, M_j^S \rangle, \tag{2.4}$$

which means that A_S is the *Gram matrix* of the row vectors of $M^S = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$. As this two matrices will have an important role in the next sections, we give a result that helps us understanding the relations between two sets of vectors with the same Gram matrix.

Theorem 2.6. *Let S_1 and S_2 be two sets of p vectors in $\mathbb{R}^q$. If they have the same Gram matrix, then there exists an orthogonal transformation that maps vectors of S_1 to S_2.*

Proof.
Let $S_1 = \{\mathbf{v}_i\}_{i\in[p]}$, $S_2 = \{\mathbf{v}_i{}'\}_{i\in[p]}$ and let Γ define the Gram matrix of the two sets; by definition, $\Gamma(i,j)$ is the inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i{}', \mathbf{v}_j{}' \rangle$. Because between vectors with the same labels the inner products are the same, we have that the Gram-Schmidt process produces exactly the same steps on both sets. Since this process is done on the matrices $M^{S_1} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$ and $M^{S_2} = [\mathbf{v}'_1 \ \cdots \ \mathbf{v}'_p]$ by subtracting to a column other columns times a scalar, it can be described by a matrix multiplication on the right and, most importantly, the multiplication is given by an invertible matrix. We call this matrix $R \in \mathrm{Mat}_{p\times p}(\mathbb{R})$ and its inverse R^{-1}.
At the end of the orthonormalization process we get two orthonormal sets Q_1 and Q_2, in which $l \leq q$ elements are non-zero and we have that $Q_1 = S_1 R$ and $Q_2 = S_2 R$. Considering the non-null vectors, both Q_1 and Q_2 are orthonormal basis of the same l-dimensional subspaces H^l of $\mathbb{R}^q$. By a simple linear algebra result, we know there exists a basis-change operator B that maps Q_2 non-zero elements to Q_1's and, restricted to l dimensions, B is orthogonal and $Q_1 = BQ_2$. We can linearly extend B to the q-dimensional space by defining it as the identity for all other vectors, granting, in this way, that its extension is still orthogonal.
Then, observing that $S_1 = Q_1 R^{-1} = BQ_2 R^{-1} = BS_2$, we have an orthogonal transformation between the two sets. $\square$

We now state the following lemma, which can be found in [2].

Lemma 2.7. *Let $S = \{\mathbf{v}_1, \ \ldots \ , \mathbf{v}_n\} \subset \mathbb{R}^k$ be a set of vectors, then the following assertions are equivalent:*

1. *S gives a unit decomposition;*

2. *S is an (n,k)-uframe;*

3. $\mathrm{Span}\{S\} = \mathbb{R}^k$ *and the Gram matrix Γ of the vectors of S is the matrix of a projection operator from $\mathbb{R}^n$ to H^k_S*

4. *$M^S \in \mathrm{Mat}_{k\times n}(\mathbb{R})$ is a sub-matrix of an orthogonal matrix of order n.*

Proof.

- 4⇒3:
  Since M^S is a $k \times n$ sub-matrix of an orthogonal matrix of order n, the rank of M^S is k, i.e. $\mathrm{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_n}\} = \mathbb{R}^k$.
  Then, let Γ be the Gram matrix of $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ and $\mathbf{P}$ the matrix of a projection onto the row space of M^S. Let $\{M^S_1, \ldots, M^S_k\}$ be the rows of M^S, by hypothesis they are orthonormal, so they form a basis for their linear hull H^k_S. In this coordinate system of H^k_S, we can identify $\mathbf{P}\mathbf{e}_i = \sum_{l=1}^k \langle \mathbf{e}_i, M^S_l \rangle M^S_l$. Recalling that $M^S_l[m] = \mathbf{v}_m[l]$, it follows

$$\mathbf{P}[i,j] = \langle \mathbf{e}_j, \mathbf{P}\mathbf{e}_i \rangle = \langle \mathbf{e}_j, \sum_{l=1}^k \langle \mathbf{e}_i, M^S_l \rangle M^S_l \rangle = \sum_{l=1}^k \langle \mathbf{e}_i, M^S_l \rangle \langle \mathbf{e}_j, M^S_l \rangle =$$
$$= \sum_{l=1}^k M^S_l[i] M^S_l[j] = \sum_{l=1}^k \mathbf{v}_i[l]\mathbf{v}_j[l] = \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \Gamma[i,j].$$

- 3⇒2:
  We need to prove that there exists an orthonormal basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ of $\mathbb{R}^n$ such that for all $\mathbf{v}_i \in S$ we have $\mathbf{v}_i = \mathbf{P}(\mathbf{f}_i)$, where $\mathbf{P}$ is the orthogonal projection onto $\mathbb{R}^k$.
  Let $\Gamma\mathbf{e}_i$ be the projection of $\mathbf{e}_i$-s, by hypothesis we have

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \Gamma[i,j] = \langle \mathbf{e}_i, \Gamma\mathbf{e}_j \rangle = \langle \Gamma\mathbf{e}_i, \Gamma\mathbf{e}_j \rangle,$$

  where in the last equality we used idempotency and symmetry of the projection operator Γ. This result implies that the two vector sets S and $\{\Gamma\mathbf{e}_i\}_{i\in[n]}$ have the same Gram matrix. Therefore, by Theorem 2.6, we can identify vectors $\mathbf{v}_i$ and $\Gamma\mathbf{e}_i$, for $i \in [n]$, and their linear hulls, up to an orthogonal transformation.

- 2⇒1:
  Let $\mathbf{P}$ be the projection from $\mathbb{R}^n$ onto $\mathbb{R}^k$ such that $\mathbf{v}_i = \mathbf{P}\mathbf{f}_i$, where $\{\mathbf{f}_i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$. Since for an arbitrary vector $\mathbf{x} \in \mathbb{R}^k \subseteq \mathbb{R}^n$ we have $\mathbf{P}\mathbf{x} = \mathbf{x}$, it follows

$$\begin{aligned} A_S\mathbf{x} &= \left(\sum_{i=n}^n \mathbf{v}_i \otimes \mathbf{v}_i\right)\mathbf{x} = \sum_{i=1}^n (\mathbf{v}_i \otimes \mathbf{v}_i)\mathbf{x} = \sum_{i=1}^n \langle \mathbf{v}_i, \mathbf{x} \rangle \mathbf{v}_i = \\ &= \sum_{i=1}^n \langle \mathbf{P}\mathbf{f}_i, \mathbf{x} \rangle \mathbf{v}_i = \sum_{i=1}^n \langle \mathbf{f}_i, \mathbf{P}\mathbf{x} \rangle \mathbf{v}_i = \mathbf{P}\left(\sum_{i=1}^n \langle \mathbf{f}_i, \mathbf{x} \rangle \mathbf{f}_i\right) \qquad \text{(I)} \\ &= \mathbf{P}\mathbf{x} = \mathbf{x}. \end{aligned}$$

  Where in line (I) we exploit that $\mathbf{P}$ is self-adjoint and that $\{\mathbf{f}_i\}_{i=1}^n$ is an orthonormal basis. Because this equality holds for every vector in $\mathbb{R}^k$, the restriction of the matrix A_S to $\mathbb{R}^k$ is the identity.

