

Math

$$x \sim 0$$

$$\sin x \sim$$

$$1 - \cos x \sim$$

$$\tan x \sim$$

$$x - \ln(1+x) \sim$$

$$\arcsin x \sim$$

$$(1+x)^{\alpha} - 1 \sim$$

$$\arctan x \sim$$

$$a^x - 1 \sim x \ln a$$

$$\ln(1+x) \sim$$

$$e^x - 1 \sim$$

$$[\ln(x + \sqrt{1+x^2})]' = \frac{1}{\sqrt{1+x^2}}$$

$$x \sim 0 \text{ if } f(x) \sim 0$$

$$t \sim 1$$

$$\ln(t) \sim t - 1$$

$$x = \underbrace{e^y - e^{-y}}_2$$

泰勒级数公式 ($x_0=0$)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\arcsin x = x + \frac{x^3}{3!}$$

$$\tan x = x + \frac{x^3}{3}$$

$$\arctan x = x - \frac{x^3}{3}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!} x^2$$

e.g. 求常数 a , 使

解: 分左右极限, 取之而往

$$\lim_{x \rightarrow 0} \int_{-x}^x \frac{1}{x} \left(1 - \frac{|t|}{x}\right) \cos(a - t) dt \text{ 存在, 并求极限}$$

$$= \lim_{x \rightarrow 0} \int_{-x}^x \frac{1}{x} \left(1 - \frac{|t|}{x}\right) (\cos a \cos t + \sin a \sin t) dt$$

$$= \lim_{x \rightarrow 0} \left[\int_{-x}^x \left(1 - \frac{|t|}{x}\right) \frac{1}{x} \cos a \cos t dt + \int_{-x}^x \left(1 - \frac{|t|}{x}\right) \frac{1}{x} \sin a \sin t dt \right]$$

$$= \lim_{x \rightarrow 0} + 2 \cos a \int_0^x \left(1 - \frac{|t|}{x}\right) \frac{1}{x} \cos t dt$$

$$= \lim_{x \rightarrow 0} + 2 \cos a \int_0^x (x - |t|) \cos t dt / x^2$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \cos a}{x^2} \frac{x \int_0^x \cos t dt - \int_0^x |t| \cos t dt}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \cos a}{x^2} \frac{\int_0^x \cos t dt + x \cos x - x \cos x}{2x}$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \cos a}{x^2} \frac{\cos x}{2}$$

$$= \lim_{x \rightarrow 0^+} \cos a$$

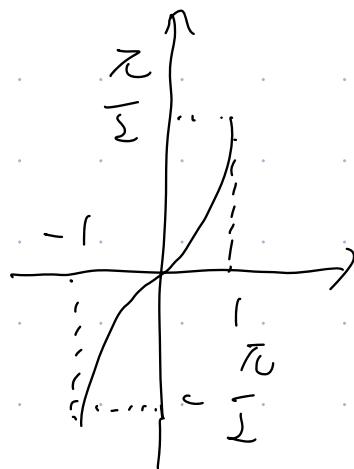
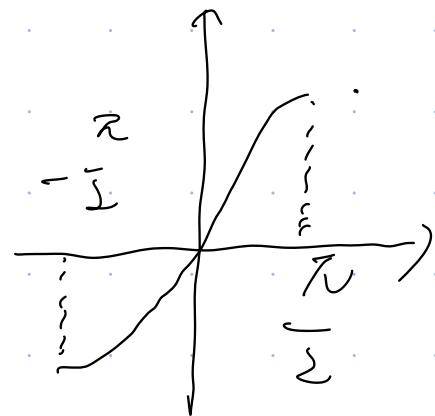
$$\text{原式} = \lim_{x \rightarrow 0^+} \frac{2 \cos a \int_0^x (x - |t|) \cos t dt}{x^2}$$

∴

$$\lim_{t \rightarrow 0} q = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$$

