Probability 1

Engineering systems are designed to operate well in the face of uncertainty of characteristics of components and operating conditions. In some case, uncertainty is introduced in the operations of the system, on purpose.

Understanding how to model uncertainty and how to analyze its effects is – or should be – an essential part of an engineer's education. Randomness is a key element of all systems we design. Communication systems are designed to compensate for noise. Internet routers are built to absorb traffic fluctuations. Building must resist the unpredictable vibrations of an earthquake. The power distribution grid carries an unpredictable load. Integrated circuit manufacturing steps are subject to unpredictable variations. Searching for genes is looking for patterns among unknown strings.

1. Sample Space and Events

Definitions

Random experiment: involves obtaining observations of some kind

Examples Toss of a coin, throw a die, polling, inspecting an assembly line, counting arrivals at emergency room, etc.

Population: Set of all possible observations. Conceptually, a population could be generated by repeating an experiment indefinitely.

Outcome of an experiment:

Elementary event (simple event): one possible outcome of an experiment

Event (Compound event): One or more possible outcomes of a random experiment

Sample space: the set of all sample points (simple events) for an experiment is called a sample space; or set of all possible outcomes for an experiment

Notation.

Sample space : S

Sample point: $E_1, E_2, \ldots$ etc.

Event: A, B, C, D, E *etc.* (any capital letter).

Example.

$S = \{E_1, E_2, \ldots, E_6\}$.

That is $S = \{1, 2, 3, 4, 5, 6\}$. We may think of S as representation of possible outcomes of a throw of a die.

More definitions

Union, Intersection and Complementation

Given A and B two events in a sample space S.

1. The *union* of A and B, $A \cup B$, is the event containing all sample points in either A *or* B or both. Sometimes we use $A or B$ for union.

2. The *intersection* of A and B, $A \cap B$, is the event containing all sample points that are both in A *and* B. Sometimes we use AB or $A and B$ for intersection.

3. The *complement* of A, A^c, is the event containing all sample points that are *not in* A. Sometimes we use $not A$ or $\overline{A}$ for complement.

Mutually Exclusive Events (Disjoint Events) Two events are said to be mutually exclusive (or disjoint) if their intersection is empty. (i.e. $A \cap B = \phi$).

Example Suppose $S = \{E_1, E_2, \ldots, E_6\}$. Let

$A = \{E_1, E_3, E_5\}$;

$B = \{E_1, E_2, E_3\}$. Then

(i)$A \cup B = \{E_1, E_2, E_3, E_5\}$.

(ii) $AB = \{E_1, E_3\}$.

(iii) $A^c = \{E_2, E_4, E_6\}$; $B^c = \{E_4, E_5, E_6\}$;

(iv) A and B are not mutually exclusive (why?)

Experiment. Toss a coin 3 times and observe the sequence of heads (h) and tails (t). Sample space consists of the 8 elements:

$$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$$

Let A be event that 2 or more heads appear consecutively, and

B that the tosses have the same outcome.

Then $A = \{hhh, hht, thh\}$,

$B = \{hhh, ttt\}$

What is $A \cap B$?

Soln:

$A \cap B = \{hhh\}$, i.e. an elementary event.

2.Probability of an Event

Consider a random experiment whose sample space is S with sample points $E_1, E_2, \ldots,$. For each event E_i of the sample space S define a number $P(E)$ that satisfies the following three conditions:

(i) $0 \leq P(E_i) \leq 1$ for all i

(ii) $P(S) = 1$

(iii) (Additive property)

$$\sum_S P(E_i) = 1,$$

where the summation is over all sample points in S.

We refer to $P(E_i)$ as the probability of the E_i.

Definition The probability of any event A is equal to the sum of the probabilities of the sample points in A.

Example. Let $S = \{E_1, \ldots, E_{10}\}$. It is known that $P(E_i) = 1/20, i = 1, \ldots, 6$ and $P(E_i) = 1/5, i = 7, 8, 9$ and $P(E_{10}) = 2/20$. In tabular form, we have

E_i	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
$p(E_i)$	1/20	1/20	1/20	1/20	1/20	1/20	1/5	1/5	1/5	1/10

Question: Calculate $P(A)$ where $A = \{E_i, i \geq 6\}$.
A:

$$\begin{aligned} P(A) &= P(E_6) + P(E_7) + P(E_8) + P(E_9) + P(E_{10}) \\ &= 1/20 + 1/5 + 1/5 + 1/5 + 1/10 = 0.75 \end{aligned}$$

Steps in calculating probabilities of events

1. Define the experiment
2. List all simple events
3. Assign probabilities to simple events
4. Determine the simple events that constitute an event
5. Add up the simple events' probabilities to obtain the probability of the event

Example Calculate the probability of observing one H in a toss of two fair coins.
Solution.
$S = \{HH, HT, TH, TT\}$
$A = \{HT, TH\}$
$P(A) = 0.5$

Example:

Toss a dice and note the number that appears on top. Then sample space consists of

$S = \{1,2,3,4,5,6\}$.

