Faculty of Engineering and Physical Sciences
Department of Physics

Year 2, Energy and Entropy
Problem Set 5

Grand canonical Ensemble

1. What is the chemical potential? Your short discussion should include one or more definitions in terms of other thermodynamic quantities, together with some interpretation and appplications.

2. Suppose there is a flat surface of a solid in contact with a vapour which acts as the thermal and particle reservoir. The atoms stick to sites on the solid surface with at most one atom per site. Each site on the surface of the solid has an energy ϵ if an atom is stuck to it, and an energy of zero if no atom is stuck to it. If there are M sites with N atoms trapped on them the energy of the state is $N\epsilon$ and then the degeneracy is $\frac{M!}{N!(M-N)!}$.

Show that the grand partition function is

$$\Xi = (1 + e^{-(\epsilon-\mu)/(k_B T)})^M.$$

Hint: Recall the binomial expansion $(1+x)^M = \sum_{N=0}^{M} \frac{M!}{N!(M-N)!} x^N$

Fermi-Dirac and Bose-Einstein distributions

3. Consider the ratio, $\langle n \rangle_B / \langle n \rangle_F$, of the number of bosons a level of energy ϵ has to the number of fermions a level of the same energy has, if in both cases the chemical potential and temperature are μ and T, respectively. Determine the asymptotic value of this ratio as $(\epsilon - \mu)/kT$ becomes very large. In this limit of large $(\epsilon - \mu)/kT$, do we need to distinguish between fermions and bosons?

4. Photons are bosons with chemical potential $\mu = 0$. Thus the number of photons in an energy level of energy ϵ at temperature T is

$$\langle n \rangle_P = \frac{1}{\exp(\epsilon/kT) - 1}$$

How many photons, on average, occupy the energy level corresponding to an, infrared, wavelength of 50μm at $T = 300K$?

5. Consider a system of quantum-mechanical particles, either bosons or fermions, at a chemical potential μ and temperature T. If the temperature $T = 300\text{K}$, and the difference between the chemical potential μ and the energy level ϵ is $\mu - \epsilon = -10^{-22} J$

(a) What is the value of the partition function Ξ_B for bosons? What is the average number, $\langle n \rangle_B$, of bosons in the energy level?

(b) What is the value of the partition function Ξ_F for fermions? What is the average number, $\langle n \rangle_F$, of fermions in the energy level?

6. An estimate of when quantum mechanical effects become important can be obtained by equating the mean spacing to the de Broglie wavelength. Using the equation produced, estimate the temperature at which quantum mechanical effects come into play in a gas of He of density $N/V = 10^{27}\text{m}^{-3}$.

7. In the lectures we have dealt with fermions and bosons under conditions of constant chemical potential, temperature and volume, because for large numbers of levels this is the easiest way to study fermions/bosons. But if there are only a few levels, fermions and bosons can be dealt with under conditions of constant number of fermions/bosons. For example, if we have two levels with energy ϵ and two fermions then there is only one possible state: one fermion in each level. This state has an energy ϵ. But if we have two bosons, then there are three possible states of the system: both in the level with 0 energy, one in each energy level, and both in the energy level with energy ϵ. Thus, at a temperature T the average energy $\langle E \rangle$ is

$$\langle E \rangle = \frac{\epsilon \exp(-\epsilon/kT) + 2\epsilon \exp(-2\epsilon/kT)}{1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)}$$

What is the average energy of:

(i) three bosons in two (non-degenerate) energy levels with energy 0 and ϵ,

(ii) two fermions and *three* (non-degenerate) energy levels with energy 0, ϵ and 2ϵ?

8. Two electrons share a potential well, in which each can have energy 0 or ϵ. By Pauli's exclusion principle they cannot both be in the same quantum state, so if, for example, both are in the same space state of energy ϵ then one must have its spin up, the other its spin down. Find an expression for the probability that the total spin in the z direction is not zero, and sketch its dependence on temperature.