反函數

當 $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $y = \sin x$ 有反函數 $x = \arcsin y$, $x \in [-1, 1]$



y 为自变量, \arcsin 为因变量, 表示 x 的值

e.g. 在下列式子中有意义的是

- A. $\arcsin \sqrt{2}$ B. $\arcsin(-\frac{\pi}{3})$ C. $\sin(\arcsin 2)$ D. $\arcsin(\sin 2)$

解:

$$\because y = \arcsin x, x \in [-1, 1]$$

\therefore D

e.g. $y = \sin x, x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, $\not\exists y = f(x)$

$$y = \sin(\pi - x) = \sin x$$

$$\arcsin y = \pi - x$$

$$\therefore x = \pi - \arcsin y$$

$$y = \pi - \arcsin x$$

$$\text{e.g. } \arccos(\cos \frac{11\pi}{6}) = \underline{\hspace{2cm}}$$

$$\cos \frac{11\pi}{6} = \cos \left[2\pi - \frac{\pi}{6} \right] = \cos \frac{\pi}{6}$$

$$\therefore \text{为 } \frac{\pi}{6}$$

$$\text{e.g. } x \in (-\frac{\pi}{2}, \frac{3}{2}\pi) \quad \arcsin(\sin x) =$$

$$\because \sin(\pi - x) = \sin x$$

$$\therefore \arcsin(\sin x) = \pi - x$$

$$\text{e.g. } x \in (-\frac{3}{2}\pi, -\frac{\pi}{2}) \quad \arcsin(\sin x) =$$

$$\because \sin(-\pi - x) = \sin x$$

$$\therefore \arcsin(\sin x) = -\pi + x$$

多元微积分

1. 微分(单变量函数)

$$dy = f'(x)dx$$

全微分(多变量函数)

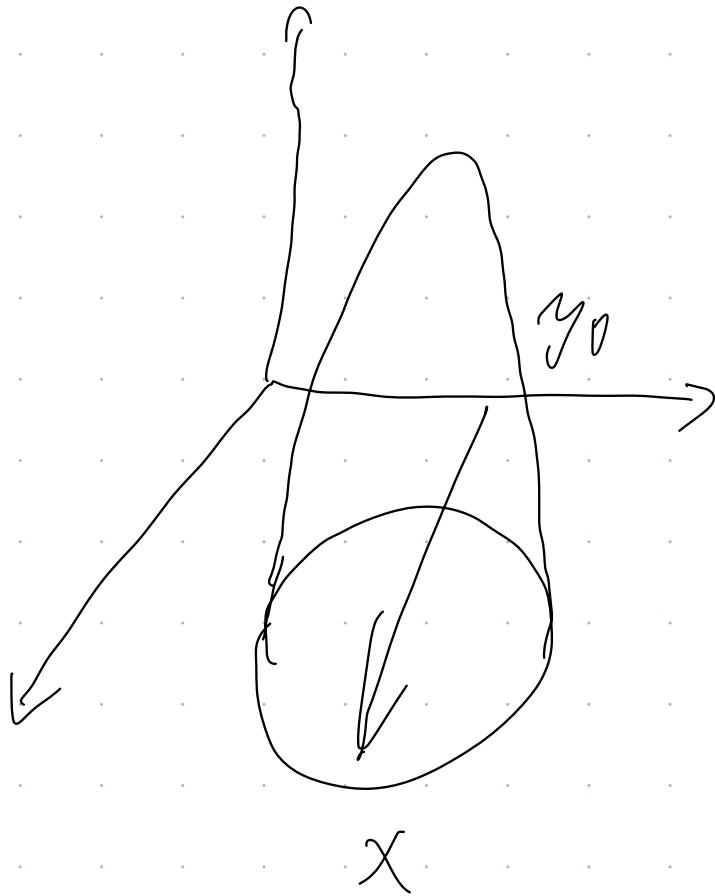
对于 $Z = f(x, y)$

$$dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$$

definition:

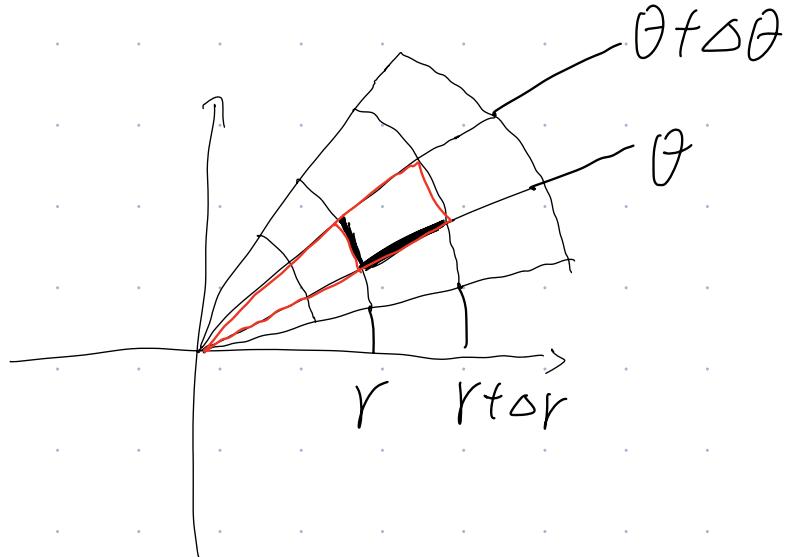
$$dZ = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} \underbrace{f(x, y) - f(x_0, y_0)}_{\sqrt{(x-x_0)^2 + (y-y_0)^2}} - \frac{\partial Z}{\partial x}(x-x_0) - \frac{\partial Z}{\partial y}(y-y_0)$$

2. 偏导数 (只对 X 或 Y 方向的导数)



对 X 求偏导
对 Y 不变

二重积分



$$\therefore S_{\vec{\theta}} = \frac{1}{2} r^2 \theta$$

$$\therefore \Delta G = S_{\vec{\theta+\Delta\theta}} - S_{\vec{\theta}}$$

$$= \frac{1}{2} [(r + \Delta r)^2 \Delta \theta - r^2 \Delta \theta]$$

$$= \frac{1}{2} \Delta \theta [2r \Delta r + \Delta r^2]$$

$$\approx \frac{1}{2} \Delta \theta \cdot 2r \Delta r$$

$$= r \Delta r \Delta \theta$$

向量代数与几何

L. f& Pk

七种未定式

$$\frac{0}{0} \quad \frac{\infty}{\infty}$$

$$0 \infty \quad \begin{array}{c} \swarrow \\ \frac{1}{0} \end{array} \quad \begin{array}{c} \uparrow \\ \frac{0}{1} \end{array}$$

$$\infty - \infty \quad \begin{array}{c} \swarrow \\ \text{通常} \end{array}$$

无分母提公因子

$$0^\infty, 0^0 \quad u^n = e^{v \ln u}$$

$$\int_1^\infty u^v, u \rightarrow 1 \text{ 时} \int \text{原式} = e^{v(u-1)}$$

$$1. x - 1 < [x] \leq x$$

2. 抓大头

3. 当 $x \rightarrow \infty$ 时, 可考虑倒带换

$$\begin{aligned}
 4. \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} - e & \quad \text{对数运算法则:} \\
 &= e^{\frac{1}{x}(\ln(1+x))} - e \\
 &= e[e^{\alpha(\ln(1+x)-1)} - 1]
 \end{aligned}$$

$$\begin{aligned}
 \log_a(MN) &= \log_a M + \log_a N \\
 \log_a\left(\frac{M}{N}\right) &= \log_a M - \log_a N
 \end{aligned}$$

