

# 1. Linearization

Code:

```
%% Linearization
syms x xdot y ydot z zdot phi theta psi p q r
syms u1 u2 u3 u4
syms m g I_x I_y I_z k_x k_y k_z k_p k_q k_r
syms z_d psi_d

states = [x y z xdot ydot zdot phi theta psi p q r];
inputs = [u1 u2 u3 u4];
equilibrium = [0 0 z_d 0 0 0 0 0 psi_d 0 0 0 m*g 0 0 0];

xddot = (1/m)*((cos(phi)*sin(theta)*cos(psi) + sin(phi)*sin(psi))*u1 - k_x*xdot);
yddot = (1/m)*((cos(phi)*sin(theta)*sin(psi) - sin(phi)*cos(psi))*u1 - k_y*ydot);
zddot = (1/m)*(cos(phi)*cos(theta)*u1 - m*g - k_z*zdot);
phi_dot = p + sin(phi)*tan(theta)*q + cos(phi)*tan(theta)*r;
theta_dot = cos(phi)*q - sin(phi)*r;
psi_dot = (sin(phi) * q)/cos(theta) + (cos(phi) * r)/cos(theta);
pdot = (1/I_x)*((I_y - I_z)*q*r + u2 - k_p*p);
qdot = (1/I_y)*((I_z - I_x)*p*r + u3 - k_q*q);
rdot = (1/I_z)*((I_x - I_y)*p*q + u4 - k_r*r);
sys = [xdot, ydot, zdot, xddot, yddot, zddot, phi_dot, theta_dot, psi_dot, pdot, qdot, rdot];
y = [x y z psi];

Asym = subs(jacobian(sys, states), [states inputs], equilibrium);
Bsym = subs(jacobian(sys, inputs), [states inputs], equilibrium);
Csym = subs(jacobian(y, states), [states inputs], equilibrium);
Dsym = subs(jacobian(y, inputs), [states inputs], equilibrium);

parameters = [m g I_x I_y I_z k_x k_y k_z k_p k_q k_r z_d psi_d];
numerics = [0.03 9.81 1.5e-5 1.5e-5 3e-5 4.5e-3 4.5e-3 4.5e-3 4.5e-4 4.5e-4 4.5e-4 2 pi/4];

A = double(vpa(subs(Asym, parameters, numerics)));
B = double(vpa(subs(Bsym, parameters, numerics)));
C = double(vpa(subs(Csym, parameters, numerics)));
D = double(vpa(subs(Dsym, parameters, numerics)));

sys = ss(A, B, C, D);
TF = tf(sys);
```



Jordan form:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linearized model at the equilibrium is unstable. Because two of the eigenvalues, namely 0, has a zero real part with corresponding Jordan block larger than  $1 \times 1$ . They are  $J([1 \ 2], [1 \ 2]) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $J([5 \ 6], [5 \ 6]) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , which show that the model is unstable.

### 3. Forced Response

Code:

```
%% Forced Response
A_heading = [A(9,9) A(9,12); A(12,9) A(12,12)];
B_heading = [B(9,4); B(12,4)];
C_heading = [C(4,9) C(4,12)];
D_heading = 0;

sys_heading = ss(A_heading, B_heading, C_heading, D_heading);
subplot(3,1,1);

[y, t] = impulse(3e-5*sys_heading, 1);
plot(t, y, 'LineWidth',2)
ylim([0, 0.07])
title("Impulse response", 'FontSize',16);
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

subplot(3,1,2);
[y,t] = step(3e-5*sys_heading, 1);
plot(t, y, 'LineWidth',2)
title("Step Response", 'FontSize',16);
```

