Question 1

(a) $h'(6) \approx \dfrac{h(7)-h(5)}{7-5} = \dfrac{18.5-15.5}{2} = 1.5$ in/min	1 : answer with units
(b) $\int_0^{10} h(t)\,dt \approx (2-0)\cdot h(2) + (5-2)\cdot h(5) + (7-5)\cdot h(7) + (10-7)\cdot h(10)$ $= 2(10.0) + 3(15.5) + 2(18.5) + 3(20.0) = 163.5$ Because h is an increasing function, the right Riemann sum approximation is greater than $\int_0^{10} h(t)\,dt$.	$3 : \begin{cases} 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{overestimate} \\ \quad \text{with reason} \end{cases}$
(c) Average depth in tank $B = \dfrac{1}{10}\int_0^{10} g(t)\,dt = 16.624$ in Average depth in tank $A = \dfrac{1}{10}\int_0^{10} h(t)\,dt < \dfrac{1}{10}(163.5) = 16.35 \text{ in} < 16.624$ in Therefore, the average depth of the water in tank B is greater than the average depth of the water in tank A.	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{average depth} \\ \quad \text{in tank } B \\ 1 : \text{answer with reason} \end{cases}$
(d) $g'(6) = 0.887$ The depth of the water in tank B is increasing at time $t = 6$ because $g'(6) > 0$.	$2 : \begin{cases} 1 : \text{uses } g'(6) \\ 1 : \text{answer with reason} \end{cases}$

Question 2

(a) Acceleration $= \langle x''(2), y''(2) \rangle$ $= \langle -1.033, 3.027 \rangle$	$2: \begin{cases} 1: x''(2) \\ 1: y''(2) \end{cases}$
(b) Distance $= \int_{1.8}^{2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ $= 0.360$ (or 0.359)	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(c) $x(1) = x(2) + \int_2^1 x'(t)\, dt$ $= 5 + \int_2^1 \left(-1 + e^{\sin t}\right) dt = 3.395$	$3: \begin{cases} 1: \text{integral} \\ 1: \text{uses } x(2) \\ 1: \text{answer} \end{cases}$
(d) Speed $= \sqrt{(x'(t))^2 + (y'(t))^2}\Big\vert_{t=\sqrt{\frac{7\pi}{2}}} = \left\vert x'\left(\sqrt{\frac{7\pi}{2}}\right)\right\vert = 0.159$ $x'\left(\sqrt{\frac{7\pi}{2}}\right) = -0.159$ Because $x'\left(\sqrt{\frac{7\pi}{2}}\right) < 0$, the particle is moving to the left at time $t = \sqrt{\frac{7\pi}{2}}$.	$2: \begin{cases} 1: \text{speed} \\ 1: \text{direction with reason} \end{cases}$

Question 3

(a) $f'(x) = g(x)$ The function f has critical points at $x = -6$ and $x = 2$. f has a local minimum at $x = -6$ because f' changes from negative to positive at that value. f has a local maximum at $x = 2$ because f' changes from positive to negative at that value.	$3 : \begin{cases} 1 : f'(x) = g(x) \\ 1 : \text{critical points} \\ 1 : \text{classifications with justification} \end{cases}$
(b) $f(0) = \int_{-8}^{0} g(t)\,dt = 8 + 4\pi$	1 : answer
(c) $\lim_{x \to -4} f(x) = f(-4) = \int_{-8}^{-4} g(t)\,dt = 0$ $\lim_{x \to -4} \left(x^2 + 4x\right) = 0$ Using L'Hospital's Rule, $\lim_{x \to -4} \frac{f(x)}{x^2 + 4x} = \lim_{x \to -4} \frac{f'(x)}{2x + 4} = \lim_{x \to -4} \frac{g(x)}{2x + 4} = \frac{2}{-4} = -\frac{1}{2}$	$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h'(x) = \frac{\left(x^2 + 1\right)g'(x) - g(x)2x}{\left(x^2 + 1\right)^2}$ $h'(1) = \frac{\left(1^2 + 1\right)g'(1) - g(1)(2)(1)}{\left(1^2 + 1\right)^2}$ $= \frac{2(-3) - 3(2)}{4} = -3$	$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

Question 4

(a) $\frac{dy}{dx} = (y-2)\left(x^2+1\right)$

$\int \frac{dy}{y-2} = \int \left(x^2+1\right) dx$

$\ln|y-2| = \frac{x^3}{3} + x + C$

$\ln 3 = \frac{0^3}{3} + 0 + C \Rightarrow C = \ln 3$

Because $y(0) = 5$, $y > 2$, so $|y-2| = y-2$.

