Examples and Computations

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For better or for worse, my perspective on mathematics leans towards the abstract—and as such, I sometimes avoid the (important) task of working carefully through examples, as well as coming up with concrete scenarios which both specialize and illustrate general proofs. To help me unlearn these habits, I was inspired by [?] and [?] to start an at-least-weekly record of concrete matters.

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1 August 26, 2023: A Presentation of the Modular Group and Explicitly Computing a Free Subgroup

I've been learning some group theory; here's an example to make sure I understand what I've been learning. In this entry, I reproduce a half-remembered proof about the finite generation of $PSL(2, \mathbf{Z})$, and an interesting application of the theory of covering spaces to the theory of groups. The argument is from Stillwell '80, which among other things has taught me that this line of reasoning harks from the 1930's!

A Mobius transformation is a function $z \mapsto (az+b)/(cz+d)$ on $\mathbf{C}\cup\{\infty\}$; transformations satisfying $ad-bc = 1$ preserve the closed upper-half plane, so the group G of Mobius transformations modulo negation acts naturally on $\mathbf{H}$, the open upper-half plane of $\mathbf{C}$. Requiring a, b, c, d be in $\mathbf{Z}$ restricts our attention to a subgroup of G known as the **modular group**, often called $PSL(2, \mathbf{Z})$ as $SL(2, \mathbf{Z})/-SL(2, \mathbf{Z})$ acts via these transformations.

I've heard the following argument (or a sibling of it) is due to Serre, by which we will show that $PSL(2, \mathbf{Z})$ is generated by two elements: a translation and an involution:

$$T(z) = z + 1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad V(z) = \frac{-1}{z} \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To see why this is true, consider $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbf{Z})$. The relation $ad - bc = 1$ implies that c and d are coprime, so we may use T to reduce d mod c so that the image of d is smaller than c. After all, $(T^k M)_{2,2} = d + kc$. We can then use V to switch $T^k M$'s columns (inverting the second, but this is no issue) and reduce c mod d'. Each step of this process strictly reduced the magnitude of the bottom-right entry of the matrix-at-hand and preserved coprimality of the bottom-row entries, so we can continue this process until $(T^{k_1} V T^{k_2} S \ldots V T^{k_n} M)_{2,2} = 1$ (if this process produces a -1 entry, recall that we work in $PSL(2, \mathbf{Z})$, not $SL(2, \mathbf{Z})$). One more switch and translation yields:

$$T^{k_0} V T^{k_1} V T^{k_2} S \ldots V T^{k_n} M = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}$$

Since all these transformations preserve the magnitude of the determinant, it follows that $\beta = \pm 1$. If $\beta = 1$, a switch, translation, and switch reduces the matrix to I. If $\beta = -1$, a switch, translation, and switch reduces the matrix to V.

We have shown that every matrix in $PSL(2, \mathbf{Z})$ is row-reducible to either I or V using only T and V, which is exactly what we desired. These elements are famous for inducing a slick presentation of the modular group, namely:

$$PSL(2, \mathbf{Z}) \cong \langle T, V\,; V^2, (VT)^3 \rangle \cong C_2 * C_3$$

where $C_i = \langle a\,; a^i\rangle$. After all, in the presence of V the implied generator VT implies T. As such, it follows that we can construct a two-dimensional surface complex $\mathcal{C}$ whose fundamental group is $PSL(2, \mathbf{Z})$ whose 1-skeleton is two loops a and b based at a single vertex, then adjoining two 2-cells with boundaries a^2 and b^3. To construct a covering complex $\widetilde{\mathcal{C}}$ whose fundamental group is a subgroup H of $\pi_1(\mathcal{C})$, we start with a 0-skeleton of the cosets mod H, adjoining them in the 1-skeleton by the lifts of the loops of $\mathcal{C}$, and then attaching 2-cells as lifts of the 2-cells in $\mathcal{C}$.

To apply this theory, we claim that a subgroup of $PSL(2, \mathbf{Z})$ is $F_2 = \langle a, b\,; \varnothing\rangle$. Using the complex $\mathcal{C}$, we let G be $PSL(2, \mathbf{Z})$'s commutator (normal!) subgroup, and construct a covering complex $\widetilde{\mathcal{C}}$ so that $\pi_1(\widetilde{\mathcal{C}}) = G$. A presentation of $PSL(2, \mathbf{Z})$ allows us to easily describe G: it is all words in $PSL(2, \mathbf{Z})$ whose total a-exponent-sum is a multiple of two, and whose total b-exponent-sum is a multiple of three (elements of G are exactly those elements which are 0 mod G, which are those elements in $PSL(2, \mathbf{Z})$ which vanish if we pretend they commute). As such, there are six minimal coset representatives of G mod $PSL(2, \mathbf{Z})$:

$$\{1, a, b, b^2, ab, ab^2\}$$

These are the vertices of $\widetilde{\mathcal{C}}$. Adjoining the edges a, b, a^{-1} and b^{-1} to $\widetilde{\mathcal{C}}$'s vertices so that the edge x goes from g to gx, we obtain two b-cycle triangles adjoined at each vertex vertically by an a-cycle. To complete the complex, we adjoin cells on each cycle to obtain the covering complex $\widetilde{\mathcal{C}}$ in full. Immediate inspection shows that $\widetilde{\mathcal{C}}$ has a deformation retract to a bouquet of two circles, so we obtain the remarkable fact that we have sought: the commutator subgroup of $PSL(2, \mathbf{Z})$ is free of rank 2.

One final note: this result is massively generalized by the Kurosh Subgroup Theorem, which is why I chose it as an example to work out concretely.