Approximately counting #BIS via containers

#BIS: open problem for general bipartite graph
#RHΠ_1 - complete.

G: bipartite graph $X \cup Y$.

G is a α-expander if

$\forall S \subseteq X$ with $|S| \le \frac{|X|}{2}$, $|N(S)| \ge (1+\alpha)|S|$

$\forall S \subseteq Y$ with $|S| \le \frac{|Y|}{2}$, $|N(S)| \ge (1+\alpha)|S|$

Thm 1. $\exists$ constant C_1 s.t. for $\alpha = \frac{C_1 \log^2 d}{d}$ - expander and d-regular bipartite graph G, there's an FPTAS for $i(G)$ and a polynomial-time sampling algorithm.

Thm 2: $\forall \alpha$, there exists constant C_2 s.t. for $d \ge 3$, $\lambda > \frac{C_2 \log d}{d^{\frac{1}{4}}}$, there's an FPTAS for $Z_G(\lambda) = \sum_I \lambda^{|I|}$ and polynomial sampling for $\mu_{G,\lambda}(I) = \frac{\lambda^{|I|}}{Z_G(\lambda)}$ for α-expander, d-regular bipartite G

Thm3. $\forall c>0$, G d-regular bipartite, there's a randomized approximation algorithm for $i(G)$ with n^{-c}-relative approximation w.p. $\geq \frac{2}{3}$ and runs in $\exp\left(O\left(\frac{n\log^3 d}{d}\right)\right)$

Polymer Model

A polymer is a vertex.

polymer a, b are incompatible if $(a,b)\in E$

compatible else

polymer model: (incompatible graph)

$$\Xi(\vec{w}) = \sum_{\substack{\Lambda:\text{ mutally compatible set of polymers}\\ \text{i.e. IS in incompatible graph}}} \prod_{\gamma\in\Lambda} w_\gamma$$

Dobrushin's Thm:

$$\ln \Xi = \sum_{\Gamma\text{ cluster}} \phi(\Gamma)\prod_{\gamma\in\Gamma} w_\gamma$$

cluster: connected subset ~~component~~ (ordered, repeating allowed)

$\Downarrow$ so R.H.S is infinite sum

$$\phi(\Gamma) = \sum_{\substack{E'\subseteq E(\Gamma)\\ E'\text{ connected}}} (-1)^{|E'|}$$

Ursell function

KP (Kotecký–Preiss) condition.

~~[illegible]~~:

~~Given a graph G,~~
~~define a polymer as the vertex set of any connected subgraph~~

If there are some functions $f, g : \{\text{all polymers}\} \to [0, +\infty)$, $\gamma \longmapsto$

Suppose for every polymer γ,

$$\sum_{\gamma' \nsim \gamma} w_{\gamma'} e^{f(\gamma')+g(\gamma')} \le f(\gamma),$$

Then the Dobrushin cluster expansion converges absolutely.

Moreover, for polymers defined for graph G,

i.e. each polymer is a subset of $V[G]$

$$\forall \text{ vertex } v \text{ of } G, \sum_{\substack{\Gamma \text{ cluster} \\ v \in \bigcup_{\gamma \in \Gamma} \gamma}} \left| \phi(\Gamma) \prod_{\gamma \in \Gamma} w_\gamma \right| e^{g(\bigcup_{\gamma \in \Gamma} \gamma)} \le 1$$

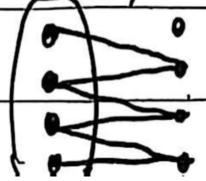
Approximating Hardcore Model ($Z_G(\lambda) = \sum_I \lambda^{|I|}$)

G: α-expander, [redacted] bipartite graph (X, Y), Δ-bounded degree

~~Define two polymer~~

2-linked: $S \subseteq X$ is connected in G^2

two step.



γ_1, γ_2 compatible if $\gamma_1 \cup \gamma_2$ is NOT 2-linked.

Define two types of polymer:

(X-type) ① $S \subseteq X$, $|S| \le \frac{|X|}{2}$, 2-linked

(Y-type) ② $S \subseteq Y$, $|S| \le \frac{|Y|}{2}$, 2-linked,

$$\omega_\gamma = \frac{\lambda^{|\gamma|}}{(1+\lambda)^{|N(\gamma)|}}$$

<u>Lemma 1.</u> For α-expander, $\forall I$ IndepSet either $|I \cap X| \le \frac{|X|}{2}$ or $|I \cap Y| \le \frac{|Y|}{2}$

