

Section 1.1

February 1, 2025

1 Exercise

(a) Is not associative since: $(a - a) - a = -a$ and $a - (a - a) = a$ i.e. $a \neq -a$.

(b) Is associative since:

$$\begin{aligned}a \star (b \star c) &= a + (b + c + bc) + a(b + c + bc) \\&= a + b + c + bc + ab + ac + abc \\&= a + b + ab + c + ac + bc + abc \\&= (a + b + ab) + c + (a + b + ab)c \\&= (a \star b) \star c\end{aligned}$$

(c) Is not associative since:

$$\begin{aligned}a \star (b \star c) &= \frac{a + \frac{b+c}{5}}{5} \\&= \frac{\frac{5a}{5} + \frac{b+c}{5}}{5} \\&= \frac{5a + b + c}{10} \\&\neq \frac{a + b + 5c}{10} \\&= \frac{\frac{a+b}{5} + \frac{5c}{5}}{5} \\&= \frac{\frac{a+b}{5} + c}{5} \\&= (a \star b) \star c\end{aligned}$$

(d) Is not associative since:

$$\begin{aligned}
 (a, b) \star ((c, d) \star (e, f)) &= (a, b) \star (cf + de, df) \\
 &= (a(cf + de) + bdf, bdf) \\
 &= (acf + ade + bdf, bdf) \\
 &= (acf + ade + bdf, bdf) \\
 &\neq (adf + bcf + bde, bdf) \\
 &= ((ad + bc)f + bde, bdf) \\
 &= (ad + bc, bd) \star (e, f) \\
 &= ((a, b) \star (c, d)) \star (e, f)
 \end{aligned}$$

(e) Is not associative since:

$$\begin{aligned}
 a \star (b \star c) &= \frac{a}{\frac{b}{c}} \\
 &= a \left(\frac{b}{c} \right)^{-1} \\
 &= a \left(\frac{c}{b} \right) \\
 &= \frac{ac}{b} \\
 &\neq \frac{a}{bc} \\
 &= \left(\frac{a}{b} \right) c^{-1} \\
 &= \frac{\frac{a}{b}}{c} \\
 &= a \star (b \star c)
 \end{aligned}$$

2 Exercise

(a) Is not commutative since $a - 2a = -a \neq a = 2a - a$.

(b) Is commutative since:

$$\begin{aligned}
 a \star b &= a + b + ab \\
 &= b + a + ba \\
 &= b \star a
 \end{aligned}$$

(c) Is commutative since:

$$\begin{aligned}
 a \star b &= \frac{a + b}{5} \\
 &= \frac{b + a}{5} \\
 &= b \star a
 \end{aligned}$$

(d) Is associative since:

$$\begin{aligned}(a, b) \star (c, d) &= (ad + bc, bd) \\ &= (cb + ad, db) \\ &= (c, d) \star (a, b)\end{aligned}$$

(e) Is not commutative since $\frac{1}{2} \neq \frac{2}{1}$

3 Exercise

Proof. Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$. Using $\overline{a + b} = \overline{a} + \overline{b}$ we can show that:

$$\begin{aligned}(\bar{a} + \bar{b}) + \bar{c} &= \overline{a + b} + \bar{c} \\ &= \overline{a + b + c} \\ &= \overline{a} + \overline{b + c} \\ &= \bar{a} + (\bar{b} + \bar{c})\end{aligned}$$

□

4 Exercise

Proof. Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$. Using $\overline{a \cdot b} = \overline{a} \cdot \overline{b}$ we can show that:

$$\begin{aligned}(\bar{a} \cdot \bar{b}) \cdot \bar{c} &= \overline{a \cdot b} \cdot \bar{c} \\ &= \overline{a \cdot b \cdot c} \\ &= \overline{a} \cdot \overline{b \cdot c} \\ &= \bar{a} \cdot (\bar{b} \cdot \bar{c})\end{aligned}$$

□

5 Exercise

Proof. We wish to show that $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ for any $n > 1$ is not a group. For $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ to be a group then there must exist an inverse \bar{a} such that $\bar{a} \cdot \bar{0} = \bar{1}$. If we take the representative of \bar{a} to be a and the representative of $\bar{0}$ to be 0 then we get $a \cdot 0 = 0$ for any a so $\bar{a} \cdot \bar{0} = \bar{0}$. Hence no inverse exists of $\bar{0}$ and therefore $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ is not a group. □

6 Exercise

- (a) It is easy to see that the inverse $\frac{-a}{2b+1}$ and identity 0 exist. Addition is also associative so remaining concern is if $+$ always results in an element with an uneven denominator.

$$\begin{aligned}\frac{a}{2b+1} + \frac{c}{2d+1} &= \frac{a(2d+1) + c(2b+1)}{(2b+1)(2d+1)} \\ &= \frac{a(2d+1) + c(2b+1)}{4bd + 2b + 2d + 1} \\ &= \frac{a(2d+1) + c(2b+1)}{2(2bd + b + d) + 1}\end{aligned}$$

The denominator is always uneven so the set is closed under addition.

