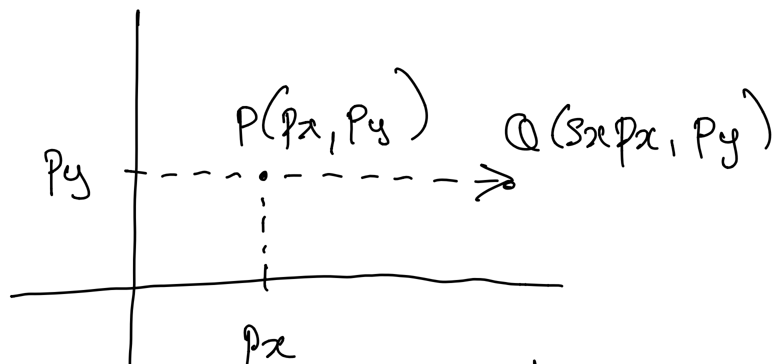
Scaling:

Consider the matrix

$$S = \begin{pmatrix} s_x & 0 \\ 0 & 1 \end{pmatrix}$$

and let's operate on the point P with coordinates (p_x, p_y).

$$\begin{pmatrix} s_x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} s_x p_x + 0 p_y \\ 0 p_x + 1 p_y \end{pmatrix} = \begin{pmatrix} s_x p_x \\ p_y \end{pmatrix}$$



The x coord. has been stretched by the factor s_x.

Eg. The unit square has vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Transform this square using the scaling matrix

$$S = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

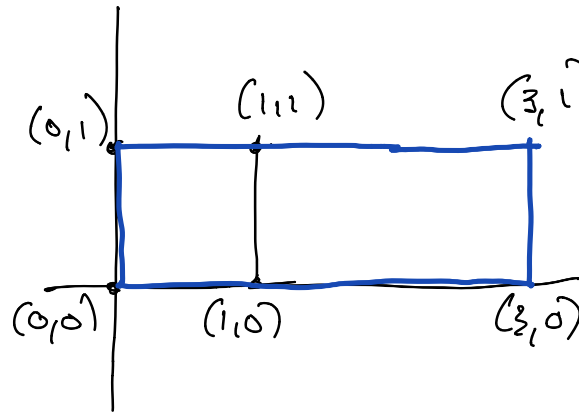
Multiply each vertex by this matrix.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



When we transform using matrices, it can be shown that straight lines get transform-ed to straight lines. This means that for shapes made out of straight lines, such as polygons, we only need to transform the vertices, ie the corners. If the factor S_x is less than 1 then there is a compression in the x-dir. by that factor. If $S_x > 1$ there is a stretching and if $S_x = 1$ there is no change.

In a similar manner the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & S_y \end{pmatrix}$$

produces the same effects in the y-dir. You can scale in the x and y-dirs. at the same time using

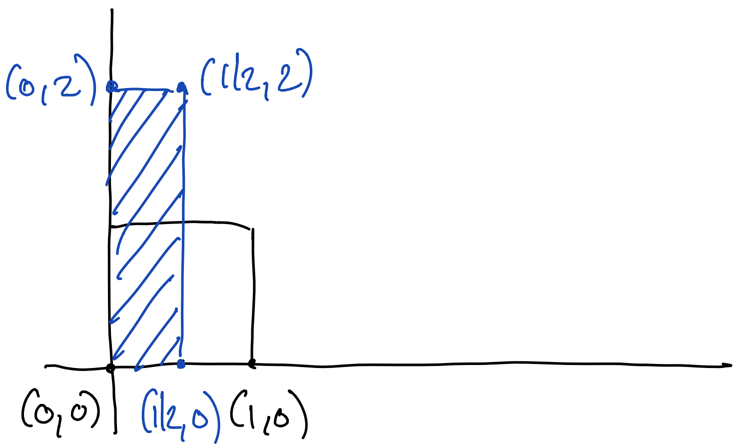
$$S = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix}$$

Eg. Scale the unit square from the previous example by a factor of $1/2$ in the x-dir. and a factor of 2 in the y-dir. The scaling matrix is

$$S = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

We can do this in the following way by combining the vertices into a single matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

