# Problem Set 2.

1.

$$\langle x\rangle = -20 \cdot \frac{3}{10} - 10 \cdot \frac{1}{5} + 30 \cdot \frac{1}{2} = \sum_i x_i p_i$$

$$= 7$$

$$\langle x^2\rangle = 400 \cdot \frac{3}{10} + 100 \cdot \frac{1}{5} + 900 \cdot \frac{1}{2} = \sum_i x_i^2 p_i$$

$$= 590$$

$$\Rightarrow \quad \sigma^2 = \langle x^2\rangle - \langle x\rangle^2$$
$$= 541$$
$$\therefore \sigma \cong 23.3.$$

2. a) Check normalisation:

$$\int_0^\infty e^{-\alpha x}\,dx = \left[-\frac{e^{-\alpha x}}{\alpha}\right]_0^\infty = \frac{1}{\alpha}$$

$\therefore \; p(x) = \alpha e^{-\alpha x}$ is the correctly normalised distribution.

$$\Rightarrow \langle x\rangle = \int_0^\infty \alpha x e^{-\alpha x}\,dx = \alpha \cdot \frac{1}{\alpha^2} = \frac{1}{\alpha}.$$

b) $$\langle x^2\rangle = \alpha \int_0^\infty x^2 e^{-\alpha x}\,dx = \alpha \cdot \frac{2}{\alpha^3} = \frac{2}{\alpha^2}.$$

c) $$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

$$\therefore \sigma = \frac{1}{\alpha}.$$

3. a) We can select 3 inks from a choice of 6 in

$$6.5.4 = \frac{6!}{3!} = 120 \text{ ways}$$

b) If we choose 5 objects from 12 ignoring the order then there are

$$12.11.10.9.8. = \frac{12!}{7!} = 95040 \text{ ways}$$

If we have 5 objects there are 5! ways of ordering them. Hence if the order the objects are chosen is import

$$\Rightarrow \frac{12!}{7!} \cdot 5! = 11404800 \text{ ways.}$$

c) $\overset{1}{S}\overset{1}{T}\overset{1}{A}\overset{2}{T}\overset{1}{I}\overset{2}{S}\overset{3}{T}\overset{2}{I}\overset{1}{C}\overset{3}{I}\overset{2}{A}N\overset{3}{S}$

$\Rightarrow$ S × 3
T × 3
A × 2
I × 3
C × 1
N × 1

$$\Rightarrow \frac{13!}{3!3!2!3!} = 14414400.$$

d) $\text{MMM}\underbrace{[\text{INIU}]}$

How many ways can remaining letters be ordered?

$$\frac{4!}{2!} = 12.$$

TOTAL orderings $\frac{7!}{3!\,2!} = 420 \Rightarrow p = \frac{1}{35}.$

4. $$\ln x! \simeq x \ln x - x$$

$$\frac{d(\ln x!)}{dx} \simeq \frac{d}{dx}\left[x \ln x - x\right]$$

$$= \ln x + x \cdot \frac{1}{x} - 1$$

$$= \ln x.$$

~~4.~~ 5. $\qquad S = k_B \ln W$

$$W = \frac{(n-1+m)!}{(n-1)!\, m!}$$

$$\Rightarrow S = k_B \ln \frac{(n-1+m)!}{(n-1)!\, m!}$$

$$\simeq k_B \ln \frac{\left(n + U/\hbar\omega\right)!}{n!\, \left(\frac{U}{\hbar\omega}\right)!}$$

Using Stirling's approximation

$$S \simeq k_B \left( \left(n + \frac{U}{\hbar\omega}\right) \ln \left(n + \frac{U}{\hbar\omega}\right) \cancel{- \left(n + \frac{U}{\hbar\omega}\right)} - n \ln n \cancel{+ n} - \frac{U}{\hbar\omega} \ln \frac{U}{\hbar\omega} \cancel{+ \frac{U}{\hbar\omega}} \right)$$

$$\frac{\partial S}{\partial U} = \frac{1}{T}$$

$$\Rightarrow \frac{1}{T} = \frac{k_B}{\hbar\omega} \left( \ln \left(n + \frac{U}{\hbar\omega}\right) - \ln \frac{U}{\hbar\omega} \right).$$

