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Data-Parallel Union-Find

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1 Introduction

This is a thesis preparation project about data-parallel union-find structure well suited for a functional array language [2]. The project is about the theory, implementation and usage of the structure. The union-find structure is structure which allows for maintaining equivalences between elements. Union-find operate on single elements but the data-parallel variant operates on multiple elements, in this project the structure will have the following operations.

- `create`: Creating an union-find structure of n elements.
- `union`: An input array of tuple pairs of elements will be unified and is therefore said to be “equivalent”.
- `find`: An input array of elements where a single unique representative of the equivalence class is returned for every element.

Three different data-parallel union-find structures will be presented, a union-find structure without any heurestics, a union-find structure using union by rank and union-find structure using union by size. These structures are implemented in the parallel functional array language Futhark and the implementations are tested and benchmarked. In the end these implementations are used in a region labeling example and type checking example.

2 Theory

This section will derive the theoretical foundation for a data-parallel union-find. The theory is based on set theory and graph theory.

2.1 Forests

To be able to define a union-find structure we first need to define some basic graph theory, specifically in relation to forests. The graphs used are directed graphs $G = (V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is a set of directed edges. Initially we need a definition of what it means for a vertex $u \in V$ to be reachable from vertex $v \in V$ meaning there is a path from u to v.

Definition 2.1 (Reachability)**.** A veterx $v \in V$ is *reachable* from a vertex $u \in V$ in a directed graph $G = (V, E)$ if there exists a sequence of directed edges $e_1, e_2, \dots, e_m \in E$ where $m \geq 1$ and $e_i = (v_{i-1}, v_i)$ for $1 \leq i \leq m$, such that $v_0 = u$ and $v_m = v$. We denote this by $u \rightsquigarrow_G v$ and may write $u \rightsquigarrow v$ if it is clear what graph is refered to.

One of the first properties of reachability is it is neither reflexive nor *irreflexive.* The reason for chosing such a definition is that the definition of an cycle in a directed graph becomes simply.

Definition 2.2 (Cycle)**.** A cycle in a directed graph $G = (V, E)$ has a cycle if there exists a $v \in V$ such that $v \rightsquigarrow v$.

With this definition of cycles a forest can be defined as following:

Definition 2.3 (Forest)**.** A forest is a directed graph $F = (V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is a set of directed edges such that:

1. There are no cycles $v \not\rightsquigarrow v$ for all $v \in V$, and
2. each vertex has at most one parent i.e. for all $(u, v_1), (u, v_2) \in E$ it holds that $v_1 = v_2$.

With the definition of a forest we can now define roots in a forest.

Definition 2.4 (Root)**.** A vertex $v \in V$ in a forest $F = (V, E)$ is a root if it has no parent. This is defined as the predicate:

$$\mathcal{R}_F(v) : v \not\rightsquigarrow u \text{ for all } u \in V$$

Using the definition of a root we can now define a tree as a special case of a forest.

Definition 2.5 (Tree)**.** A tree is a forest $T = (V, E)$ where there exists a unique root $r \in V$ such that $v \rightsquigarrow r$ for all $v \in V \backslash \{r\}$.

Futhermore we will now work towards seeing a forest as a collection of trees. To do this we first need to establish how many roots a forest has.

Proposition 2.1 (Forest Root Count)**.** A forest $F = (V, E)$ where $|V| = n$ and $|E| = n - k$ has k roots.

Proof. Let $F = (V, E)$ be a forest where $|V| = n$ and $|E| = n - k$. By the second property of a forest then $n - k$ vertices must have a parent. Since there are n vertices in total it follows that there are exactly k vertices $r_1, r_2, \dots, r_k \in V$ that has no parent. Hence there are exactly k roots in F. $\square$

Knowing how many roots a forest has does not finish the picture of how a forest is a collection of trees. We also need to show that each vertex also has a path to a root.

Proposition 2.2 (Root Path Exist)**.** In a forest $F = (V, E)$ for each element $v \in V$ then there exists a root $r \in V$ such that $\mathcal{R}_F(r)$ and either $v \rightsquigarrow r$ or $v = r$.

Proof. Let $F = (V, E)$ be a forest and $v \in V$ be an arbitrary element in V. By proposition 2.1 there exists at least one root $r \in V$ such that $\mathcal{R}_F(r)$. This can be shown by structural induction on a vertex $v \in V$ that either $v = r$ or $(v, p) \in E$ such that $p \rightsquigarrow r$.

- If $(v, p) \notin E$ then $v \not\rightsquigarrow u$ for all $u \in V$ so $v = r$ and $\mathcal{R}_F(r)$.
- If $(v, p) \in E$ then by induction hypothesis $p \rightsquigarrow r$ such that $r \in V$ and $\mathcal{R}_F(r)$. It follows that since $v \rightsquigarrow p \rightsquigarrow r$ so $v \rightsquigarrow r$.

□

Lastly we can finish the picture of a forest being a collection of trees by showing that the path from a vertex to a root is unique. So that there is only one tree for each vertex in the forest.

Proposition 2.3 (Unique Path)**.** Let $F = (V, E)$ be a forest and $v, u \in V$. If $v \rightsquigarrow u$ then the path from v to u is unique.

Proof. Let $F = (V, E)$ be a forest, $v, u \in V$ and $v \rightsquigarrow u$. Since every vertex has at most one parent by the second property of a forest it follows that there is only one out going edge from each vertex in the path from v to u. Hence the path from v to u is unique. □

2.2 Union-Find Structure

Using the definition of a forest we can now define a union-find structure, the way it will represented is as a forest. Here the forest represents an equivalence on V where each tree in the forest represents an equivalence class.

Definition 2.6 (Union-Find Structure)**.** A union-find structure is a forest $F = (V, E)$.

The way this equivalence relation is defined is by the representative of each element in the forest. The representative of a vertex is found by traversing the edges in the forest until a root is found. Using the notion of a root we can now define the representative of an element in a forest.

Definition 2.7 (Representative)**.** The representative of an element $v \in V$ in a forest $F = (V, E)$ is the root $r \in V$ such that there is a path from v to r. This is defined as the function:

$$\rho_F(v) := r \text{ where } r \in V \text{ such that } \mathcal{R}_F(r) \wedge (v \rightsquigarrow r \vee v = r)$$

With the notion of a representative it is possible to define the set of vertices in the same tree in a forest.

Definition 2.8 (Tree Set)**.** The set of vertices of the same tree $\mathcal{E}_F(v)$ in a forest $F = (V, E)$ is defined as:

$$\mathcal{E}_F(v) := \{u : u \in V \text{ where } \rho_F(u) = \rho_F(v)\}$$

The notion of equivalence classes can now be formalized using the definition of a partition. This will be needed to show properties of the union-find structure.

Definition 2.9 (Partition)**.** The set $P \subseteq \mathbb{P}(S)$ is a partition of a set S if:

1. $a \neq \emptyset$ for all $a \in P$
2. $a \cap b = \emptyset$ for all $a, b \in P$ where $a \neq b$
3. $\bigcup_{a \in P} a = S$

We say that P is a partition of S and P forms an equivalence relation on S where each $a \in P$ is an equivalence class.

Using the definition of a partition we can now show that a forest is a partition of its vertices based on the tree sets.

Proposition 2.4 (Forest Partition)**.** A forest $F = (V, E)$ is a partition of V for the following set:

$$\{\mathcal{E}_F(v) : v \in V\}$$

Proof. Let $F = (V, E)$ be a forest. We will show that the set in the proposition is a partition of V by showing that it satisfies the three properties in definition 2.9.

1. By definition of $\mathcal{E}_F(v)$ it can not be empty since for $\mathcal{E}_F(v)$ then $\rho_F(v) = \rho_F(v)$. Hence $\mathcal{E}_F(v) \neq \emptyset$ for all $v \in V$.

2. Let a and b be two arbitrary elements in the set such that $a \neq b$. By definition of a and b there exists $v_1, v_2 \in V$ such that $a = \{u : u \in V \wedge \rho_F(u) = \rho_F(v_1)\}$ and $b = \{u : u \in V \wedge \rho_F(u) = \rho_F(v_2)\}$. Since $a \neq b$ it follows that $\rho_F(v_1) \neq \rho_F(v_2)$ since otherwise $a = b$, hence $a \cap b = \emptyset$.

3. Let v be an arbitrary element in V. By proposition 2.2 there exists a root $r \in V$ such that $\mathcal{R}_F(r)$ and $v \rightsquigarrow r$ or $v = r$. By definition of the representative it follows that $\rho_F(v) = r$. Now let $a = \{u : u \in V \wedge \rho_F(u) = \rho_F(v)\}$. By definition of a it follows that $v \in a$. Since v was arbitrary it follows that $\bigcup_{a \in P} a = V$.

□

With the notion of a partition of a forest we can now define the equivalence relation on the forest.

Definition 2.10 (Same Tree Relation)**.** The relation $\sim_F$ on a forest F is defined as:

$$u \sim_F v :\iff u \in \mathcal{E}_F(v)$$

Trivially we can now show that the same tree relation is an equivalence relation.

Corollary 2.1 (Same Tree Relation is an Equivalence Relation)**.** The relation $\sim_F$ on a forest F is an equivalence relation due to $\{\mathcal{E}_F(v) : v \in V\}$ being a partition of V.

Proof. From proposition 2.4 we directly get that $(V, \sim_F)$ is an equivalence relation since the set $\{\mathcal{E}_F(v) : v \in V\}$ is a partition of V. □

Now using the notion of partitions we can determine if two forests are equivalent in the sense that they have the same tree sets. This is useful when showing that two union-find structures are equivalent.

Definition 2.11 (Forests with Equivalent Tree Sets)**.** Two forests $F = (V, E)$ and $F' = (V', E')$ have equivalent tree sets $F \cong F'$ if:

- Vertices are the same $V = V'$.
- The tree sets are equivalent $\mathcal{E}_F(v) = \mathcal{E}'_F(v)$ for all $v \in V$.

It is also possible to define the property of uniting two trees in a forest should satisfy. We would want the resulting forest to have that the two trees are now one tree containing all the elements from both trees. And all other trees in the forest should remain unchanged.

Definition 2.12 (Tree Union Property)**.** The tree union of two elements v and u for a forest $F = (V, E)$ is such that $v \sim_{F'} u$ in a new forest $F' = (V', E')$ and F' satify the following properties:

1. $\mathcal{E}_{F'}(v) = \mathcal{E}_{F'}(u) = \mathcal{E}_F(v) \cup \mathcal{E}_F(u)$ and
2. $\mathcal{E}_{F'}(w) = \mathcal{E}_F(w)$ for all $w \in V \backslash (\mathcal{E}_F(v) \cup \mathcal{E}_F(u))$.

It is now possible to define a tree union operation that satisfies the tree union property. It uses the fact if with assign one root as a parent to a another root then we still get a forest. But we actually only care about the trees still being trees containing the same vertices and not the structure of the trees themselves. Hence it does not matter which representative becomes the parent of the other.

