COMP SCI 6O03

Linear Optimization

Fall 2018 report

Combinatorial optimization formulations with applications to theoretical physics

The project focuses on the analysis of a combinatorial structure that is associated to a problem in theoretical physics. Given an integer d, we consider all the $2^d - 1$ nonzero vectors with $\{0,1\}$-valued coordinates. Let $S(d)$ denote the set of all possible 2^{2^d-1} sub-sums of these $2^d - 1$ vectors. The combinatorial structure we aim to investigate is the convex hull $H(d)$ of all these 2^{2^d-1} sub-sums. Let $a(d)$ be the number of vertices, i.e. extreme points, of $H(d)$.

For instance for $d = 2$, there are $2^2 - 1 = 3$ $\{0,1\}$-valued nonzero vectors: $(0,1),(1,0)$, and $(1,1)$. There are $2^{2^2-1} = 8$ possible sub-sums and $H(2)$ forms an hexagon whose 6 vertices are $(0,0),(1,0),(0,1),(1,2)$, $(2,1)$, and $(2,2)$. Thus $a(2) = 6$. Similarly, $a(3) = 32$ and, up to the symmetries of $H(d)$, the vertices of $H(3)$ are $(0,0,0),(1,0,0),(2,1,0),(2,2,0)$, and $(3,1,1)$.

The quantity $a(d)$ is known to be the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from Euclidean time/energy ($d+1$ is the number of energy/time variables). These are also known as Generalized Retarded Functions. The set $S(d)$ of all 2^{2^d-1} sub-sums can be interpreted as the edge-vertex incidence relationships on an hypergraph on d nodes.

The project aims at investigating the connections between edge-vertex incidence relationships for relatively small d and at proposing a visualization and providing a tool to facilitate further studies. For example, for $d = 2$ the only two elements of $S(2)$ which are not vertices of $H(2)$ are $(0,1)+(1,0)$ and $(1,1)$. They correspond, respectively, to a loop on each of the 2 nodes, and the edge between the 2 nodes.

The report should provide a compact graphic representation of the vertices of $H(d)$ for $d \leq 5$ and explain the underlying combinatorial structure and proposed computational framework.

Relevant readings include:

Deza, Manoussakis, and Onn. *Primitive zonotopes.* Discrete and Computational Geometry (2018)
Evans: *QFT pages at Imperial College.* `plato.tp.ph.ic.ac.uk/~time/qft/`
Sloane: *On-Line Encyclopedia of Integer Sequences.* `oeis.org/A034997`