$$\Rightarrow e^{\hbar\omega/k_B T} = \frac{n + U/\hbar\omega}{U/\hbar\omega}.$$

$$\Rightarrow U/\hbar\omega \left( e^{\hbar\omega/k_B T} - 1 \right) = n$$

$$\Rightarrow \quad U = \frac{n \hbar\omega}{e^{\hbar\omega/k_B T} - 1}.$$

~~5~~. 6.~~4~~ $N+n$ sites in total, with $N$ filled & $n$ vacant

$$\Rightarrow \quad W = \frac{(N+n)!}{N!\,n!}$$

$$S = k_B \ln W$$

$$= k_B \Big( (N+n)\ln(N+n) - \cancel{(N+n)} - N\ln N + \cancel{N} - n\ln n + \cancel{n} \Big)$$

$$= k_B \big( (N+n)\ln(N+n) - n\ln n - N\ln N \big).$$

$$F = U - TS$$

$$= n\epsilon - k_B T \big( (N+n)\ln(N+n) - n\ln n - N\ln N \big)$$

$$\frac{dF}{dn} = 0 \quad \text{in equilibrium}$$

$$\Rightarrow \quad 0 = \epsilon - k_B T \big( \ln(N+n) - \ln n \big)$$

$$\Rightarrow \quad \epsilon / k_B T = \ln \frac{N+n}{n}$$

$$\Rightarrow \quad \left(e^{\epsilon/k_B T} - 1\right) n = N$$

$$\therefore \quad n = \frac{N}{e^{\epsilon/k_B T} - 1}.$$

If $T = 1300\,K \Rightarrow k_B T = 0.112\ eV$

$$\therefore \quad \frac{\epsilon}{k_B T} \simeq 8.93$$

$$\therefore \quad \frac{n}{N} \simeq 1.33 \times 10^{-4}$$

If the crystal is large then the additional entropy for the surface atoms is negligible compared to the entropy of the bulk.

Q. 7.8 In the spin system there are $N$ positions between the spins to put domain walls.

$$\uparrow\uparrow\uparrow \mid \downarrow\downarrow\downarrow \quad \cdots$$

There are $n$ of these walls & $(N-n)$ that are not domain walls.

The first spin can be $\uparrow$ or $\downarrow$, & the remaining orientations follow from the wall positions.

Hence:

$$W = 2 \cdot \frac{N!}{n!\,(N-n)!}$$

$$S = k_B \ln W$$

$$\simeq k_B \left( \ln 2 + N \ln N - N - n \ln n + n - (N-n)\ln(N-n) + (N-n) \right)$$

$$= k_B \left( \ln 2 + N \ln N - n \ln n - (N-n)\ln(N-n) \right).$$

Using $\frac{1}{T} = \frac{\partial S}{\partial U}$ with $U = n\epsilon$.

$$\frac{\partial S}{\partial U} = \frac{dS}{dn}\frac{dn}{dU} = \frac{dS}{dn}\cdot\frac{1}{\epsilon}$$

$$\Rightarrow \quad \frac{1}{T} = \frac{k_B}{\epsilon}\left(-\ln n + \ln(N-n)\right)$$

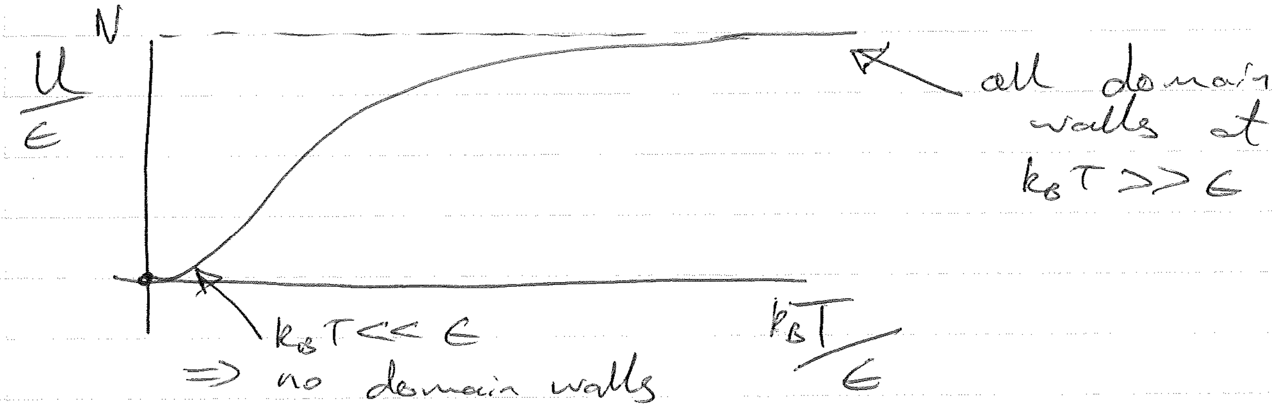
$$\Rightarrow \quad \frac{\epsilon}{k_BT} = \ln\left(\frac{N-n}{n}\right)$$

