We have a follow on result, namely that if $a \not\sim b$ then $[a] \cap [b] = \emptyset$.

Pf: Suppose that this is false, ie there exist $c \in [a] \cap [b]$ then $c \in [a]$ and $c \in [b]$ ie $c \sim a$ and $c \sim b$ and by transitivity $a \sim b$ and this contradicts $a \not\sim b$. Our assumption that $[a] \cap [b] \neq \emptyset$ is false.

Theorem: If $\sim$ is an equivalence on a non-empty set A then

$$\{[a] : a \in A\}$$

is a partition of A.

Pf:

$$\bigcup_{a \in A} [a] = A$$

Note: $\bigcup_{a \in A}$ means take the union of all possible equivalence classes for every a in A.

$\bigcup_{a \in A} [a] = A$ because $a \in [a]$ for $a \in A$. Also $[a] \cap [b] = \emptyset$ if $a \not\sim b$ (shown earlier) so the set of of equivalence classes are disjoint.

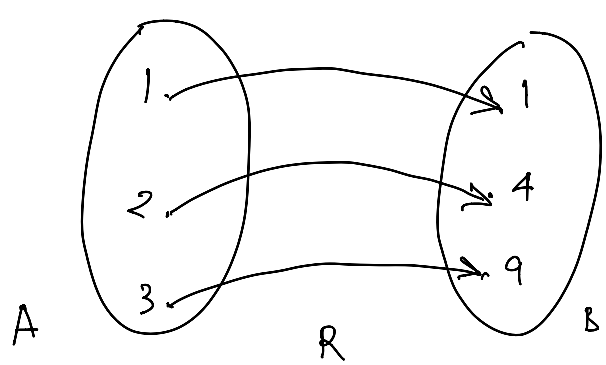
The set $\{[a] : a \in A\}$ is called the <u>quotient of A by $\sim$</u> and it's denoted by $A/\sim$

Functions: A second important class of binary relation is furnished by functions.

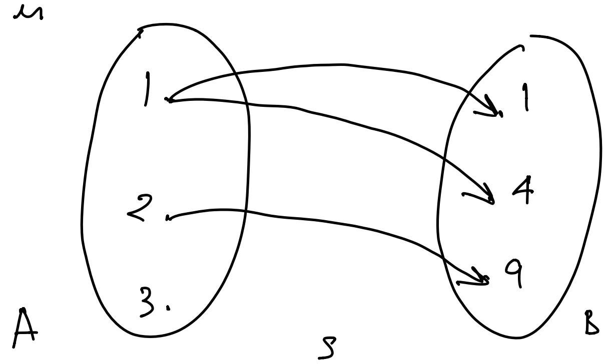
A binary relation $R: A \longrightarrow B$ is said to be a function if for every $a \in A$ there exists a unique pair (a, b) in R.

Eg. Suppose $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$ and $R: A \longrightarrow B$ is given by $R = \{(1,1), (2,4), (3,9)\}$ then R is relation and a function. If $S: A \longrightarrow B$ is given by $S = \{(1,1), (1,4), (2,9)\}$ then S isn't a function because there are two pairs in S with 1 in the domain position and there is no pair mapping 3 to an element of B. It's still a relation of course.

It's useful to visualise a relation in a diagram using arrows to indicate which elements get mapped to which. The function R is illustrated as



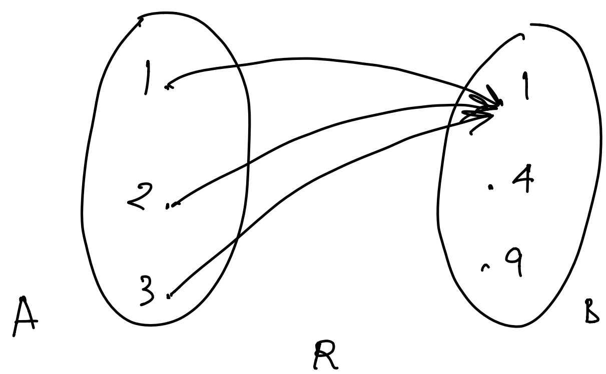
while S is



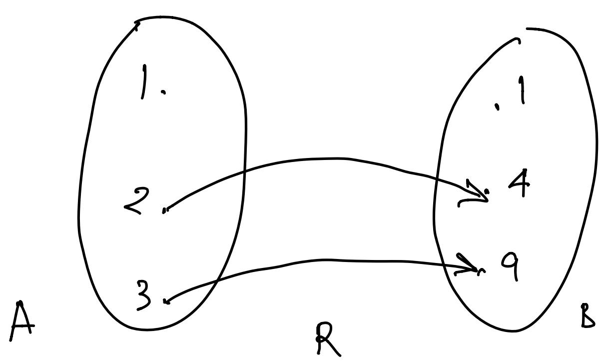
S isn't a function because 1 has two images and 3 has no image.