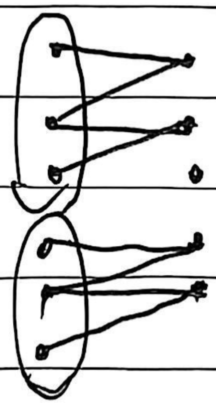
<u>Lemma 2.</u> For $\lambda > e^{\frac{11}{\alpha}}$,

$$\tilde{Z}_G(\lambda) \triangleq (1+\lambda)^{|Y|} \Xi^X(G) + (1+\lambda)^{|X|} \Xi^Y(G)$$

is an $\exp(-n)$-relative approximation of $Z_G(\lambda)$

Intuition:

mutually compatible



we can arbitrary choose or not choose a vertex in $Y \setminus \bigcup_{\gamma \in \Lambda} N(\gamma)$ to form an IS with $\bigcup_{\gamma \in \Lambda} \gamma$.

contribution to $Z_G(\lambda)$

$$= \lambda^{\sum_{\gamma \in \Lambda} |\gamma|} \cdot (1+\lambda)^{|Y| - \sum_{\gamma \in \Lambda} |N(\gamma)|}$$ ✓

Pf.

$\mathcal{I}^X(G) \triangleq \{ I : I \cap X$ is composed by X-type polymers$\}$

$\mathcal{I}^Y(G) \triangleq \{ I : I \cap Y$ is composed by Y-type polymers$\}$

Then, $\forall I$ either $I \in \mathcal{I}^X(G)$ or $I \in \mathcal{I}^Y(G)$

$$\tilde{Z}_G(\lambda) = \sum_{I \in \mathcal{I}^X} \lambda^{|I|} + \sum_{I \in \mathcal{I}^Y} \lambda^{|I|}$$

$$= Z_G(\lambda) + \underbrace{\sum_{I \in \mathcal{I}^X \cap \mathcal{I}^Y} \lambda^{|I|}}_{\exp(-n)\text{-relative}}$$

$$\ln \Xi = \sum_{\Gamma \text{ cluster}} \phi(\Gamma) \prod_{\gamma \in \Gamma} w_\gamma$$

$$\ln \Xi(l) = \sum_{\substack{\Gamma \text{ cluster} \\ |\Gamma| \le l}} \phi(\Gamma) \prod_{\gamma \in \Gamma} w_\gamma$$, can be computed in $O(n \cdot \Delta^l)$

If KP-condition is satisfied,

$l = O(\log n)$ $\quad \left| \ln \Xi - \ln \Xi(l) \right| \le \varepsilon.$

Approximating $i(G)$

$G = X \cup Y$ $\quad |X| = |Y| = n$

G: α-expander, d-regular bipartite graph

~~expanding~~ $[A]$: 2-linked closure of $A \subseteq X$

i.e. $\{ x \in X : N(x) \subseteq N(A) \}$

$A \subseteq X$ is expanding if $|N(A)| - |[A]| \ge \frac{C_1}{2} \frac{\log^2 d}{d} |N(A)|$

Define polymer:

1. $S \subseteq X$, 2-linked, expanding, $w_\gamma = 2^{-|N(\gamma)|}$
2. .. Y — — — — —

γ_1, γ_2 compatible if $\gamma_1 \cup \gamma_2$ is NOT 2-linked

Lemma. For n sufficiently large, $2^n(\Xi^X+\Xi^Y)$ is an ε-relative approximation to $i(G)$, where $\varepsilon = 2^{-\frac{n\log^2 d}{60d}}$

Pf. $\mathcal{I}^X(G) = \{I : I\cap X \text{ consists of } X\text{-type polymers}\}$

$\mathcal{I}^Y(G) = \{I : I\cap Y \text{ consists of } Y\text{-type polymers}\}$

$2^n(\Xi^X+\Xi^Y) = |\mathcal{I}^X|+|\mathcal{I}^Y| = i(G) + |\mathcal{I}^X\cap\mathcal{I}^Y|$

Goal: ① to bound $|\mathcal{I}^X\cap\mathcal{I}^Y|$

② efficiently calculate Ξ

Property 1. If G α-expander with $\alpha = G\frac{\log^2 d}{d}$ and d-regular then $\forall A\subseteq X$ s.t. $|[A]|\le\frac{n}{2}$, A is expanding.

Pf. $|N(A)| = |N([A])| \ge (1+\alpha)|[A]|$

For d sufficiently large, $\alpha \le 1$, $|N(A)|-|[A]| \ge \frac{\alpha}{2}|N$

Let $\mathcal{G}(v,a,w) = \{\gamma : v\in\gamma,\ |[\gamma]| = a,\ |N(\gamma)| = w\}$

Lemma 4. $|\mathcal{G}(v,a,w)| \le 2^{w-\varepsilon_1(w-a)}$

Pf. $F\subseteq N(A)$ is essential for A if

① $F \supseteq \underbrace{\{v\in N(A) : v \text{ has } \ge\frac{d}{2} \text{ neighbours in } [A]\}}_{= W_{d/2}}$

② $[A]\subseteq N(F)$

Lemma 7. $\exists$ family $\mathcal{C}(v,a,w) \subset 2^{Y}$ of size $\le 2^{\frac{16 w \log^2 d}{d}}$ s.t. $\forall$ 2-linked $A \subseteq X$, with $v \in A$, $|[A]| = a$, $|N(A)| = w$, $\mathcal{C}(v,a,w)$ contains an essential subset of A.