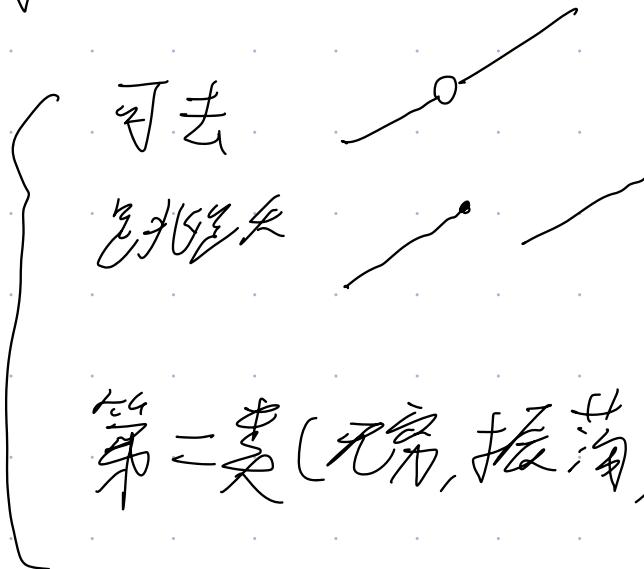
$$= e \left[\frac{1}{x} \ln(1+x) - 1 \right]$$

$$= e \left[\frac{\ln(1+x) - x}{x} \right]$$

$$= e \left[-\frac{\frac{1}{2}x^2}{x} \right]$$

$$= -\frac{e}{2}x$$

连续与间断点



第二类(无穷, 振荡)

找无定义点求左右 limit

$\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0} f(x)$ 为可去

$\lim_{x \rightarrow x_0^+} f(x) \neq \lim_{x \rightarrow x_0^-} f(x)$ 为跳跃

2. Differential

Definition:

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

保持一致

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

可导充分条件

$f(x)$ 可导, $f'(x)$ 奇

$$f'(-x) = -f'(x)$$

$$f'(-x) = \lim_{\Delta x \rightarrow 0} \frac{f(-x+\Delta x) - f(-x)}{\Delta x}$$

$$\begin{aligned} &= - \lim_{-\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{-\Delta x} \\ &= -f'(x) \end{aligned}$$

$|x - x_0|$ 在 x_0 处一定不等于

当 $f(x)$ $\left\{ \begin{array}{l} 1. 在 x_0 处 连 续 \\ 2. f(x_0) = 0 \end{array} \right.$, $\exists \forall f(x) |x - x_0|$ 在 $x = x_0$ 处 不 等 于

e.g. 求 $f(x) = (x^2 - 1)(x^3 + x^2 - 2x - 2)$ 在 $\sqrt{2}$ 处不连续点

$$= (x^2 - 1) | (x+1)(x^2 - 2) |$$

$$= (x+1)(x-1) | (x+1)(x+\sqrt{2})(x-\sqrt{2}) |$$

不连续点: $-1, -\sqrt{2}, \sqrt{2}$

$\therefore (x^2 - 1) |_{x=-1} = 0$ 且在 $x = -1$ 处连续

\therefore 有 2 个不连续点

property

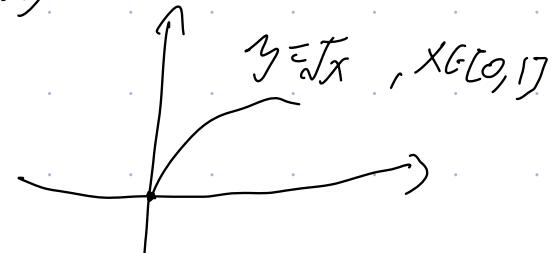
1. 改变奇偶性

2. $f(x+T) = f(x) \Rightarrow f'(x+T) = f'(x)$

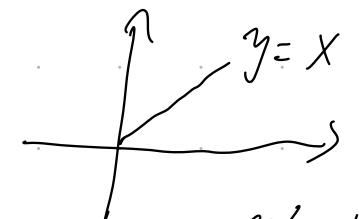
$f(x)$ 有界 $\Leftrightarrow f'(x)$ 有界

$f'(x)$ 在闭区间有界 $\rightarrow f(x)$ 有界

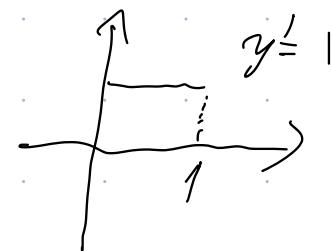
$y'_{x=0}$ ∞



$y = x$



y' 无界



可微性

$$\Delta y = \underbrace{A\Delta x}_\downarrow + O(\Delta x)$$

$$dy|_{x=x_0} = f'(x) \Delta x = f'(x) dx$$

反微分 求 $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

$$\frac{d^2x}{dy^2} = \frac{d\left(\frac{dx}{dy}\right)}{dy} = \frac{d\left(\frac{dx}{dy}\right)}{dx dy} \cdot dx$$