Proposition 2.5 (Root Union)**.** Let forest $F = (V, E)$, $p = \rho_F(u)$ be the representative of u and let $q = \rho_F(v)$ be the representative of v where $q \neq p$. Then define F' as:

$$F' := (V, E \cup \{(q, p)\})$$

Then $u \sim_{F'} v$ in F' and F' will satisfy the properties of a tree union.

Proof. Let $F = (V, E)$, $p = \rho_F(u)$ be the representative of u and let $q = \rho_F(v)$ be the representative of v. By definition q will have parent p in F' and since q is a root it has no parent then F' is a forest. Now for all $w \in \mathcal{E}_F(q)$ it holds that $w \rightsquigarrow q$ or $q = w$ and since $q \rightsquigarrow p$ it follows that $w \rightsquigarrow p$. Hence $w \in \mathcal{E}_{F'}(p)$ for all $w \in \mathcal{E}_F(q)$ and trivially $w \in \mathcal{E}_{F'}(p)$ for all $w \in \mathcal{E}_F(p)$ so it follows that $\mathcal{E}_{F'}(v) = \mathcal{E}_{F'}(u) = \mathcal{E}_F(v) \cup \mathcal{E}_F(u)$. Now let $w \in V \backslash (\mathcal{E}_F(v) \cup \mathcal{E}_F(u))$ be an arbitrary element. Since $w \not\rightsquigarrow p$, $w \neq p$, $w \not\rightsquigarrow q$, and $w \neq q$ it follows that w has the same representative in F' as in F hence $\mathcal{E}_{F'}(w) = \mathcal{E}_F(w)$. □

This operation can now be used to define the union operation on a union-find structure. This operation corresponds to the union operation in a sequential union-find data structure. The efficient implementations of union-find will do things like path compression or path halving to optimize the distance from a vertex to its representative. And these operations would also fulfill the tree union property since they do not change the equivalence classes of the forest.

2.3 Conflict-Free Sets

The problem now is how to define a parallel union operation on a union-find structure. The challenge is that multiple union operations might try to change the same part of the forest at the same time. And due to the nature of working in a data-parallel model we can not have atomic operations that would solve these problems in an concurrent manner. The way this problem will be solve is we consider a set of pairs of vertices that must be unified in the forest. We can find the representative of all of these vertex pairs, these representative pairs forms an graph of roots in the forest. We can simply pick out a subset of edges from this graph of roots that forms a forest and glue these edges onto the original forest. This will ensure that there are no conflicts when adding the edges to the original forest. We will call such a set of edges a conflict-free set.

An example of this process can be seen in figures 1 here we have a forest with three trees. In figure 2 we see the graph of roots on the left and on the right a conflict-free set highlighted in red. Finally in figure 3 we see the resulting forest after adding the conflict-free set to the original forest.

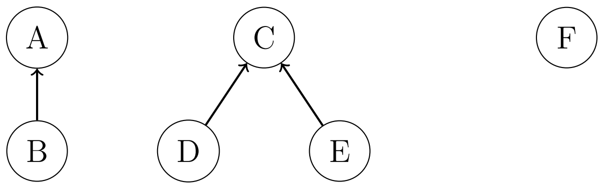

Figure 1: A forest where there are three trees with representatives A, C and F.

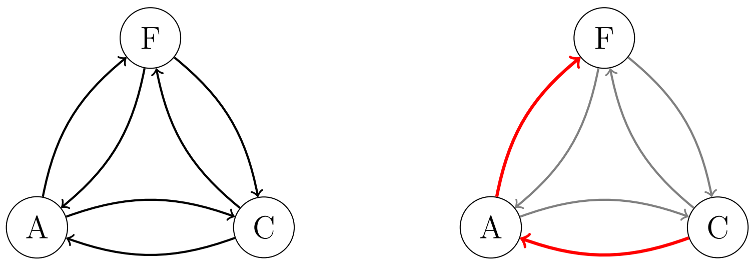

Figure 2: On the left a graph of roots from Figure 1. On the right a conflict-free set highlighted in red.

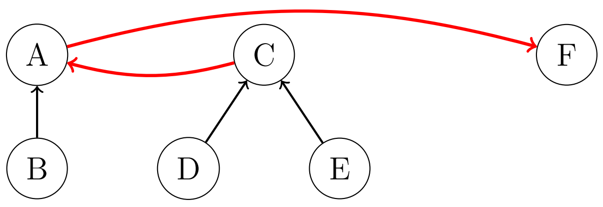

Figure 3: The forest from Figure 1 after adding the conflict-free set from Figure 2.

The formal definition of a conflict-free set is as previous described a set of root pairs that forms a forest.

Definition 2.13 (Conflict-free Set)**.** Let F be a forest, $X \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs. Then X is a conflict-free set in F if (V, Y) is a forest.

With this definition of a conflict-free set we now wish to show that we can add all edges in a conflict-free set to a forest and still have a forest.

Proposition 2.6 (Conflict-free Forest Union)**.** Let forest $F = (V, E)$ be a forest and let $X \subseteq V \times V$ be a conflict-free set in F where $|X| = n$. Then defining the following forests:

$$
\begin{aligned}
&F_0 := F \\
&F_i := (V, E_{i-1} \cup \{(v_i, u_i)\}) \text{ for } (v_i, u_i) \in X \text{ and } 1 \leq i \leq n
\end{aligned}
$$

Then F_n is a forest.

Proof. Let forest $F = (V, E)$ be a forest, $X \subseteq V \times V$ be a conflict-free set in F. We will show that F_n is a forest by induction on i.

- Base case: If $i = 0$ then $F_i = F_0 = F$ which is a forest.
- Induction hypothesis: Assume that F_{i-1} is a forest for all $1 \leq i < n$. Let $(v_i, u_i) \in X$, we know that $v_i \neq v_j$ for all $(v_j, u_j) \in Y \setminus \{(v_i, u_i)\}$ since otherwise (V, X) would not be a forest and X would not be a conflict-free set in F. So v_i will only have one parent in F_i since it only appears once as a child in (V, X). By definition all of the edges in X consists of roots in F, and since (V, X) is a forest there are no cycles $y \not\rightsquigarrow y$ for all $y \in V$ in F_i. Hence F_i is a forest.

Thus by induction F_n is a forest. $\square$

We also need to show that Conflict-free Forest Union 2.6 satisfies Tree Union Property 2.12. This is done by showing that adding edges from the conflict-free set to a forest in any order is equivalent to adding each edge in the same order using Root Union 2.5.

Proposition 2.7 (Conflict-free Set Equivalence)**.** Let forest F be a forest and let $X \subseteq V \times V$ be a conflict-free set in F where $|X| = n$. Then defining the following forests:

$$\begin{aligned} F_0 &:= F \\ F_i &:= (V, E_{i-1} \cup \{(v_i, u_i)\}) \text{ for } (v_i, u_i) \in X \text{ and } 1 \leq i \leq n \\ G_0 &:= F \\ G_j &:= (V, E_{i-1} \cup \{(\rho_{G_{j-1}}(v_j), \rho_{G_{j-1}}(u_j))\}) \text{ for } (v_j, u_j) \in X \text{ and } 1 \leq j \leq n \end{aligned}$$

Then $F_n \cong G_n$.

Proof. Let F be a forest, $X \subseteq V \times V$ be a conflict-free set in F. We will show that $F_n \cong G_n$. We know that for some $(v_i, u_i) \in X$ then $v_i \neq y$ for all $(y, w) \in X \backslash \{(v_i, u_i)\}$ since otherwise (V, X) would not be a forest and X would not be a conflict-free set in F. So all edge set unions will only give a root v_i a new parent u_i once. So $\rho_{F_n}(v_i) = \rho_{F_n}(u_i)$ and $\rho_{G_n}(u_i) = \rho_{G_n}(v_i)$ hence v_i remains in the same tree in both F_n and G_n. Since this holds for all $(v_i, u_i) \in X$ it follows that all elements in V remains in the same tree in both F_n and G_n. Hence $F_n \cong G_n$. □

From this equivalence it will be shown that Conflict-free Forest Union 2.6 satisfies Tree Union Property 2.12.

Corollary 2.2 (Conflict-free Union Satisfies Tree Union Property)**.** Let forest F be a forest and let $X \subseteq V \times V$ be a conflict-free set in F where $|X| = n$. Then defining the following forests:

$$\begin{aligned} F_0 &:= F \\ F_i &:= (V, E_{i-1} \cup \{(v_i, u_i)\}) \text{ for } (v_i, u_i) \in X \text{ and } 1 \leq i \leq n \end{aligned}$$

Then for all $(v_i, u_i) \in X$ it holds that $v_i \sim_{F_n} u_i$ and F_n satisfies the properties of a tree union.

Proof. Let forest F be a forest, $X \subseteq V \times V$ be a conflict-free set in F. By proposition 2.7 it holds that $F_n \cong G_n$ where G_n is defined as in proposition 2.7. By proposition 2.5 it holds that for all $(v_i, u_i) \in X$ then $v_i \sim_{G_n} u_i$ and G_n satisfies the properties of a tree union. Since $F_n \cong G_n$ it follows that for all $(v_i, u_i) \in X$ then $v_i \sim_{F_n} u_i$ and F_n satisfies the properties of a tree union. □

Now that we have established the properties of conflict-free sets we can now define a method to find a conflict-free set from a set of root pairs. The method chosen to find a conflict-free set is to consider an directed acyclic graph. Such a graph fulfills one property of a forest, namely that there are no cycles. This is ensured by ordering the edges in the graph such that for all edges (v, u) it holds that $v < u$ for some strict total order $(V, <)$. We may do this since the equivalence classes in a forest can be represented by any representative in the tree. Hence we can alwauys pick a total order on the vertices in the forest.

Proposition 2.8 (Ordered Edges Implies Acyclicity)**.** Let $G = (V, E)$ be a directed graph where for all $(v, u) \in E$ it holds that $v < u$ for some strict total order $(V, <)$. Then G has no cycles.

Proof. Let $G = (V, E)$ be a directed graph where for all $(u, v) \in E$ it holds that $u < v$ for some total order $(V, <)$. Let edges $e_1, e_2, \dots, e_m \in E$ where $m \geq 1$ and $e_i = (v_{i-1}, v_i)$ for $1 \leq i \leq m$ be some path in G. Since the edges are ordered it follows that:

$$v_0 < v_1 < v_2 < \cdots < v_{m-1} < v_m$$

Hence by transitivity of the total order it follows that $v_0 < v_m$. So $v_0 \neq v_m$ hence there are no cycles in G. □

The nice property of using a directed acyclic graph is we can now pick out any edges from the graph such that no two edges have the same child. This will ensure that the picked out edges forms a forest.