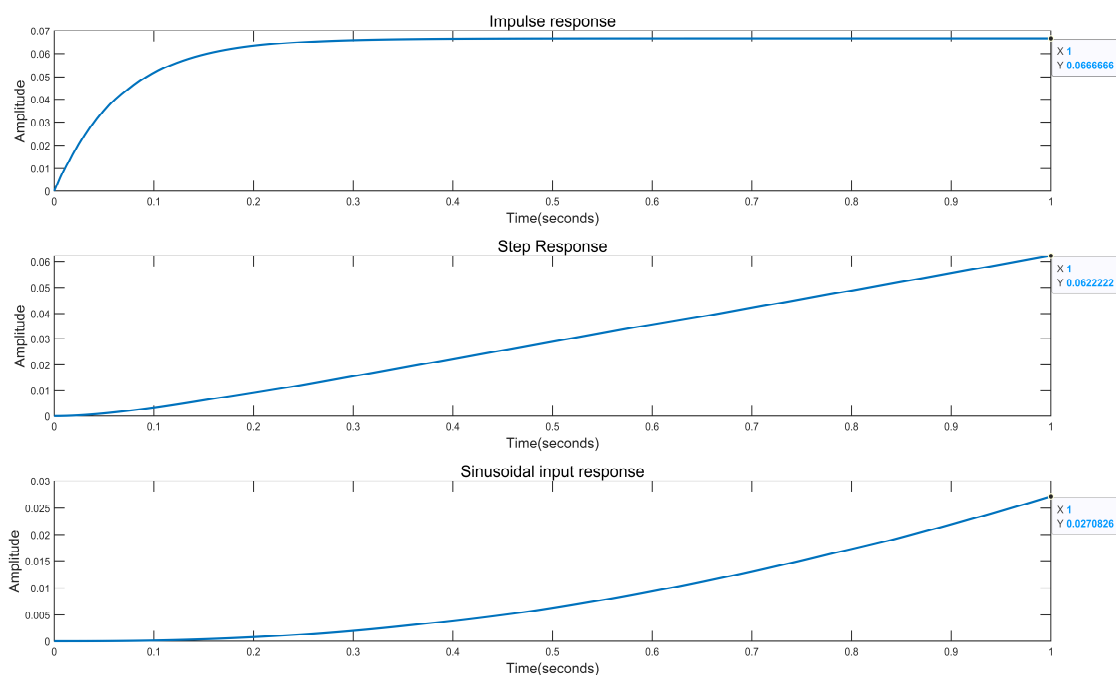
```

xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

data_points = 10000;
stoptime = 1;
dt = 1/data_points;
t = (0:dt:stoptime);
u = 3e-5*sin(t);
y = lsim(sys_heading, u, t);
subplot(3,1,3);
plot(t,y, 'Linewidth', 2);
title("Sinusoidal input response", 'FontSize',16);
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

```

Plots:



Verification using final value theorem:

$$G_{\psi}(s) = \frac{\frac{10}{3} * 10^4}{s^2 + 0.15s}$$

$$Y(s) = G_{\psi}(s) * U(s) = \frac{\frac{10}{3} * 10^4}{s^2 + 0.15s} * 3 * 10^{-5} * 1 = \frac{1}{s^2 + 15s}$$

$$\lim_{s \rightarrow 0} s * F(s) = \lim_{s \rightarrow 0} s * \frac{1}{s^2 + 15s} = 1/15 \approx 0.0667$$

The result of final value theorem coincides with the result shown by MATLAB.

## 4. Proportional Control

Code:

```
% Proportional Control
A_altitude = [A(3,3) A(3,6); A(6,3) A(6,6)];
B_altitude = [B(3,1); B(6,1)];
C_altitude = [1 0];
D_altitude = 0;

x0 = [-1;0];

k = -1;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,1)
[y, t] = initial(sys_altitude, x0, 20);
plot(t,y, 'linewidth',2)
title("k=-1: Divergence", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

k = 0;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,2)
[y,t]=initial(sys_altitude, x0, 1);
plot(t,y, 'linewidth', 2)
title("k=0: Converge to non-zero(-1)", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

k = 0.0001;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,3)
[y,t]=initial(sys_altitude, x0, 300);
plot(t,y, 'linewidth', 2)
```

