Section 1.1

January 31, 2025

1 Exercise

(a) Is not associative since: $(a-a)-a=-a$ and $a-(a-a)=a$ i.e. $a \neq -a$.

(b) Is associative since:

$$\begin{aligned} a \star (b \star c) &= a + (b+c+bc) + a(b+c+bc) \\ &= a+b+c+bc+ab+ac+abc \\ &= a+b+ab+c+ac+bc+abc \\ &= (a+b+ab)+c+(a+b+ab)c \\ &= (a \star b) \star c \end{aligned}$$

(c) Is not associative since:

$$\begin{aligned} a \star (b \star c) &= \frac{a + \frac{b+c}{5}}{5} \\ &= \frac{\frac{5a}{5} + \frac{b+c}{5}}{5} \\ &= \frac{5a+b+c}{10} \\ &\neq \frac{a+b+5c}{10} \\ &= \frac{\frac{a+b}{5} + \frac{5c}{5}}{5} \\ &= \frac{\frac{a+b}{5} + c}{5} \\ &= (a \star b) \star c \end{aligned}$$

(d) Is not associative since:

$$\begin{aligned}(a,b)\star((c,d)\star(e,f)) &= (a,b)\star(cf+de,df)\\ &= (a(cf+de)+bdf,bdf)\\ &= (acf+ade+bdf,bdf)\\ &= (acf+ade+bdf,bdf)\\ &\neq (adf+bcf+bde,bdf)\\ &= ((ad+bc)f+bde,bdf)\\ &= (ad+bc,bd)\star(e,f)\\ &= ((a,b)\star(c,d))\star(e,f)\end{aligned}$$

(e) Is not associative since:

$$\begin{aligned}a\star(b\star c) &= \frac{a}{\frac{b}{c}}\\ &= a\left(\frac{b}{c}\right)^{-1}\\ &= a\left(\frac{c}{b}\right)\\ &= \frac{ac}{b}\\ &\neq \frac{a}{bc}\\ &= \left(\frac{a}{b}\right)c^{-1}\\ &= \frac{\frac{a}{b}}{c}\\ &= a\star(b\star c)\end{aligned}$$

2 Exercise

(a) Is not commutative since $a-2a=-a\neq a=2a-a$.

(b) Is commutative since:

$$\begin{aligned}a\star b &= a+b+ab\\ &= b+a+ba\\ &= b\star a\end{aligned}$$

(c) Is commutative since:

$$\begin{aligned}a\star b &= \frac{a+b}{5}\\ &= \frac{b+a}{5}\\ &= b\star a\end{aligned}$$

(d) Is associative since:

$$\begin{aligned}(a,b)\star(c,d) &= (ad+bc,bd)\\ &= (cb+ad,db)\\ &= (c,d)\star(a,b)\end{aligned}$$

(e) Is not commutative since $\frac{1}{2}\neq\frac{2}{1}$

3 Exercise

Proof. Let $\overline{a},\overline{b},\overline{c}\in\mathbb{Z}/n\mathbb{Z}$. Using $\overline{a}+\overline{b}=\overline{a+b}$ we can show that:

$$\begin{aligned}(\overline{a}+\overline{b})+\overline{c} &= \overline{a+b}+\overline{c}\\ &= \overline{a+b+c}\\ &= \overline{a}+\overline{b+c}\\ &= \overline{a}+(\overline{b}+\overline{c})\end{aligned}$$

□

4 Exercise

Proof. Let $\overline{a},\overline{b},\overline{c}\in\mathbb{Z}/n\mathbb{Z}$. Using $\overline{a}\cdot\overline{b}=\overline{a\cdot b}$ we can show that:

$$\begin{aligned}(\overline{a}\cdot\overline{b})\cdot\overline{c} &= \overline{a\cdot b}\cdot\overline{c}\\ &= \overline{a\cdot b\cdot c}\\ &= \overline{a}\cdot\overline{b\cdot c}\\ &= \overline{a}\cdot(\overline{b}\cdot\overline{c})\end{aligned}$$

□

5 Exercise

Proof. We wish to show that $(\mathbb{Z}/n\mathbb{Z},\cdot)$ for any $n>1$ is not a group. For $(\mathbb{Z}/n\mathbb{Z},\cdot)$ to be a group then there must exists an inverse $\overline{a}$ such that $\overline{a}\cdot\overline{0}=\overline{1}$. If we take the representative of $\overline{a}$ to be a and the representative of $\overline{0}$ to be 0 then we get $a\cdot 0=0$ for any a so $\overline{a}\cdot\overline{0}=\overline{0}$. Hence no inverse exists of $\overline{0}$ and therefore $(\mathbb{Z}/n\mathbb{Z},\cdot)$ is not a group. □

6 Exercise

(a) Is not a group since the inverse to $\frac{2a}{2b+1}$ is $\frac{2b+1}{2a}$ which does no have an odd denominator so it does not belong in the original set meaning $\frac{2a}{2b+1}$ has no inverse .

(b) Is not a group since the inverse to $\frac{2a+1}{2b}$ is $\frac{2b}{2a+1}$ which does no have an even denominator.

(c) Is not a group since $\frac{1}{n} < 1$ is in the set but $1 \leq n$ is its inverse.

(d) Is not a group since $n \geq 1$ is in the set but $\frac{1}{n} < 1$ is its inverse.