Lemma 8. $\forall F \subseteq Y$, a, w.

$$\mathcal{G}(F,a,w) \triangleq \{A \mid A \models a, w,\ F \text{ is essential to } A\}$$

then $|\mathcal{G}(F,a,w)| \le 2^{w - c_3(w-a)}$

Pf of Lemma 4.

$$|\mathcal{G}(v,a,w)| \le \sum_{F \in \mathcal{C}(v,a,w)} |\mathcal{G}(F,a,w)|$$

$$\cancel{\le |\mathcal{C}(v,a,w)| \cdot \max}$$

$$\le 2^{\frac{16 w \log^2 d}{d}} \cdot 2^{w - c_3(w-a)}$$

$$\le 2^{w - c_1(w-a)}$$

by $w - a \ge \frac{c_1}{2} \frac{\log^2 d}{d} \cdot w$ and taking $c_1 > \frac{64}{c_3}$ $\square$

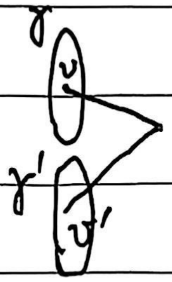
Lemma 12. Let $f(\gamma) = \ln 2 \cdot \frac{|\gamma| \log^2 d}{d}$.

$$g(\gamma) = 2 \ln 2 \cdot \frac{|N(\gamma)| \log^2 d}{d}$$

Then KP condition: $\forall \gamma$,

$$\sum_{\gamma' \nsim \gamma} w_{\gamma'} e^{f(\gamma') + g(\gamma')} \le f(\gamma) \quad \text{holds.}$$

Pf. $\sum_{\gamma' \not\sim \gamma} w_{\gamma'} e^{f(\gamma')+g(\gamma')} \le \sum_{v\in\gamma} \sum_{v' \text{ two steps from } v} \sum_{\gamma' \ni v'} w_{\gamma'} e^{f(\gamma')+g(\gamma')}$



$$\le |\gamma| \cdot d^2 \max_{\substack{v\in\gamma \\ v'}} \sum_{\gamma'\ni v'} 2^{-N(\gamma')} e^{f(\gamma')+g(\gamma')}$$

$$\le |\gamma| d^2 \max_{\substack{v\in\gamma \\ v'}} \sum_{w\ge d} \sum_{t \ge \frac{c_1}{2}\frac{w\log^2 d}{d}} |G(v', w-t, w)| \, 2^{-w+\frac{3w\log^2 d}{d}}$$

$$\le |\gamma| d^2 \sum_{w\ge d} 2^{\frac{3w\log^2 d}{d}} \sum_{t\ge \frac{c_1}{2}\frac{w\log^2 d}{d}} 2^{-c_1 t}$$

$$\le d^2|\gamma| \sum_{w\ge d} 2^{\frac{3w\log^2 d}{d} - \frac{8w\log^2 d}{d}}$$

$$\le d^3 |\gamma| \, 2^{-5\log^2 d} \ll f(\gamma)$$

KP condition holds.

So $\sum_{\substack{\Gamma \text{ cluster} \\ \bigcup_{\gamma\in\Gamma}\gamma \ni v}} \left|\phi(\Gamma)\prod_{\gamma\in\Gamma} w_\gamma\right| e^{g(\bigcup_{\gamma\in\Gamma}\gamma)} \le 1.$

Now we approximate Ξ by $\Xi(\ell) := \exp\left(\sum_{\substack{\Gamma \dots \\ |\bigcup\Gamma| = \ell}} \dots\right)$

<u>Lemma</u>: $|\ln \Xi^X - \ln \Xi^X(\ell)| \le n \cdot 2^{-2\frac{\ell\log^2 d}{d}}$

In particular, if $\ell \ge \frac{d}{2\log^2 d}\ln\left(\frac{n}{\varepsilon}\right)$, then

$$|\ln\Xi^X - \ln\Xi^X(\ell)| \le \varepsilon.$$

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Pf. $$\left| \ln \Xi^X - \ln \Xi^X(\ell) \right| = \left| \sum_{\substack{\Gamma \text{ cluster} \\ \text{with} \\ |\bigcup_{\gamma\in\Gamma}\gamma| > \ell}} \phi(\Gamma) \prod_{\gamma\in\Gamma} w_\gamma \right|.$$

$$\leq \sum_v \sum_{\substack{\Gamma \cdots \\ \cdots \\ \Gamma \ni v}} \left| \cdots \right|$$

$$\leq n \cdot e^{-2\ln 2\, \ell \frac{\log^2 d}{d}}$$

$\square$