微分形式梳理

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2024 年 8 月 11 日

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1 前言

本系列的目的是梳理一下微分形式的一些概念。学习的时候发现这些知识没过几个礼拜就会遗忘，而且记号有些混乱，因此做一下梳理。

2 Differentiable Manifolds

2.1 Differentiable Manifolds

Definition 1 (Differentiable Manifolds)**.** *M is a set (even without topological structure). M is an **n-dimensional differentiable manifold**, if there exists a family of injective maps $f_\alpha : U_\alpha \subset \mathbf{R}^n \to M$, where all U_α are open sets in $\mathbf{R}^n$ and,*

(1) $\bigcup_\alpha f_\alpha(U_\alpha) = M$

(2) for each pair $\alpha \neq \beta$, if $W := f_\alpha(U_\alpha) \bigcap f_\beta(U_\beta) \neq \phi$, then $f_\alpha^{-1}(W)$ and $f_\beta^{-1}(W)$ are also open sets in $\mathbf{R}^n$, satisfying $f_\alpha^{-1} \circ f_\beta$ and $f_\beta^{-1} \circ f_\alpha$

are differentiable on their own domain(which are open sets by the previous restriction).

Remark 1. *A differentiable manifold intrinsically induces a topological structure. A set $A \subset M$ is an open set iff $f_\alpha^{-1}(A \bigcap f_\alpha(U_\alpha))$ is an open set in $\mathbf{R}^n$ for all α.*

It follows that this defines a topology in M and all f_α are homeomorphisms.

Remark 2. *If M is born with a topological structure, how can we prove that M is a differentiable manifold while matching its induced topology with its inborn topology (which is often we hope). If we are able to additionally prove that all f_α are homeomorphisms, the induced topology can be fit into the inborn topology.*

Remark 3. *From now on, to develop calculus, we fix a Hausdorff and second countable n-dim differentiable manifold M.*

2.2 Tangent Space

Definition 2 (Differentiable)**.** *M, N are differentiable manifolds. $p \in M$. A map $F : M \to N$ is* ***differentiable at p*** *if for the parameterization of p (U_α, f_α) and the parameterization of $F(p)$ (V_α, g_α), $g_\alpha^{-1} \circ F \circ f_\alpha$ is differentiable at $f_\alpha^{-1}(p)$.*

It's necessary to check that this definition is independent of the choice of the parameterization.

Definition 3 (tangent space)**.** *Fix a point p in M, denote the set of all curves passing p as $\mathcal{C}$, where a curve passing p is a differentiable map $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. Two curves γ_1, γ_2 are* ***tangent at p*** *if for every differentiable function $F : M \to \mathbf{R}$, $\frac{d}{dt}|_{t=0}(F \circ \gamma_1) = \frac{d}{dt}|_{t=0}(F \circ \gamma_2)$, which is an equivalence relation. The* ***tangent space of M at p*** *T_pM is defined by the quotient space of $\mathcal{C}$ under the tangent relation.*

Remark 4 (The inspiration)**.** *In normal geometry in $\mathbf{R}^n$, we define tangent vector v of a curve $\gamma(t) = (x^1(t), x^2(t), ..., x^n(t))$ by letting $v = (x^{1\prime}(t), x^{2\prime}(t), ..., x^{n\prime}(t))$.*

It follows that for every differentiable function $F : \mathbf{R}^n \to \mathbf{R}$,

$$\begin{aligned}
\frac{d}{dt}|_{t=0}(F \circ \gamma)(t) &= \sum_{i=1}^{n} \partial_i F \cdot x^{i\prime}(0) \\
&= (\sum_{i=1}^{n} x^{i\prime}(0) \cdot \frac{\partial}{\partial x^i})(F) \\
&= \frac{\partial}{\partial v} F
\end{aligned} \tag{1}$$

which turns out to be the same when two curves have the same tangent vector v, i.e. two curves are tangent.

Remark 5. *From Remark 4, we have an explicit definition for T_pM. Let D be the set of all differentiable functions $F : M \to \mathbf{R}$. γ is a curve passing p. The **tangent vector to** γ **at p** is the map $v : D \to R$ defined by*

$$vF = \frac{d}{dt}|_{t=0}(F \circ \gamma)$$

T_pM is defined by all tangent vectors to all curves passing p.

Remark 6. *The two definitions are equivalent in the sense of isomorphism. Moreover, the second definition holds a linear structure naturally. To figure out its dimension, given a parameterization (U_α, f_α) at p, the **coordinate curves** are curves*

$$\gamma_i(t) = f_\alpha(f_\alpha^{-1}(p) + (0, .., \underset{i-th}{t}, ..., 0))$$

γ_i play the same role as The tangent vector of γ_i can be denoted as $\frac{\partial}{\partial x^i}$ under the second definition and (1). T_pM has a basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, so it's n-dimensional.

Definition 4 (Differential)**.** *Let M, N be differentiable manifolds*