Proof By Induction

Induction

Induction is a powerful method for showing a property is true for all non-negative integers. Induction plays a central role in discrete mathematics and computer science.

To understand how induction works, suppose there is a professor who brings to class a bottomless bag of assorted miniature candy bars. She offers to share the candy in the following way. First, she lines the students up in order. Next she states two rules:

1. The student at the beginning of the line gets a candy bar.

2. If a student gets a candy bar, then the following student in line also gets a candy bar.

Let's number the students by their order in line, starting the count with 0, as usual in computer science. Now we can understand the second rule as a short description of a whole sequence of statements:

- If student 0 gets a candy bar, then student 1 also gets one.
- If student 1 gets a candy bar, then student 2 also gets one.
- If student 2 gets a candy bar, then student 3 also gets one.

$$\vdots$$

Of course this sequence has a more concise mathematical description:

> If student n gets a candy bar, then student $n + 1$ gets a candy bar, for all nonnegative integers n.

So suppose you are student 17. By these rules, are you entitled to a miniature candy bar? Well, student 0 gets a candy bar by the first rule. Therefore, by the second rule, student 1 also gets one, which means student 2 gets one, which means student 3 gets one as well, and so on. By 17 applications of the professor's second rule, you get your candy bar! Of course the rules actually guarantee a candy bar to *every* student, no matter how far back in line they may be.

The reasoning that led us to conclude that every student gets a candy bar is essentially all there is to induction.

The Principle of Induction.

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- $P(n)$ IMPLIES $P(n+1)$ for all nonnegative integers, n,

then

- $P(m)$ is true for all nonnegative integers, m.

Rule. *Induction Rule*

$$\frac{P(0), \quad \forall n \in \mathbb{N}.\, P(n) \text{ IMPLIES } P(n+1)}{\forall m \in \mathbb{N}.\, P(m)}$$

This general induction rule works for the same intuitive reason that all the students get candy bars, and we hope the explanation using candy bars makes it clear why the soundness of the ordinary induction can be taken for granted. In fact, the rule is so obvious that it's hard to see what more basic principle could be used to justify it.[1] What's not so obvious is how much mileage we get by using it.

Example 1

Prove by mathematical induction the following theorem

For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof:

- $\underline{P(n)}$ is the predicate that $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$
- $\underline{\text{Show } P(1) \text{ is true}:}$

$LHS: \ P(1) = 1$

$RHS: P(1) = \dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$

$\Rightarrow \quad LHS = RHS$

$\Rightarrow P(1)$ is true

• $\underline{P(n)}$ is the predicate that $1+2+3+...+n=\dfrac{n(n+1)}{2}$

• $\underline{\text{Show P(n+1)}}$

The basic plan for proving validity of any implication is to assume the statement on the left and prove the statement on the right.

In this case, we assume $P(n)$ in order to prove $P(n+1)$ which is the equation

$$\underbrace{1+2+3+...+n}_{P(n)}+(n+1)=\frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)}{2}+(n+1)$$

$$\frac{n(n+1)+2(n+1)}{2}$$

$$\frac{n^2+n+2n+2}{2}$$

$$\frac{(n+1)(n+2)}{2} \quad = \quad RHS$$

Thus if $P(n)$ is true then so is $P(n+1)$.

Therefore the induction principle says that the predicate

$P(m)$ is true for all non-negative intergers, m.

Example 2

Prove by mathematical induction the following theorem

For all $n \in \mathbb{N}$,

$$3 + 6 + 9 + \ldots (3n) = \frac{3n(n+1)}{2}$$

Proof:

- $\underline{P(n)}$ is the predicate that $3 + 6 + 9 + \ldots + (3n) = \dfrac{3n(n+1)}{2}$
- $\underline{\text{Show } P(1) \text{ is true:}}$

$LHS : P(1) = 3$

$RHS : P(1) = \dfrac{3(1)(1+1)}{2} = \dfrac{6}{2} = 3$

$\Rightarrow \quad LHS = RHS$

$\Rightarrow P(1)$ is true

- $\underline{P(n)}$ is the predicate that $3+6+9+...+(3n)=\dfrac{3n(n+1)}{2}$
- $\underline{\text{Show P(n+1)}}$

The basic plan for proving validity of any implication is to assume the statement on the left and prove the statement on the right.

In this case, we assume $P(n)$ in order to prove $P(n+1)$ which is the equation

$$\underbrace{3+6+9+...+(3n)}_{P(n)}+3(n+1)=\frac{3n(n+1)(n+2)}{2}$$

$$\frac{3n(n+1)}{2}+3(n+1)$$

$$\frac{3n(n+1)+6(n+1)}{2}$$

$$\frac{3n^2+3n+6n+6}{2}$$

$$\frac{3n(n+1)(n+2)}{2} \quad = \quad RHS$$

Thus if $P(n)$ is true then so is $P(n+1)$.

Therefore the induction principle says that the predicate

$P(m)$ is true for all non-negative intergers, m.