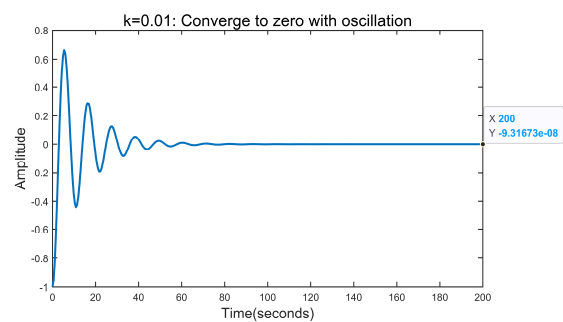
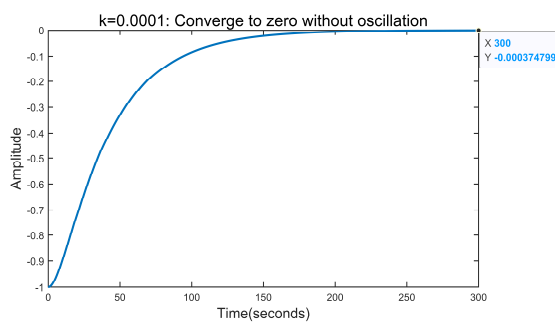
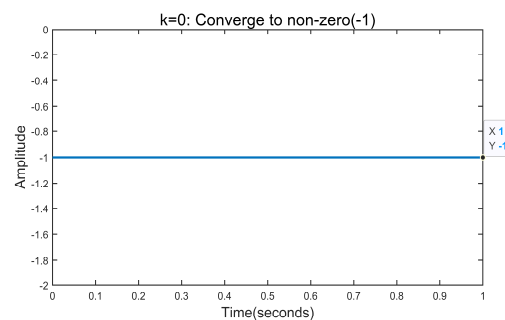
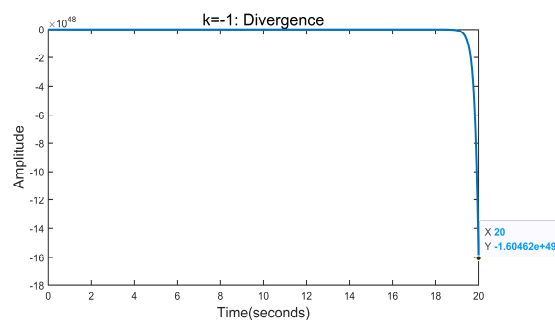
```

title("k=0.0001: Converge to zero without oscillation", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);

k = 0.01;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,4)
[y,t]=initial(sys_altitude, x0, 200);
plot(t,y,'linewidth',2)
title("k=0.01: Converge to zero with oscillation", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);

```

Plots:



Case Divergence: choose  $k = -1$

Case Converge to non-zero: choose  $k = 0$

Case Converge to zero without oscillation: choose  $k = 0.0001$

Case Converge to zero with oscillation: choose  $k = 0.01$

## Full MATLAB code:

```

close all
clear all
clc

%% Linearization
syms x xdot y ydot z zdot phi theta psi p q r
syms u1 u2 u3 u4
syms m g I_x I_y I_z k_x k_y k_z k_p k_q k_r
syms z_d psi_d

states = [x y z xdot ydot zdot phi theta psi p q r];
inputs = [u1 u2 u3 u4];
equilibrium = [0 0 z_d 0 0 0 0 psi_d 0 0 0 m*g 0 0 0];

xddot = (1/m)*((cos(phi)*sin(theta)*cos(psi) + sin(phi)*sin(psi))*u1 - k_x*xdot);
yddot = (1/m)*((cos(phi)*sin(theta)*sin(psi) - sin(phi)*cos(psi))*u1 - k_y*ydot);
zddot = (1/m)*(cos(phi)*cos(theta)*u1 - m*g - k_z*zdot);
phi_dot = p + sin(phi)*tan(theta)*q + cos(phi)*tan(theta)*r;
theta_dot = cos(phi)*q - sin(phi)*r;
psi_dot = (sin(phi) * q)/cos(theta) + (cos(phi) * r)/cos(theta);
pdot = (1/I_x)*((I_y - I_z)*q*r + u2 - k_p*p);
qdot = (1/I_y)*((I_z - I_x)*p*r + u3 - k_q*q);
rdot = (1/I_z)*((I_x - I_y)*p*q + u4 - k_r*r);
sys = [xdot, ydot, zdot, xddot, yddot, zddot, phi_dot, theta_dot, psi_dot, pdot, qdot, rdot];
y = [x y z psi];

Asym = subs(jacobian(sys, states), [states inputs], equilibrium);
Bsym = subs(jacobian(sys, inputs), [states inputs], equilibrium);
Csym = subs(jacobian(y, states), [states inputs], equilibrium);
Dsym = subs(jacobian(y, inputs), [states inputs], equilibrium);

parameters = [m g I_x I_y I_z k_x k_y k_z k_p k_q k_r z_d psi_d];
numerics = [0.03 9.81 1.5e-5 1.5e-5 3e-5 4.5e-3 4.5e-3 4.5e-3 4.5e-4 4.5e-4 4.5e-4 2 pi/4];

A = double(vpa(subs(Asym, parameters, numerics)));
B = double(vpa(subs(Bsym, parameters, numerics)));
C = double(vpa(subs(Csym, parameters, numerics)));
D = double(vpa(subs(Dsym, parameters, numerics)));

sys = ss(A, B, C, D);
TF = tf(sys)

```

```
%% Stability
[V,D,W] = eig(A); % V consists of the eigenvectors in its columns, D consists of the eigenvalues on its
diagonals
[V_Jordan, J] = jordan(A);% J is the Jordan form of A

%% Forced Response
A_heading = [A(9,9) A(9,12); A(12,9) A(12,12)];
B_heading = [B(9,4); B(12,4)];
C_heading = [C(4,9) C(4,12)];
D_heading = 0;

sys_heading = ss(A_heading, B_heading, C_heading, D_heading);
subplot(3,1,1);

[y, t] = impulse(3e-5*sys_heading, 1);
plot(t, y, 'LineWidth',2)
ylim([0, 0.07])
title("Impulse response", 'FontSize',16);
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);

subplot(3,1,2);
[y,t] = step(3e-5*sys_heading, 1);
plot(t, y, 'LineWidth',2)
title("Step Response", 'FontSize',16);
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);

data_points = 10000;
stoptime = 1;
dt = 1/data_points;
t = (0:dt:stoptime);
u = 3e-5*sin(t);
y = lsim(sys_heading, u, t);
subplot(3,1,3);
plot(t,y, 'LineWidth', 2);
title("Sinusoidal input response", 'FontSize',16);
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);
```



```
%% Proportional Control
A_altitude = [A(3,3) A(3,6); A(6,3) A(6,6)];
B_altitude = [B(3,1); B(6,1)];
C_altitude = [1 0];
D_altitude = 0;

x0 = [-1;0];

k = -1;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,1)
[y, t] = initial(sys_altitude, x0, 20);
plot(t,y, 'linewidth',2)
title("k=-1: Divergence", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
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set(gca, 'linewidth',1);

k = 0;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,2)
[y,t]=initial(sys_altitude, x0, 1);
plot(t,y, 'linewidth', 2)
title("k=0: Converge to non-zero(-1)", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

k = 0.0001;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
subplot(2,2,3)
[y,t]=initial(sys_altitude, x0, 300);
plot(t,y, 'linewidth', 2)
title("k=0.0001: Converge to zero without oscillation", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca, 'linewidth',1);

k = 0.01;
A_controlled_altitude = A_altitude - k*B_altitude*C_altitude;
sys_altitude = ss(A_controlled_altitude, [], C_altitude, []);
```

```
subplot(2,2,4)
[y,t]=initial(sys_altitude, x0, 200);
plot(t,y,'linewidth',2)
title("k=0.01: Converge to zero with oscillation", 'FontSize',16)
xlabel("Time(seconds)", 'FontSize',14)
ylabel("Amplitude", 'FontSize',14)
set(gca,'linewidth',1);
```