- 1⇒4 Let $\{\mathbf{e}_i\}_{i\in[k]}$ be the standard orthonormal basis of $\mathbb{R}^k$. By hypothesis we can write

$$\mathbf{e}_j = \mathrm{Id}_k\,\mathbf{e}_j = \left(\sum_{i=n}^{n}\mathbf{v}_i\otimes\mathbf{v}_i\right)\mathbf{e}_j = \sum_{i=1}^{n}\langle\mathbf{v}_i,\mathbf{e}_j\rangle\mathbf{v}_i.$$

Therefore,

$$1 = |\mathbf{e}_j|^2 = \langle\mathbf{e}_j,\mathbf{e}_j\rangle = \langle\mathrm{Id}_k\,\mathbf{e}_j,\mathbf{e}_j\rangle = \left\langle\sum_{i=1}^{n}\langle\mathbf{v}_i,\mathbf{e}_j\rangle\mathbf{v}_i,\ \mathbf{e}_j\right\rangle = \sum_{i=1}^{n}\langle\mathbf{e}_j,\mathbf{v}_i\rangle^2 = \sum_{i=1}^{n}\mathbf{v}_i[j]^2.$$

Furthermore, for $j\neq l$,

$$0 = \langle\mathbf{e}_j,\mathbf{e}_l\rangle = \sum_{i=1}^{n}\langle\mathbf{e}_j,\mathbf{v}_i\rangle\langle\mathbf{e}_l,\mathbf{v}_i\rangle = \sum_{i=1}^{n}\mathbf{v}_i[j]\mathbf{v}_i[l].$$

Recall that for $M = [\mathbf{v}_1\ \cdots\ \mathbf{v}_n]$ rows we have $M_i[j] = \mathbf{v}_j[i]$, i.e. the last two equations are

$$\langle M_i, M_j\rangle = \begin{cases} 1 & i = j \\ 0 & i\neq j \end{cases}.$$

This completes the proof. □

Remark. Thanks to this Lemma, we have that the maximizers for problem (1.2), are the same for the following problem:

$$\mathrm{Vol}(Z_S),\qquad \text{for } S \text{ an } (n,k)\text{-uframe.} \tag{2.5}$$

In reason of this equivalence and in order not to create confusion with the notation, we refer to both problems using (1.2).

A direct geometrical consequence of Lemma 2.7 is the following result on the sum of S vectors' squared norms.

Corollary 2.7.1. *Let $S = \{\mathbf{v}_i\}_{i=1}^n$ be an (n,k)-uframe, then*

$$\sum_{i=1}^{n}|\mathbf{v}_i|^2 = \mathrm{Tr}(A_S) = k. \tag{2.6}$$

Proof.
By Lemma 2.7, $M = [\mathbf{v}_1\ \cdots\ \mathbf{v}_n]$ is a sub-matrix of an orthogonal matrix of order n, we thus have

$$\sum_{i=1}^{n}|\mathbf{v}_i|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k}(\mathbf{v}_i[j])^2 = \sum_{j=1}^{k}\sum_{i=1}^{n}(\mathbf{v}_i[j])^2 = \sum_{j=1}^{k}|M_j|^2 = \sum_{j=1}^{k}1 = k.$$

And, by (2.4), we also have $\sum_{j=1}^{k}|M_j|^2 = \mathrm{Tr}(A_S)$. □

Thanks to Lemma 2.7, we are able to understand the link between (n,k)-frames and projections of I^n.

Theorem 2.8. *Let $\tilde{S} = \{\mathbf{v}_i\}_{i=1}^n \subseteq \mathbb{R}^k$ be an (n,k)-frame, then $B_{\tilde{S}} \cdot \tilde{S} = \{B_{\tilde{S}}\mathbf{v}_i\}_{i=1}^n$ is an (n,k)-uframe.*

Proof.
By Lemma 2.7, it is enough to show that $A_{(B_{\tilde{S}} \cdot \tilde{S})} = \mathrm{Id}_k$. We verify it through

$$\begin{aligned} A_{(B_{\tilde{S}} \cdot \tilde{S})} &= \sum_{i=1}^n \mathbf{v}_i{}' \otimes \mathbf{v}_i{}' = \sum_{i=1}^n (B_{\tilde{S}}\mathbf{v}_i \otimes B_{\tilde{S}}\mathbf{v}_i) = B_{\tilde{S}} \left(\sum_{i=1}^n \mathbf{v}_i \otimes \mathbf{v}_i \right) B_{\tilde{S}}{}^t = \\ &= B_{\tilde{S}} A_{\tilde{S}} B_{\tilde{S}}{}^t = B_{\tilde{S}} A_{\tilde{S}} B_{\tilde{S}} = \mathrm{Id}_k \, . \end{aligned}$$

□

Theorem 2.9. *Every zonotope defined by n line segments is an affine transformation of a projection of I^n.*

Proof.
Let Z_S be a non degenerate zonotope in $\mathbb{R}^k$ generated by $S = \{\mathbf{v}_1, \ \dots \ , \mathbf{v}_n\}$, then $\mathrm{Span}(S) = \mathbb{R}^k$ and thus S is an (n,k)-frame. By Theorem 2.8, $B_S \cdot S$ is an (n,k)-uframe and in particular there is an orthonormal basis $F = \{\mathbf{f}_1, \ \dots \ , \mathbf{f}_n\}$ of $\mathbb{R}^n$ such that $B_S \cdot S = \mathbf{P} \cdot F$, where $\mathbf{P}$ is the projection $\mathbb{R}^n \mapsto \mathbb{R}^k$. Since F is orthonormal, the Minkowski sum of the linear segments with endpoints in F gives an n-dimensional unit cube $C^n = Z_F$.
We recall that, by definition, any affine transformation of a Minkowski sum is the Minkowski sum of the affine transformations of the addend sets. Consequently,

$$B_S^{-1}\mathbf{P} \cdot C^n = B_S^{-1}\mathbf{P}\sum_{i=1}^n [\mathbf{0}, \mathbf{f}_i] = \sum_{i=1}^n [\mathbf{0}, B_S^{-1}\mathbf{P} \cdot \mathbf{f}_i] = \sum_{i=1}^n [\mathbf{0}, \mathbf{v}_i] = Z_S,$$

where we used the fact that $B_S|_{\mathbb{R}^k}$ is invertible by definition. Indeed, $B_S^{-1}\mathbf{P}$ is an affine transformation by composition of affine transformations, proving the theorem. □

The computation of the volume of a projection can be done using a well-known explicit formula for the volume of Z_S, and it uses the previously introduced matrix M^S. It is Shephard's formula, proved in [6]:

Lemma 2.10 (Shephard, 1974, [6])**.**

$$\mathrm{Vol}(Z_S) = \sum_{I \in \binom{[n]}{k}} |\det(M_I^S)|. \tag{2.7}$$

Where, for any index set $I \subseteq [n]$, M_I^S represents M^S submatrix $M_I^S = [\mathbf{v}_{i_1} \ \cdots \ \mathbf{v}_{i_{|I|}}]$, with $i_j \in I$ and $i_1 \le i_2 \le \ldots \le i_{|I|}$. We use $\binom{[n]}{k}$ to describe the set of all k elements subsets of $[n]$.

