Research Note - Quantum Entanglement

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1 Basical Concepts

1.1 Density matrix

As defination of density matrix, for a quantum complete set of states $\{|n\rangle\}$ which satisfy $\sum_n |n\rangle \langle n| = 1$, the density matrix is:

$$\rho = |n\rangle \langle n|$$

and it's trace is tr $\rho = 1$. It can be used to caculate the average value of an observable G:

$$\langle G \rangle = \langle \psi | G | \psi \rangle = \sum_{n,n'} \langle \psi | n \rangle \langle n | G | n' \rangle \langle n' | \psi \rangle = \mathrm{tr}(\rho G)$$

Funny fact:

$$\nabla \cdot A\vec{x} = \mathrm{tr}(A)$$

If quantum state cannot be described by ONE wave function, in other words, the quantum state is a mixed state$|\psi\rangle = \sum_k c_k |k\rangle = \sum_k p_k |k\rangle, \sum_k p_k = 1$, the density matrix is:

$$\rho = \sum_k p_k |k\rangle \langle k|$$

1.2 Helicity

we define the helicity operator as:

$$h = \vec{S} \cdot \frac{\vec{p}}{|\vec{p}|} = \vec{S} \cdot \hat{p}$$

where $\vec{S}$ is the spin operator, and $\hat{p}$ is the momentum direction. The eigenvalue of helicity operator is the helicity of the particle. In 3-dim, the helicity eigenvector is:

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} (|1,0\rangle \pm i\,|0,1\rangle)$$

$$|\psi_0\rangle = |0,0\rangle$$

where $|1,0\rangle$ and $|0,1\rangle$ are the eigenvector of S_z.

1.3 Irreducible tensor

On the textbook

In *Quantum Mechanics* by J.Y Zeng, in Section 7.3 Part 2, or *Modern Quantum Mechanics* by J.J Sakurai, in Section 3.11

1.4 $SU(3)$ symmetry matrix

For a $SU(3)$ symmetry matrix, the matrixλ_i is: we define transformation matrix U as:

$$\phi' = U\phi$$

which:

$$U = \exp\left[\frac{1}{2} i\hat{\theta}\hat{n} \cdot \lambda\right]$$

Table 1: Gell-Mann matrix

$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$	$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$	$\lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	-

2 Bell inequalities

2.1 Density matrix in spin-1 particle

In the paper[1], we define the normalized helicity eigenvector $\psi_{\pm,0}$ of the massive spin-1 particle of mass M, corresponding respectively to eigenvalues $\lambda = \pm 1, 0$. In order to describe the helicity of the spin-1 particle in a more general reference frame and in a covariant manner, we first promote the three basis vectors to four-vectors by extending them with a null temporal component and then perform a Lorentz boost along the $-\hat{k}$ direction. we difineL

- velocity: $\beta = \sqrt{1 - M^2/E^2}$
- 4-momentum: $p^\mu = E(1, \beta\hat{k})$
- boost basis vector:

$$n_1^\mu = (0, \hat{n}), n_2^\mu = (0, \hat{r}), n_3^\mu = \frac{E}{M}(\beta, \hat{k})$$

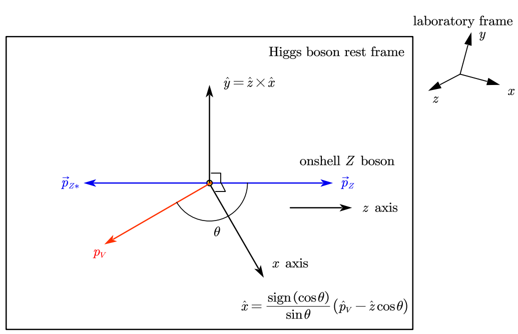

Figure 1: Reference System in 3-dim

2.2 CHSH inequality derivation

The proof[2] of the CHSH inequality is based on the following assumptions: **Local hidden variable** theories: the measurement results of the two particles are determined by the hidden variables of the particles themselves, and the measurement results of the two particles are independent of each other.

We assume there a pair of entangled particles for a, a', b, b' are different measurement directions, and A, A', B, B' are the measurement results which's possibe value are $\{-1, 0, 1\}$.

In other words, A are observables in $\mathcal{H}_A$, and B are observables in $\mathcal{H}_B$. If the local hidden variable theories are correct, the possiblity of pairs of outcomes $P(a, b)$ is:

$$P(a,b) = \int d\lambda A(a,\lambda)B(b,\lambda)\rho(\lambda)$$

where λ is the hidden variable, and $\rho(\lambda)$ is the probability distribution of the hidden variable. So that:

$$\begin{aligned}
&P(a,b) - P(a,b') \\
&= \int d\lambda\rho(\lambda)[A(a,\lambda)B(b,\lambda) - A(a,\lambda)B(b',\lambda)] \\
&= \int d\lambda\rho(\lambda)[A(a,\lambda)B(b,\lambda) - A(a,\lambda)B(b',\lambda) \pm A(a,\lambda)B(b',\lambda)A(a',\lambda)B(b',\lambda) \mp A(a,\lambda)B(b',\lambda)A(a',\lambda)B(b',\lambda)] \\
&= \int d\lambda\rho(\lambda)A(a,\lambda)B(b,\lambda)[1 \pm A(a',\lambda)B(b',\lambda)] - A(a,\lambda)B(b',\lambda)[1 \pm A(a',\lambda)B(b,\lambda)]
\end{aligned}$$

apply the triangle inequality ($c = a + b \Rightarrow |c| \leqslant |a| + |b|$):

$$\begin{aligned}
&|P(a,b) - P(a,b')| \\
&\leqslant \left|\int d\lambda\rho(\lambda)A(a,\lambda)B(b,\lambda)[1 \pm A(a',\lambda)B(b',\lambda)]\right| + \left|\int d\lambda\rho(\lambda)A(a,\lambda)B(b',\lambda)[1 \pm A(a',\lambda)B(b,\lambda)]\right| \\
&\leqslant \int d\lambda\, |A(a,\lambda)B(b,\lambda)|\, |1 \pm A(a',\lambda)B(b',\lambda)| + \int d\lambda\, |A(a,\lambda)B(b',\lambda)|\, |1 \pm A(a',\lambda)B(b,\lambda)|
\end{aligned}$$

cause $[1 \pm A(a',\lambda)B(b',\lambda)]\rho(\lambda) \geqslant 0$, and $[1 \pm A(a',\lambda)B(b,\lambda)]\rho(\lambda) \geqslant 0$ they are **non-negative**. Then we have:

Because both A, B, A', B' are $\{-1, 0, 1\}$, the absolute value of the product of them is 1.

$$\begin{aligned}
&|P(a,b) - P(a,b')| \\
&\leqslant \int d\lambda\rho(\lambda)[1 \pm A(a',\lambda)B(b',\lambda)] + \int d\lambda\rho(\lambda)[1 \pm A(a',\lambda)B(b,\lambda)] \\
&\leqslant 2 \pm \left[\int d\lambda\rho(\lambda)A(a',\lambda)B(b',\lambda) + \int d\lambda\rho(\lambda)A(a',\lambda)B(b,\lambda)\right] \\
&\leqslant 2 \pm [P(a',b') + P(a',b)]
\end{aligned}$$

where the last inequality is the definition of $P(a', b')$ and $P(a', b)$. So that:

We use triangle inequality again

$$\begin{aligned}
&|P(a,b) - P(a,b')| \leqslant 2 \pm [P(a',b') + P(a',b)] \\
&\Rightarrow |P(a,b) - P(a,b')| + |P(a,b) + P(a,b')| \leqslant 2 \\
&\Rightarrow |P(a,b) - P(a,b') + P(a,b) + P(a,b')| \leqslant \text{Left.} \leqslant 2
\end{aligned}$$

Q.E.D.

But for the quantum mechanics, the measurement results of the two particles are not independent of each other, so that the CHSH inequality is violated. for two particles with spin 1, the state of particles is: $\{|+\rangle, |0\rangle, |-\rangle\}$, so pair of particles are in the state:

$$\{|+\rangle, |0\rangle, |-\rangle\}_A \otimes \{|+\rangle, |0\rangle, |-\rangle\}_B$$

Cause limit of spin momentum conservation, the state of the two particles must be:

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}(|+-\rangle - |00\rangle + |-+\rangle)$$

2.3 CGLMP inequality

Our measurement is based on 3 dimension (spin 1)In Spin-1 case (3 dimension), the density matrix

Detiel of proof in [1] section 2.3

For pherhaps of local hidden variable theories, CGLMP inequality[3], in the case of two particles with spin 1, the inequality[4] is:

$$\begin{aligned} & I_3 \leq 2 \\ \text{which:} \quad & I_3 = + [P(A_1 = B_1) + P(B_1 = A_2 + 1) \\ & \quad + P(A_2 = B_2) + P(B_2 = A_1)] \\ & \quad - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ & \quad + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \end{aligned}$$

where A_i and B_i are the measurement results of the two particles, and $P(A_i = B_j)$ is the probability of the two particles having the same measurement results. We can use Bell operator $\mathcal{O}_{Bell}$ for the inequality:

$$I_3 = \langle \mathcal{O}_{Bell} \rangle = \mathrm{tr}\{\rho \mathcal{O}_{Bell}\} \leqslant 2$$

For our $H \to ZZ$ process, ZZ state must be:

$$|\psi_{ZZ}\rangle = a_1 |+-\rangle + a_2 |00\rangle + a_3 |-+\rangle$$

where $a_1^2 + a_2^2 + a_3^2 = 1$.

$$e^{\mu}_{\sigma}(m, \vec{k}) = \begin{array}{c|c} & \begin{matrix} S_- & S_0 & S_0 \end{matrix} \\ \hline \begin{matrix} k_0 \\ k_x \\ k_y \\ k_z \end{matrix} & \begin{bmatrix} 0 & \frac{|\vec{k}|}{m} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{\sqrt{k^2}}{m} & 0 \end{bmatrix} \end{array}$$

$$|\psi_{ZZ}\rangle = \eta_{\mu\nu} e^{\mu}_{\sigma}(m_1, \vec{k}) e^{\nu}_{\lambda}(m_2, -\vec{k}) \left|\vec{k}, \sigma\right\rangle_A \left|-\vec{k}, \lambda\right\rangle_B$$

this time, the ψ can be written in Spin-1 basis:

$$\psi = (0, 0, 1, 0, -\beta, 0, 1, 0, 0)$$

and the density matrix is tensor product of the state:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & \beta^2 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

References

[1] M. Fabbrichesi, R. Floreanini, E. Gabrielli, and L. Marzola, "Bell inequalities and quantum entanglement in weak gauge bosons production at the LHC and future colliders," *The European Physical Journal C*, vol. 83, p. 823, Sept. 2023. arXiv:2302.00683 [hep-ex, physics:hep-ph, physics:quant-ph].

[2] J. F. Clauser and M. A. Horne, “Experimental consequences of objective local theories,” *Physical review D*, vol. 10, no. 2, p. 526, 1974.

[3] B. Dalton, “The cglmp bell inequalities,” *The European Physical Journal Special Topics*, vol. 230, pp. 903–914, 2021.

[4] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, “Bell inequalities for arbitrarily high-dimensional systems,” *Physical Review Letters*, vol. 88, Jan. 2002.