$y - 2 = 3e^{\frac{x^3}{3}+x}$

$y = 2 + 3e^{\frac{x^3}{3}+x}$

Note: this solution is valid for all real numbers.

$$5: \begin{cases} 1: \text{separation of variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1: \text{solves for } y \end{cases}$$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\lim\limits_{x\to-\infty}\left(2 + 3e^{\frac{x^3}{3}+x}\right) = 2$

1 : answer

(c) $\left.\frac{dy}{dx}\right|_{(1,\,3)} = (3-2)\left(1^2+1\right) = 2$

$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(x^2+1\right) + (y-2)(2x)$

$\left.\frac{d^2y}{dx^2}\right|_{(1,\,3)} = (2)\left(1^2+1\right) + (3-2)(2) = 6$

Because $\left.\frac{d^2y}{dx^2}\right|_{(1,\,3)} > 0$ and $\frac{d^2y}{dx^2}$ is continuous, the graph of $y = f(x)$ is concave up at the point $(1, 3)$.

$$3: \begin{cases} 2: \left.\frac{d^2y}{dx^2}\right|_{(1,\,3)} \\ 1: \text{concave up with reason} \end{cases}$$

Question 5

(a) $g'(x) = 2f(x) + (2x - 1)f'(x)$

$$g'(3) = 2f(3) + 5f'(3) = (2)(7) + (5)(-1) = 14 - 5 = 9$$

$$2 : \begin{cases} 1 : g'(x) \\ 1 : g'(3) \end{cases}$$

(b) $h'(x) = f'(f(x)) \cdot f'(x)$

$$h'(4) = f'(f(4)) \cdot f'(4) = f'(3) \cdot 6 = (-1)(6) = -6$$

$$2 : \begin{cases} 1 : h'(x) \\ 1 : h'(4) \end{cases}$$

(c) $u = x \quad dv = f''(x)\,dx$

$du = dx \quad v = f'(x)$

$$\int x f''(x)\,dx = x f'(x) - \int f'(x)\,dx = x f'(x) - f(x)$$

$$\begin{aligned}\int_1^5 x f''(x)\,dx &= \left[x f'(x) - f(x)\right]_1^5 \\ &= (5f'(5) - f(5)) - (f'(1) - f(1)) \\ &= (5 \cdot 2 - (-1)) - (5 - 4) = 10\end{aligned}$$

$$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(d) f'' is continuous. $\Rightarrow$ f' is differentiable and continuous on $3 \le x \le 4$.

$$\frac{f'(4) - f'(3)}{4 - 3} = \frac{6 - (-1)}{1} = 7$$

Therefore, by the Mean Value Theorem, there is a value c, for $3 < c < 4$, such that $f''(c) = 7$.

$$2 : \begin{cases} 1 : \dfrac{f'(4) - f'(3)}{4 - 3} \\ 1 : \text{conclusion, using MVT} \end{cases}$$

Question 6

(a) The Taylor series for e^x about $x = 0$ is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$. First four nonzero terms for f : $x\left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!}\right) = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!}$ General term for f : $\frac{x^{3n+1}}{n!}$	$4 : \begin{cases} 1 : \text{terms for } e^x \\ 2 : \text{first four terms for } f \\ 1 : \text{general term for } f \end{cases}$
(b) $\int_0^x f(t)\,dt = \left[\frac{t^2}{2} + \frac{t^5}{5} + \frac{t^8}{16} + \frac{t^{11}}{66} + \cdots\right]_0^x$ $= \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{16} + \frac{x^{11}}{66} + \cdots$ First four nonzero terms for g : $\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{16} + \frac{x^{11}}{66}$	$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$
(c) $\frac{g^{(5)}(0)}{5!} = \frac{1}{5} \Rightarrow g^{(5)}(0) = \frac{5!}{5} = 24$	1 : answer
(d) $\left\lvert P_5\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\rvert \le \frac{\max\limits_{0\le x\le 1/2}\left\lvert g^{(6)}(x)\right\rvert}{6!}\left(\frac{1}{2}\right)^6$ $= \frac{\max\limits_{0\le x\le 1/2}\left\lvert f^{(5)}(x)\right\rvert}{6!\,2^6} \le \frac{500}{6!\,2^6}$	$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{upper bound} \end{cases}$