2.4 Parallel Union-Find

The conflict-free sets makes it is possible to define how unification in the union-find strucutre works. If we consider that we have some forest $F = (V, E)$ and then are given a set of vertex pairs $A \subseteq V \times V$ we wish to unify these pairs such that they are equivalent in some forest F. We can turn A into an directed acyclic graph (V, Z) where Z only consists of root pairs from F. We want to pick out a subset of Z such that we unify as many of the vertex pairs of Z in F as possible leading to a good time complexity of the algorithm. All of these pairs can not be unified immediately so an algorithm will be first derived which picks out a subset of the directed acyclic graph (V, Z) and then unifies them in F. To determine what a large set is, we first need to define a notion of an edge cover that can be serve as a measure of how many vertex pairs in Z can be unified:

Definition 2.14 (Edge Cover)**.** Let V be a set and $E \subseteq V \times V$ such that $\pi_1(E) \cup \pi_2(E) = V$ then E is an edge cover of V.

Using this definition we know that $V' = \pi_1(Z) \cup \pi_2(E)$ is an edge cover of the subgraph (V', Z) of (V, Z). Why this is relevant is that vertices in $V \backslash V'$ will not be unified so they are not relevant during unification. Futhermore, the bound that will be established is for every iteration then atleast $\frac{|V'|}{2}$ vertices must be unified. The intuition behind this is that every time a vertex is given a parent then it can not be given a parent later so it has been dealt with. We can show that when we have such an edge cover then the following inequality holds:

Proposition 2.9 (Edge Cover Inequality)**.** Let V be a set and $E \subseteq V \times V$ be a edge cover of V then:

$$|\pi_1(E)| < \frac{|V|}{2} \implies |\pi_2(E)| > \frac{|V|}{2}$$

Proof. Let V be a set and $E \subseteq V \times V$ be a edge cover of V, and $|\pi_1(E)| < \frac{|V|}{2}$. Let $A = \pi_1(E) = \{v : (v, u) \in E\}$, $B = \pi_2(E) = \{u : (v, u) \in E\}$ and $C = B \backslash A$. By definition of C we have $A \cap C = \emptyset$ so $|A| + |C| = |V|$ and since $|B| \geq |C|$ we can conclude that:

$$|B| \geq |C| = |V| - |A| > |V| - \frac{|V|}{2} = \frac{|V|}{2}$$

Hence $|\pi_1(E)| > \frac{|V|}{2}$. □

This inequality tells us that if Z does not resolve enough vertices, then if we invert the edges in Z then it would be possible to resolve enough vertices. It just remains to show that inverting these edges direction will still give an directed acyclic graph.

Proposition 2.10 (Inverted Acyclic Graph is Acyclic)**.** Let $G = (V, E)$ be a directed acyclic graph. Then the inverted graph $G' = (V, E')$ where $E' = \{(u, v) : (v, u) \in E\}$ is also acyclic.

Proof. Let $G = (V, E)$ be a directed acyclic graph and $G' = (V, E')$ where $E' = \{(u, v) : (v, u) \in E\}$ is the inverted graph. Let edges $e_1, e_2, \dots, e_m \in E'$ where $m \geq 1$ and $e_i = (v_{i-1}, v_i)$ for $1 \leq i \leq m$ be some path in G'. By definition of E' it follows that there exists edges $e'_1, e'_2, \dots, e'_m \in E$ where $e'_i = (v_i, v_{i-1})$ for $1 \leq i \leq m$. If there was a cycle in G' then it would hold that $v_0 = v_m$. But since G is acyclic it follows that $v_0 \neq v_m$. Hence there are no cycles in G'. □

The algorithm which tries to unifies atleast $\frac{|V'|}{2}$ can now be defined. It starts by determining if the directed acyclic subgraph (V', Z) should be inverted as to give $\frac{|V'|}{2}$ vertices a parent. Afterwards just pick out as many pairs with a unique child.

Algorithm 2.1 (Maximal Union)**.** Let forest $F = (V, E)$ be a forest and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) is an acyclic directed graph. The maximal union algorithm is defined as:

```
    $MaximalUnion(F, Z)$
1.    $(V, E) \leftarrow F$
2.    $V' \leftarrow \pi_1(Z) \cup \pi_2(Z)$
3.    $Z \leftarrow \begin{cases} \{(u, v) : (v, u) \in Z\} & \pi_1(Z) < \frac{|V'|}{2} \\ Z & \pi_1(Z) \geq \frac{|V'|}{2} \end{cases}$
4.    $X \leftarrow Y \subseteq Z$ where $|Y| = |\pi_1(Z)|$ and $\pi_1(Y) = \pi_1(Z)$
5.    $G \leftarrow (V, X)$
6.    $E \leftarrow E \cup \{(v, \rho_G(u)) : (v, u) \in X\}$
7.    return $((V, E), Z \backslash X)$
```

By definition we can clearly see atleast $\frac{|V'|}{2}$ vertices are given a parent as we wanted. It will now be shown that the added edges is a conflict-free set so a forest is still the result.

Proposition 2.11 (Maximal Union Correctness)**.** Let forest $F = (V, E)$ be a forest and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) is an acyclic directed graph. Then the maximal union algorithm results in a forest $F' = (V, E')$ which satifies the Tree Union Property 2.12 for the conflict-free set $X \subseteq Z$ in F.

Proof. Let forest $F = (V, E)$ be a forest and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) is an acyclic directed graph. From proposition 2.10 we know that (V, Z) remains an acyclic graph throughout the algorithm. Since X is defined such that $\pi_1(X) = \pi_1(Z)$ and $|X| = |\pi_1(Z)|$ it follows that X is a conflict-free set in F since no vertex v appears more than once as a child in X i.e. $G = (V, X)$ is a forest. We can also conclude that $(V, \{(v, \rho_G(u)) : (v, u) \in X\}) \cong (V, X)$ since every child directly points to its root. So adding the edges in $\{(v, \rho_G(u)) : (v, u) \in X\}$ to E it follows by corollary 2.2 that the algorithm fulfills the Tree Union Property 2.12 and $F' = (V, E')$ where $E' = E \cup \{(v, \rho_G(u)) : (v, u) \in X\}$. □

Before the time complexity of maximal union can be shown the time complexity of path comopression has to be discussed. A problem that occur in the analysis is the computation of $\{(v, \rho_G(u)) : (v, u) \in X\}$. Since the forest (V, X) that may be constructed could be just a tree which is one long chain so $\rho_G(u)$ does $O(|X|)$ work to find its parent. This problem can be solved using pointer jumping, specifically Wyllie's List Ranking algorithm [8, p. 59] can be used directly on a forests to do $O(n \log n)$ work with $O(\log n)$ span on a forest of n vertices. This is not work-efficient, you would want to do $O(n)$ work. There are list ranking algorithms [1] which are work-efficient with $O(\log^2 n)$ span[1]. List ranking can be used to construct an euler tour of an edge list[2], this euler tour represents a V-Tree [3, pp. 84–91] and this method will work on a forests. It is not clear if this method can be avoided and instead just directly applying the list ranking on the forest as to avoid alot of work. This can atleast be done with Wyllies List ranking. It also seems that using a connected components algorithm by Shiloach and Vishkin [6] could be used which will give you a representative, it claims to be work-efficient with $O(\log n)$ span but uses a computational model with concurrent write. This may or may not be a problem to express this in a parallel functional array language but there are implementations in NESL [4].

Proposition 2.12 (Maximal Union Time Complexity)**.** Let forest $F = (V, E)$ be a forest and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs in F and (V, Z) is an acyclic directed graph. Then the maximal union algorithm runs in $O(|Z|)$ work and $O(\log^2 |Z|)$ depth.

Proof. For step 1. it takes $O(1)$ work and $O(1)$ if we assume that E is only used once in this function. Step 2-3. finds unique elements and can be computed with a parallel integer sort and a filter, assuming the encoding of vertices uses a fixed number of k-bits then this part is $O(|Z|)$ work and $O(\log |Z|)$ span. Step 4. can be implemented by a parallel integer sort on the first element of each pair in Z followed by a parallel filter that selects the first occurrence of each unique first element, this takes $O(|Z|)$ work and $O(\log |Z|)$ depth. Step 6. takes $O(|X|)$ work and $O((\log |X|)^2)$ depth to do path compression and add these edges to E. Step 7. takes $O(|Z|)$ work and $O(\log |Z|)$ depth to compute the set difference $Z \backslash X$ by a filter. Hence the total work is $O(|Z|)$ and the total depth is $O(\log |Z|)$. Hence the maximal conflict-free set algorithm runs in $O(|Z|)$ work and $O((\log |Z|)^2)$ depth. $\square$

Now the time complexity is known we can finally give the algorithm for parallel union which performs bulk unification making a set of A pair vertices

[1]They assume scan has $O(1)$ span which is not reasonable anymore.

[2]https://www.cs.cmu.edu/~scandal/nesl/algorithms.html#trees

become equivalent in the final forest. The way the algorithm works is by constructing a directed acyclic graph of roots from A which will be called Z. Then have a loop with the invariant that (V, Z) is an directed acyclic graph. Then in the loop body simply perform maximal union and turn the remaining uninserted edge of Z into an directed acyclic graph. Continue till Z is empty and then F is the final forest where all pairs of A has been unified.

Algorithm 2.2 (Parallel Tree Union)**.** Let forest $F = (V, E)$ be a tree and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. The parallel tree union algorithm is defined as:

```
    $ParallelTreeUnion(F, A)$
1.  $Z_p \leftarrow \{(\rho_F(v), \rho_F(u)) : (v, u) \in A \wedge \rho_F(v) \neq \rho_F(u)\}$
2.  $Z \leftarrow \{(\min\{v, u\}, \max\{v, u\}) : (v, u) \in Z_p\}$
3.  while $|Z| > 0$ do
4.     $(F, Z_q) \leftarrow MaximalUnion(F, Z)$
5.     $Z_r \leftarrow \{(\rho_F(v), \rho_F(u)) : (v, u) \in Z_q \wedge \rho_F(v) \neq \rho_F(u)\}$
6.     $Z \leftarrow \{(\min\{v, u\}, \max\{v, u\}) : (v, u) \in Z_r\}$
7.  return $F$
```

First of all we have to establish that the actual algorithm produces the correct output, as in it fullfills the Tree Union property 2.12.

Proposition 2.13 (Parallel Tree Union Correctness)**.** Let forest $F = (V, E)$ be a tree and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Then the parallel tree union algorithm results in a forest $F' = (V, E')$ which satifies the Tree Union Property 2.12 for A.

Proof. To show that the algorithm returns a forest $F' = (V, E')$ where for all $(v, u) \in A$ it holds that $v \sim_{F'} u$ it will be shown that the steps in the algorithm does not remove unification problems $(v, u) \in A$ which do not hold at some step in the final forest F.

First in step. 1-2 creates a directed acyclic graph (V, Z) which represents the same unification problems as in A. Since if $\rho_F(v) = \rho_F(u)$ where $(v, u) \in A$ then $v \sim_F u$ so u and v have already been unified. Secondly reordering the components of (v, u) does not change the since adding an edge (u, v) or (v, u) to a forest commutes since $v \sim_F u$ commutes.

