One of Diophantus's Texts (I.1)

To separate a given number into two numbers having a given difference.

Given number: 100, given difference 40.

[I] Let the lesser number required be x.

[II] The greater number is then $x + 40$.

[III] Then $2x + 40 = 100$.

[IV] Therefore $x = 30$.

Thus the required numbers are 30 and 70.

One of Euclid's Texts (VII.17)

If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

[I] Let the number A multiplied by the two numbers B and C make D and E.

[II] I say that B is to C as D is to E.

[III] Since A multiplied by B makes D, therefore B measures D according to the units in A. (VII.Def.20)

[IV] But the unit F also measures the number A according to the units in it, therefore the unit F measures the number A the same number of times that B measures D. Therefore the unit F is to the number A as B is to D. (VII.Def.20) (V.11)

[V] For the same reason the unit F is to the number A as C is to E, therefore B is to D as C is to E. (VII.13)

[VI] Therefore, alternately B is to C as D is to E.

Therefore, if a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

(*I find Euclid very hard to think about without diagrams. Most of the Editions of the Elements I've seen use diagrams of magnitudes – lines – for the propositions in Book VII. I think this is a mistake and that the numbers are better represented by collections of dots, as Euclid thinks of multitude in a very discrete way.*)

Euclid's Algebraic Tools

Common Notion 1 : Things which equal the same thing also equal one another.

(The transitive property)

Common Notion 2 : If equals are added to equals, then the wholes are equal.

(Adding the same thing to both sides of an equation)

Common Notion 3 : If equals are subtracted from equals, then the remainders are equal.

(Subtracting the same thing from both sides of an equation)

V.7 : Equal magnitudes have to the same [magnitude] the same ratio; and the same has to equal magnitudes the same ratio.

(Dividing both sides of an equation by the same thing)

V.9 : Magnitudes which have the same ratio to the same [magnitude] equal one another; and magnitudes to which the same [magnitude] has the same ratio are equal.

(Multiplying both sides of an equation by the same thing)

VII.5 : If a number is part of a number, and another is the same part of another, then the sum is also the same part of the sum that the one is of the one.

$(\frac{a}{n} + \frac{b}{n} = \frac{a+b}{n})$

VII.16 : If two numbers multiplied by one another make certain numbers, then the numbers so produced equal one another.

(Commutativity of multiplication)

VII:17 : If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

(Another way of multiplying both sides of an equation by the same thing)

II.1 : If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

(Distribution)

II.4 : If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

(Product of Binomials)

1. To separate a given number into two numbers having a given difference.

Diophantus's Argument

Given number: 100, given difference 40.

[I] Let the lesser number required be x.

[II] The greater number is then $x + 40$.

[III] Then $2x + 40 = 100$.

[IV] Therefore $x = 30$.

Thus the required numbers are 30 and 70.

Questions to Ponder

How many givens does this argument require?

When Diophantus represents the greater number as $x+40$, which of the givens is he making use of?

Diophantus does not explain how he moves from step three to step four? Can you explain that in terms Euclid would use?

Could you fulfill the requirements if you made the *greater* number x? How would the argument change?

Constructing a Proof

We can expand Diophantus's argument to include some of the bits he leaves as implicit and supply a justification for each of his claims. In the reasoning section, please indicate which of our tools convinces us of each step in the argument.

{G1}	Given number: 100,
{G2}	given difference 40.

	Statement	Reasoning
[I]	Let the lesser number required be x.	Assumption of the unknown.
[II]	The greater number is then $x + 40$	
(IIa)	Then $x + x + 40 = 100$,	
[III]	and $2x + 40 = 100$,	
(IIIa)	and $2x = 60$.	
[IV]	Therefore $x = 30$	

Practice With the Method

To separate a given number into two numbers having a given difference:

Given number: 80, given difference 20. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 70, given difference 15. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 50, given difference 12. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

Given number: 19, given difference $2\frac{3}{4}$. Let the lesser number required be _________ . The greater number is then _____________ . Then _____________ = _________ . Therefore _________ . Thus the required numbers are _____ and _____ .