$$= \frac{d\left(\frac{dx}{dy}\right)}{dx} \cdot \frac{dx}{dy} = \left(\frac{1}{f'(x)}\right)' \cdot \frac{1}{f'(x)} = -\frac{f''(x)}{\left(f'(x)\right)^3}$$

e.g. 设 $f(x)$ 单调可导, $f'(x) \neq 0$, 且 $f(x)$ 在 $x=1$ 处 = P_f 可导,

$f(1) = -2$, $f'(1) = -\frac{\sqrt{2}}{2}$, $f''(1) = 2$, 记 $y = f(x)$ 的反函数

数为 $x = \varphi(y)$, 求 $\frac{dx}{dy} \Big|_{y=-2}$, $\frac{d^2x}{dy^2} \Big|_{y=-2}$

$$\frac{dx}{dy} \Big|_{y=-2} = \frac{1}{f'(1)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\frac{d^2x}{dy^2} = \frac{d(\frac{dx}{dy})}{dy} = \frac{d(\frac{dx}{dy})}{dx} \cdot \frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2} \cdot \frac{1}{f'(x)} \Big|_{x=1}$$

$$= -\frac{1}{2} x(-\sqrt{2}) \leq 4\sqrt{2}$$

e.g.

$$y = f(x) = 3x^2 + e^x, \quad x = \varphi(y), \quad \varphi'(3+e) = \underline{\hspace{2cm}}$$

if: $3+e = 3x^2 + e^x$ $f'(x)|_{x=1} = 6x + e^x|_{x=1} = 6+e$

$$\therefore x = 1 \quad f''(x)|_{x=1} = 6+e^x|_{x=1} = 6+e$$

$$\frac{dx}{dy^2} = \frac{d\frac{dx}{dy}}{dy} = \frac{f''(x)}{(f'(x))^2} \Big|_{x=1}$$

$$= -\frac{6+e}{(6+e)^2} = -\frac{1}{(6+e)^2}$$

$$\text{e.g. } y = y(x) \begin{cases} x = \arctant \\ 2y - ty^2 + e^t = 5 \end{cases} \quad 2 \sqrt{\frac{dy}{dx}}$$

$$\frac{dy}{dt} = 2y' - y^2 - 2tyy' + e^t = 0$$

$$y' = \frac{y^2 - e^t}{2 - 2ty}$$

$$\frac{dx}{dt} = \frac{1}{1 + t^2}$$

$$\frac{dy}{dx} = \frac{(y^2 - e^t)(1 + t^2)}{2 - 2ty}$$

高阶导数

1. 例题

2. Taylor

e.g.

$$f(x) = x^2 \ln(1-x), \quad \forall n \geq 3, f^{(n)}(0) =$$

$$\ln(1-x) = -x - \frac{x^2}{2} + \frac{-x^3}{3} + \dots$$

$$\therefore f(x) = -x^3 - \frac{x^4}{2} - \frac{x^5}{3} - \dots$$

$$= \underbrace{\frac{n!}{n-2}}$$

3. 級数展開

$$(uv)^{(n)} = C_n^0 u^{(n)} v^{(0)} + C_n^1 u^{(n-1)} v^{(1)} + \dots + C_n^n u^{(0)} v^{(n)}$$

e.g. $f(x) = xe^x$, $f(x)$

$$(xe^x)^{(n)} = e^x + C_n^1 e^x + 0$$

$$= xe^x + ne^x$$

中值定理

-元微积分学

$F(x)$ 在某区间内处处可导, 且 $F'(x) = f(x)$, 则 $F(x)$ 为 $f(x)$ 的原函数

e.g.