3 Transformations of the (n, k)-uframes

In this section, we focus on how to transform a given (n, k)-*uframe* S into a new one S' in such a way that S' is still a projection of an higher dimensional cube. Our first strategy is to use the results obtained in the previous Section to transform the (n, k)-uframe S into an (n, k)-frame $\tilde{S}$ and then obtain from the latter a new (n, k)-uframe S'. Motivated by Theorem 2.8, we use for this last step the operator $B_{\tilde{S}}$, defined by (2.3).

More precisely, we let $\mathbf{T}$ be a transformation of $\mathbb{R}^k$, and we define a new transformation $\mathbf{T}'$ from S to S' as the composition of $\mathbf{T}$ and $B_{\mathbf{T}(S)}$.

$$S \quad \xrightarrow{\mathbf{T}(.)} \quad \tilde{S} \quad \xrightarrow{B_{\tilde{S}}\cdot} \quad S' \tag{3.1}$$

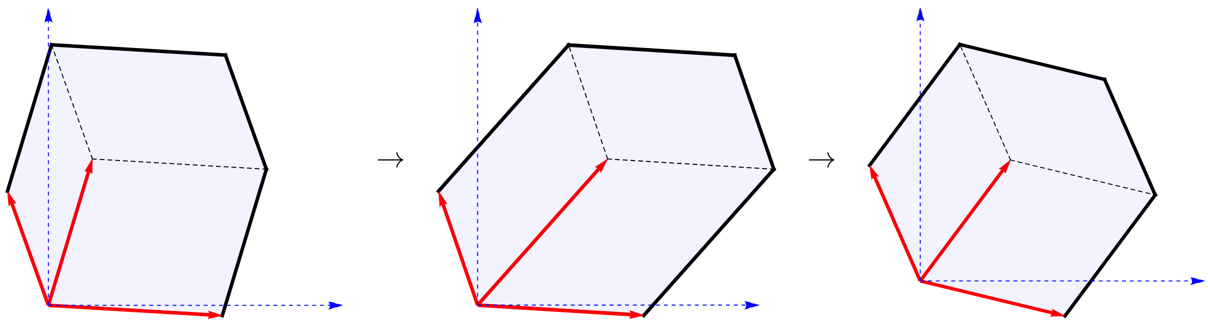

Figure 2: Example of application of scheme (3.1) in case $n = 3, k = 2$.

Aside of this chain of transformations, we give in next Lemma an important property of all substitutions that preserves the operator A_S, to apply it later in the case in which S gives a unit decomposition.

Lemma 3.1. *Let S be an (n, k)-frame. A substitution $S \mapsto S'$ preserves A_S (i.e- $A_S = A_{S'}$) if and only if it can be expressed through a multiplication on the right by an* orthogonal matrix *$R \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, i.e. $S' = S \cdot R$.*

Proof.

- $\Leftarrow$ Suppose $R \in O(\mathbb{R}^n)$ and let S' be the set $\{\mathbf{v}_i{}'\}_{i=1}^n$, with $\mathbf{v}_i{}' = \mathbf{v}_i R$. By definition of $O(\mathbb{R}^n)$, we have that $R^t = R^{-1}$ and, therefore,

$$A_{S'} = \sum_{i=1}^{n} \mathbf{v}_i{}' \otimes \mathbf{v}_i{}' = \sum_{i=1}^{n} (\mathbf{v}_i R \otimes \mathbf{v}_i R) = \sum_{i=1}^{n} (\mathbf{v}_i R R^t \otimes \mathbf{v}_i) = \sum_{i=1}^{n} \mathbf{v}_i \otimes \mathbf{v}_i = A_S.$$

- $\Rightarrow$ Suppose that both S and S' have $A_S = A_{S'}$. Let M^S and $M^{S'}$ be the matrices whose columns are respectively S and S' vectors; by (2.4) we have that the rows of M^S and $M^{S'}$ have the same Gram matrix. Using Theorem 2.6 we know that there exist an orthogonal transformation

$$R: \quad \{M_i^{S'}\}_{i\in[k]} \longmapsto \{M_i^S\}_{i\in[k]} = \{R(M_i^{S'})\}_{i\in[k]}. \tag{3.2}$$

Since those rows are vectors in $\mathbb{R}^n$, R can be expressed by a $n \times n$ orthogonal matrix. Clearly, (3.2) is equivalent to

$$M^S = M^{S'} \cdot R^t \quad \Leftrightarrow \quad M^{S'} = M^S \cdot R \quad \Leftrightarrow \quad \mathbf{v}_i{}' = \mathbf{v}_i \cdot R \quad \forall i \in [n],$$

hence, we prove that the transformation can be expressed through an orthogonal matrix multiplication on the right.

□

Corollary 3.1.1. *The transformations on two vectors of S, namely $\mathbf{v}_i$ and $\mathbf{v}_j$, are, up to the sign, of the form :*

$$\begin{aligned} \mathbf{v}_i{}' &= \cos\alpha\,\mathbf{v}_i - \sin\alpha\,\mathbf{v}_j, \\ \mathbf{v}_j{}' &= \sin\alpha\,\mathbf{v}_i + \cos\alpha\,\mathbf{v}_j. \end{aligned} \tag{3.3}$$

We call the transformations of the form (3.3) elliptic rotations.

Proof.
By Lemma 3.1, the transformation R, in order to preserve all the vectors of S aside $\mathbf{v}_i$ and $\mathbf{v}_j$, should modify, in $\mathbb{R}^n$, only $\mathbf{e}_i$ and $\mathbf{e}_j$ (i.e it should be a transformation of the 2-dimensional hyperplane spanned by those vectors). Expressed in $\mathrm{Span}\{\mathbf{e}_i, \mathbf{e}_j\}$ coordinate system, R is the composition of a rotation and a symmetry, so is given by

$$R = \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^r,$$

where $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$ and $r \in \{0, 1\}$. Therefore, by linearity of the projection $\mathbf{P}$ onto $\mathbb{R}^k$ and dropping the symmetry part (as it does not influence the volume), we obtain the desired relation

$$\begin{aligned} \mathbf{P}(R\mathbf{e}_i) &= \mathbf{P}(c_\alpha \mathbf{e}_i - s_\alpha \mathbf{e}_j) = c_\alpha \mathbf{P}(\mathbf{e}_i) - s_\alpha \mathbf{P}(\mathbf{e}_j) = c_\alpha \mathbf{v}_i - s_\alpha \mathbf{v}_j = \mathbf{v}_i{}', \\ \mathbf{P}(R\mathbf{e}_j) &= \mathbf{P}(s_\alpha \mathbf{e}_i + c_\alpha \mathbf{e}_j) = s_\alpha \mathbf{P}(\mathbf{e}_i) + c_\alpha \mathbf{P}(\mathbf{e}_j) = s_\alpha \mathbf{v}_i + c_\alpha \mathbf{v}_j = \mathbf{v}_j{}'. \end{aligned}$$

□

We define the metric space $\Omega(n, k)$ of all (n, k)-frames, in which the distance between $S = \{\mathbf{v}_i\}_{i \in [n]}$ and $S' = \{\mathbf{v}_i{}'\}_{i \in [n]}$ is given by

$$\mathrm{dist}(S, S') = \sqrt{\sum_{i=1}^{n} |\mathbf{v}_i - \mathbf{v}_i{}'|^2}. \tag{3.4}$$

Evidently, we can consider the (n, k)-frames as nk-dimensional vectors and thus $\Omega(n, k)$ can be seen as a subset of $\mathbb{R}^{nk}$ inheriting its standard metric. Using this distance we can discuss continuous transformations of $S \in \Omega(n, k)$ and by definition we see that both A_S and B_S change continuously as functions of S. Therefore, following the scheme (3.1) we can continuously transform an (n, k)-uframe $S \mapsto S'$ to get another one, as long as $\mathbf{T}(\cdot)$ is continuous. We can thus define *global* and *local maximizers* for the problem (1.2).

