- Linear/undetermined systems
  - Matrix multiplication is $\rightarrow\downarrow$.
  - **Row echelon form** is the result of Gaussian elimination.
  - **Pivots** are first nonzero value in a row.
  - **Free variables** are those columns without pivots.
  - **Dependent variables** are columns with pivots.
  - **Rank** of a matrix is the number of pivots.
  - $Ax = b$ for multiple solutions is the **general (total) solution**: $A(x_s + \beta x_n) = b$ where x_s is s.t. $Ax_s = b$ and x_n s.t. $Ax_n = 0$
  - **Specific (particular) solutions** are solutions x_s, and **vectors in the null space** are solutions x_n.
    To find specific solutions, set free variables equal to 0 and solve for $Ax_s = b$.
  - Vectors form a basis for a **column space** of A if they're columns in A and linearly independent. $\mathcal{C}(A) = \text{Span}(a_0, \ldots, a_n - 1)$. So just the columns in the original matrix in which the pivots appear. Note: Also valid are columns from row echelon form, as they have the same span.
    Dimension is the number of dependent variables
  - Vectors in a **row space** are column vectors. The rows in which the pivots appear (both in row echelon and initial, though we usually use row echelon) transversed.
  - To find vectors in the **null space** set the first free variable to 1 and the second to 0 for the first vector and solve for $Ax_n = 0$, then flip for the second.
    Null space dimension is the number of columns - the number of pivots.
  - **General solution** is $x_s + \beta_k x_k$ all k vectors in the null space.
- QR factorization:
  - **Normal equation**: $A^T A\hat{x} = A^T b$ or $(A^T A)^{-1} A^T b = \hat{x}$
    Plug in A and b to find $\hat{x}$ = best approximate solution (**linear least-squares solution**)
  - **Compute projection** of b onto A, call it $\hat{b}$: $A(A^T A)^{-1} A^T b = \hat{b}$.
    Note that this is just the $\hat{x}$ from the normal equation multiplied by A, so $\hat{b} = A\hat{x}$.
  - Orthonormal vectors:
    $$q_k = \frac{a_k^\perp}{\rho_{k,k}} = \frac{a_k^\perp}{\| a_k^\perp \|_2}$$
    where
    $$a_k^\perp = a_k - \rho_{0,k} q_0 - \ldots - \rho_{k-1,k} q_{k-1}$$
    $$a_k^\perp = a_k - q_0^T a_k q_0 - \ldots - q_{k-1}^T a_k q_{k-1}$$
  - $A = QR$ where $Q = (q_0 | \ldots | q_{n-1})$
    and $R = \begin{pmatrix} \| a_0 \|_2 & q_0^T a_1 & \cdots & q_0^T a_{n-1} \\ & \| a_1^\perp \|_2 & \ddots & \vdots \\ & & \ddots & q_{n-2}^T a_{n-1} \\ 0 & & & \| a_{n-1}^\perp \|_2 \end{pmatrix}$. Basically any $\rho_{k,k} = \| a_k^\perp \|_2$, $\rho_{(i<k),k} = q_i^T a_k$, and $\rho_{(i>k),k} = 0$.
  - $Q^T Q = I$
- Space spanned by vectors: $A = \{a_0 | \ldots | a_{n-1}\}$ where A is the space and a_k is a vector.
- **Eigenvalues** are scalars λ. λ is an eigenvalue of $A \iff Ax = \lambda x$ for some non-zero vector x. So:
  For 2x2 $M = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$, and for 3x3 $M = \begin{pmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{pmatrix}$,
  and $\det(A - \lambda I) = 0$.
- A vector x is an **eigenvector** of A if $Ax = \lambda x$ where λ is some scalar. You get them by doing $(A - \lambda I)x = 0$ and solving for x.

- Determinants:
  - For 2x2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(M) = ad - bc$
  - For 3x3 matrices $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det(M) = a(ei - hf) - b(di - fg) + c(dh - eg)$, or $\det(M) = aei + bfg + cdh - (afh + bdi + ceg)$
- Inverses:
  - For 2x2 matrices $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  - For 3x3 matrices and up don't try to be tricky. Use Gauss-Jordan.
- **Diagonalization**: We have $X^{-1}AX = D$ where:

  $X = (x_0|...|x_{n-1})$ and $D = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{pmatrix}$ where each x_k and λ_k are an eigenvector and its eigenvalue, respectively.
- **LU Factorization**: $U =$ result of Gaussian elimination, L is all the transformations you made combined. $LU = A$

  To do $Ax = b$...

  $$Ax = b$$

  $$LUx = b$$

  $$Ly = b \text{ and } Ux = y$$

  solve for y in the first equation, and then use y to solve for x in the second.
- Equivalent to "A is nonsingular"
  - A is invertible.
  - A^{-1} exists.
  - $AA^{-1} = A^{-1}A = I$.
  - A represents a linear transformation that is a bijection.
  - $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.
  - $Ax = 0$ implies that x=0.
  - $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$
  - The determinant of A is nonzero: $det(A) \neq 0$.
  - LU with partial pivoting does not break down.
  - $\mathcal{N}(A) = 0$.
  - $\mathcal{C}(A) = \mathbb{R}^n$.
  - $\mathcal{R}(A) = \mathbb{R}^n$.
  - A has linearly independent columns.
  - A has linearly independent rows.