It's clear in this example that congruence of triangles is an equivalence.

Eg. Define the relation $R : \mathbb{Z} \longrightarrow \mathbb{Z}$ by aRb if $a \leq b$. This is reflexive and transitive but not symmetric since $2 \leq 3$ but $3 \nleq 2$. R is not an equivalence.

Eg. Define a relation $R : \mathbb{Z} \longrightarrow \mathbb{Z}$ by aRb if $b-a$ is divisible by 3. Is this an equivalence? We test each property

(i) Reflexivity: If $a \in \mathbb{Z}$, $a-a = 0$ and 3 divides 0. Hence R is reflexive.

(ii) Symmetry: Suppose aRb, then 3 divides $b-a$. But then 3 also divides $-(b-a) = a-b$ ie bRa. Hence symmetric.

(iii) Transitivity: Suppose aRb and bRc. Then 3 divides $b-a$ and 3 also divides $c-b$. Then we need to show $c-a$ is divisible by 3. But

$$c-a = (c-b)+(b-a)$$

which is divisible by 3 since it's a sum of pieces divisible by 3. Hence transitive and an equivalence.

This relation is called congruence modulo 3. If aRb we say a is congruent to b mod 3.

Eg. Define $R: \mathbb{Z} \longrightarrow \mathbb{Z}$ by aRb if and only if $a=b$ or $a=-b$. This is an equivalence. It's reflexive since $a=a$ for every $a \in \mathbb{Z}$. If aRb then $a=b$ or $a=-b$. If $a=b$ then $b=a$ and bRa. If $a=-b$ then $b=-a$ and bRa. Thus it's symmetric. If aRb and bRc then if $a=b$, $b=c$ then $a=c$ and aRc

$b=-c$ " $a=-c$ " aRc

if $a=-b$, $b=c$ " $a=-c$ " aRc

$b=-c$ " $a=c$ " aRc

hence transitive and an equivalence.

Equivalence Relations and Partitions:

Recall: A collection $A_1, A_2, \cdots, A_n$ of subsets of a set A, is a partition of A if and only if

(i) The $A_1, A_2, \cdots, A_n$ are pairwise disjoint, ie

$$A_i \cap A_j = \emptyset$$

for all $i, j = 1, \cdots, n$, $i \neq j$.

(ii) The union of $A_1, \cdots, A_n$ is A ie $\bigcup_{k=1}^{n} A_k = A$

[Note: $\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n$]

For equivalence relations it's common to use $\sim$ instead of R. Let $A \neq \emptyset$ and let $\sim : A \longrightarrow A$ be an equivalence (relation) then the <u>equivalence class</u> of a, denoted $[a]$ is the set

$$[a] = \{b \in A : a \sim b\}$$

If $a \sim b$ then we say a <u>is equivalent to</u> b. Thus $[a]$ is the set of all elements in A which are equivalent to a.

If $[a]$ is an equiv. class then a is called a representative of that class. The equiv. class is independent of it's representative ie we can use any member of the equiv. class as it's representative. Let's show this

Suppose $c \in [a]$ then we claim $[c] = [a]$.

Pf: Suppose $d \in [a]$ then $a \sim d$. Since $c \in [a]$ we also have $a \sim c$. By transitivity $d \sim c$ ie $d \in [c]$. Thus $[a] \subseteq [c]$. Similarly if $e \in [c]$ then $c \sim e$ and since $c \sim a$ we have by trans. again $a \sim e$ ie $e \in [a]$ and therefore $[c] \subseteq [a]$. Therefore $[c] = [a]$.

Let $\sim$ be an equivalence on A, $A \neq \emptyset$, then the following are equivalent:

① $a \sim b$

② $[a] = [b]$

③ $[a] \cap [b] \neq \emptyset$

Pf: ① $\Rightarrow$ ② Suppose $a \sim b$ then $b \in [a]$ and our previous result implies $[a]=[b]$.

(2) $\Rightarrow$ (3) If $[a]=[b]$ then $a\in[a]$ and $a\in[b]$ hence $[a]\cap[b]\neq\emptyset$.

(3) $\Rightarrow$ (1) Suppose $[a]\cap[b]\neq\emptyset$. Then there must exist some element $c\in[a]$ and $c\in[b]$. That is $a\sim c$ and $b\sim c$ and by trans. $a\sim b$.