We say that an (n,k)-uframe S a *global maximizer* for (1.2) if for any other (n,k)-uframe S', we have that $\mathrm{Vol}(Z_S) \geq \mathrm{Vol}(Z_{S'})$. Similarly, S is a *local maximizer* if for every (n,k)-uframe S' "close enough" to S in $\Omega(n,k)$ we have $\mathrm{Vol}(Z_S) \geq \mathrm{Vol}(Z_{S'})$. It was explained in [3] that the study of extrema of (1.2) for (n,k)-uframes in $\Omega(n,k)$ is equivalent to the study of extrema for (1.2) through the Grassmannian of $\mathbb{R}^n$. In particular, the local maxima are the same for both problems.

Using the previously stated results, we give an equivalent condition for an (n,k)-uframe to be a maximizer.

Lemma 3.2. *Let S be an (n,k)-uframe and S' be an (n,k)-frame, then S is a maximizer for* (1.2) *if and only if*

$$\frac{\mathrm{Vol}(Z_{S'})}{\mathrm{Vol}(Z_S)} \leq \sqrt{\det(A_{S'})} \tag{3.5}$$

Proof.
Let $B_{S'}$ be like defined in (2.3), then, by Theorem 2.8, we have that $B_{S'}S'$ is an (n,k)-uframe. S is a maximizer for (1.2) iff it volume is greater than that of any other uframe, thus

$$\mathrm{Vol}(B_{S'}Z_{S'}) = \det(B_{S'})\,\mathrm{Vol}(Z_{S'}) \leq \mathrm{Vol}(Z_S)$$

Given this, both directions trivially follow from

$$\frac{\mathrm{Vol}(Z_{S'})}{\mathrm{Vol}(Z_S)} \leq \frac{1}{\det(B_{S'})} = \frac{1}{\det(A_{S'}^{-1/2})} = \sqrt{\det(A_{S'})}$$

□

Remark. We can get an equivalent condition for an (n,k)-uframe to be a local maximizer of (1.2) by adding in Lemma 3.2 the hypothesis for S' to be in a small enough neighbourhood of S.

4 Necessary conditions

In this section, we apply previous results to specific transformations, in order to give some explicit conditions on the (n,k)-uframes to be maximizers for (1.2). Firstly, we study what happens if we substitute $\mathbf{v}_i \in S$ by $\mathbf{v}_i{}' = \mathbf{v}_i + \lambda\mathbf{v}_i$ and then obtain a new uframe using $B_{S'}$. Afterwards, we examine the consequences of performing an elliptic rotation on two distinct vectors. In both cases, we work with small perturbations of the original set, to keep the newly generated (n,k)-uframe in a small enough neighbourhood.

4.1 Single vector transformation

We focus here on how the volume of the zonotope varies when we modify the length of just one vector along its direction.

Lemma 4.1. *Let an (n,k)-uframe $S = \{\mathbf{v}_1, \ \dots \ , \mathbf{v}_n\}$ be a local maximizer for* (1.2)*. Let S' be the set given by $\{\mathbf{v}_1, \dots, \mathbf{v}_i + \lambda\mathbf{v}_i, \dots, \mathbf{v}_n\}$, for $\lambda \in \mathbb{R}$, then*

$$A_{S'} = A_S + (\lambda^2 + 2\lambda)\mathbf{v}_i \otimes \mathbf{v}_i \tag{4.1}$$

Proof.
By a straightforward computation, we have

$$\begin{aligned}
A_{S'} &= \sum_{j=1}^{n} \mathbf{v}_j{}' \otimes \mathbf{v}_j{}' = \sum_{j\neq i} \mathbf{v}_j \otimes \mathbf{v}_j + (\mathbf{v}_i + \lambda \mathbf{v}_i) \otimes (\mathbf{v}_i + \lambda \mathbf{v}_i) = \\
&= \sum_{j\neq i} \mathbf{v}_j \otimes \mathbf{v}_j + \mathbf{v}_i \otimes \mathbf{v}_i + (\lambda \mathbf{v}_i) \otimes \mathbf{v}_i + \mathbf{v}_i \otimes (\lambda \mathbf{v}_i) + (\lambda \mathbf{v}_i) \otimes (\lambda \mathbf{v}_i) = \\
&= A_S + (\lambda^2 + 2\lambda)\mathbf{v}_i \otimes \mathbf{v}_i.
\end{aligned}$$

□

Lemma 4.2. *Let S be an (n, k)-uframe and S' be the zonotope obtained by scaling $\mathbf{v}_i \in S$ by a factor $1 + \lambda$, i.e $\mathbf{v}_i{}' = \mathbf{v}_i + \lambda \mathbf{v}_i$. Then, S is a local maximizer for the volume if, for λ close to zero, the following holds:*

$$\frac{\operatorname{Vol}(Z_{S'})}{\operatorname{Vol}(Z_S)} \leq 1 + \lambda |\mathbf{v}_i|^2 + o(\lambda). \tag{4.2}$$

Proof.
Using (4.2), Lemma 3.2 and the fact that $\mathbf{v}_i \otimes \mathbf{v}_i$ gives a rank 1 symmetric matrix (and thus by diagonalization we can find a base in which its only term is on the diagonal and it is its eigenvalue $|\mathbf{v}_i|^2$), we can compute

$$\det(A_{S'}) = \det(\mathrm{Id}_k + (\lambda^2 + 2\lambda)\mathbf{v}_i \otimes \mathbf{v}_i)) = 1 + (\lambda^2 + 2\lambda)|\mathbf{v}_i|^2,$$

or $\det(A_{S'}) = 1 + 2\lambda |\mathbf{v}_i|^2 + o(\lambda^2)$, for λ small enough. Taking its square root we prove the wanted inequality. □

Corollary 4.2.1. *Let S be an (n, k)-uframe. If S is a (local) maximizer for* (1.2)*, then, for every $\mathbf{v}_i$ generating vector of S ($i \in [n]$), we have*

$$\sum_{I \in \binom{[n]}{k}, i \in I} |\det(M_I^S)| = |\mathbf{v}_i| \operatorname{Vol}(Z_S | \{\mathbf{v}_i\}^\perp) = |\mathbf{v}_i|^2 \operatorname{Vol}(Z_S), \tag{4.3}$$

where $\{\mathbf{v}_i\}^\perp$ is the orthogonal complement of $\{\mathbf{v}_i\}$ in $\mathbb{R}^k$.

Proof.
Define S' as in the previous Lemma. Using (2.7) and determinant's linearity we compute

$$\begin{aligned}
\frac{\operatorname{Vol}(Z_{S'})}{\operatorname{Vol}(Z_S)} &= \frac{\sum_I |\det(M_i^{S'})|}{\sum_I |\,\mathrm{d}(I)|} = \frac{\sum_{i \notin I} |\,\mathrm{d}(I)| + \sum_{i \in I} (1 + \lambda) |\,\mathrm{d}(I)|}{\sum_I |\,\mathrm{d}(I)|} = \\
&= \frac{\sum_I |\,\mathrm{d}(I)| + \lambda \sum_{i \in I} |\,\mathrm{d}(I)|}{\sum_I |\,\mathrm{d}(I)|} = 1 + \lambda \frac{\sum_{i \in I} |\det(M_I^S)|}{\operatorname{Vol}(Z_S)},
\end{aligned}$$

where the summation is over all $I \in \binom{[n]}{k}$ and we get rid of the absolute value of $1 + \lambda$ because we assume λ close to zero (and thus smaller than 1 in absolute value).