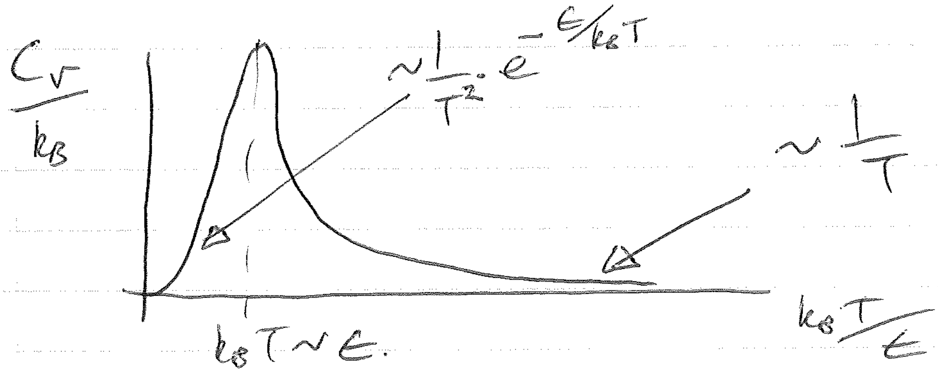
$$\Rightarrow \quad \left(e^{\epsilon/k_BT} + 1\right) n = N$$

$$\therefore \quad n = \frac{N}{e^{\epsilon/k_BT}+1}$$

$$\& \quad U = n\epsilon = \frac{N\epsilon}{e^{\epsilon/k_BT}+1}.$$

$$C_v = \frac{dU}{dT} = \frac{\epsilon}{k_BT^2}\frac{N\epsilon\, e^{\epsilon/k_BT}}{\left(e^{\epsilon/k_BT}+1\right)^2}$$



8.

$$W = \frac{(2N)!}{(N+r)!\,(N-r)!}$$

$$S = k_B \ln W$$

$$\simeq k_B \Big( (2N)\ln(2N) - \cancel{2N} - (N+r)\ln(N+r) + \cancel{(N+r)} - (N-r)\ln(N-r) + \cancel{(N-r)} \Big)$$

Since $\quad TdS = dU - f\,dl$

$$\Rightarrow \quad f = -T \left.\frac{\partial S}{\partial l}\right|_{U}$$

$$\frac{\partial S}{\partial l} = \frac{dS}{dr} \cdot \frac{dr}{dl} = \frac{dS}{dr} \cdot \frac{1}{2a} \qquad (\text{as } l = 2ra)$$

$$\therefore \frac{dS}{dr} = k_B \left( -\ln(N+r) + \ln(N-r) \right)$$

$$= k_B \ln\left(\frac{N-r}{N+r}\right) = k_B \ln\left(\frac{1-r/N}{1+r/N}\right)$$

Using the Taylor expansion

$$\ln\left(\frac{1-x}{1+x}\right) \simeq -x - x + O(x^2) = -2x.$$

$$\Rightarrow \frac{dS}{dr} \simeq -2k_B \frac{r}{N} = \frac{-k_B}{Na} \cdot l.$$

$$\therefore \frac{\partial S}{\partial l} = -\frac{k_B\, l}{2Na^2} \quad \& \quad f = \frac{k_B T l}{2Na^2}.$$