The loop in Step. 3-6 has the following invariant that Z is an directed acyclic graph. It will be shown that this invariant is fulfilled and that implies F fulfills $v \sim_F u$ for all $(v, u) \in A$ in the final F. In the start of the loop the acylic directed graph invariant holds. Then in step 4. using Maximal

Union 2.1 achieves a forest with a proper subset of Z being unified and Z_q is the remaining pairs that have not been unified. In step 5-6. the directed acyclic graph Z is constructed (by the same arguments as in step 1-2.) from Z_q fulfilling the loop invariants.

Since the loop continuesly unifies using Maximal Union on the remaining unification problems then the algorithm does fulfill the Tree Union property 2.12. □

Before giving the general analysis of union find a specicial case of when the initial forest is $F = (V, \emptyset)$. Since this is a likely case that can happen for certain algorithms.

Proposition 2.14 (Empty Parallel Tree Union Time Complexity)**.** Let forest $F = (V, \emptyset)$ be a tree, and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Then the parallel tree union algorithm does $O(|A|)$ work and has $O(\log^3 |A|)$ span.

Proof. Let forest $F = (V, \emptyset)$ be a tree, step 1-2. a filter and an ordering is applied which can be computed in $O(|A|)$ work and $O(\log |A|)$ span. This is the time complexity since $\rho_F(v)$ and $\rho_F(u)$ is $O(1)$ work and span so the maximum traversals to root is constant.

The body of the loop at step 4-6 does at first $O(|Z|)$ work and $O(\log^2 |Z|)$ span where $|Z|$. The factor is dominated by Maximal Unions time complexity 2.12 due to 5-6 being a filter and the map. Since the search of the representative element is $O(1)$ because the path compression in maximal union will make the distance to the representative be at most 1.

The amount of vertices removed from $V' = \pi_1(Z) \cup \pi_2(Z)$ in the next iteration is lowerbounded by $\frac{|V'|}{2}$ due to maximal union 2.1. These remaining vertices are used to construct a directed acyclic graph (V', Z') where $Z' \subseteq V' \times V'$. We can establish the following bound of Z' using the fact that $\binom{n}{2}$ is the upperbound for a directed acyclic graphs size of n vertices.

$$|Z'| \leq \binom{\frac{|V'|}{2}}{2} = \frac{\left(\frac{|V'|}{2}\right)^2 - \frac{|V'|}{2}}{2} = \frac{\frac{1}{2}|V'|^2 - |V'|}{4}$$

Now if we consider the half amount of edges that (V', Z') can maximally have $\frac{\binom{|V'|}{2}}{2}$ the following inequality arises.

$$|Z'| \leq \frac{\frac{1}{2}|V'|^2 - |V'|}{4} \leq \frac{|V'|^2 - |V'|}{4} = \frac{\binom{|V'|}{2}}{2}$$

Meaning that by halving the amount of V' every iteration is bounded by halving the amount of Z' worked on each iteration of the loop. Since $|Z'|$ is upperbounded by $|A|$ we get the total work done by the loop is:

$$\sum_{k=0}^{\lfloor \log |A| \rfloor} \frac{|A|}{2^k} = |A| \sum_{k=0}^{\lfloor \log |A| \rfloor} \frac{1}{2^k} < |A| \sum_{k=0}^{\infty} \frac{1}{2^k} = 2|A|$$

Meaning the work of the function is $O(|A|)$. The span is $O(\log^3 |A|)$ since the worst span is by maximal union $O(\log^2 |A|)$ which is done $O(\log |A|)$ times. □

The time complexity is good for certain cases, but in the general case for any forests, the time complexity becomes way worse.

Proposition 2.15 (Parallel Tree Union Time Complexity)**.** Let forest $F = (V, E)$ be a tree, and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Given k applications of Parallel Tree Union 2.2 before then Parallel Tree Union does $O(|A| k \log |V|)$ work and has $O(k \log |V| + \log^3 |A|)$ span.

Proof. The analysis is the same as in the empty case 2.14 beside in step 1. Considering a sequence of unification problems that have been unified $A_1, A_2, \dots, A_k$. In the worst case then $|A_i|$ will extened upon a trees height by $\lfloor \log |\pi(A_i) \cup \pi_2(A_i)| \rfloor$ since the alogrithm half the number of vertices every iteration of $|A|$ in the worst case. since $|\pi(A_i) \cup \pi_2(A_i)| \leq |V|$ the following bound can be derived on a sequence of unification problems.

$$\sum_{i=1}^{k} \lfloor \log |\pi(A_i) \cup \pi_2(A_i)| \rfloor \leq \sum_{i=1}^{k} \lfloor \log |V| \rfloor = k \lfloor \log |V| \rfloor$$

Therefore we can conclude the amount the work is $O(|A| k \log |V|)$ and the span is $O(k \log |V| + \log^3 |A|)$ □

Now we can go on to consider the asymptotics of performing a find.

Proposition 2.16 (Parallel Tree Find Time Complexity)**.** Let forest $F = (V, E)$ be a tree and let $v \in V$ be a element in V which representative will be found. Given k applications of Parallel Tree Union 2.2 then find does $O(k \log |V|)$ work and has $O(k \log |V|)$ span.

Proof. Using the proof of Parallel Tree Find Time Complexity 2.15 where it was shown that the tree height is upperbounded by $k \lfloor \log |V| \rfloor$ after k applications of parallel tree union. Hence finding a single elements representative will have $O(k \log |V|)$ work and span. □

2.5 Union by Size and Rank

The next problem is to extend the data-parallel union-find structure such that it would allow the usage of the union by size or rank heurestic. To do this graph (V, E) with weighted vertices is introduced where the weight is given by a function $f : V \to \mathbb{N}$. We may update a functions range by $f' = f \oplus S$ where $S \subseteq V \times \mathbb{N}$ where the pair $(v, n) \in S$ with the maximum n becomes the result of v i.e. $f'(v) = n$. Now using this we wish to show that it is possible to establish an total ordering on $f(v)$ where if $f(v) = f(u)$ then it default to ordering by the values given to f.

Proposition 2.17 (Weighted Total Ordering)**.** Let V be a total ordering set $(V, \lesssim)$ and $f : V \to \mathbb{N}$ then $(v, \preceq_f)$ is a total ordering where

$$v \preceq_f u \begin{cases} f(v) < f(u) & f(v) < f(u) \\ v \lesssim u & f(v) = f(u) \end{cases}$$

Proof. To show it is a total ordering it must be shown that the relation is reflexive, transitive, antisymmetric, and total.

1. If $v \preceq_f v$ holds then $f(v) = f(v)$ holds and so does $v \lesssim v$.
2. If $v \preceq_f u$ and $u \preceq_f w$ holds then:
   - If $f(v) < f(u) < f(w)$ holds then $v \preceq_f w$.
   - If $f(v) = f(u) = f(w)$ holds then $v \preceq_f w$.
   - If $f(v) < f(u) = f(w)$ holds then $v \preceq_f w$.
   - If $f(v) = f(u) < f(w)$ holds then $v \preceq_f w$.
3. If $v \preceq_f u$ and $u \preceq_f v$ holds then $f(v) < f(u)$ and $f(v) > f(u)$ can not hold so $f(v) = f(u)$. So $v \lesssim u$ and $u \lesssim v$ both holds and since $(V, \lesssim)$ is a total ordering then $v = u$.
4. Since $(\mathbb{N}, \leq)$ then either $f(v) < f(u)$, $f(u) < f(v)$, or $f(v) = f(u)$ holds. First case yields $v \preceq_f u$, second case yields $u \preceq_f v$. Last case yields either $v \lesssim u$ or $v \lesssim u$, since $(V, \lesssim)$ is total then either $v \preceq_f u$ or $v \preceq_f u$ holds making the relation total.

□

The idea now is to have f be a function which reflects the rank/size of a vertex and then create the directed acyclic graph from a set of root pairs. Then it is possible to pick conflict-free set and insert this into the forest which will respect the weights limiting the height of the trees. For this algorithm it has two different was of being instantiated, either with the by rank or by size heurestic. This modifies maximal union and parallel union quite a bit.

Algorithm 2.3 (Weighted Maximal Union)**.** Let forest $F = (V, E)$ be a forest, $f : V \to \mathbb{N}$ and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) where $v \preceq_f u$ and $v \neq u$ for all $(v, u) \in Z$. The weighted maximal union algorithm is defined as:

```
    $WeightedMaximalUnion(F, Z, f)$
1.      $(V, E) \leftarrow F$
2.      $A \leftarrow \{(v, u) : (v, u) \in Z \wedge f(v) = f(u)\}$
3.      $B \leftarrow \{(v, u) : (v, u) \in Z \wedge f(v) \neq f(u)\}$
4.      $V' \leftarrow \pi_1(A) \cup \pi_2(A)$
5.      $A \leftarrow \begin{cases} \{(u, v) : (v, u) \in A\} & \pi_1(A) < \frac{|V'|}{2} \\ A & \pi_1(A) \geq \frac{|V'|}{2} \end{cases}$
6.      $Z \leftarrow A \cup B$
7.      $X \leftarrow Y \subseteq Z$ where $|Y| = |\pi_1(Z)|$ and $\pi_1(Y) = \pi_1(Z)$
8.      $G \leftarrow (V, X)$
9.      $X \leftarrow \{(v, \rho_G(u)) : (v, u) \in X\}$
10.     $E \leftarrow E \cup X$
11.     $f \leftarrow f \oplus \{(v, f(u) + 1) : (v, u) \in X \wedge f(v) = f(u)\}$   (By Rank)
11.     $f \leftarrow f \oplus \left\{ \left( v, \sum_{(v,u) \in X} f(u) \right) : v \in V \right\}$   (By Size)
12.     return $((V, E), Z \backslash X, f)$
```

Proposition 2.18 (Weighted Maximal Union Correctness)**.** Let forest $F = (V, E)$ be a forest, $f : V \to \mathbb{N}$ and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) where $v \preceq_f u$ and $v \neq u$ for all $(v, u) \in Z$. Then the weighted maximal union algorithm results in a forest $F' = (V, E')$ which satifies the Tree Union Property 2.12 for the conflict-free set $X \subseteq Z$ in F.

Proof. Let forest $F = (V, E)$ be a forest and let $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs F and (V, Z) be a set of root pairs F and (V, Z) where $v \preceq_f u$ and $v \neq u$ for all $(v, u) \in Z$. By Proposition 2.8 we have that (V, Z) is acyclic, and by partitioning Z in to A and B and swapping edges in A changes the relation from $(V, \lesssim)$ to $(V, \gtrsim)$ in $(V, \preceq_f)$ hence it is still a total ordering so $(V, A \cup B)$ is still acyclic. The algorithm will always remove at least one vertex from $\pi_1(Z) \cup \pi_2(Z)$ since there must be atleast one unique vertex on the left hand side of any $(v, u) \in Z$. Therefore, the remaining of the algorithm does the same as in Maximal Union 2.1 hence the algorithm results in a forest $F' = (V, E')$ which satifies the Tree Union Property 2.12 for the conflict-free set $X \subseteq Z$ in F by Proposition 2.11. The only way they differ is changing the function f which is not of importance to the correctness. $\square$

Now we find the time complexity of Weighted Maximal Union.

