Grover in a simulator

The homework called *Simon and Grover, classically* defined Grover's problem.

Implement three cases of Grover's algorithm and run the implementations on the quantum circuit simulator called Quirk, linked here: `https://algassert.com/quirk`. Use only gates in the Toolbox shown above the circuit in the grey area; avoid using the Make Gate facility.

The first two implementations should be for $n = 2$, with one implementations using an oracle for the case where $f(00) = 1$, and the other implementation using an oracle for the case where $f(11) = 1$. The third implementation should be for $n = 3$, using an oracle for the case where $f(101) = 1$.

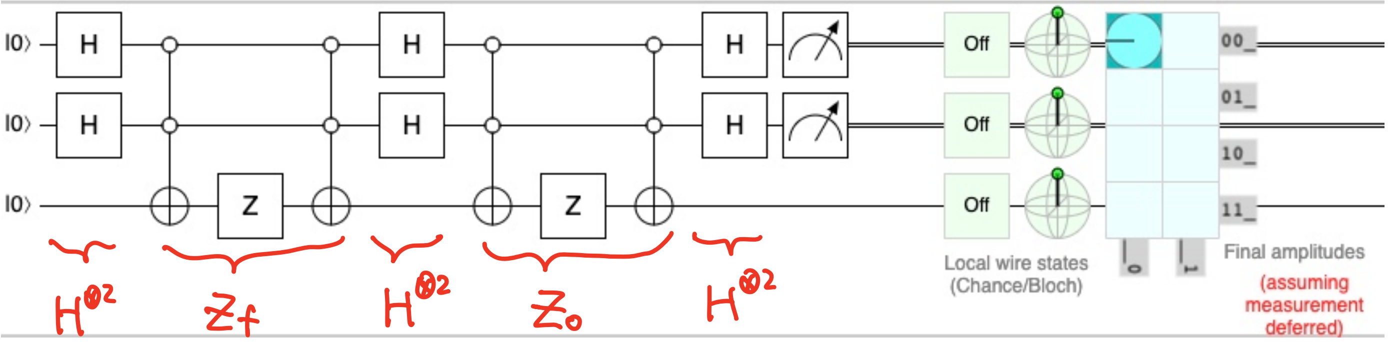
Submit screenshots that show your implementations and illustrate that they work, and submit a file with an explanation of how it works.

Grover's algorithm : ① Apply $H^{\otimes n}$ to X

② Repeat {apply G to X} $O(\sqrt{2^n})$ times

③ Measure X and output the result.

$f(00) = 1$: $G H^{\otimes 2} |00\rangle$ which $G = -H^{\otimes 2} Z_0 H^{\otimes 2} Z_f$



Explanation:

The circuit has two main qubits and one helper qubit.
It first applies $H^{\otimes 2}$ to the two main qubits X which is $|00\rangle$.
Then it applies G to X one time since

$$k \approx \frac{\pi}{4}\sqrt{N} - \frac{1}{2} = \frac{\pi}{4}\sqrt{2^n} - \frac{1}{2} = \frac{\pi}{4}\sqrt{4} - \frac{1}{2} = \frac{\pi}{2} - \frac{1}{2} \approx 1.$$

G is constructed by 4 parts:

① Z_f : since $f(00) = 1$, $Z_f|00\rangle = (-1)^{f(x)}|00\rangle = (-1)^1|00\rangle = -|00\rangle$

$\therefore$ Z_f is used to change the sign of X, which is $|00\rangle$ in this case

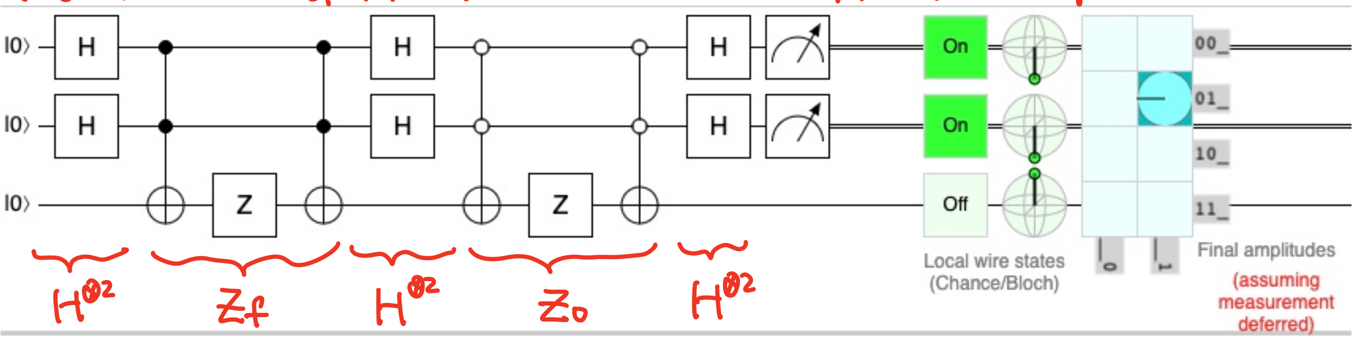
② It applies two H gates $H^{\otimes 2}$ to the two main qubits.

③ Z_0 : $Z_0|x\rangle = \begin{cases} -|x\rangle, & \text{for } x = 0^n \\ |x\rangle, & \text{for } x \neq 0^n \end{cases}$. $\therefore$ Z_0 is used to change the sign of $|00\rangle$.