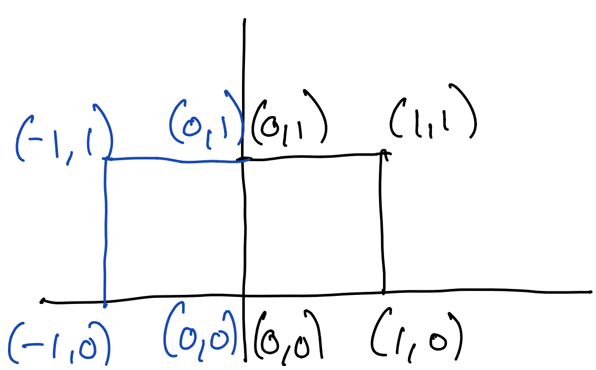


<u>Reflections</u>: We can reflect in the y-axis using the matrix

$$M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let's reflect our unit square.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



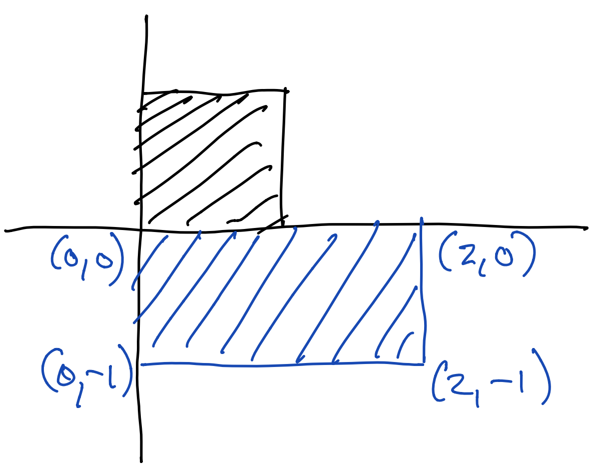
Similarly $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ will reflect in the x-axis.

We can combine two or more operations by performing them successively. For eg. If want to scale by a factor of 2 in the x-dir. and then reflect in the x-axis we do the following

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

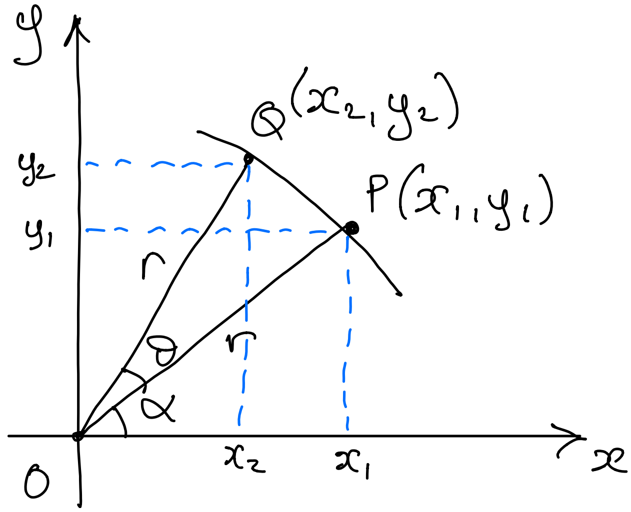
Recall that $A(BC) = (AB)C$. We can mult. the LH two 1st. to get

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$



The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ has the same effect as the original scaling and reflecting matrices. Multiple transformations can always be combined in this way to obtain a single matrix that does the same job. Just remember that the order matters.

<u>Rotation</u>: Suppose we want to rotate a point P counter-clockwise about the origin.



Want to rotate P through an angle θ about O in the ccw (counterclockwise) dir. α is angle the line through P and O makes. Q is the resulting pt. Let r be the dist. from O to P. In a rotation about a pt. the dist. from the pt. of rotation is unchanged so OQ also has length r.

From the dia. (diagram) above

$$x_1 = r\cos\alpha$$
$$y_1 = r\sin\alpha$$

Similarly

$$x_2 = r\cos(\alpha+\theta)$$
$$y_2 = r\sin(\alpha+\theta)$$

Basic trig. identities:

$$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$$
$$\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$$

Therefore

$$x_2 = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$y_2 = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$$

ie

$$x_2 = x_1\cos\theta - y_1\sin\theta$$
$$y_2 = y_1\cos\theta + x_1\sin\theta$$

ie

$$x_2 = x_1\cos\theta - y_1\sin\theta$$
$$y_2 = x_1\sin\theta + y_1\cos\theta$$

In matrix form

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

The matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the transformation matrix for a ccw rot. (rotation) by θ about O.