

Research Note - Quantum Entanglement

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July 23, 2024

1 Basic Concepts

1.1 Density matrix

As definition of density matrix, for a quantum complete set of states $\{|n\rangle\}$ which satisfy $\sum_n |n\rangle \langle n| = 1$, the density matrix is:

$$\rho = |n\rangle \langle n|$$

and it's trace is $\text{tr } \rho = 1$. It can be used to caculate the average value of an observable G : Funny fact:

$$\langle G \rangle = \langle \psi | G | \psi \rangle = \sum_{n,n'} \langle \psi | n \rangle \langle n | G | n' \rangle \langle n' | \psi \rangle = \text{tr}(\rho G) \quad \nabla \cdot A \vec{x} = \text{tr}(A)$$

If quantum state cannot be described by ONE wave function, in other words, the quantum state is a mixed state $|\psi\rangle = \sum_k c_k |k\rangle = \sum_k p_k |k\rangle$, $\sum_k p_k = 1$, the density matrix is:

$$\rho = \sum_k p_k |k\rangle \langle k|$$

1.2 Irreducible tensor

1.3 $SU(3)$ symmetry matrix

For a $SU(3)$ symmetry matrix, the matrix λ_i is: we define transformation matrix U as:

Table 1: Gell-Mann matrix

$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$	$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$	$\lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	-

$$\phi' = U \phi$$

which:

$$U = \exp \left[\frac{1}{2} i \theta \hat{n} \cdot \lambda \right]$$

2 Bell inequalities

2.1 CHSH inequality derivation

The proof[1] of the CHSH inequality is based on the following assumptions: **Local hidden variable** theories: the measurement results of the two particles are determined by the hidden variables of the particles themselves, and the measurement results of the two particles are independent of each other.

We assume there a pair of entangled particles for a, a', b, b' are different measurement directions, and A, A', B, B' are the measurement results which's possible value are $\{-1, 0, 1\}$. In other words, A are observables in \mathcal{H}_A , and B are observables in \mathcal{H}_B . If the local hidden variable theories are correct, the possibility of pairs of outcomes $P(a, b)$ is:

$$P(a, b) = \int d\lambda A(a, \lambda) B(b, \lambda) \rho(\lambda)$$

where λ is the hidden variable, and $\rho(\lambda)$ is the probability distribution of the hidden variable. So that:

$$\begin{aligned} & P(a, b) - P(a, b') \\ &= \int d\lambda \rho(\lambda) [A(a, \lambda) B(b, \lambda) - A(a, \lambda) B(b', \lambda)] \\ &= \int d\lambda \rho(\lambda) [A(a, \lambda) B(b, \lambda) - A(a, \lambda) B(b', \lambda) \pm A(a, \lambda) B(b', \lambda) A(a', \lambda) B(b', \lambda) \mp A(a, \lambda) B(b', \lambda) A(a', \lambda) B(b, \lambda)] \\ &= \int d\lambda \rho(\lambda) A(a, \lambda) B(b, \lambda) [1 \pm A(a', \lambda) B(b', \lambda)] - A(a, \lambda) B(b', \lambda) [1 \pm A(a', \lambda) B(b, \lambda)] \end{aligned}$$

apply the triangle inequality ($c = a + b \Rightarrow |c| \leq |a| + |b|$):

$$\begin{aligned} & |P(a, b) - P(a, b')| \\ &\leq \left| \int d\lambda \rho(\lambda) A(a, \lambda) B(b, \lambda) [1 \pm A(a', \lambda) B(b', \lambda)] \right| + \left| \int d\lambda \rho(\lambda) A(a, \lambda) B(b', \lambda) [1 \pm A(a', \lambda) B(b, \lambda)] \right| \\ &\leq \int d\lambda |A(a, \lambda) B(b, \lambda)| |1 \pm A(a', \lambda) B(b', \lambda)| + \int d\lambda |A(a, \lambda) B(b', \lambda)| |1 \pm A(a', \lambda) B(b, \lambda)| \end{aligned}$$

cause $[1 \pm A(a', \lambda) B(b', \lambda)] \rho(\lambda) \geq 0$, and $[1 \pm A(a', \lambda) B(b, \lambda)] \rho(\lambda) \geq 0$ they are **non-negative**. Then we have:

$$\begin{aligned} & |P(a, b) - P(a, b')| \\ &\leq \int d\lambda \rho(\lambda) [1 \pm A(a', \lambda) B(b', \lambda)] + \int d\lambda \rho(\lambda) [1 \pm A(a', \lambda) B(b, \lambda)] \\ &\leq 2 \pm \left[\int d\lambda \rho(\lambda) A(a', \lambda) B(b', \lambda) + \int d\lambda \rho(\lambda) A(a', \lambda) B(b, \lambda) \right] \\ &\leq 2 \pm [P(a', b') + P(a', b)] \end{aligned}$$