- (b) Once again we only need to show the denominator is even.

$$\frac{a}{2b} + \frac{c}{2d} = \frac{a2d + c2b}{2(b2d)}$$

The denominator is always even so the set is closed under addition.

- (c) This is not a valid group since $\frac{1}{2}$ is an element in the set but $\frac{1}{2} + \frac{1}{2} = \frac{1}{1}$ is too large to be in set, hence the set is not closed under addition.
- (d) This is not a valid group since both 2 and $\frac{-2}{3}$ are members but $2 + \frac{-2}{3} = \frac{1}{2}$ is too small to be an element in the set, hence the set is not closed under addition.
- (e) Clearly an inverse always exists $\frac{-a}{1}$ or $\frac{-a}{2}$ and we have the identity 0 and an associative operation. It remains to show that the set is closed under addition.

$$\begin{aligned}\frac{a}{1} + \frac{b}{1} &= \frac{a+b}{1} \\ \frac{a}{2} + \frac{b}{2} &= \frac{a+b}{2} \\ \frac{a}{2} + \frac{b}{1} &= \frac{a+2b}{2}\end{aligned}$$

As one can see it is closed under addition.

- (f) First we observe that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, since there does not exist a number which both divide 5 and 6 then the fraction can not be simplified and we have an element which does not belong to the original set. Meaning this set is not closed under addition and therefore not a group.

7 Exercise

Proof. By definition we have that G must be closed under \star since we always remove the integral part leading to $0 \leq x \star y < 1$.

Now we wish to show that the operation is associative, first we realize:

$$[x - [y]] = [x] - [y]$$

This holds since $[y]$ is already an integer so we can subtract it later. Now we can derive the following and show it is associative:

$$\begin{aligned} (x \star y) \star z &= (x \star y) + z - [x \star y + z] \\ &= (x + y - [x + y]) + z - [(x + y - [x + y]) + z] \\ &= x + y + z - [x + y + z - [x + y]] - [x + y] \\ &= x + y + z - ([x + y + z] - [x + y]) - [x + y] \\ &= x + y + z - [x + y + z] + [x + y] - [x + y] \\ &= x + y + z - [x + y + z] \\ &= x + y + z - [x + y + z] + [y + z] - [y + z] \\ &= x + y + z - ([x + y + z] - [y + z]) - [y + z] \\ &= x + y + z - [x + y + z - [y + z]] - [y + z] \\ &= x + (y + z - [y + z]) - [x + (y + z - [y + z])] \\ &= x \star (y + z - [y + z]) \\ &= x \star (y \star z) \end{aligned}$$

We have an identity 0 since:

$$x \star 0 = x + 0 - [x + 0] = x - [x] = 0 + x - [0 + x] = 0 \star x$$

And we have an inverse $-x$ since:

$$\begin{aligned} x \star -x &= x + (-x) - [x + (-x)] \\ &= 0 - [0] \\ &= 0 \\ &= 0 - [0] \\ &= (-x) + x - [(-x) + x] \\ &= -x \star x \end{aligned}$$

Hence (G, \star) must be a group. \square

8 Exercise

- (a) (G, \cdot) must be closed under multiplication since given $z, w \in G$ then it must hold for some $n, m \in \mathbb{Z}^+$ that $(z^n)^x \cdot (w^m)^y = 1$ for all $x, y \in \mathbb{R}$.

Since $z^n = w^m = 1$ and $1^x = 1$ for all $x \in \mathbb{R}$ we can derive:

$$\begin{aligned} 1 &= (z^n)^m \cdot (w^m)^n \\ &= z^{mn} \cdot w^{mn} \\ &= (zw)^{nm} \end{aligned}$$

Hence G is closed under multiplication because $(zw)^{nm} = 1$, therefore $zw \in G$. Lastly (G, \cdot) is a group since $1 \in G$ is the identity. The inverse to z is z^{n-1} since $z \cdot z^{n-1} = z^n = 1$ for some $n \in \mathbb{Z}^+$ and $z^{n-1} \in G$ because $(z^{n-1})^n = z^{n \cdot n - n} = \frac{z^{n \cdot n}}{z^n} = z^{n \cdot n - n} = z^{n \cdot n - n} = (z^n)^n = 1$. And multiplication is associative.

- (b) $(G, +)$ is not a group since $1 \in G$ but $1 + 1 = 2 \notin G$ since there exists no $n \in \mathbb{Z}^+$ such that $2^n = 1$ hence G is not closed under addition and therefore not a group.