By the last identity and (4.2), we get

$$\begin{aligned} 1+\lambda\frac{\sum_{i\in I}|\det(M_I^S)|}{\mathrm{Vol}(Z_S)} &\leq 1+\lambda|\mathbf{v}_i|^2+o(\lambda)\\ \Leftrightarrow\quad \lambda\frac{\sum_{i\in I}|\det(M_I^S)|}{\mathrm{Vol}(Z_S)} &\leq \lambda|\mathbf{v}_i|^2+o(\lambda). \end{aligned}$$

Then, we see that

$$\frac{\sum_{i\in I}|\det(M_I^S)|}{\mathrm{Vol}(Z_S)} \leq \lambda|\mathbf{v}_i|^2+o(1) \quad \text{for } \lambda\downarrow 0,$$

or, equivalently,

$$\sum_{i\in I}|\det(M_I^S)| \leq |\mathbf{v}_i|^2\,\mathrm{Vol}(Z_S).$$

By the same argument, for $\lambda\uparrow 0$,

$$\begin{aligned} \lambda\frac{\sum_{i\in I}|\det(M_I^S)|}{\mathrm{Vol}(Z_S)} \leq \lambda|\mathbf{v}_i|^2+o(\lambda) &\Leftrightarrow \frac{\sum_{i\in I}|\det(M_I^S)|}{\mathrm{Vol}(Z_S)} \geq |\mathbf{v}_i|^2+o(1)\\ &\Leftrightarrow \sum_{i\in I}|\det(M_I^S)| \geq |\mathbf{v}_i|^2\,\mathrm{Vol}(Z_S). \end{aligned}$$

Putting together the two inequalities, we prove the statement. □

Next Theorem has been proved in [3, Section 7] and is a generalization of Lemma 4.2.

Theorem 4.3 (Ivanov, [3])**.** *Let H_k be such that a (local) maximum for $\mathrm{Vol}(I^n|H_k)$ is attained. Let $\mathbf{v}_i$ be the projection of $\mathbf{e}_i$ onto H_k for $i\in[n]$. Then, for any vectors $\{\mathbf{x}_i\}_{i=1}^n\subset H_k$ and scalars $\{t_1,\dots,t_n\}\subset\mathbb{R}$, the function $f(t_1,\dots,t_n)=\mathrm{Vol}\left(\sum_{i=1}^n[\mathbf{0},\mathbf{v}_i+t_i\mathbf{x}_i]\right)$ is differentiable at the origin and the following identity holds*

$$\frac{f(t_1,\dots,t_n)}{\mathrm{Vol}(I^n|H_k)}=1+\sum_{i=1}^n t_i\langle\mathbf{v}_i,\mathbf{x_i}\rangle+o\left(\sqrt{t_1^2+\cdots+t_n^2}\right). \tag{4.4}$$

Thanks to this result, we can prove that all maximizers have the following geometrical property.

Lemma 4.4. *If S is an optimal (n-k)-uframe, then for every $I\in\binom{[n]}{k}$ the set $\{\mathbf{v}_i\}_{i\in I}$ is linearly independent.*
Or, equivalently, all determinants in (2.7) are non-zero.

Proof.
Towards contradiction, suppose that there is a $I\in\binom{[n]}{k}$ such that $\det(M_I^S)=0$, that implies $\mathbf{v}_{i_k}\in\mathrm{Span}\{\mathbf{v}_i\}_{i\in I,i\neq i_k}$ (supposing wlog $i_1<\ldots<i_k$, for $i_l\in I$). Let $\mathbf{x}\in(\mathrm{Span}\{\mathbf{v}_i\}_{i\in I,i\neq i_l})^\perp$, it is linearly independent from the other vectors. Therefore,

$$|\det[\mathbf{v}_{i_1}\,\dots\,(\mathbf{v}_{i_k}+t\mathbf{x})]|=|\det[\mathbf{v}_{i_1}\,\dots\,\mathbf{v}_{i_{k-1}}\;t\mathbf{x}]|=|t|\underbrace{|\det[\mathbf{v}_{i_1}\,\dots\,\mathbf{v}_{i_{k-1}}\;\mathbf{x}]|}_{\neq 0 \text{ if } \mathbf{x}\neq\mathbf{0}}=|t|C.$$

Then, transforming S in S' by $\mathbf{v}_{i_k} \mapsto \mathbf{v}_{i_k} + t\mathbf{x}$ as in Lemma 4.3 and noting $f(t) = \mathrm{Vol}(S')$. Using (2.7) and (4.4), we obtain

$$1 + |t|\frac{C}{\mathrm{Vol}(S)} \leq \frac{f(t)}{\mathrm{Vol}(S)} \leq 1 + t \underbrace{\langle \mathbf{v}_{i_k}, \mathbf{x} \rangle}_{=0} + o(t) = 1 + o(t)$$

and thus is not differentiable in $t = 0$, contradicting Lemma 4.3. Therefore all determinants are non-zero. □

4.2 Elliptic rotations of two vectors

Since the formulation of (2.7) would be quite cumbersome due the number of times we use it, we simplify the notation for the cases in which the $k \times n$ matrix M^S is "mostly fixed", i.e. we only change a fixed numbers of columns. For $I \in \binom{[n]}{k}$ a set of indices and M_I^S the $k \times k$ submatrix of M^S previously defined, we write

$$\mathrm{d}(I) \;=\; \det(M_I^S)$$

for its determinant and, for $i \in [n]$, $J \in \binom{[n]}{k-1}$ we define

$$\mathrm{d}(J, \mathbf{v}_i) = \det[\mathbf{v}_{j_1} \;\cdots\; \mathbf{v}_i \;\cdots\; \mathbf{v}_{j_{k-1}}], \qquad j_l \in J,\; j_1 \leq \ldots \leq i \leq \ldots \leq j_{k-1}, \tag{4.5}$$

where one can see that the vector $\mathbf{v}_i$ is in its correct position, label-wise speaking. By extension, we suppose that for any transformation $\mathbf{v}_i{}'$ of $\mathbf{v}_i$, $\mathrm{d}(I, \mathbf{v}_i{}')$ will "place" the vector in $\mathbf{v}_i$ position and we can use the same notation to highlight two or more different vectors (e.g $\mathrm{d}(I, \mathbf{v}_i, \mathbf{v}_j)$, where $|I| = k-2$). As one can see, we allow the presence of equal indices; since in this case the determinant would be zero, the column in which the vector is placed does not matter, thus our notation is well defined.

Definition 4.5. We define the *δ-sign* function δ by

$$\delta(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}. \tag{4.6}$$

Going back to the previously introduced notation, we will write

$$\delta(I, \mathbf{v}) \;=\; \delta(\mathrm{d}(I, \mathbf{v})) \tag{4.7}$$

for the determinants sign.