Proposition 2.19 (Weighted Maximal Union Time Complexity)**.** Let forest $F = (V, E)$ be a forest, $f : V \to \mathbb{N}$ be a function, and $Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$ be a set of root pairs in F and (V, Z) is an acyclic directed graph. Then the maximal union algorithm runs in $O(|Z|)$ work and $O(\log^2 |Z|)$ depth.

Proof. The arguments are almost the same as in Proposition 2.12. The step 2-3 is just a partition and either of step 11. can be done using integer sorts and segmented reduce. $\square$

Now we have to consider its corresponding parallel tree union algorithm.

Algorithm 2.4 (Weighted Parallel Tree Union)**.** Let forest $F = (V, E)$ be a tree, $f : V \to \mathbb{N}$ and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. The weighted parallel tree union algorithm is defined as:

```
   $WeightedParallelTreeUnion(F, A, f)$
1.    $Z_p \leftarrow \{(\rho_F(v), \rho_F(u)) : (v, u) \in A \wedge \rho_F(v) \neq \rho_F(u)\}$
2.    $Z \leftarrow \{(v, u) : (v, u) \in Z_p \wedge v \preceq_f u\} \cup \{(u, v) : (v, u) \in Z_p \wedge u \preceq_f v\}$
3.    while $|Z| > 0$ do
4.       $(F, Z_q, f) \leftarrow WeightedMaximalUnion(F, Z, f)$
5.       $Z_r \leftarrow \{(\rho_F(v), \rho_F(u)) : (v, u) \in Z_q \wedge \rho_F(v) \neq \rho_F(u)\}$
6.       $Z \leftarrow \{(v, u) : (v, u) \in Z_r \wedge v \preceq_f u\} \cup \{(u, v) : (v, u) \in Z_r \wedge u \preceq_f v\}$
7.    return $F$
```

The correctness of weighted parallel tree union should be quite evident but the proposition will be stated for completeness.

Proposition 2.20 (Weighted Parallel Tree Union Correctness)**.** Let forest $F = (V, E)$ be a tree, $f : V \to \mathbb{N}$ and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Then the weighted parallel tree union algorithm results in a forest $F' = (V, E')$ which satifies the Tree Union Property 2.12 for A.

Proof. The proof follows mutatis mutandis from that of Proposition 2.13. $\square$

Now the time complexity of Weighted Parallel Tree Union will be considered in regards to union by rank. Here we initially have $f(v) = 0$ for all $v \in V$ and we now realize the following from definition of Weighted Maximal Union by Rank that the weight of a representative u only increase in weight if $f(v) = f(u)$ for (v, u) from a set of root pairs.

Remark. From the definition of the Weighted Maximal Union 2.3 the following follows. Let $F = (V, E)$, $f : V \to \mathbb{N}$, $(v, u) \in Z \subseteq \{(\rho_F(v), \rho_F(u)) : (v, u) \in V \times V\}$, $G = (V, Z)$ such that $\rho_{F'}(u) = \rho_{F'}(v) = u$ where $F' = (V', E')$ is the forest as a result of the algorithm then the following holds.

- If $f(v) < f(u)$ for all $(v, u) \in Z$ then $(v, u) \in E'$ becomes the new forest and f' is the new weight function where $f'(u) = f(u)$
- If $f(v) = f(u)$ for some $(v, u) \in Z$ then $(v, u) \in E'$ becomes the new forest and and f' is the new weight function where $f'(u) = f(u) + 1$.

Now to show that the time complexity is work-efficient first a bound on the rank of every element is needed.

Proposition 2.21 (Rank Bound)**.** Let $F = (V, E)$, $f : V \to \mathbb{N}$, and $v \in V$, then the follow bound holds for all elements v when used for union by rank.

$$f(v) \leq \lfloor \log |V| \rfloor$$

Proof. It can be shown that the tree is bounded by $\lfloor \log k \rfloor$ given $k = |V|$ vertices using induction. The base case holds since given 1 vertex v then $f(v) = 0 \leq \lfloor \log k \rfloor$. Assume now the claim holds for k vertices it will now be shown it holds for $k + 1$ vertices. So by the induction hypothesis $f(v), f(u) \leq \lfloor \log k \rfloor$ if $f(u) < f(v)$ then for the new weight function f' it must hold that $f'(v) = f(v) \leq \lfloor \log k \rfloor \leq \lfloor \log(k + 1) \rfloor$. If $f(v) = f(u) = h$ then $f'(v) = f(v) + 1$. By the induction hypothesis for $v, u \in V$ it holds that $f(v), f(u) \leq \lfloor \log k \rfloor$ so the height is bounded by h meaning the subtrees v and u has at most 2^h vertices. Therefore, when unifying v and u with v as the parent the final tree contains at least $2^h + 2^h = 2^{h+1}$ vertices. Since the final tree has at most $k + 1$ vertices it follows that $2^{h+1} \leq k + 1$ and therefore $h + 1 = f(v) + 1 = f'(v) \leq \lfloor \log(k + 1) \rfloor$. $\square$

Using the bound on ranks it is possible to derive the the time complexity of the Weighted Parallel Tree Union algorithm when using the rank heurestic.

Proposition 2.22 (Weighted Parallel Tree Union (By Rank) Time Complexity)**.** Let forest $F = (V, E)$ be a tree, and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Then Weighted Parallel Tree Union does $O(|A| \log |V|)$ work and has $O((\log |V|)(\log^2 |A|))$ span.

Proof. In step 1. $O(|A| \log |V|)$ work is done and have $O(\log |V|)$ span due to trees bounded height. Now the key The key to determining the time complexity is to show that the loop runs at most $1 + \lfloor \log |V| \rfloor$ times. This can be done by induction where we say that after the kth iteration $k \leq f(v)$ for all $v \in V$.

- The base case holds since for all $v \in V$ it holds that $k = 0 \leq f(v)$.
- By induction hypothesis $k \leq f(v), f(u)$, for some $(v, u) \in A$ then v will be removed and replaced by u since it is the first component so $v \notin \pi_1(A') \cup \pi_2(A')$ where A' is A in the next iteration. Now if $f(v) < f(u)$ then $k + 1 \leq f'(v)$ and if $f(v) = f(u)$ then $k + 1 \leq f(u) + 1 = f'(u)$. Now if $(u, v) \in A$ then u will be removed and replaced by v since it is the first component so $u \notin \pi_1(A') \cup \pi_2(A')$. Now if $f(u) < f(v)$ then $k + 1 \leq f'(v)$ and if $f(u) = f(v)$ then $k + 1 \leq f'(v) = f(v) + 1$.

So after $1 + \lfloor \log |V| \rfloor$ iterations iterations $1 + \lfloor \log |V| \rfloor \leq f(v)$ but since $f(v) \leq \lfloor \log |V| \rfloor$ then A must be empty in the end. So the work done is at most $O(|A| \log |V|)$ work and the span is $O((\log |V|)(\log^2 |A|))$. □

Lastly I did not have time prove union by size is work-efficient in the sense that it does as much work as a sequential union-find structure with union by size so instead conjecture.

Conjecture 2.1 (Weighted Parallel Tree Union (By Size) Time Complexity)**.** Let forest $F = (V, E)$ be a tree, and let $A \subseteq V \times V$ be a set of pairs of elements in V that will be unified in parallel. Then Weighted Parallel Tree Union does $O(|A| \log |V|)$ work and has $O((\log |V|)(\log^2 |A|))$ span.

3 Implementation

The theoretical algorithms have been formulated in a manner that is well suited for a functional data-parallel array language and they will be in this section implemented in such a language. The programming language they will be implemented in is the Futhark programming language and the following section will describe how to implement such algorithms.

```
module type unionfind = {
  type unionfind [n]
  type handle
  val create : (n: i64) -> *unionfind [n]
  val find [n] [u] : *unionfind [n] -> [u]handle -> *(
     unionfind [n], [u]handle)
  val union [n] [u] : *unionfind [n] -> [u](handle,
     handle) -> *unionfind [n]
  val handles [n] : unionfind [n] -> *[n]handle
  val from_i64 [n] : unionfind [n] -> i64 -> *handle
  val to_i64 [n] : unionfind [n] -> handle -> i64
}
```

Figure 4: Module type definition of union-find.

3.1 Interface

The interface of the union-find structure is defined as a module type in figure 4, this is a common interface for any data-parallel union-find implementation. It consists of a data type which is the union-find structure itself. The elements in the union-find structure are represented as integers. These integers are called *handles* and are used to refer to the elements in the union-find structure. The union-find structure is initialized with a fixed number of elements n where the handles are in the range $[0, n-1]$ so they can be used as indices in an array of size n. When exposing these elements to a user then they are abstract data types such that the the user cannot do uninteneded operations on the handles or give invalid handles to the union-find structures operations.

The operations supported are *create*, *find*, and *union*. The create function creates a union-find structure which is empty such that no element is “equivalent” with any other element. The find operation takes an array of handles and results in a array of each elements representative. This can be used to check if two elements are in the same set or is equivalent by checking if their representatives are the same. The union operation takes an array of 2-tuple handles, all tuples in the array will be unified such that they are equivalent in the resulting union-find structure.

There are also the function *handles* which gives an array of all available unique handles. There are also function which are able to cast a signed 64-bit integer *to* and *from* a handle. These are nice since in the handles array is an bijection between handles and their integer representation.

```
type unionfind [n] = {parents: [n]handle}

def create (n: i64) : *unionfind [n] =
  {parents = rep none}
```

Figure 5: The type of union-find structure and a function for creating such a structure.

3.2 Union-find

The first and simplest implementation of the union-find structure is based on the basic parallel tree union algorithm 2.2. And as seen from the asymptotic analysis 2.14 it can be asymptotically efficient for certain inputs.

3.2.1 Data type

A simple union-find structure can be implemented using an array indices which where the index represents the handle and the value at that index is the parent of that element. If the value at that index is a special value which is the highest possible integer value then that element is a root and therefore its own representative. The type of the union-find structure and the creation of the structure can be seen in Figure 5

3.2.2 Find

To implement the find operation we can simply do a parallel map over all elements to find their representative by a simple loop that follows the parent pointers until a root is found. The implementation has an auxiliary function which finds the children of a vector, this auxiliary function is called in the actual implementation of find. This auxiliary function is helpful in other implementation due to how futhark handles uniqueness types and records. The implementation can be seen in Figure 6.

