- Linear/undetermined systems

    - Matrix multiplication is $\rightarrow\downarrow$.
    - **Row echelon form** is the result of Gaussian elimination.
    - **Pivots** are first nonzero value in a row. **Free variables** are those columns without pivots. **Dependent variables** are columns with pivots.
    - **Specific (particular) solutions** are solutions $x_s$, and **vectors in the null space** are solutions $x_n$
    - $Ax = b$ for multiple solutions is the **general (total) solution**: $A(x_s + \beta x_n) = b$
      where $x_s$ is s.t. $Ax_s = b$ and $x_n$ s.t. $Ax_n = 0$
    - Vectors form a basis for a column space of $A$ if they're in $A$ and linearly independent.
    - Vectors in a row space are column vectors.

- QR factorization:

    - **Normal equation**: $A^T A \hat{x} = A^T b$ or $(A^T A)^{-1} A^T b = \hat{x}$
      Plug in $A$ and $b$ to find $\hat{x} = $ best approximate solution (**linear least-squares solution**)
    - **Compute projection** of $b$ onto $A$, call it $\hat{b}$: $A(A^T A)^{-1} A^T b = \hat{b}$.
      Note that this is just the $\hat{x}$ from the normal equation multiplied by $A$, so $\hat{b} = A\hat{x}$.
    - Orthonormal vectors:
      $$q_k = \frac{a_k^\perp}{\rho_{k,k}} = \frac{a_k^\perp}{\| a_k^\perp \|_2}$$
      where
      $$a_k^\perp = a_k - \rho_{0,k} q_0 - \ldots - \rho_{k-1,k} q_{k-1}$$
      $$a_k^\perp = a_k - q_0^T a_k q_0 - \ldots - q_{k-1}^T a_k q_{k-1}$$
    - $A = QR$ where $Q = (q_0 | \ldots | q_{n-1})$

      and $R = \begin{pmatrix} \| a_0 \|_2 & \rho_{0,1} & \cdots & \rho_{0,n-1} \\ & \| a_1^\perp \|_2 & \ddots & \vdots \\ & & \ddots & \rho_{n-2,n-1} \\ 0 & & & \| a_{n-1}^\perp \|_2 \end{pmatrix}$. Basically any $\rho_{k,k} = \| a_k^\perp \|_2$, $\rho_{(i<k),k} = q_i^T a_k$, and $\rho_{(i>k),k} = 0$.

    - $Q^T Q = I$

- Space spanned by vectors: $A = \{a_0 | \ldots | a_{n-1}\}$ where A is the space

- **Eigenvalues** are scalars $\lambda$. $\lambda$ is an eigenvalue of $A \iff Ax = \lambda x$ for some non-zero vector $x$. So:

  For 2x2 $M = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$, and for 3x3 $M = \begin{pmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{pmatrix}$,

  and $\det(A - \lambda I) = 0$.

- **Eigenvectors**

- Determinants:

    - For 2x2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(M) = ad - bc$
    - For 3x3 matrices $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det(M) = a(ei - hf) - b(di - fg) + c(dh - eg)$, or
      $\det(M) = aei + bfg + cdh - (afh + bdi + ceg)$

- Equivalent to "A is nonsingular"

    - $A$ is invertible.
    - $A^{-1}$ exists.
    - $AA^{-1} = A^{-1}A = I$.
    - $A$ represents a linear transformation that is a bijection.
    - $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.
    - $Ax = 0$ implies that x=0.
    - $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$
    - The determinant of $A$ is nonzero: $\det(A) \neq 0$.

- $LU$ with partial pivoting does not break down.
- $\mathcal{N}(A) = 0$.
- $\mathcal{C}(A) = \mathbb{R}^n$.
- $\mathcal{R}(A) = \mathbb{R}^n$.
- $A$ has linearly independent columns.
- $A$ has linearly independent rows.

- $LU$ with partial pivoting does not break down.
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- $\mathcal{R}(A) = \mathbb{R}^n$.