Though δ-sign function, we have the following trivial result

Lemma 4.6. *Let A, B, $t \in \mathbb{R}$, then for t close to 0 and $A \neq 0$ the following holds:*

$$|A + tB| = |A| + \delta(A)tB + o(t). \tag{4.8}$$

We recall that in Lemma 3.1 we defined by (3.3) the *elliptic rotation* of two vectors $\mathbf{v}_i, \mathbf{v}_j$ as a function of α. We now compute how it changes the volume of S.

Lemma 4.7. *Let S be an (n,k)-uframe and S' obtained by S through the elliptic rotation of two vectors $\mathbf{v}_i$ and $\mathbf{v}_j$, as defined in* (3.3). *Then, for α close to zero, we have*

$$\operatorname{Vol}(S') = \operatorname{Vol}(S) + \alpha \sum_{I \in \binom{[n]\setminus\{i,j\}}{k-1}} \delta(I,\mathbf{v}_i)\delta(I,\mathbf{v}_j)\left(|\operatorname{d}(I,\mathbf{v}_i)| - |\operatorname{d}(I,\mathbf{v}_j)|\right) + o(\alpha). \tag{4.9}$$

Proof.

The proof of this lemma is just a computational work. We assume i and j to be respectively $n-1,\ n$ and we write

$$\operatorname{Vol}(S') \overset{\text{Lemma 2.10}}{=} \sum_{I \in \binom{[n]}{k}} |\operatorname{d}(I)| =$$

$$= \underbrace{\sum_{I \in \binom{[n-2]}{k}} |\operatorname{d}(I)|}_{\mathbf{A}'} + \underbrace{\sum_{I \in \binom{[n-2]}{k-1}} \left[|\operatorname{d}(I,\mathbf{v}_i{}')| + |\operatorname{d}(I,\mathbf{v}_j{}')|\right]}_{\mathbf{B}'} + \underbrace{\sum_{I \in \binom{[n-2]}{k-2}} |\operatorname{d}(I,\mathbf{v}_i{}',\mathbf{v}_j{}')|}_{\mathbf{C}'}.$$

Likewise, we decompose S volume formula in $\operatorname{Vol}(S) = \mathbf{A}+\mathbf{B}+\mathbf{C}$, in which the sums are the same and we just use $\mathbf{v}_i, \mathbf{v}_j$ instead of their images. We immediately see that in the term $\mathbf{A}'$ there is no $\mathbf{v}_i{}'$, nor $\mathbf{v}_j{}'$ and is thus the same also in $\operatorname{Vol}(S)$.

We now study the other two terms, using determinant's multilinearity

$$\begin{aligned}
\mathbf{C}' &= \sum_{I \in \binom{[n-2]}{k-2}} |\operatorname{d}(I,(c_\alpha \mathbf{v}_i - s_\alpha \mathbf{v}_j),(s_\alpha \mathbf{v}_i + c_\alpha \mathbf{v}_j))| = \\
&= \sum_I |c_\alpha \operatorname{d}(I,\mathbf{v}_i,(s_\alpha \mathbf{v}_i + c_\alpha \mathbf{v}_j)) - s_\alpha \operatorname{d}(I,\mathbf{v}_j,(c_\alpha \mathbf{v}_i - s_\alpha \mathbf{v}_j))| = \\
&= \sum_I |c_\alpha{}^2 \operatorname{d}(I,\mathbf{v}_i,\mathbf{v}_j) - s_\alpha{}^2 \operatorname{d}(I,\mathbf{v}_j,\mathbf{v}_i)| = \sum_I |(c_\alpha{}^2 + s_\alpha{}^2)\operatorname{d}(I,\mathbf{v}_i,\mathbf{v}_j)| = \mathbf{C}.
\end{aligned}$$

And, finally,

$$\begin{aligned}
\mathbf{B}' &= \sum_{I \in \binom{[n-2]}{k-1}} [|\operatorname{d}(I,(c_\alpha \mathbf{v}_i - s_\alpha \mathbf{v}_j)|) + |\operatorname{d}(I,\ (s_\alpha \mathbf{v}_i + c_\alpha \mathbf{v}_j))|] \overset{\alpha\sim 0}{=} \\
&= \sum_I [|\operatorname{d}(I,\mathbf{v}_i) - \alpha \operatorname{d}(I,\mathbf{v}_j) - o(\alpha)| + |\alpha \operatorname{d}(I,\mathbf{v}_i) + \operatorname{d}(I,\mathbf{v}_j) + o(\alpha)|] = && \text{Lemma 4.6} \\
&= \sum_I [|\operatorname{d}(I,\mathbf{v}_i)| - \delta(I,\mathbf{v}_i)\alpha \operatorname{d}(I,\mathbf{v}_j) + |\operatorname{d}(I,\mathbf{v}_j)| + \delta(I,\mathbf{v}_j)\alpha \operatorname{d}(I,\mathbf{v}_i)] + o(\alpha) = \\
&= \underbrace{\sum_I [|\operatorname{d}(I,\mathbf{v}_i)| + |\operatorname{d}(I,\mathbf{v}_j)|]}_{\mathbf{B}} + \alpha \sum_I [\delta(I,\mathbf{v}_j)\operatorname{d}(I,\mathbf{v}_i) - \delta(I,\mathbf{v}_i)\operatorname{d}(I,vj)] + o(\alpha) = && \text{(II)} \\
&= \mathbf{B} + \alpha \sum_I \left[\frac{\operatorname{d}(I,\mathbf{v}_j)}{|\operatorname{d}(I,\mathbf{v}_j)|}\operatorname{d}(I,\mathbf{v}_i) - \frac{\operatorname{d}(I,\mathbf{v}_i)}{|\operatorname{d}(I,\mathbf{v}_i)|}\operatorname{d}(I,\mathbf{v}_j)\right] + o(\alpha) = \\
&= \mathbf{B} + \alpha \sum_I \left[(|\operatorname{d}(I,\mathbf{v}_i)| - |\operatorname{d}(I,\mathbf{v}_j)|)\frac{\operatorname{d}(I,\mathbf{v}_i)\operatorname{d}(I,\mathbf{v}_j)}{|\operatorname{d}(I,\mathbf{v}_i)||\operatorname{d}(I,\mathbf{v}_j)|}\right] + o(\alpha) = \\
&= \mathbf{B} + \alpha \sum_I (|\operatorname{d}(I,\mathbf{v}_i)| - |\operatorname{d}(I,\mathbf{v}_j)|)\delta(I,\mathbf{v}_i)\delta(I,\mathbf{v}_j) + o(\alpha),
\end{aligned}$$

where the assumption (II) on fourth line uses that, from Lemma 4.4, $\mathrm{d}(I, \mathbf{v}_k) \neq 0$ for $k = i, j$. □

5 Optimal volume for $k = 2$

In this section we use the obtained results to set the global maxima for the volume of (n,2)-uframe. We prove that the only maximizer is, up to orthogonal transformations, the regular 2n-gone. In order to do so, we firstly need to introduce some more definitions, which will be crucial in the proofs to come.