The key fact about a function is that every element in the domain must have exactly one image in the range.

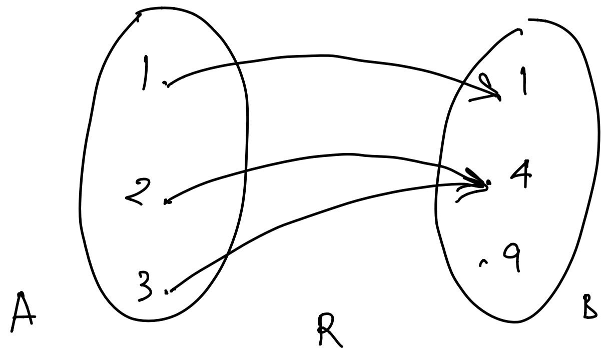
Examples:



This is a func. It's a constant function.



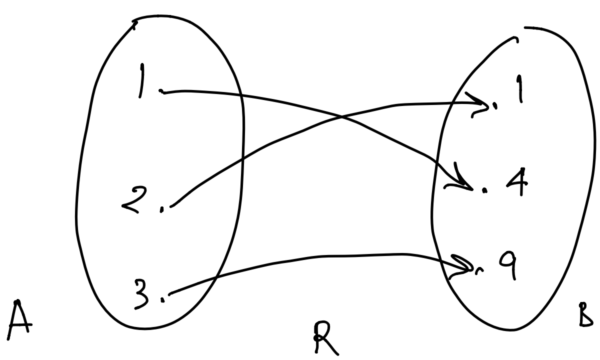
This isn't a func. as 1 has no image.



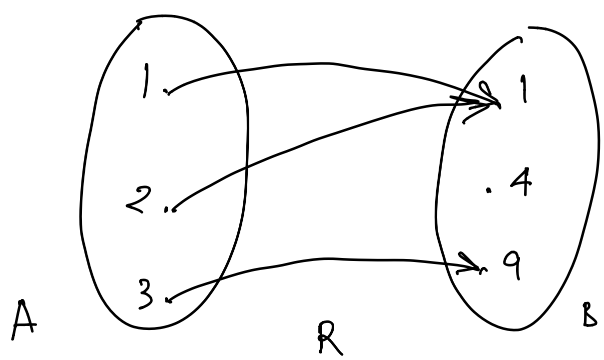
This is a function.

<u>Types of Functions:</u>

A function is <u>injective</u> or <u>one-to-one</u> if and only if no two distinct elements in the domain have the same image.



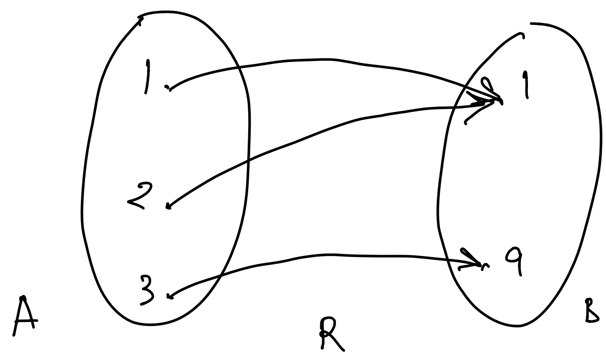
This is a func. and it's injective.



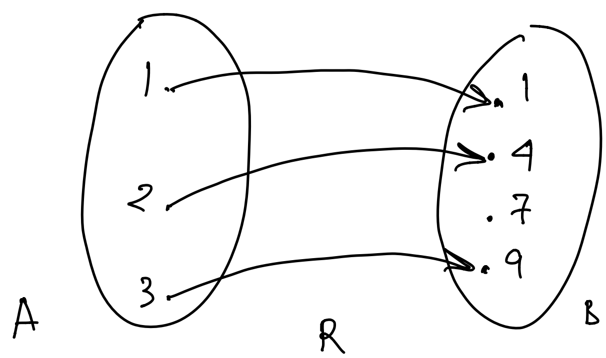
This is a func. but it's not injective because 1 and 2 have the same image.

A function is <u>surjective</u> or <u>onto</u> if every element in the range is the image of some element in the domain.

In the previous example, 4 isn't the image of anything in A so it isn't surjective.



This func. is surjective but not injective.



This is a func. which is injective but not surjective.

If a function is injective and surjective we say it's <u>bijective</u>. Such functions are important because they have inverses.

<u>Inverses</u>: We say two functions $f: A \longrightarrow B$ and $g: B \longrightarrow A$ are inverses if whenever $a \in A$ and b is an element of b, then

$$g(f(a)) = a$$

and

$$f(g(b)) = b$$

Note: By $f(a)$ we mean the element in B which is the image of a under the function f.

A function has an inverse if and only if it is bijective.