3.2.3 Union

The union operation can be implemented using the parallel tree union algorithm 2.2 but due to its abstract nature one must figure out how to make it work in the concrete implementation. The initial step is to consider maximal union. First step of maximal union is it is given an directed acyclic graph and the question is if there the number of unique vertices in outgoing edges is more than half the total number of unique vertices occuring in all edges.

```
def find_by_vector [n] [u]
                   (parents: *[n]handle)
                   (hs: [u]handle) : *([n]handle, [u]
                      handle) =
  let ps =
    map (\h ->
           loop h
           while parents[h] != none do
             parents[h])
        hs
  in (parents, ps)

def find [n] [u]
         ({parents}: *unionfind [n])
         (hs: [u]handle) : *(unionfind [n], [u]handle)
            =
  let (new_parents, ps) = find_by_vector parents hs
  in ({parents = new_parents}, ps)
```

Figure 6: A function to find the parents of an array of handles using the array of parents and a function to find the parents of an array of handles using the union-find structure.

This question is actually just whether there are more unique outgoing vertices that have an outgoing edge rather than an incoming one. This is used to determine if the directed acyclic graph should be inverted or not. In Futhark this can be computed using the builtin histogram or a integer sort with a segmented reduce. You can also modify the algorithm to have random asymptotics by randomly choosing to flip the edges. The actual implementation will use HyperLogLog++[5] which estimates the number of unique vertices. Meaning the implementation does not have the actual asymptotics as the one described in the theory section. Due to how good HyperLogLog++ is at giving an estimate then it should make little to no difference. Then based on this estimate this edges can be inverted if needed.

The next problem is picking out a forest from the directed acyclic graph. This can again be done using histogram where the minimum (or even maximum) parent is selected. It is important to note this again strays away from the original algorithms asymptotic but it is unlikely it will perform badly. Now using the new parent vector simply partition the the directed acyclic graph into the edges that have been inserted and not added. The ones that have been added are compressed using pointer jumping with Wyllies List Ranking algorithm [8, p. 59]. This strays from the asymptotics of the theoretical algorithm but it is still a very fast implementation. This was chosen since this is not at the core of this project ant it seems also likely that the constants of a work-efficient implementation will be too large for pratical use. All of these details culminates into the the implementation of maximal union found in Figure 7

It is now possible to implement the union operation, the implementation is a loop that check if there are no more pairs to be unified then continue trying to use maximal union. The implementation has an auxiliary function like *find* but it operation on an array of tuples. The function takes the parent array and the tuples and find the representative of every component in the tuple pair and returns the parent array unchanged with an array of the pairs representatives. This function is used in the loop body and the then turned into a directed acyclic graph by ordering them and filtering otu any pairs with the same representative. Then finally maximal union can be applied until the loop terminates. The final implementation can be seen in Figure 8.

3.3 Union by Rank

Union-find with union by rank follows almost the same pattern as union-find but now with an added array to keep track of the roots rank. The definition of this can be seen in Figure 9. The find operation will be exactly the same as before but the union operation becomes more complex.

```
module hll = mk_hyperloglog_plusplus i64key

def maximal_union [n] [u]
                  (parents: *[n]handle)
                  (eqs: [u](handle, handle)) : ?[m].(*[
                     n]handle, [m](handle, handle)) =
  let (l, r) = unzip eqs
  let unique_l = hll.insert () (hll.create 10) l |> hll
     .count
  let unique_r = hll.insert () (hll.create 10) r |> hll
     .count
  let (l, r) = if unique_l < unique_r then (r, l) else
     (l, r)
  let parents = reduce_by_index parents i64.min none l
     r
  let (eqs, done) =
    copy (partition (\(i, p) -> parents[i] != p) eqs)
  let parents = compression parents (map (.0) done)
  in (parents, eqs)
```

Figure 7: The implementation of maximal union using HyperLogLog++ with Wyllie List ranking to compress the added vertices.

```
def union [n] [u]
          ({parents}: *unionfind [n])
          (eqs: [u](handle, handle)) : *unionfind [n] =
  let (parents, _) =
    loop (parents, eqs)
    while length eqs != 0 do
      let (parents, eqs) = find_pairs parents eqs
      let eqs =
        map (\(a, b) -> if a < b then (a, b) else (b, a
           )) eqs
        |> filter (\(a, b) -> a != b)
      let (parents, eqs) = maximal_union parents eqs
      in (parents, eqs)
  in {parents}
```

Figure 8: The implementation union.

```
type unionfind [n] =
  { parents: [n]handle
  , ranks: [n]u8
  }

def create (n: i64) : *unionfind [n] =
    { parents = rep none
    , ranks = rep 0
    }
```

Figure 9: The type of union-find structure using union by rank and a function for creating such a structure.

3.3.1 Union

The outer loop of union is almost the same as in the union operation without heurestics. Some changes were made to make the code easier to read so the Weighted Parallel Tree Union and Weighted Maximal Union has been adjusted. The tree union algorithm consists of only a loop where it "normalizes" the equalities and then gives them to a maximal union algorhtm. This structure can be seen in Figure 10.

How the order works is it first of all finds the representatives of the equalities and then it orders them by rank if they differ otherwise the integer value. Lastly it removes if the equalities would create a cycle, and create to arrays where one of them consists of cases where the rank differ and the other array are for equalities where the ranks differs. Now just count the number of unique elements for each component and invert the constraints with the same rank if needed based on the number of unique elements. The implementation can be see in Figure 11 and should look quite familiar to Figure 7.

Now maximal union becomes almost the same as in for union-find without heurestics. You have to pick some constraint and remove the constraints that got picked. There is one small difference and that is if a constraint contained elements with the same rank and one of the elements becomes the new root then the rank should be incremented by one. This is done using a reduce by index and the implementation can be seen in Figure 12.

3.4 Union by Size

Union by size needs still a parent vector, the sizes of a root and some temporary indices. The temporary indices are needed to keep track of what element got selected as a parent without allocating a whole array for every call to

```
def union [n] [u]
          ({parents, ranks}: *unionfind [n])
          (eqs: [u](handle, handle)) : *unionfind [n] =
  let (new_parents, new_ranks, _) =
    loop (parents, ranks, eqs)
    while not (null eqs) do
      let (parents, ranks, eqs) = order parents ranks
         eqs
      let (parents, ranks, eqs) = maximal_union parents
          ranks eqs
      in (parents, ranks, eqs)
  in { parents = new_parents
      , ranks = new_ranks
      }
```

Figure 10: The union operation used for Union by Rank.

union. The precise data type and function for creation can be see in Figure 13.

3.4.1 Union

Union has the same loop structure as in union by rank but with sizes and the added temporary indices. Order is also still an auxiliary function but it now order on size instead of rank. The big change is how maximal union works, to pick a constraint the index of the constraint is used to guarantee a unique constraint. Then the constraints are partitioned into constraints that are done when the function is called and the remaining constraints that have not been solved. Now do path compression to find the representative and find the sizes of the representatives children. Then these can be used to update the size of the parent using a reduce by index. Lastly clear the content of the temporary indices such that it can be used for the next time. This implementation can be seen in Figure 14.

4 Discussion

4.1 Tests

The approach for testing the implementation is property based testing. The property tested for is if we have a sequential union-find structure then the

```
def order [n] [u]
          (parents: *[n]handle)
          (ranks: *[n]u8)
          (eqs: [u](handle, handle)) : ?[m].( *[n]
             handle, *[n]u8, [m](handle, handle)) =
  let eqs_elems = unzip eqs |> uncurry (++)
  let (new_parents, new_eqs_elems) = find_by_vector
     none parents eqs_elems
  let (_, value_eqs, rank_eqs) =
    split new_eqs_elems
    |> uncurry zip
    |> map (\(v, u) ->
              if ranks[v] == ranks[u]
              then if v < u then (v, u) else (u, v)
              else if ranks[v] < ranks[u] then (v, u)
                 else (u, v))
    |> partition2 (uncurry (==)) (\(v, u) -> ranks[v]
       == ranks[u])
  let (vs, us) = unzip value_eqs
  let unique_vs = hll.insert () (hll.create 10) vs |>
     hll.count
  let unique_us = hll.insert () (hll.create 10) us |>
     hll.count
  let value_eqs = if unique_vs < unique_us then zip us
     vs else zip vs us
  let eqs = value_eqs ++ rank_eqs
  in (new_parents, ranks, eqs)
```

Figure 11: The order auxiliary used for Union by Rank.

```
def maximal_union [n] [u]
                  (parents: *[n]handle)
                  (ranks: *[n]u8)
                  (eqs: [u](handle, handle)) : ?[m].(
                     *[n]handle, *[n]u8, [m](handle,
                     handle)) =
  let (ls, rs) = unzip eqs
  let parents = reduce_by_index parents i64.min none ls
      rs
  let (new_eqs, done) =
    copy (partition (\(i, p) -> parents[i] != p) eqs)
  let is = map (.0) done
  let (new_parents, new_ps) = compression none parents
     is
  let new_ranks_done =
    copy
    <| map2 (\l p ->
                u8.bool (ranks[l] u8.== ranks[p]) +
                   ranks[p])
            is
            new_ps
  let new_ranks = reduce_by_index ranks u8.max 0 new_ps
      new_ranks_done
  in (new_parents, new_ranks, new_eqs)
```

Figure 12: The maximal union used for Union by Rank.

```
type unionfind [n] =
  { parents: [n]handle
  , sizes: [n]i64
  , temporary_indices: [n]i64
  }

def create (n: i64) : *unionfind [n] =
  { parents = rep none
  , sizes = rep 1
  , temporary_indices = rep none
  }
```

Figure 13: The type of union-find structure using union by size and a function for creating such a structure.

equivalence classes should be the same for all vertices that must be unified in the original input. The reason for doing this test is it is very easy to reason about the sequential union-find algorithm without any optimizations unlike the parallel algorithms.

Now define $F = (V, E)$ as an initial union-find structure where every element is only equal to itself and A as a set of unification problems. Then define that U is the sequential union algorithm and V as some other union algorithm. The result of running the two algorithms must produce forests which represent the same equivalence classes.

$$F' \cong F'' \text{ where } F' = U(F, A) \text{ and } F'' = V(F, A)$$

The implementation detail of doing this comparison is asserting that the minimum element of an elements equivalence class is the same in both F' and F''.

$$\min \mathcal{E}_{F'}(v) = \min \mathcal{E}_{F''}(v) \text{ for all } v \in V$$

This can be implemented using a map over V with a nested reduce over V which is asymptotically slow but for testing it suffice.

The tests inputs is generated by creating an even lengthed sequence of V which are picked uniformily. Different number of $|V|$ is chosen and a different number of sequences are picked to achieve confidence in the implementation.