Definition 5.1. Let $S = \{\mathbf{v}_i\}_{i=1}^n$ a set of vectors in $\mathbb{R}^2$ and $\mathbf{v}_i, \mathbf{v}_j \in S$, linearly independent, we partition the plane in two internally disjoint sets:

$$C_1 = \{\alpha\mathbf{v}_i + \beta\mathbf{v}_j : \alpha\beta \geq 0,\ \alpha, \beta \in \mathbb{R}\},$$

$$C_2 = \{\alpha\mathbf{v}_i + \beta\mathbf{v}_j : \alpha\beta \leq 0,\ \alpha, \beta \in \mathbb{R}\}.$$

We then say that $\mathbf{v}_i$ and $\mathbf{v}_j$ are *adjacent* if the other vectors in S are either all in C_1 or all in C_2.

Lemma 5.2. *Let $S = \{\mathbf{v}_i\}_{i=1}^n \subset \mathbb{R}^2$ such that its elements are two by two linearly independent , fix $\mathbf{v}_i, \mathbf{v}_j$ in S and define $f : [n] \to \{\pm 1, 0\}$ as $f(I) := \delta(I, i)\delta(I, j)$. Then, if $\mathbf{v}_i$ and $\mathbf{v}_j$ are adjacent, f is constant.*

Proof.
Let at first write for all $\mathbf{v}_l \in S$,

$$\mathbf{v}_l = \alpha_l\mathbf{v}_i + \beta_l\mathbf{v}_j.$$

This is possible by linear independence of the two vectors. In the same way $\mathrm{d}(l, m) \neq 0,\ \forall l \neq m \in [n]$, so we can apply delta-sign definition and get the following inequality

$$\begin{aligned} f(l) &= \frac{\mathrm{d}(l,i)\,\mathrm{d}(l,j)}{|\,\mathrm{d}(l,i)||\,\mathrm{d}(l,j)|} = \frac{\det[\mathbf{v}_l\ \mathbf{v}_i]\det[\mathbf{v}_l\ \mathbf{v}_j]}{|\det[\mathbf{v}_l\ \mathbf{v}_i]\det[\mathbf{v}_l\ \mathbf{v}_j]|} = \\ &= \frac{\det[(\alpha_l\mathbf{v}_i + \beta_l\mathbf{v}_j)\ \mathbf{v}_i]\det[(\alpha_l\mathbf{v}_i + \beta_l\mathbf{v}_j)\ \mathbf{v}_j]}{|\det[(\alpha_l\mathbf{v}_i + \beta_l\mathbf{v}_j)\ \mathbf{v}_i]\det[(\alpha_l\mathbf{v}_i + \beta_l\mathbf{v}_j)\ \mathbf{v}_j]|} = \\ &= \frac{\beta_l\det[\mathbf{v}_j\ \mathbf{v}_i]\cdot\alpha_l\det[\mathbf{v}_i\ \mathbf{v}_j]}{|\beta_l\det[\mathbf{v}_j\ \mathbf{v}_i]\cdot\alpha_l\det[\mathbf{v}_i\ \mathbf{v}_j]|} = -\frac{\alpha_l\beta_l}{|\alpha_l\beta_l|}, \end{aligned}$$

which is constant over $l \in [n]$ by adjacency definition. □

Lemma 5.3. *If S is an (n,2)-uframe and is a local maximizer of the volume, then for all $\mathbf{v}_i, \mathbf{v}_j$ generating vectors, $|\mathbf{v}_i|^2 = |\mathbf{v}_j|^2$ and they all have length $\sqrt{2/n}$.*

Proof.
We can label the vectors in such a way that consecutive labelled vectors are adjacent (considering n and 1 as consecutive, too). Let $j = i + 1 \mod n$ and S' be like in Lemma 3.3, i.e $S' = S \setminus \{\mathbf{v}_i, \mathbf{v}_j\} \bigcup \{\mathbf{v}_i{}' = c_\alpha\mathbf{v}_i - s_\alpha\mathbf{v}_j,\ \mathbf{v}_j{}' = s_\alpha\mathbf{v}_i + c_\alpha\mathbf{v}_j\}$. By the same Lemma, S' is an

(n,2)-uframe and thus $A_{S'} = B_{S'} = \mathrm{Id}_2$.
Through the restriction of Lemma 4.3 to our case, we obtain

$$\begin{aligned}\frac{\mathrm{Vol}(S')}{\mathrm{Vol}(S)} &\leq 1 + \sum_{i=1}^{n} t_i \langle \mathbf{v}_i, \mathbf{x}_i \rangle + o(\sqrt{t_1^2 + \ldots + t_n^2}) = \\ &= 1 + \alpha \langle \mathbf{v}_i, \mathbf{v}_j \rangle - \alpha \langle \mathbf{v}_j, \mathbf{v}_i \rangle + o(\alpha) = \\ &= 1 + o(\alpha).\end{aligned}$$

Applying this result along with (4.9), we get, for α close to 0,

$$\begin{aligned} & 0 \leq \alpha \frac{\sum_I \delta(I,i)\delta(I,j)[|\,\mathrm{d}(I,i)| - |\,\mathrm{d}(I,j)|]}{\mathrm{Vol}(S)} + o(\alpha) \leq o(\alpha) \\ \Leftrightarrow \quad & 0 \leq \sum_{I \in [n] \setminus \{i,j\}} \delta(I,i)\delta(I,j)[|\,\mathrm{d}(I,i)| - |\,\mathrm{d}(I,j)|] \quad \leq o(1) = 0, \end{aligned}$$

where the summation is over $I \in \binom{[n]\setminus\{i,j\}}{k-1} = [n]\setminus\{i,j\}$. By the choice of i, j, $\mathbf{v}_i$ and $\mathbf{v}_j$ are adjacent. Therefore we can get rid of $\delta(I,i)\delta(I,j)$ and obtain

$$\begin{aligned} \sum_{I \in [n] \setminus \{i,j\}} (|\,\mathrm{d}(I,i)| - |\,\mathrm{d}(I,j)|) = 0 \quad & \Leftrightarrow \\ \sum_{I \in [n]} |\,\mathrm{d}(I,i)| = \sum_{I \in [n]} |\,\mathrm{d}(I,j)| \quad & \overset{(4.3)}{\Leftrightarrow} \\ |\mathbf{v}_i|^2 \,\mathrm{Vol}(S) = |\mathbf{v}_j|^2 \,\mathrm{Vol}(S) \quad & \Leftrightarrow \\ |\mathbf{v}_i| = |\mathbf{v}_j|, \quad & \end{aligned}$$

which proves the first part of the lemma.
Finally, by the corollary of Lemma 2.7, we have that

$$2 = k = \sum_{i=1}^{n} |\mathbf{v}_i|^2 = n|\mathbf{v}_i|^2, \qquad \forall i \in [n],$$

and thus $\forall i \in [n]$, $|\mathbf{v}_i| = \sqrt{\frac{2}{n}}$. □

Theorem 5.4. *Every local maximizer of* (1.2) *is the global maximizer and it is unique up orthogonal transformations.*

The maximizer is the regular $(2n)$*-gone with side length* $\sqrt{2/n}$*.*

Proof.
Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a local maximizer of (1.2), where the vectors are labelled counter-clockwise according to their line span $L_i = \{\lambda \mathbf{v}_i : \lambda \in \mathbb{R}\}$, for $i \in [n]$. In Theorem 5.3, we proved that all vectors of S have the same length $c = \sqrt{2/n}$. Defining $\alpha_{i,j}$ as the angle between L_i and L_j counter-clockwise speaking, we recall that

$$|\det[\mathbf{v}_i \ \mathbf{v}_j]| = |\mathbf{v}_i||\mathbf{v}_j||\sin(\alpha_{i,j})| = c^2 |\sin(\alpha_{i,j})|.$$

