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Union-Find

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1 Definitions

Definition 1.1 (Union-Find)**.** The union-find data structure U represents a partition of a set S if:

1. $a \neq \emptyset$ for all $a \in \mathcal{C}(U)$
2. $a \cap b = \emptyset$ for all $a, b \in \mathcal{C}(U)$ where $a \neq b$
3. $\bigcup_{a \in \mathcal{C}(U)} a = S$

where $\mathcal{C} : \mathbb{P}(U) \to \mathbb{P}(\mathbb{P}(S))$ converts the data structure U into a partition of S.

Definition 1.2 (Union-Find Set)**.** The set $\mathbb{U}_n := \mathbb{P}(\mathbb{N}_{\leq n} \times (\mathbb{N}_{\leq n} \cup \{0\}))$ is the set of all union-find data structures of size n. Each element $U \in \mathbb{U}_n$ is a set containing tuples (i, p) where:

1. $i \in \mathbb{N}_{\leq n}$ is a unique index identifying the element i.e. $|\{\pi_1(u) : u \in U\}| = n$.
2. $p \in \mathbb{N}_{\leq n} \cup \{0\}$ is the index of the parent element. If $p = 0$ then the element is a root.

Definition 1.3 (Cycle)**.** A cycle exists in a set $U \in \mathbb{U}_n$ if:

1. $i = p$ for some $(i, p) \in U$ or
2. $p_1 = i_2, p_2 = i_3, \ldots, p_m = i_1$ for some $(i_1, p_1), (i_2, p_2), \ldots, (i_m, p_m) \in U$ where $m > 1$.

Definition 1.4 (Find Root)**.** The root of an element i in a cycleless union-find data structure $U \in \mathbb{U}_n$ is denoted $\mathcal{P}_U : \mathbb{N}_{\leq n} \to \mathbb{N}_{\leq n}$ defined as:

$$\mathcal{P}_U(i) := \begin{cases} i & \text{if } p = 0 \\ \mathcal{P}_U(p) & \text{if } p \neq 0 \end{cases} \text{ where } (j, p) \in U \land i = j.$$

The root represents the set that the element i belongs to, i.e. all elements with the same root belong to the same set.

Proposition 1.1 (Termination of Find Root)**.** The function $\mathcal{P}_U(i)$ defined in definition 1.3 terminates for all i where $1 \leq i \leq n$ if $U \in \mathbb{U}_n$ has no cycles.

Proof. The proof is trivially true. □

Proposition 1.2 (Representation of Union-Find)**.** A cycleless union-find structure $U \in \mathbb{U}_n$ fulfills definition 1.1 for the following conversion function:

$$\mathcal{C}(U) := \{\{f(\pi_1(u)) : u \in U \wedge \mathcal{P}_U(\pi_1(u)) = p\} : p \in \mathbb{N}_{\leq n}\} \backslash \{\emptyset\}$$

where $f : \mathbb{N}_{\leq n} \rightarrow S$ is a bijective function mapping indices to elements in the set S.

Proof. Let $U \in \mathbb{U}_n$ be a union-find data structure fulfilling the three criteria in proposition 1.2. We will show that U fulfills the three criteria in definition 1.1.

1. By definition of $\mathcal{C}$ it is clear that $a \neq \emptyset$ for all $a \in \mathcal{C}(D)$ since $\mathcal{C}$ only includes non-empty sets.

2. Let $a, b \in \mathcal{C}(D)$ where $a \neq b$ then by definition $a = \{f(\pi_1(u)) : u \in U \wedge \mathcal{P}_U(\pi_1(u)) = p_a\}$ and $b = \{f(\pi_1(u)) : u \in U \wedge \mathcal{P}_U(\pi_1(u)) = p_b\}$ for some $p_a, p_b \in \mathbb{N}_{\leq n}$ where $p_a \neq p_b$. Since if $p_a = p_b$ then $a = b$ it follows that $a \cap b = \emptyset$.

3. Let $s \in S$ then since U is a union-find data structure of size n there exists $(i, p) \in U$ for some i and p. Let $r = \mathcal{P}_U(i)$ then by definition of $\mathcal{C}$ it follows that $s \in \{f(\pi_1(u)) : u \in U \wedge \mathcal{P}_U(\pi_1(u)) = r\} \in \mathcal{C}(U)$. Since s was arbitrary it follows that $\bigcup_{a \in \mathcal{C}(U)} a = S$.