1) What is the probability of an even number appearing?
2) What is the probability of an even or prime number appearing?
3) What is the probability of an odd number appearing?
4) What is the probability of an odd prime number appearing?

Soln:

Let A = event that an even no. occurs, A = {2,4,6}

Let B = event that an odd number occurs,

B = {1,3,5}

Let C = event that a prime no. occurs C = {2,3,5}

Then

1. P(A) = 3/6 = ½ = 0.5
2. $P(A \cup C)$ = 5/6 = 0.833
3. P(B) = 3/6 = ½ = 0.5
4. P(Odd Prime) = P(3, 5) = 2/6 = 0.33

Example

Three horses A,B,C race;
A is twice as likely to win as B,
and B is twice as likely to win as C.

- Find their probabilities of winning;
  denoted P(A), P(B) and P(C).

- Also find the probability that B or C wins.

Soln:

Let $p = P(C)$.

Since B is twice as likely to win

$$P(B) = 2p.$$

Since A is twice as likely as B to win

$$P(A) = 2P(B) = 4p.$$

Since the sum must be equal 1, we get

$$p + 2p + 4p = 1, \text{ so } p = 1/7.$$

Then

$$P(A) = 4/7, P(B) = 2/7 \text{ and } P(C) = 1/7.$$

Also $P(B \cup C) = P(B) + P(C) = 2/7 + 1/7 = 3/7$
since B and C are **mutually exclusive** events

Equally Likely Outcomes

The equally likely probability P defined on a finite sample space $S = \{E_1, \dots, E_N\}$, assigns the same probability $P(E_i) = 1/N$ for all E_i.

In this case, for any event A

$$P(A) = \frac{N_A}{N} = \frac{\text{sample points in } A}{\text{sample points in } S} = \frac{\#(A)}{\#(S)}$$

where N is the number of the sample points in S and N_A is the number of the sample points in A.

Example. Toss a fair coin 3 times.

(i) List all the sample points in the sample space

Solution: $S = \{HHH, \cdots TTT\}$ (Complete this)

(ii) Find the probability of observing exactly two heads, at most one head.

3. Laws of Probability

Conditional Probability

The *conditional probability* of the event A given that event B has occurred is denoted by $P(A|B)$. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$. Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Independent Events

Definitions. (i) Two events A and B are said to be *independent* if

$$P(A \cap B) = P(A)P(B).$$

(ii) Two events A and B that are not independent are said to be *dependent*.

Remarks. (i) If A and B are independent, then
$P(A|B) = P(A)$ and $P(B|A) = P(B)$.
(ii) If A is independent of B then B is independent of A.

Probability Laws

Complementation law:

$$P(A) = 1 - P(A^c)$$

Additive law:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Moreover, if A and B are mutually exclusive, then $P(AB) = 0$ and

$$P(A \cup B) = P(A) + P(B)$$

Multiplicative law (Product rule)

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ &= P(B|A)P(A) \end{aligned}$$

Moreover, if A and B are independent

$$P(AB) = P(A)P(B)$$

Example Let $S = \{E_1, E_2, \ldots, E_6\}$; $A = \{E_1, E_3, E_5\}$; $B = \{E_1, E_2, E_3\}$; $C = \{E_2, E_4, E_6\}$;$D = \{E_6\}$. Suppose that all elementary events are equally likely.

(i) What does it mean that all elementary events are equally likely?

(ii) Use the complementation rule to find $P(A^c)$.

(iii) Find $P(A|B)$ and $P(B|A)$

(iv) Find $P(D)$ and $P(D|C)$

Law of total probability Let the B, B^c be complementary events and let A denote an arbitrary event. Then

$$P(A) = P(A \cap B) + P(A \cap B^c)\ ,$$

or

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Bayes' Law

Let the B, B^c be complementary events and let A denote an arbitrary event. Then

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Example:
Suppose E is the event that a randomly generated bit string of length 4 begins with a 1, and F is the event that a randomly generated bit string contains an even number of 0s. Are E and F independent?

Solution: Obviously $p(E) = ½ = p(F)$.

$E \cap F = \{1111,1001,1010,1100\}$
$p(E \cap F) = 4/16 = 1/4$
$p(E \cap F) = p(E)p(F)$.
Conclusion: E and F are independent

Examples

1) A random experiment consists of rolling two fair dice once. Define the events

   A: the numbers rolled are not the same

   B: the sum of the two dice is 7

   C: the sum of the two dice is 8

   D: both numbers rolled are odd

   E: one number rolled is odd and the other is even

   And calculate $P(B|A)$, $P(C|A)$, $P(D|A)$ and $P(E|A)$ giving your answer as a fraction in each case. Is event C independent of A?

2) In a used car sales forecourt there are 9 cars. 5 of these cars are red, 3 are Volkswagen and 2 are both. If a car is selected at random from the forecourt then answer the following giving each answer as a fraction:

1) What is the probability the car is red?
2) What is the probability the car is not a Volkswagen?
3) What is the probability the car is Volkswagen but not red?
4) What is the probability that the car is red or Volkswagen?
5) If the car is red what is the probability that it is also a Volkswagen?