Mathematics 2 **Proof By Induction**

Q3. Using proof by induction prove each of the following are true, for all $n \in \mathbb{Z}_+$:

i. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

ii. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$

iii. $1 + 8 + 16 + \cdots + 8n = (2n + 1)^2$

iv. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

v. $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \dfrac{(n+1)(2n+1)(2n+3)}{3}$

vi. $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = \dfrac{3}{4}(5^{n+1} - 1)$

vii. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

viii. $8 | (9^n - 1)$

ix. $7 | (8^n - 1)$

Q3 In the following solutions I have omitted the step where you show the results is true for the case $n = 1$. You must include that part of the solution in your answers otherwise you will lose marks. I've omitted it here because it's generally very simple and it would just add to the length of the solutions. For the same reasons I've omitted the final step where you conclude that the result is true for all $n \in \mathbb{Z}_+$. I do the first one completely however, by way of an example.

(i). Prove by induction

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

If $n = 1$, the LHS is just 1 and the RHS is 1^2 and since

$$1 = 1^2$$

the result is true when $n = 1$.

Assume the result is true when $n = k$, for some positive integer k, i.e.

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

is true.

Then, if $n = k + 1$, we have

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

That is

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$$

is also true. But this is just the statement we want to prove with n set to $k+1$. Thus, if the statement is true for $n = k$, then it is also true for $n = k+1$ and since the statement is true when $n = 1$, by induction it is true for all $n \in \mathbb{Z}_+$. In other words, we've shown that if the statement if true for any value of n then it is also true for the next value. Since we showed the statement is true for the value $n = 1$ we can conclude that it is true for $n = 2, n = 3$ and so on and therefore it is true for every positive integer n.

(ii). Prove by induction

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

Assume true for $n = k$.

$$
\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right) + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{2^2} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1 + 2 + 3 + \cdots + (k+1))^2
\end{aligned}
$$

(iii). Prove by induction

$$1 + 8 + 16 + \cdots + 8n = (2n+1)^2$$

Assume true for $n = k$.

$$
\begin{aligned}
1 + 8 + 16 + \cdots + 8k + 8(k+1) &= (2k+1)^2 + 8(k+1) \\
&= 4k^2 + 4k + 1 + 8k + 8 \\
&= 4k^2 + 12k + 9.
\end{aligned}
$$

Now

$$
\begin{aligned}
(2(k+1)+1)^2 &= 2^2(k+1)^2 + 2(2)(k+1) + 1^2 \\
&= 4(k^2 + 2k + 1) + 4(k+1) + 1 \\
&= 4k^2 + 8k + 4 + 4k + 4 + 1 \\
&= 4k^2 + 12k + 9.
\end{aligned}
$$

Other variations are possible. For example, having found

$$1 + 8 + 16 + \cdots + 8k + 8(k+1) = 4k^2 + 12k + 9,$$

we could continue as follows.

$$
\begin{aligned}
4k^2 + 12k + 9 &= (2k)^2 + 2(2k)(3) + 3^2 \\
&= (2k+3)^2 \\
&= (2k+2+1)^2 \\
&= (2(k+1)+1)^2
\end{aligned}
$$

as required.

(iv). Prove by induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Assume true for $n = k$.

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

(v). Prove by induction

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Assume true for $n = k$.

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2(k+1)+1)^2 &= \frac{(k+1)(2k+1)(2k+3)}{3} + (2(k+1)+1)^2 \\
&= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3} \\
&= \frac{(2k+3)[(k+1)(2k+1) + 3(2k+3)]}{3} \\
&= \frac{(2k+3)[2k^2 + k + 2k + 1 + 6k + 9]}{3} \\
&= \frac{(2k+3)(2k^2 + 9k + 10)}{3} \\
&= \frac{(2k+3)(k+2)(2k+5)}{3} \\
&= \frac{(k+2)(2k+3)(2k+5)}{3} \\
&= \frac{((k+1)+1)(2(k+1)+1)(2(k+1)+3)}{3}
\end{aligned}$$

(vi). Prove by induction

$$3 + 3.5 + 3.5^2 + \cdots + 3.5^n = \frac{3(5^{n+1} - 1)}{4}$$

Divide out the common factor of 3. Assume true for $n = k$.

$$\begin{aligned} 1 + 5 + 5^2 + \cdots + 5^k + 5^{k+1} &= \frac{5^{k+1} - 1}{4} + 5^{k+1} \\ &= \frac{5^{k+1} + 4.5^{k+1} - 1}{4} \\ &= \frac{5.5^{k+1} - 1}{4} \\ &= \frac{5^{k+2} - 1}{4} \end{aligned}$$

(vii). Prove by induction

$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Assume true for $n = k$.

$$\begin{aligned} 1.2 + 2.3 + 3.4 + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)[k(k+2) + 3(k+2)]}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

(viii). I'll do this one in a bit more detail. The rest are similar.

Prove by induction that

$$9^n - 1$$

is divisible by 8 for all positive integers n.

First, when $n = 1$ the expression $9^n - 1$ is $9^1 - 1 = 8$ which is divisible by 8 so the statement is true when $n = 1$.

Now assume the statement is true when $n = k$, where k is a fixed, but unspecified positive integer, i.e.

$$9^k - 1$$

is divisible by 8.

Then, if $n = k + 1$, we have

$$\begin{aligned} 9^{k+1} - 1 &= 9 \cdot 9^k - 1 \\ &= (8+1) \cdot 9^k - 1 \\ &= 8 \cdot 9^k + 9^k - 1 \end{aligned}$$

This is a sum of two pieces, namely $8 \cdot 9^k$ and $9^k - 1$. The first of these is 8 times something and therefore is divisible by 8. The second is divisible by 8 because we assumed it in our starting hypothesis, namely that $9^k - 1$ is divisible by 8. Each piece in the sum is divisible by 8 and therefore the sum is also divisible by 8. But this sum is just $9^{k+1} - 1$ so it is divisible by 8. Thus, if the statement is true for $n = k$, then it is also true for $n = k + 1$ and since the statement is true when $n = 1$, by induction it is true for all $n \in \mathbb{Z}_+$. In other words, we've shown that if the statement if true for any value of n then it is also true for the next value. Since we showed the statement is true for the value $n = 1$ we can conclude that it is true for $n = 2, n = 3$ and so on and therefore it is true for every positive integer n.

(ix). Prove by induction that

$$8^n - 1$$

is divisible by 7 for all positive integers n. Assume true when $n = k$. Then

$$\begin{aligned} 8^{k+1} - 1 &= 8 \cdot 8^k - 1 \\ &= 7 \cdot 8^k + 8^k - 1 \end{aligned}$$

and this is divisible by 7 since $7 \cdot 8^k$ is divisible by 7, and by hypothesis, so is $8^k - 1$.