④ Finally, it applies two H gates $H^{\otimes 2}$ to the two main qubits such that we can measure them.

After measurement, the circuit outputs $|00\rangle$ with the probability 1.
It matches $f(00) = 1$, so the Grover's algorithm works perfectly.

$f(11)=1$: $G\, H^{\otimes 2}|11\rangle$ which $G = -H^{\otimes 2} Z_0 H^{\otimes 2} Z_f$



Explanation:

The circuit has two main qubits and one helper qubit.
It first applies $H^{\otimes 2}$ to the two main qubits X which is $|11\rangle$.
Then it applies G to X one time since

$$k \approx \frac{\pi}{4}\sqrt{N} - \frac{1}{2} = \frac{\pi}{4}\sqrt{2^n} - \frac{1}{2} = \frac{\pi}{4}\sqrt{4} - \frac{1}{2} = \frac{\pi}{2} - \frac{1}{2} \approx 1.$$

G is constructed by 4 parts:

① Z_f : since $f(11)=1$, $Z_f|11\rangle = (-1)^{f(x)}|11\rangle = (-1)^1|11\rangle = -|11\rangle$

$\therefore$ Z_f is used to change the sign of X, which is $|11\rangle$ in this case.

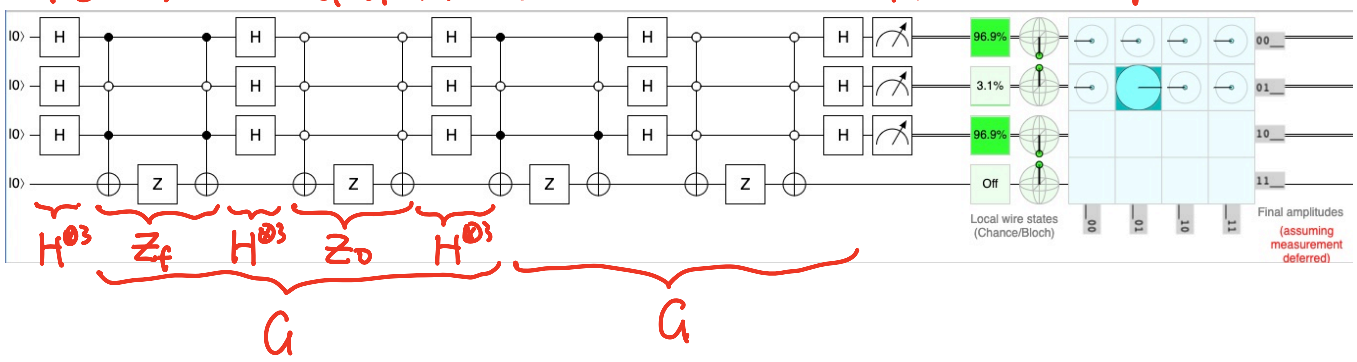
② It applies two H gates $H^{\otimes 2}$ to the two main qubits

③ Z_0 : $Z_0|x\rangle = \begin{cases} -|x\rangle, & \text{for } x = 0^n \\ |x\rangle, & \text{for } x \neq 0^n \end{cases}$. $\therefore$ Z_0 is used to change the sign of $|00\rangle$.

④ Finally, it applies two H gates $H^{\otimes 2}$ to the two main qubits such that we can measure them.

After measurement, the circuit outputs $|11\rangle$ with the probability 1.
It matches $f(11)=1$, so the Grover's algorithm works perfectly.

$f(101) = 1$: $G\,G\,H^{\otimes 3}|00\rangle$ which $G = -H^{\otimes 3} Z_0 H^{\otimes 3} Z_f$



Explanation:

The circuit has three main qubits and one helper qubit.
It first applies $H^{\otimes 3}$ to the three main qubits X which is $|101\rangle$
Then it applies G to X two times since

$$k \approx \frac{\pi}{4}\sqrt{N} - \frac{1}{2} = \frac{\pi}{4}\sqrt{2^n} - \frac{1}{2} = \frac{\pi}{4}\sqrt{8} - \frac{1}{2} = \frac{\sqrt{2}}{2}\pi - \frac{1}{2} \approx 1.72 \approx 2$$

(round to the nearest integer)

G is constructed by 4 parts:

① Z_f : since $f(101) = 1$, $Z_f|101\rangle = (-1)^{f(x)}|101\rangle = (-1)^1|101\rangle = -|101\rangle$

∴ Z_f is used to change the sign of X, which is $|101\rangle$ in this case

② It applies three H gates $H^{\otimes 3}$ to the three main qubits

③ Z_0 : $Z_0|x\rangle = \begin{cases} -|x\rangle, & \text{for } x = 0^n \\ |x\rangle, & \text{for } x \neq 0^n \end{cases}$ ∴ Z_0 is used to change the sign of $|000\rangle$

④ Finally, it applies three H gates $H^{\otimes 3}$ to the three main qubits

After two G, we can measure the three main qubits.

After measurement, the circuit outputs $|101\rangle$ with the highest probability 0.945 and the probability is really closed to 1 and much larger than the probability of all the other states.
Therefore, we can see $|101\rangle$ as the output of the Grover's algorithm.
It matches $f(101) = 1$, so the Grover's algorithm works perfectly.