Quantum states

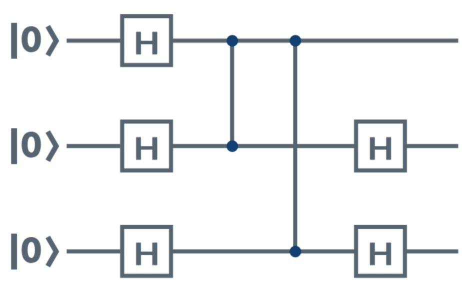
For a ket $|k_1 \dots k_n\rangle$, we index the qubits from left to right, starting with index 1.

Question 1. For the following state, suppose we measure the qubit with index 2 in the standard basis and get 0. Show the resulting state. Justify your answer.

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{4}|10\rangle - \frac{\sqrt{7}}{4}|11\rangle$$

Question 2. Suppose we apply $H^{\otimes 3}$ to the state $|101\rangle$, after which we measure the two qubits with indexes 1,2 in the standard basis. What is the probability that we get 11 ?

Question 3. Consider the following circuit with three qubits.



Here H is the Hadamard gate, while each 2-qubit connection is CZ $= C(Z)$.

Suppose that at the end, we measure all three qubits in the standard basis. What is the probability that we will get 000 ? Justify your answer.

Question 4. Consider the following state.

$$\frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Suppose we measure the qubit with index 1 in the standard basis. What is the probability of getting 0, and if that happens, what is the state of the other qubit? Also, suppose we measure the qubit with index 2 in the standard basis. What is the probability of getting 1, and if that happens, what is the state of the other qubit?

Question 1. For the following state, suppose we measure the qubit with index 2 in the standard basis and get 0. Show the resulting state. Justify your answer.

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{4}|10\rangle - \frac{\sqrt{7}}{4}|11\rangle$$

Since we measure the second qubit and we get 0, the probability of measuring $|0\rangle$ is:

$$\left|\frac{1}{2}\right|^2 + \left|\frac{1}{4}\right|^2 = \frac{5}{16}$$

The resulting state will be:

$$\frac{\frac{1}{2}|00\rangle + \frac{1}{4}|10\rangle}{\sqrt{(\frac{1}{2})^2 + (\frac{1}{4})^2}} = \frac{\frac{1}{2}|0\rangle + \frac{1}{4}|1\rangle}{\sqrt{\frac{5}{16}}} \otimes |0\rangle$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{5}}|00\rangle + \frac{1}{4}\cdot\frac{4}{\sqrt{5}}|10\rangle$$

$$= \frac{2\sqrt{5}}{5}|00\rangle + \frac{\sqrt{5}}{5}|10\rangle$$

Question 2. Suppose we apply $H^{\otimes 3}$ to the state $|101\rangle$, after which we measure the two qubits with indexes 1,2 in the standard basis. What is the probability that we get 11 ?

$$|x\rangle = |101\rangle$$

$$H^{\otimes 3}|101\rangle = H|1\rangle H|0\rangle H|1\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2^3}}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)(|0\rangle - |1\rangle)$$

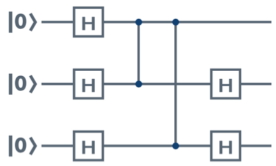
$$= \frac{1}{\sqrt{2^3}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle - |110\rangle + |111\rangle)$$

$$= \frac{1}{\sqrt{2^3}}\sum_{y\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle \quad \text{(Lemma 5)}$$

The states that have 11 as the first two qubits are $|110\rangle$ and $|111\rangle$

$$\therefore \Pr(\text{Get 11 in the first two qubits}) = \left|\frac{1}{\sqrt{2^3}}\right|^2 + \left|\frac{1}{\sqrt{2^3}}\right|^2 = \frac{1}{4}$$

Question 3. Consider the following circuit with three qubits.



$X = |000\rangle$

Here H is the Hadamard gate, while each 2-qubit connection is $\mathrm{CZ} = C(Z)$.

Suppose that at the end, we measure all three qubits in the standard basis. What is the probability that we will get 000 ? Justify your answer.

$$(I \otimes H \otimes H)\, C(Z)[k_1, k_3]\, C(Z)[k_1, k_2]\, H^{\otimes 3} |000\rangle$$

$$= (I \otimes H \otimes H)\, C(Z)\, C(Z) \frac{1}{\sqrt{2^3}} \sum_{y \in \{0,1\}^3} (-1)^{x \cdot y} |y\rangle \qquad \text{(Lemma 5)}$$

$$= (I \otimes H \otimes H)\, C(Z)\, C(Z) \frac{1}{\sqrt{2^3}} \sum_{y \in \{0,1\}^3} |y\rangle$$

Apply $C(Z)$: either qubit can technically be the controlled qubit. since the output will be the same whether the controlled qubit is the first qubit or the second qubit

Assume the first qubit is controlled qubit for the first $C(Z)$:

it only affects the states : $|110\rangle + |111\rangle \Rightarrow -|110\rangle - |111\rangle$

$$= (I \otimes H \otimes H)\, C(Z) \frac{1}{\sqrt{2^3}}$$
$$(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle)$$

Assume the first qubit is controlled qubit for the second $C(Z)$:

$$= (I \otimes H \otimes H) \frac{1}{\sqrt{2^3}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle)$$

Since we are calculating the probability that we will get 000, we only need to calculate the amplitude of $|000\rangle$:

The states that will get $|000\rangle$ after applying H gates are $|000\rangle$, $|001\rangle$, $|010\rangle$, and $|011\rangle$, and the amplitude of $|000\rangle$ in these states are the same since

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\qquad$ $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

There is no possibilites that we will get a negative amplitude for $|000\rangle$.

$$(I \otimes H \otimes H) \frac{1}{\sqrt{2^3}} |000\rangle = \frac{1}{\sqrt{2^3}} \cdot |0\rangle \otimes H|0\rangle \otimes H|0\rangle$$

$$= \frac{1}{\sqrt{2^3}} \cdot |0\rangle \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2^5}} |0\rangle (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$\therefore$ The amplitude of $|000\rangle = \frac{1}{\sqrt{2^5}} \cdot 4 = \frac{1}{\sqrt{2}}$

$\therefore$ $Pr(\text{Get } 000) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Question 4. Consider the following state.

$$\frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Suppose we measure the qubit with index 1 in the standard basis. What is the probability of getting 0, and if that happens, what is the state of the other qubit? Also, suppose we measure the qubit with index 2 in the standard basis. What is the probability of getting 1, and if that happens, what is the state of the other qubit?

Measure the first qubit,

$$\Pr(\text{Getting } 0) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

Since there is only one basic state that has 0 as the first qubit which is $|01\rangle$,

if we get 0 when we measure the first qubit,

the state of the other qubit is $|1\rangle$

Measure the second qubit,

$$\Pr(\text{Getting } 1) = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{3}{4}$$

If we get 1 when we measure the second qubit,

the resulting state will be

$$\frac{\frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{\frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}{\sqrt{\frac{3}{4}}} \otimes |1\rangle$$

$$= \left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) \otimes |1\rangle$$

$$= \left(\frac{\sqrt{3}}{3}|0\rangle + \frac{\sqrt{6}}{3}|1\rangle\right) \otimes |1\rangle$$

$\therefore$ The state of the other qubit is $\frac{\sqrt{3}}{3}|0\rangle + \frac{\sqrt{6}}{3}|1\rangle$.