This approach does not catch problems like there may be a cycle in the union-find structure. The hope is that with the random input no cycle should be producible since otherwise the test would go on forever. The approach

```
def maximal_union [n] [u]
                  (parents: *[n]handle)
                  (sizes: *[n]i64)
                  (temporary_indices: *[n]i64)
                  (eqs: [u](handle, handle)) : ?[m].(
                    *[n]handle, *[n]i64, *[n]i64, [m](
                    handle, handle)) =
  let lefts = map (.0) eqs
  let eq_is = indices eqs
  let temporary_indices =
    reduce_by_index temporary_indices i64.min i64.
       highest lefts eq_is
  let (done, eqs) =
    zip (indices eqs) eqs
    |> partition (\(i, (l, _)) -> i ==
       temporary_indices[l])
    |> bimap (map (.1)) (map (.1))
  let (is, ps) = unzip done
  let parents = scatter parents is ps
  let (new_parents, new_ps) = compression none parents
     is
  let children_sizes = map (\i -> sizes[i]) is
  let new_sizes = reduce_by_index sizes (+) 0 new_ps
     children_sizes
  let new_eqs = copy eqs
  let new_temporary_indices = scatter temporary_indices
      lefts (rep i64.highest)
  in (new_parents, new_sizes, new_temporary_indices,
     new_eqs)
```

Figure 14: The maximal union used for Union by Size.

does also not assert any guarantees about the height of the tree. The problem with being able to test this is the interface would have to expose the inner wrokings of the union-find structure and/or add functionalities to the interface that would only be useful for testing. Both of these cases would be confusing to a user of the library.

A pitfall of the test suite is it should probably have more different sequences of union on different datasets. It is quite clear that is is easy to create a bug where a cycle is introduced leading to an infinite loop. These cases were usually captured by the benchmark suite instead.

The implementations passes this test suite so it is seen as the implementations work as intended.

4.2 Benchmarks

To benchmark the implementations different *datasets* and *unification strategies* for unifying subsets of the datasets is used. A dataset is made up of different constraints, here the constraints are of the form $(v, u) \in V^2$ where V is the set of vertices or the elements that can become equivalent by unification. The unification strategy is how the subsets of the dataset should be unified. The idea is that when using union-find it probably is not the case that all unification problems can be solve immediately and some find operations will be interspersed between union operaions. So to see how this effects the performance of union by combining different datasets and strategies. And also see how the find operations performance is effected by the sequence of union operaions defined by the dataset and strategy.

The following are the different Futhark implementations of union-find being compared.

- **Data-Parallel:** This implementation uses Parallel Tree Union 2.2 and is the simplest data-parallel version.
- **Data-Parallel (by Rank):** This implementation uses Weighted Parallel Tree Union by Rank 2.4 which corresponds to a union-find implementation with only the union by rank heurestic.
- **Data-Parallel (by Size):** This implementation uses Weighted Parallel Tree Union by Size 2.4 which corresponds to a union-find implementation with only the union by size heurestic.
- **Sequential (by Rank and Path Halving)** This is a classical implementation of union-find which uses Union by Rank and Path Halving. This implementation is also made in Futhark only using sequential code

and the C code has been inspected and the implementation is deemed reasonable. It was chosen to write this in Futhark so it was more comparable to the other implementations.

4.2.1 Datasets

The following is the list of different ways datasets are generated, their construction is explained and afterwards why they are chosen is explained.

- **Random:** From a set of vertices V select uniformily random a sequence $2m$ vertices and create tuples by pairing vertices with an even index with the next vertex in the sequence. This allows for duplicate unification problems. This dataset should be a case where most operations will act nicely and show a good example of a best case scenario.
- **Linear:** From a set of vertices V let $W \subseteq V$ where $|W| = m + 1$ then chain them together into unification problems such that v_i, v_{i+1} where $v_i, v_{i+1} \in W$. This forms one long chain of unification problems of size m. The implementation uses integers for vertices and the chain will be formed from a integer and its successor so this will effect caching for this benchmark. This long chain effects the list ranking the most but is also a good scenario for histograms.
- **Single:** From a set of vertice V let $W \subseteq V$ where $|W| = m$. Now pick $v \in W$ and create a set of unification problems of (v, w) for all $w \in V$. This makes a problem where every thing is in the same equivalence class. This can be a worst case scenario for histograms in the algorithms.
- **Inverse Single:** From a set of vertice V let $W \subseteq V$ where $|W| = m$. Now pick $v \in W$ and create a set of unification problems of (w, v) for all $w \in V$. This is just the same as single but inverting it makes the problem also account for if the algorithm poorly chooses which side to use as children in the forest that is the union-find structure. Once again a worst case scenario for histograms possibly.

4.2.2 Unification Strategy

The following is the list of different ways subsets of the datasets are unified, after the strategy is explained and the choice is clarified. explained.

- **All:** Try resolve all m unification problems at once. This leads to $O(1)$ calls to union. This is an absolute best case scenario for union-find.

	All	**Halving**	**Reverse Halving**	**Chunked**
Random	1342	11806	12443	17152
Linear	1281	5507	5766	15811
Single	819	7888	7318	8521
Inverse Single	818	7783	7097	8129

Table 1: Data-Parallel Union-Find Union Operation Benchmark Results (times in μs, m = 1,000,000, n = 200,000)

	All	**Halving**	**Reverse Halving**	**Chunked**
Random	2283	17279	18501	26252
Linear	1566	9176	10138	10897
Single	1273	5636	5621	6205
Inverse Single	1264	5827	5399	6122

Table 2: Data-Parallel Union-Find by Size Union Operation Benchmark Results (times in μs, m = 1,000,000, n = 200,000)

- **Halving:** Try by recursively resolving half of the m unification problems. This leads ot $O(\log m)$ calls to union. This seems like a likely scenario for a sequence of calls for union-find. Commonly we want logarithmic depth.
- **Reverse Halving:** This is the same as halving but in reverse order. So now starting with one unification problem double the amount of unification problems to resolve each iteration until all m problems are resolved. This leads to $O(\log m)$ calls to union. Same reason as halving but is here just as a sanity check.
- **Chunked:** Resolve 10000 unification problems of m at a time until all m have been resolved. This is to see what happens what happens to the find operation incase the union-find structure has poor asymptotics for find after many union applications.

4.2.3 Sortware and Hardware Details

The benchmarks are made in Futhark and are described in th Section 3, the data-parallel implementations are benchmarked with the OpenCL backend on an RTX 3060. The sequential implementation uses the C backend on an 12th Gen Intel(R) Core(TM) i7-12700 CPU.

	All	**Halving**	**Reverse Halving**	**Chunked**
Random	1939	13750	14638	22719
Linear	1637	9983	10540	10323
Single	1361	5800	5618	6615
Inverse Single	1362	5651	5469	6766

Table 3: Data-Parallel Union-Find by Rank Union Operation Benchmark Results (times in μs, $m = 1{,}000{,}000$, $n = 200{,}000$)

Test Type	**All**	**Halving**	**Reverse Halving**	**Chunked**
Random	409323	412848	411768	396458
Linear	412875	409061	402656	415118
Single	411821	422046	402506	407269
Inverse Single	403909	407411	403082	410958

Table 4: Sequential Union-Find Union Operation Benchmark Results (times in μs, $m = 1{,}000{,}000$, $n = 200{,}000$)

	All	**Halving**	**Reverse Halving**	**Chunked**
Random	9406	9215	8960	9206
Linear	8329	9257	8301	506588
Single	7617	8248	7949	7931
Inverse Single	8015	7952	8130	7977

Table 5: Data-Parallel Union-Find Find Operation Benchmark Results (times in μs, $m = 10{,}000{,}000$, $n = 1{,}000{,}000$, 10 finds per element)

	All	**Halving**	**Reverse Halving**	**Chunked**
Random	9484	9752	9385	9497
Linear	8160	8151	8195	8069
Single	8142	7939	8275	8031
Inverse Single	8187	7991	8247	8339

Table 6: Data-Parallel Union-Find by Size Find Operation Benchmark Results (times in μs, $m = 10{,}000{,}000$, $n = 1{,}000{,}000$, 10 finds per element)

	All	Halving	Reverse Halving	Chunked
Random	9618	9498	9580	9765
Linear	8423	8182	8355	8072
Single	8321	8188	8293	7873
Inverse Single	8198	8163	8301	7988

Table 7: Data-Parallel Union-Find by Rank Find Operation Benchmark Results (times in μs, m = 10,000,000, n = 1,000,000, 10 finds per element)

Test Type	All	Halving	Reverse Halving	Chunked
Random	20502732	20217976	20192048	20195360
Linear	20288659	20537245	20518061	20532433
Single	23240794	23227048	23192741	20183378
Inverse Single	23196742	23256317	23255364	20191462

Table 8: Sequential Union-Find Find Operation Benchmark Results (times in μs, m = 10,000,000, n = 1,000,000, 10 finds per element)

4.2.4 Discussion

It is clear to see the data-parallel implementations are much faster in regards to union and find operation. But since all the data-parallel algorithms does more work than the sequential version and we have a finite amount of processors so the sequential implementation should become faster in theory. It seems like this will happen on very incrediable large inputs so these data-parallel implementation seems more than well off. Generally the sequential implementation does well in the sense it is consistent in the benchmark result.

It is also arguably bad faith to do a purely sequential version, a concurrent union-find would be more reasonable to compare against. The sequential implementation was chosen since this is the common implementation found in the real world. And the sequential algorithm does few instructions so it was the hope that it would still be very fast but clearly a sequential version is very slow despite the little overhead. It could be that the futhark C backend was bad at writing C but the code seems reasonable so it is assumed to perform well.

When comparing the data-parallel implementations union operation we generally see that union by size performs worst, then union rank rank and then union-find without heurestics is quickest. There are some outliers like for when the unification strategy is chunked. This should make sense since union by size and union by rank will bound the tree height.

The biggest difference in regards to the performance of the data-parallel union-find structures is for the find operation. When the unification strategy is chunked then for the linear input the performance difference becomes clear. While using union by size or union by rank will keep the performance reasonable.

Something that should probably also had been benchmarked is a work-efficient parallel union-find [7]. It was known it existed in the beginning of the projekt but it was early dismissed since it seemed highly task parallel. But now looking back there are parts of it that could had been implemented like a union by rank or union by size implementation.

5 Examples

5.1 Region Labeling

Region Labeling, or Connected-component labeling is an image analysis problem where we want to give neighbouring pixel with the color the same label. A way to solve this problem is by producing a graph where a vertex is a pixel with edges to neighbouring pixels. So we create a graph $G = (V, E)$ consiting of every pixel in an image of width and height $w, h \in \mathbb{N}$. To define this graph we have a function $n : V \to \mathbb{P}(V)$ which gives the set of neighbours. For this specific example it would be:

$$n((i, j)) = V \cap \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$$

And a second function $c : V \to C$ where C is the set of colors a pixel can have. Then it is possible to define the following graph from the image.

$$\begin{aligned} V &= \{(i, j) : i, j \in \mathbb{N} \land 0 \le i < h \land 0 \le j < w\} \\ E &= \bigcup_{(i,j) \in V} \{v : v \in n((i, j)) \land c(v) = c((i, j))\} \end{aligned}$$

Then one could use breath first search or depth first search to solve for strongly connected components where each component is given a label. Another approach is to use union-find where the graph is seen as a set of constraints that must be solved. These constraint are solved simply by unifying two pixels such that they are equivalent.