□

Definition 1.5 (Substitute Parent)**.** The function for substitutin a parent of an element i to p in a cycleless union-find data structure $U \in \mathbb{U}_n$ is denoted $\mathcal{S}_U : \mathbb{N}_{\leq n} \times \mathbb{N}_{\leq n} \rightarrow \mathbb{P}(\mathbb{U}_n)$ and defined as:

$$\mathcal{S}_U(i, p) := (U \backslash \{(i, p')\}) \cup \{(i, p)\} \text{ where } (i, p') \in U$$

Definition 1.6 (Equivalence Set)**.** The set of equivalent indices of an element i in a cycleless union-find data structure $U \in \mathbb{U}_n$ is defined as:

$$\mathcal{E}_U(i) := \{j : j \in \mathbb{N}_{\leq n} \wedge \mathcal{P}_U(j) = \mathcal{P}_U(i)\}$$

Definition 1.7 (Equivalent Union-Find Structures)**.** Two cycleless union-find data structures $U, U' \in \mathbb{U}_n$ are equivalent if:

$$\mathcal{E}_{U'}(i) = \mathcal{E}_U(i) \text{ for all } i \in \mathbb{N}_{\leq n}$$

Definition 1.8 (Well-formed Union)**.** For a cycleless union-find data structure $U \in \mathbb{U}_n$ the union of two elements $i \sim j$ where $i, j \in \mathbb{N}_{\leq n}$ is well-formed if it results in a cycleless union-find data structure $U' \in \mathbb{U}_n$ such that:

1. $\mathcal{E}_{U'}(i) = \mathcal{E}_U(i) \cup \mathcal{E}_U(j)$ and
2. $\mathcal{E}_{U'}(k) = \mathcal{E}_U(k)$ for all $k \in \mathbb{N}_{\leq n} \backslash (\mathcal{E}_U(i) \cup \mathcal{E}_U(j))$.

Definition 1.9 (Sequential Unions)**.** Given a sequence of union relations $i_1 \sim j_1, i_2 \sim j_2, \ldots, i_m \sim j_m$ where $i_k, j_k \in \mathbb{N}_{\leq n}$ for all $1 \leq k \leq m$ and a initial $U \in \mathbb{U}_n$. The final union-find data structure $U' \in \mathbb{U}_n$ is defined as:

$$\begin{aligned} U_0 &:= \{(i, 0) : i \in \mathbb{N}_{\leq n}\} \\ (q_k, p_k) &:= (\mathcal{P}_{U_{k-1}}(i_k), \mathcal{P}_{U_{k-1}}(j_k)) \text{ for } 1 \leq k \leq m \\ U_k &:= \begin{cases} U_{k-1} & \text{if } p_k = q_k \\ \mathcal{S}_{U_{k-1}}(q_k, p_k) & \text{if } p_k \neq q_k \end{cases} \text{ for } 1 \leq k \leq m \\ U' &:= U_m \end{aligned}$$

Proposition 1.3 (Sequential Unions is Well-formed)**.** Given a sequence of union relationss $i_1 \sim j_1, i_2 \sim j_2, \ldots, i_m \sim j_m$ where $i_k, j_k \in \mathbb{N}_{\leq n}$ for all $1 \leq k \leq m$ and a cycleless union-find data structure $U \in \mathbb{U}_n$. The final union-find data structure $U' \in \mathbb{U}_n$ defined in definition 1.9 is well-formed.

Proof. To prove that U' is well-formed we need to show that each union in the sequence is well-formed. This can be shown by induction on the index k of the union in the sequence.

- For U_0 it is clear that $\mathcal{E}_{U_0}(i) = \{i\}$ for all $i \in \mathbb{N}_{\leq n}$ since all elements are their own root.
- Assume that U_{k-1} is well-formed for some $1 \leq k \leq m$. We will show that U_k is well-formed. Let $q_k = \mathcal{P}_{U_{k-1}}(i_k)$ and $p_k = \mathcal{P}_{U_{k-1}}(i_k)$.
  1. If $p_k = q_k$ then by definition $U_k = U_{k-1}$ and thus U_k is well-formed by the inductive hypothesis.
  2. If $p_k \neq q_k$ then since q_k is a root $(q_k, 0) \in U_{k-1}$ and by definition of $\mathcal{S}$.

     $$U_k := (U_{k-1} \backslash \{(q_k, 0)\}) \cup \{(q_k, p_k)\} \text{ where } (q_k, 0) \in U_{k-1}$$

     Now $p_k = \mathcal{P}_{U_k}(q_k)$ hence $\mathcal{P}_{U_k}(q) = p_k$ for all $q \in \mathcal{E}_{U_{k-1}}(q_k)$. And by definition of p_k being a root it follows that $\mathcal{P}_{U_k}(p) = p_k$ for all $p \in \mathcal{E}_{U_{k-1}}(p_k)$. Thus $\mathcal{E}_{U_k}(i_k) = \mathcal{E}_{U_{k-1}}(i_k) \cup \mathcal{E}_{U_{k-1}}(j_k)$ and since all other elements are unaffected by the union it follows that $\mathcal{E}_{U_k}(l) = \mathcal{E}_{U_{k-1}}(l)$ for all $l \in \mathbb{N}_{\leq n} \backslash (\mathcal{E}_U(i) \cup \mathcal{E}_U(j))$.
     Finally since U_{k-1} is cycleless and q_k is a root it follows that U_k is cycleless and thus U_k is well-formed.

□