Predicate Logic

Motivation

Consider the following examples of ‘reasoning’:

1(a)
$$\frac{\textit{10 is a number which is the sum of 4 squares}}{\therefore\ \textit{There is a number which is the sum of 4 squares}}$$

2(a)
$$\frac{\begin{array}{c}\textit{Every student at this University pays fees}\\ \textit{Monica is a student at this University}\end{array}}{\therefore\ \textit{Monica pays fees}}$$

In each case the conclusion seems to ‘follow’ from the assumptions/premises. But in what sense? What do we mean by ‘follow’?
Since such arguments are common in our everyday lives, it would seem worthwhile to understand and answer this question, and that’s what logic is all about, it’s the study of ‘valid reasoning or argument’.

In both the above examples the reasoning seems to be 'valid' (which right now just equates with 'OK'), but what does this mean? A first guess here is that it means: The conclusion is true given that the premises are true. This is close, but we have to be careful here. Consider for example the argument:

$$3(a) \qquad \frac{\textit{There is a number which is the sum of 4 squares}}{\therefore \textit{Every number is the sum of 4 squares}}$$

This does not seem to be 'valid' in the sense of the first two examples, despite the fact that the assumption and conclusion are actually true.

The reason the first two arguments are valid and the last is not is that they do not actually depend on the meaning of 'sum of 4 squares', 'Monica', '10', 'student at this university', 'pays fees' nor what universe of objects (natural numbers in the first and last, students in the second) we are referring to, whereas in the last the <u>meaning</u> of 'is the sum of 4 squares' does matter. For example if we change 'sum of 4 squares' to 'sum of 3 squares' then the premiss is true but the conclusion false.

To see this let's write

$\forall$ for 'for all'
$\exists$ for 'there exists'
c for 10
$P(x)$ for 'x is the sum of 4 squares'

Then our first and last examples become:

1(b) $$\frac{P(c)}{\therefore\ \exists x\, P(x)}$$

3(b) $$\frac{\exists x\, P(x)}{\therefore\ \forall x\, P(x)}$$

Clearly the conclusion in the first of these 'follows' no matter what universe the x ranges over, no matter what element in that universe c stands for and no matter what property of x P(x) stands for.
In other words, no matter what they stand for, if the premises are true then so is the conclusion.

For example, if we take this universe to be the set of all buses along O’Connell St, c stands for the number 747 bus and P(x) means that bus x goes to the airport, then the first argument would become

$$1(c) \quad \frac{\text{The 747 bus goes to the airport}}{\therefore \text{ There is a bus on O'Connell St that goes to the airport}}$$

which we would surely accept as 'OK'.

However, in the second case we obtain

$$3(\text{c}) \quad \frac{\text{There is a bus on O'Connell St that goes to the airport}}{\therefore \text{ All buses along O'Connell St go to the airport}}$$

and now the conclusion is false, while the premiss is true, so this clearly is not an 'OK' argument.

Similarly in the Monica example if we let

m stand for Monica
$S(x)$ stand for 'x is a student at this university'
$F(x)$ stand for 'x pays fees'
$\rightarrow$ stand for 'if . . . then', equivalently 'implies',

then the example becomes

$$
2(\mathrm{b}) \qquad \begin{array}{l} \forall x\,(S(x) \rightarrow F(x)) \\ S(m) \\ \hline \therefore F(m) \end{array}
$$

and again this looks an OK argument no matter what universe of objects the variable x ranges over, no matter what element of this universe m stands for and no matter what properties of such x, $S(x)$ and $F(x)$ stand for.

In other words, no matter what meaning (or *interpretation*) we give to this universe, m and $S(x), F(x)$, if the premises are true then so is the conclusion. The validity of the Monica example 2 derives from this fact. The non-validity of our 'all numbers are the sum of 4 squares' example 3 is a consequence of this failing in this case, despite the fact that in this interpretation the conclusion of 3(a) is true.