From equation (2.7), it follows that

$$\mathrm{Vol}(Z_S) = \sum_{i<j\in[n]} |\det[\mathbf{v}_i\ \mathbf{v}_j]| = c^2 \sum_{i<j\in[n]} |\sin(\alpha_{i,j})|. \tag{5.1}$$

Because any angle between two lines in a plane is smaller than π, we have that $\alpha_{i,j} \in [0.\pi]$ for all $i < j \in [n]$. Therefore, we can express the last identity as a function $f : [0,\pi]^{\binom{n}{2}} \mapsto \mathbb{R}$. Obviously, $\sin(\alpha)$ is concave and positive for $\alpha \in [0,\pi]$ and, thus, it is $|\sin(\alpha)|$. We observe that equality (5.1) gives us a result for the volume only under the constraint

$$\alpha_{i,j} = \alpha_{i,i+1} + \alpha_{i+1,i+2} + \cdots + \alpha_{j-1,j}, \qquad \forall i,j \in [n]. \tag{*}$$

This gives us a subset of $[0,\pi]^{\binom{n}{2}}$ which is convex by linearity of the constraint.

Since f is the finite sum of concave functions, it is as well a concave function. Moreover, it is a strictly concave function on $(0,\pi)^{\binom{n}{k}}$. Therefore,the restriction of f on (*) admits one unique maxima, which is global. By Theorem 2.6 and the definition of Z_S, the maximizer of problem (1.2) is unique up to orthogonal transformation of its vectors.

In order to show that the maximal zonotope is the regular $2n$-gone, it is enough to prove that the angles $\alpha_{i,j}$ between two adjacent vectors are all equals. To do so, we define a distance on the labels by

$$\Delta(i,j) = \min\{j-i,\ n-(j-i)\}, \tag{5.2}$$

which is the smallest number of angles $\alpha_{l,l+1}$ in which we can partition the one between the lines L_i and L_j. We remark that Δ takes integer values in $\left[\left\lfloor \frac{n}{2} \right\rfloor\right]$ and that it let us rewrite (5.1) as

$$K^2 \sum_{i<j} |sin(\alpha_{i,j})| = K^2 \left(\sum_{\Delta(i,j)=1} |sin(\alpha_{i,j})| + \cdots + \sum_{\Delta(i,j)=\lfloor n/2 \rfloor} |sin(\alpha_{i,j})| \right), \tag{5.3}$$

where $i,j \in [n]$. Since in each combination of i,j with $\Delta(i,j) = d$ we use d angles $\alpha_{l,l+1}$ and each angle between adjacent vectors has to be counted as much as the others, we have

$$\sum_{\Delta(i,j)=d} \alpha_{i,j} = \frac{d}{n}\pi N, \tag{5.4}$$

in which N is the number of combination of $i,j \in [n]$ with $\Delta(i,j) = d$. Since for each sinus sum in the right-hand side of (5.3) the sum over the variables is fixed by the last identity, we complete the proof by a consequence of *Karamata's inequality* ([4]).

This inequality states that, for an interval $I \subseteq \mathbb{R}$ and f a concave function, if $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are numbers in I such that *majorizes* $(y_1, \ldots, y_N)$, then

$$f(x_i) + \cdots + f(x_N) \leq f(y_1) + \cdots + f(y_N).$$

Where we say that $(x_1, \ldots, x_N)$ *majorizes* $(y_1, \ldots, y_N)$ if

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n$$

and, after relabelling the numbers by $x_{(1)}, \dots, x_{(N)}$ and $y_{(1)}, \dots, y_{(N)}$, respectively, in decreasing order, i.e.

$$x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(N)}, \qquad y_{(1)} \geq y_{(2)} \geq \cdots \geq y_{(N)},$$

we have

$$x_{(1)} + \cdots + x_{(i)} \geq y_{(1)} + \cdots + y_{(i)}, \qquad \forall i \in [N].$$

A direct consequence of this inequality is that, letting $a = \frac{x_1+\cdots+x_N}{N}$, every $(x_1, \dots, x_N)$ majorize $(a, \dots, a)$ and, thus, we have

$$f(x_i) + \cdots + f(x_N) \leq N f(a).$$

Therefore, applying this last result to our specific case, where $f(x) = |sin(x)|$, the sum is given by (5.4) and $I = [0, \pi]$, we get that for every sum

$$\sum_{\Delta(i,j)=d} |\sin(\alpha_{i,j})|,$$

the maximum is attained when $\alpha_{i,j} = \frac{d}{n}\pi$. Merging together this identity for every $d \in [\lfloor n/2 \rfloor]$, we obtain that $\alpha_{i,i+1} = \pi/n$, for all $i \in [n]$, maximizes each sum in the right-hand side of (5.3). Since all sums are maximized by the same solution, this also maximizes (5.1), thus proving the theorem. □

Thanks to Lemma 3.1 and the corollary of Lemma 2.7, we can continuously transform any (n, k)-uframe into one with all equals vectors lengths, because the squared length sum is fixed and the transformations changes several vectors at time we can bring them to the desired length. Moreover, since by uniqueness of the maxima we proved that all zonotopes with different vectors length have volume strictly smaller than the optimal case, we can choose the transformation in order to monotonously increase the volume. Furthermore, in the proof of last lemma we saw that, when in this case the volume is given by a strictly concave function of the angles between the vectors, therefore, by a convex optimisation argument, the following is proved

Theorem 5.5. *Let S be an (n, k)-uframe, then we can continuously transform it into S^* maximizer of* (1.2) *while increasing the volume of Z_S in a monotonous way.*

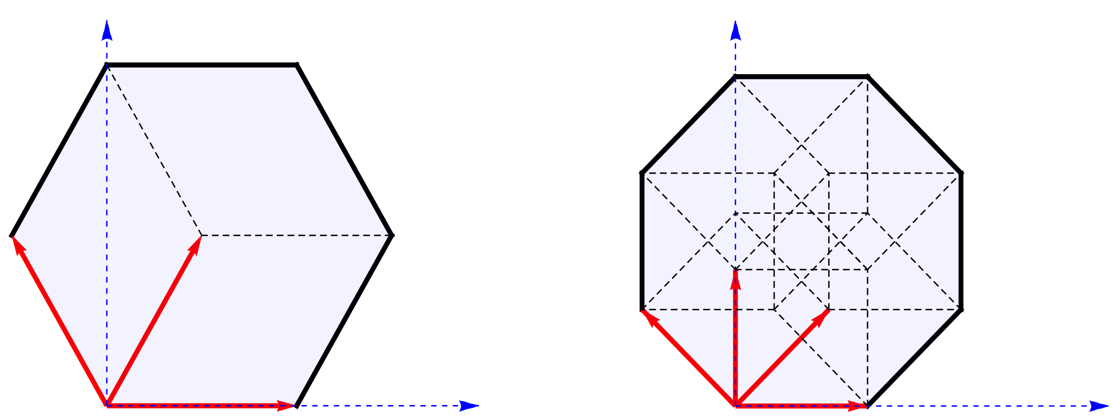

Figure 3: Optimal projections onto $\mathbb{R}^2$ of I^3 and I^4 respectively.

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