The implementation of this was done in futhark and is simply mapping over all pixel indices giving them an integer label which is the flat position in the array that can be used for union-find. Then inspect if the neighbours have equivalent colors, keep these edges and filter out the ones that are not equivalent. The implementation of creating these equivalences can be seen

```
type dir = #n | #w | #e | #s

def mk_equivalences [h] [w] (img: [h][w]u32) : ?[n].[n](i64, i64) =
  tabulate_2d h
              w
              (\i j ->
                 let p = (i, j)
                 let flat_p = flat_pos w p
                 in map (\n ->
                           let p' = move n p
                           in if in_bounds img p' &&
                                 get p img == get p' img
                              then (flat_p, flat_pos w p')
                              else (flat_p, -1))
                        [#n, #w, #e, #s])
  |> flatten_3d
  |> filter ((>= 0) <-< (.1))
```

Figure 15: The construction of equivalences of neighbouring pixels.

in Figure 15. The next step is very simple, create the union find structure, try to unify all equivalences, and then look up the labels of all equivalences.

From the persepective of a user this is not work-efficient. The solution would do $O(hw\log(hw))$ work and have $O(\log^3(hw))$ span since $k = 0$. It is likely this would be $O(hw)$ work due to how the constraints are formed but it seems tricky to analyze. A much easier way to do it work-efficient implementation is to make constraints of horizontal and vertical neighbours seperately:

$$n_v((i,j)) = V \cap \{(i+1,j)\} \qquad n_h((i,j)) = V \cap \{(i,j+1)\}$$

Now when trying to solve any of these the loop in Parallel Tree Union 2.2 will only be ran once. Since there are no conflicts in which constraint to pick, all the constraints form a directed acyclic graph that just so happen to be a forest. So just unify the first the horizontal constraints and then the vertical or vice versa which leads to $O(hw)$ work and $O(\log^2(hw))$ span. This is still not optimal, it is possible to solve this problem using a connected components algorithm by Shiloach and Vishkin [6] which would do $O(hw)$ work and have $O(\log(hw))$ span.

```
entry region_label_unionfind [h] [w] (img: [h][w]u32) =
  let uf = u.create (h * w)
  let eqs =
    copy (mk_equivalences img
          |> map (\(i, j) ->
                    ( u.from_i64 uf i
                    , u.from_i64 uf j
                    )))
  let uf = u.union uf eqs
  let labels = u.find' uf (u.handles uf)
  in unflatten (map (u.to_i64 uf) labels :> [h * w]i64)
```

Figure 16: Construct equivalences and then solve the constraints.

5.2 Type Constraints

A use case for union-find is to type check programming languages, the following section will discuss type checking lambda calculus in Futhark. The syntax of lambda calculus consists of a variables, lambda function, and function application.

$$e ::= x \mid \lambda x.e \mid e_0\ e_1$$

And the type of an expression is either a type variable, or has an arrow type meaning the expression is an function from some type variable to another.

$$\tau ::= \alpha \mid \tau_0 \to \tau_1$$

The expressions and types can quite naturally be expressed in a language with recursive data types. A problem is Futhark does not have recursive data types and most functional array languages does not support this either. The way this is instead expression is by type constructors which has indices which points to its subexpressions. To do so first define vname as the expression variable names, tname as the type variable names, and e as the pointer from an expression constructor to its subexpressions. From here the type definitions should be straightforward, these definitions can be seen in Figure 17.

Now if we wish to express the application of the identity function to a variable $(\lambda v_0 \to v_0)\ v_1$. Then we create an array of four type constructors, the first is the application type constructor #app 1 3. The first argument of the type constructor points to the function argument at index 1 and the second is pointing to the value given to the function at index 3. The next

```
type vname = i64
type tname = i64
type e = i64

type exp =
     #var vname
   | #lam vname e
   | #app e e

type typ =
     #tvar tname
   | #tarrow tname tname
```

Figure 17: The types defined for expressions of lambda calculus and the type of lambda calculus expressions in Futhark.

is #lam 0 1 where the first arugment is the function argument name v_0 and the second argument is pointing to the function body at index 2. Next the body is just v_0 (i.e. #var 0). And lastly is the argument given to the identity function v_1 (i.e. #var 1). This result in the following encoding.

[#app 1 3, #lam 0 2, #var 0, #var 1]

Now the next step is to generate constraints from the lambda expressions to do type checking. To generate the type constraints we define inference rules where we see that only function application gives rise to constraints.

$$\frac{\Gamma(x) = \alpha}{\Gamma \vdash x : \alpha, \emptyset}$$

$$\frac{\Gamma[x \mapsto \alpha] \vdash e : \tau_1, C}{\Gamma \vdash \lambda x.e : \alpha \to \tau_1, C} \quad (\text{Fresh } \alpha)$$

$$\frac{\Gamma \vdash e_0 : \tau_0 \to \tau_1, C_0 \quad \Gamma \vdash e_1 : \tau_0, C_1}{\Gamma \vdash e_0\ e_1 : \alpha, C_0 \cup C_1 \cup \{\tau_0 \sim \tau_1 \to \alpha\}} \quad (\text{Fresh } \alpha)$$

When implementing constraint generation in any common functional language you would traverse the the abstract syntax tree while keeping track of the type environment and generate constraints from this. It seems like it should be possible to emulate the type environment using an array of open-closed parenthesis. Open parenthesis introduces a new variable while closed remove a variable and computing the depth of these it should be possible to determine if a variable is in scope by solving for the previous or smaller

```
type constraint = (typ, typ)
def exp_tname (i: e) : tname = i
def var_tname (n: i64) (v: vname) : tname = n + v

def constraint [n]
               (exps: [n]exp)
               (i: i64) : constraint =
  match exps[i]
  case #var x ->
    (#tvar (exp_tname i), #tvar (var_tname n x))
  case #lam x e ->
    (#tvar (exp_tname i), #tarrow (var_tname n x) (
       exp_tname e))
  case #app e0 e1 ->
    (#tvar (exp_tname e0), #tarrow (exp_tname e1) (
       exp_tname i))

def constraints [n] (exps: [n]exp) =
  tabulate n (constraint exps)
```

Figure 18: The types defined for expressions of lambda calculus and the type of lambda calculus expressions in Futhark.

element. Since the type system is so simple an approach can be used where every expression maps to a type variable and every expression variable name is unique so it easily maps to a type variable easily. Here every expressions type variable is its index and expression variables is the name plus the number of expression.

The type constraints are defined by an equality on type $\tau_0 \sim \tau_1$ and a function t which gives the type of an expression and a function vt which gives the type of a expression variables name. Using this scheme a single type constraint arises for every expression.

$$x \Rightarrow t(x) \sim vt(x) \tag{1}$$
$$\lambda x.e \Rightarrow t(\lambda x.e) \sim vt(x) \sim t(e) \tag{2}$$
$$e_0\ e_1 \Rightarrow t(e_0) \sim t(e_1) \to t(e_0\ e_1) \tag{3}$$

The implementation of generating all these rules can be seen in Figure 18. Now before we can go ahead and solve these constraints some helper functions are needed, top-level equality on types is needed. It does not need to recurse down and check if they are equivalent. A way to compute the number of type

```
def eq (t: typ) (t': typ) : bool =
  match (t, t')
  case (#tarrow _ _, #tvar _) -> false
  case (#tvar _, #tarrow _ _) -> false
  case (#tvar a, #tvar a') -> a == a'
  case (#tarrow a b, #tarrow a' b') -> a == a' && b ==
     b'

def vname (exp: exp) : vname =
  match exp
  case #app _ _ -> 0
  case #lam v _ -> v
  case #var v -> v

def num_tvars [n] (exps: [n]exp) : i64 =
  n + i64.maximum (map vname exps)

def num_types [n] (exp: [n]exp) : i64 =
  let n = num_tvars exp
  in n + n * n
```

Figure 19: The helper function for type checking.

variables and type variables combined with arrows between type variables. A way of encoding types into an integer that can be used for union-find.

Now the big problem is to actually solve the constraints, to implement this you would have to encode the constraints then unify them. Then for the arrow types you would have to recurse down the arrow type and create type equalities. In a sequential implementation with recursion this is easy but for Futhark you would have to flatten this recursive function. I sadly ran out of time and have therefore only made a version which check top level equalities. So the type checker does not owrk but there is possibly good idea behind this which would make good use of the union-find structure. The implementation can be seen in Figure 20

6 Conclusion

The project succesfully derived a data-parallel union-find structures which should be ideal for functional array languages. Specifically an union-find

```
def solve [n] (exps: [n]exp) =
  let cons = constraints exps
  let num_ts = num_types exps
  let num_tvars = num_tvars exps
  let u = uf.create num_ts
  let types = unzip cons |> uncurry (++)
  let table = map (\i -> if i < num_tvars then tvar i
     else tvar (-1)) (iota num_ts)
  let table = scatter table (map (encode n) types)
     types
  let (u, _) =
    loop (u, cons) = (u, cons)
    while length cons != 0 do
      let econs =
        map (\(t, t') -> (uf.from_i64 u (encode n t),
            uf.from_i64 u (encode n t'))) cons
      let u = uf.union u econs
      let is = map (.0) econs |> uf.find' u |> map (uf.
         to_i64 u)
      let cons =
        map2 (\i (_, t) -> (table[i], t)) is cons
        |> filter (not <-< uncurry eq)
        |> map new_constraints
        |> filter (is_valid <-< (.0) <-< (.0))
        |> unzip
        |> uncurry (++)
      in (u, cons)
  let exp_types = tabulate n (uf.from_i64 u) |> uf.find
     ' u |> map (table[uf.to_i64 u]) |> zip exps
  let type_eqs =
    tabulate num_ts (uf.from_i64 u)
    |> uf.find' u
    |> map (uf.to_i64 u)
    |> map2 (\j i -> (table[j], table[i])) (iota num_ts
       )
    |> filter (is_valid <-< (.1))
    |> filter (not <-< uncurry eq)
  in (exp_types, type_eqs)
```

Figure 20: The constraint solver.

structure using a union by rank heurestic was work-efficient in regards to union-find structure only using this heurestic would do $O(m \log |V|)$ work where m is the number of elements pairs being unified. It was not possible to find a way to get a work-efficient algorithm which does amortized $O(m\alpha(|V|))$ where m is the number of elements being unified or found. The data-parallel union-find structure using union by rank has the added advantage that it is quite simple to implement in a functional array language context. The union by size variant was conjectured to be work-efficient but from a practical standpoint the implementation is complicated compared to union by rank and uses alot of extra memory. Lastly a union-find structure without any heurestics was presented but it may be the case it is not a useful structure due to its little difference to the union by rank variant. Besides this some use cases for these union-find structures was shown, specifically region labeling has a less strong case for the structure being useful. The stronger case is for type checking but the example was of poor quality.

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