where the last inequality is the definition of $P(a', b')$ and $P(a', b)$. So that:

$$\begin{aligned} & |P(a, b) - P(a, b')| \leq 2 \pm [P(a', b') + P(a', b)] \\ &\Rightarrow |P(a, b) - P(a, b')| + |P(a, b) + P(a, b')| \leq 2 \\ &\Rightarrow |P(a, b) - P(a, b') + P(a, b) + P(a, b')| \leq \text{Left} \leq 2 \end{aligned}$$

Q.E.D.

Because both A, B, A', B' are $\{-1, 0, 1\}$, the absolute value of the product of them is 1.

We use triangle inequality again

But for the quantum mechanics, the measurement results of the two particles are not independent of each other, so that the CHSH inequality is violated. for two particles with spin 1, the state of particles is: $\{|+\rangle, |0\rangle, |-\rangle\}$, so pair of particles are in the state:

$$\{|+\rangle, |0\rangle, |-\rangle\}_A \otimes \{|+\rangle, |0\rangle, |-\rangle\}_B$$

Cause limit of spin momentum conservation, the state of the two particles must be:

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}(|+-\rangle - |00\rangle + |-+\rangle)$$

2.2 CGLMP inequality

Our measurement is based on 3 dimension (spin 1) In Spin-1 case (3 dimension), the density matrix

Detail of proof in [2] section 2.3

For perhaps of local hidden variable theories, CGLMP inequality[3], in the case of two particles with spin 1, the inequality[4] is:

$$\begin{aligned} I_3 &\leq 2 \\ \text{which: } I_3 &= + [P(A_1 = B_1) + P(B_1 = A_2 + 1) \\ &\quad + P(A_2 = B_2) + P(B_2 = A_1)] \\ &\quad - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ &\quad + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \end{aligned}$$

where A_i and B_i are the measurement results of the two particles, and $P(A_i = B_j)$ is the probability of the two particles having the same measurement results. We can use Bell operator \mathcal{O}_{Bell} for the inequality:

$$I_3 = \langle \mathcal{O}_{Bell} \rangle = \text{tr}\{\rho \mathcal{O}_{Bell}\} \leq 2$$

For our $H \rightarrow ZZ$ process, ZZ state must be:

$$|\psi_{ZZ}\rangle = a_1 |+-\rangle + a_2 |00\rangle + a_3 |-+\rangle$$

where $a_1^2 + a_2^2 + a_3^2 = 1$.

$$e_\sigma^\mu(m, \vec{k}) = \begin{array}{c} \begin{array}{c} k_0 \\ k_x \\ k_y \\ k_z \end{array} \end{array} \begin{array}{c} \begin{array}{c} S_- \\ S_0 \\ S_0 \end{array} \end{array} \begin{array}{c} \begin{array}{c} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \end{array} \end{array} \begin{array}{c} \begin{array}{c} \frac{|\vec{k}|}{m} \\ 0 \\ 0 \\ -\frac{\sqrt{k^2}}{m} \end{array} \end{array} \begin{array}{c} \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \end{array} \end{array}$$

$$|\psi_{ZZ}\rangle = \eta_{\mu\nu} e_\sigma^\mu(m_1, \vec{k}) e_\lambda^\nu(m_2, -\vec{k}) \left| \vec{k}, \sigma \right\rangle_A \left| -\vec{k}, \lambda \right\rangle_B$$

this time, the ψ can be written in Spin-1 basis:

$$\psi = (0, 0, 1, 0, -\beta, 0, 1, 0, 0)$$

and the density matrix is tensor product of the state:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & \beta^2 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

References

- [1] J. F. Clauser and M. A. Horne, “Experimental consequences of objective local theories,” *Physical review D*, vol. 10, no. 2, p. 526, 1974.
- [2] M. Fabbrichesi, R. Floreanini, E. Gabrielli, and L. Marzola, “Bell inequalities and quantum entanglement in weak gauge bosons production at the LHC and future colliders,” *The European Physical Journal C*, vol. 83, p. 823, Sept. 2023. arXiv:2302.00683 [hep-ex, physics:hep-ph, physics:quant-ph].
- [3] B. Dalton, “The cglmp bell inequalities,” *The European Physical Journal Special Topics*, vol. 230, pp. 903–914, 2021.
- [4] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, “Bell inequalities for arbitrarily high-dimensional systems,” *Physical Review Letters*, vol. 88, Jan. 2002.