例 8.2 函数 $f(x) = \begin{cases} \frac{1}{\sqrt{1+x^2}}, & x \leq 0, \\ (x+1)\cos x, & x > 0 \end{cases}$ 的一个原函数为 () .

(A) $F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x), & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0 \end{cases}$

(B) $F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x) + 1, & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0 \end{cases}$

(C) $F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x), & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0 \end{cases}$

(D) $F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x) + 1, & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0 \end{cases}$

原函数 : 连续

A. $\lim_{x \rightarrow 0^-} f(x) = 1$

B. $f(0) = 1$

$\lim_{x \rightarrow 0^+} f(x) = 1$

$f(0) = 0$

✗

$\cos x - (x+1)\sin x - \sin x$

$= \cos x - x \sin x$

$$C. f(0) = 0$$

$$D. f(0) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

✓

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = 1 \quad \checkmark$$

$$\sin x + (x+1)\cos x - \sin x$$

$$\int f(x) dx = F(x) + C, \text{ 不连续的 function cluster}$$

连续分支函数，一个分支

原函数存在定理

$\left\{ \begin{array}{l} f(x) \text{ 连续} \rightarrow \text{一定有原函数} \Rightarrow \int f(x) dx = \int_a^x f(x) dx + C \\ f(x) \text{ 不连续} \left\{ \begin{array}{l} \text{第一类间断点, 无穷} \Rightarrow \text{无 } F(x) \\ \text{瑕点} \Rightarrow \text{可能有} \end{array} \right. \end{array} \right.$

定积分存在定理

必要条件：区间有限，函数有界

充分条件：
 $[a, b]$ $\left\{ \begin{array}{l} f(x) \text{ 连续} \\ f(x) \text{ 单调} \end{array} \right.$

| 有限个间断点

积分中值定理 $\int_a^b f(x) dx = f(\xi)(b-a)$

变限积分

表格法

$$\int_0^\pi e^{-t} \sin t dt$$

$$\begin{array}{ccccc} \sin t & & \cos t & & -\sin t \\ & f & & & -) + \\ e^{-t} & & -e^{-t} & & e^{-t} \end{array}$$

$$I = \left[-\sin t e^{-t} - e^{-t} \cos t \right]_0^\pi - \int_0^\pi \sin t e^{-t} dt$$

$$I = -\frac{1}{2} e^{-t} (\cos t + \sin t) \Big|_0^\pi$$

$$= -\frac{1}{2} \left(e^{-\pi} (\cos \pi - \sin \pi) - 1 \right)$$

$$= \frac{1}{2} (e^{-\pi} + 1)$$

$$I = \int_0^{\pi^2} \sqrt{x} \cos \sqrt{x} dx$$

$$\sqrt{x} = t, x = t^2, dx = 2t dt$$

$$I = \int_0^{\pi} t \cos t \cdot 2t dt$$

$$= 2 \int_0^{\pi} t^2 \cos t dt$$

$$\begin{array}{ccccccc} t^2 & & 2t & & 2 & & 0 \\ \swarrow + & & \searrow - & & \nearrow + & & \\ \cos t & & \sin t & -\cos t & -\sin t & & \end{array}$$

$$2 \left(t^2 \sin t + 2t \cos t - 2 \sin t \right) \Big|_0^{\pi}$$

$$= 2(-2\pi) = -4\pi$$

完稿

$$\int_a^b f(x) dx = \int f(a+b-t) dt$$

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\int_0^{\frac{\pi}{2}} x f(\sin x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin 2x) dx = \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{4}{5} \cdot \frac{2}{3} \cdot 1, & n \in \text{odd} \end{cases}$$

$$\left\{ \frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{\pi}{2}, n \text{ even} \right.$$

Mis C

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \sum_{k=1}^n k(k+1)$$

$$S_{\text{半透}} = \pi a b$$

$$S_{\text{扇}} = \frac{1}{2} r^2 \theta$$

$$\int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(-1) = 2$$

等比級數

$$\sum_{n=0}^{\infty} aq^n \quad (a \neq 0)$$

$$\left\{ \begin{array}{l} \text{發散, } |q| \geq 1 \\ \text{收斂, } |q| < 1 \end{array} \right.$$