What we have learnt here is that to understand and investigate 'valid' arguments we need to study formal examples like the one above where all meaning has been stripped away, where we have been left with just the essential bare bones.

What is a Predicate?

A **predicate** (or propositional function) is a statement involving a variable. It usually describes a property about something.

Examples:

- S(x) = “x is a student”
- P(x) = “ x is a professor”
- B(x) = “ x is blue”
- H(x,y) = “ x has y”

A predicate is a **proposition with variables**

Predicates

Suppose we have

$$P(x, y) = [x + 2 = y]$$

If

$x = 1$ and $y = 3$: $P(1,3)$ is true

If

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Example

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Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Domain of a Predicate

In the predicate

P(x) = “2x is an even integer”,

it is important that the domain D of P(x) is defined, for instance D={Integers}

The **domain, D,** is the set where the x’s come from i.e. the set of possible x values.

Quantifiers

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

What is a Quantifier?

Most statements in maths and computer science use terms such as ‘for every’ and ‘for some’.

<u>Example:</u>

- For every triangle T, the sum of the angles of T is 180°.
- For *every* integer n, n is less than p, for *some* prime number p.

A <u>**quantifier**</u> is a logical symbol which makes an assertion about the set of values which make one or more formulas true. We will discuss 2 important quantifiers.

Quantifiers

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Example 1:

Suppose

$P(x)$ = "x has a mobile phone"

and D = {set of students}

Then

1. $\forall x, P(x)$ means "Every student has a mobile phone".
2. $\exists x, P(x)$ means "There exists some student(s) who has a mobile phone".

Example 2:

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all DIT students.
- Express the statement "Every computer science student must take a discrete mathematics course".

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Example 3:

Express the statement "for every x and for every y, $x + y > 10$"

- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x(Q(x) \to P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

$$\forall x(Q(x) \vee P(x))$$

Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Example 4:

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x)$$

mean?

Example 5:

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Example 4:

$\exists x \exists y$ P(x) means "there exists some x and some y such that x+y=5".

Example 5:

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Quantifiers
Truth Values

In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table: Truth Values of Quantifiers

Mixing Quantifiers

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$. Thus, ordering is important.

For example:

- $\forall x \exists y Loves(x, y)$: everybody loves somebody
- $\exists y \forall x Loves(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you *can* commute *similar* quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Example 1:

Express, in predicate logic, the statement that there are an infinite number of integers.

Solution:

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Example 2:

Express the *commutative law of addition* for $\mathbb{R}$.

Solution: We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Rules of inference for quantified statements

1. Universal instantiation

- $\forall\, x \in D, P(x)$
- $d \in D$
- Therefore $P(d)$

2. Universal generalization

- $P(d)$ for any $d \in D$
- Therefore $\forall x,\ P(x)$

3. Existential instantiation

- $\exists\, x \in D, P(x)$
- Therefore $P(d)$ for some $d \in D$

4. Existential generalization

- $P(d)$ for some $d \in D$
- Therefore $\exists\, x,\ P(x)$

Example 1: Write the following as a quantified statement

English:

Everyone likes either Microsoft or Apple.

Jill does not like Microsoft.

Therefore Jill likes Apple.

Maths:

M(x) = "x likes Microsoft"

A(x) = "x likes Apple"

$$\forall x\ M(x) \vee A(x)$$

$M(\text{Jill}) \vee A(\text{Jill})$.

$\neg M(\text{Jill})$.

$\Rightarrow A(\text{Jill})$.

Example 2: Write the following as a quantified statement

English: Every student is intelligent

Maths: Let $S(x)$ = "x is a student"

Let $I(x)$ = "x is intelligent"

$\forall x\ S(x) \rightarrow I(x)$

Example 3: Write the following as a quantified statement

English: Some student(s) is intelligent

Maths: Let $S(x)$ = "x is a student"

Let $I(x)$ = "x is intelligent"

$\exists x\ S(x) \wedge I(x)$

Negation

Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).