1. Given number: 10, given difference: 4. Lesser number: _______ Greater number: _______
2. Given number: 20, given difference: 8. Lesser number: _______ Greater number: _______
3. Given number: 9, given difference: 3. Lesser number: _______ Greater number: _______
4. Given number: 11, given difference: 3. Lesser number: _______ Greater number: _______
5. Given number: 13, given difference: 3. Lesser number: _______ Greater number: _______
6. Given number: 13, given difference: 5. Lesser number: _______ Greater number: _______
7. Given number: 15, given difference: 5. Lesser number: _______ Greater number: _______
8. Given number: 9, given difference: 2. Lesser number: _______ Greater number: _______

9. Given number: 21, given difference: 6. Lesser number: ______ Greater number: ______

10. Given number: 20, given difference: $\frac{1}{2}$. Lesser number: ______ Greater number: ______

11. Given number: $\frac{7}{8}$, given difference: $\frac{1}{5}$. Lesser number: ______ Greater number: ______

12. Given number: $4\frac{3}{8}$, given difference: $1\frac{2}{3}$. Lesser number: ______ Greater number: ______

0.1. Does it Generalize?

When we tackle problems like this, we're often interested in whether we can solve *all* problems that have the same shape. In other words, can we generalize this problem? Let's start by generalizing over the difference:

To separate a given number into two numbers having an unknown given difference:

Given number: 10, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

Given number: 20, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

Given number: 100, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

We could generalize even further! Let's make *both* the given number and the given difference general.

Given number: **s**, given difference **d**. Let the lesser number required be ________ . The greater number is then ____________ . Then ____________ = ________ . Therefore ________ . Thus the required numbers are ________ and ________ .

Mapping Diophantus to a Modern Curriculum

Book One

1. To divide a given number into two having a given difference. $(2x + d = s)$

2. To divide a given number into two having a given ratio. $(kx = s)$

3. To divide a given number into two such that one is a given ratio of the other plus a given difference. $(x + k(x - d) = s)$

4. To find two numbers in a given ratio such that their difference is also given. $(kx - x = d)$

8. To two given numbers to add the same (required) number so as to make the resulting numbers have to one another a given ratio. $(\frac{x+k}{y+k} = s)$

9. From two given numbers to subtract the same (required) number so as to make the remainders have to one another a given ratio. $(\frac{x-k}{y-k} = s)$

11. Given two numbers, to add the first to, and subtract the second, from the same (required) number so as to make the resulting numbers have to one another a given ratio. $(\frac{p+x}{q-x} = s)$

14. To find two numbers such that their product has to their sum a given ratio. [One is arbitrarily assumed.] $(\frac{xy}{x+y} = k)$

27. To find two numbers such that their sum and product are given numbers. $(xy = k; x+y = q)$

28. To find two numbers such that their sum and sum of squares are given numbers. $(xy = k; x^2 + y^2 = q)$

29. To find two numbers such that their sum and difference of squares are given numbers. $(x^2 + y^2 = p; x^2 - y^2 = q)$

30. To find two numbers such that their difference and product are given numbers. $(x - y = p; xy = q)$

Book Two

8. To divide a given square into two squares. [Fermat!] $(a^2 + b^2 = c^2)$

9. To divide a given number which is the sum of two squares into two other squares. $(a^2 + b^2 = p^2 + q^2)$

10. To find two square numbers having a given difference. $(a^2 - b^2 = k)$

21. To find two numbers such that the square of either minus the other number gives a square. $(a^2 - b = p^2; b^2 - a = q^2)$

Towards a Renaissance of Classical Mathematics

The state of Classical Christian Education in mathematics right now is like the state of Europe at the beginning of the Renaissance: we're rediscovering the richness of the ancient texts, and figuring out how they can shape and inform our culture today. There's a lot of work yet to be done on that project, and lots of opportunities to be a part of that community.

Classical Mathematical Community Website: `https://williamcarey.github.io/cmc/`

Bill Carey – `billcarey@mac.com`

Materials for Teachers and Students

As I work through Diophantus annotating more of his puzzles/arguments for use with students, I'll post the materials at the classical mathematical community website.

Discussion Based Mathematics

Over the past year a group of classical math teachers have been gathering via zoom to do math together. We've looked at classical texts and played with mathematical puzzles. We've had debate style discussions, problem-centric discussions, and worked through conic sections and parabolas. And had a lot of fun. If you're interested in joining our mathematical discussion community, please reach out to me at `billcarey@mac.com` and let me know!