等比數列

$$a_n = a_1 q^{n-1}$$

$$S_n = \frac{a_1(1-q^n)}{1-q}$$

當收斂 ($|q| < 1$ 時)

$$S = \frac{a_1}{1-q}$$

數列的求和

series 累加求和

重要级数

1. p级数

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

发, $p < 1$
收, $p > 1$

2. 等比级数

\propto

$$(等) |q| \geq 1$$

$$\sum_{n=0}^{\infty} aq^n \quad (a \neq 0) \quad \left. \begin{array}{l} \text{收斂} \\ |q| < 1 \end{array} \right\}$$

当收敛 ($|q| < 1$)

$$S = \frac{a_1}{1-q}$$

3. $\bar{f} \times P$

$$\sum \quad \begin{array}{c} l \\ \beta \end{array} \quad \left. \begin{array}{l} \text{收斂}, \alpha > 1 \\ \text{发散}, \alpha < 1 \end{array} \right\}$$

$$\tilde{n} = n^\alpha \ln^\beta n \quad \left| \begin{array}{l} \alpha > 1, \text{发散} \\ \alpha = 1, \quad \left\{ \begin{array}{l} \beta > 1, \text{发散} \\ \beta < 1, \text{收敛} \end{array} \right. \end{array} \right.$$

二重积分

1. D 关于 x-axis 对称

2. $f(x, y)$ 关于 x 对称

当 $f(x, -y) = -f(x, y)$ 时 $\iint_D f(x, y) d\sigma = 0$

$$f(x, -y) = f(x, y) \quad \text{R} \quad \int \int f(x, y) d$$

y 同理

$$\text{e.g. } D = \{(x, y) \mid x^2 + y^2 \leq 1, (x-1)^2 + y^2 \geq 1\}, \text{ 求}$$

$$\int \int (5y^3 + x^2 + y^2 - 2x + y + 1) dx dy$$

积分区域关于 x 轴对称

轮换对称

1. D 关于 $y=x$ 对称

$$2, \iint_D f(x, y) d\sigma = \iint_D f(y, x) d\sigma$$

$$\therefore \iint_D f(x, y) d\sigma = \frac{1}{2} \left(\iint_D f(x, y) d\sigma + \iint_D f(y, x) d\sigma \right)$$

e.g. $Z = f(x^y, y^x)$, $\text{try } \frac{\partial Z}{\partial x} =$

解:

$$Z = f(u, v), u = x^y, v = y^x$$

$$\frac{\partial Z}{\partial x} = f_1' y x^{y-1} + f_2' y^x \ln y$$

e.g. $f(x, y)$ 可微, $f(1, 2) = 2, f_x'(1, 2) = 3, f_y'(1, 2) = 3, Z(x) = f(x, f(x, 2x))$

$$f_z Z'(1)$$

$$Z'(x) = f_1'(x, f(x, 2x)) + f_2'(x, f(x, 2x)) \underset{x}{f_x'(x, 2x)}$$

$$= f_1'(x, f(x, 2x)) + f_2'(x, f(x, 2x)) [f_1'(x, 2x) + f_2'(x, 2x) \cdot 2]$$

$$= f_1'(1, f(1, 2)) + f_2'(1, f(1, 2)) [f_1'(1, 2) + f_2'(1, 2) \cdot 2]$$

$$= f_1'(1,2) + f_2'(1,2) \{ f_1'(1,2) + 2f_2'(1,2) \}$$

$$= 3 + 3 \times (3 + 2 \times 3) = 30$$

$$f_1'(1,2) = f_x'(1,2)$$

e.g. $f(xy, x+y) = x^2y^2 + xy$

1. 求 $z = f(x, y)$

2. 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$f(xy, x+y) = (x+y)^2 - xy \quad \frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = 2y$$

$$\therefore f(x, y) = y^2 - x$$

e.g.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = A\Delta x + B\Delta y + O(\rho)$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - A\Delta x - B\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = D$$

Δz