Math

$x \sim 0$

$\sin x \sim$

$\tan x \sim$

$\arcsin x \sim$

$\arctan x \sim$

$\ln(1+x) \sim$

$e^x - 1 \sim$

$1 - \cos x \sim$

$x - \ln(1+x) \sim$

$(1+x)^\alpha - 1 \sim$

$a^x - 1 \sim x\ln a$

$$\left[\ln\left(x+\sqrt{1+x^2}\right)\right]' = \frac{1}{\sqrt{1+x^2}}$$

$x \sim 0$时 $f(x) \sim 0$

$t \sim 1$

$\ln(t) \sim t - 1$

$$x = \frac{e^y - e^{-y}}{2}$$

常见麦克劳林公式 $(x_0=0)$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

$$\arcsin x = x+\frac{x^3}{3!}$$

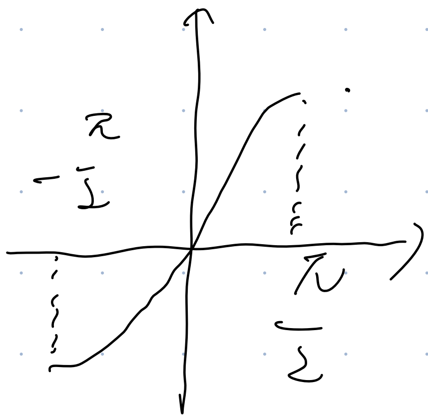
$$\tan x = x+\frac{x^3}{3}$$

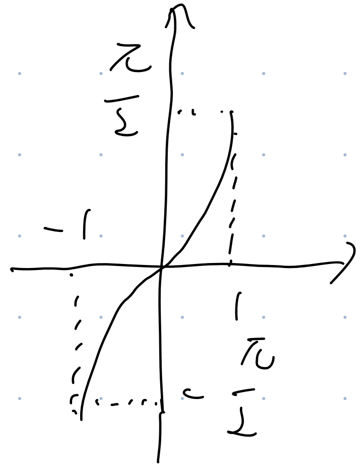
$$\arctan x = x-\frac{x^3}{3}$$

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha+1)}{2!}x^2$$

反函数

当 $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$，$y=\sin x$ 有反函数 $x=\arcsin y$，$x\in[-1,1]$





y 为自变量，arc 表弧，输出 x 弧度

e.g. 1、下列式子中有意义的是

A、$\arcsin\sqrt{2}$ B、$\arcsin(-\frac{\pi}{3})$ C、$\sin(\arcsin 2)$ D、$\arcsin(\sin 2)$

答：

$\because y=\arcsin x$，$x\in[-1,1]$

$\therefore$ D

e.g. $y=\sin x$, $x\in[\frac{\pi}{2},\frac{3\pi}{2}]$，求 $y=f^{-1}(x)$

$y=\sin(\pi-x)=\sin x$

$\arcsin y=\pi-x$

$\therefore x=\pi-\arcsin y$

$y=\pi-\arcsin x$

e.g. $\arccos(\cos\frac{11\pi}{6}) =$ ______

$\cos\frac{11\pi}{6} = \cos[2\pi - \frac{\pi}{6}] = \cos\frac{\pi}{6}$

$\therefore$ 为 $\frac{\pi}{6}$

e.g. $x \in (\frac{\pi}{2}, \frac{3}{2}\pi)$ $\arcsin(\sin x) =$

$\because \sin(\pi - x) = \sin x$

$\therefore \arcsin(\sin x) = \pi - x$

e.g. $x \in (-\frac{3}{2}\pi, -\frac{\pi}{2})$ $\arcsin(\sin x) =$

$\because \sin(-\pi - x) = \sin x$

$\therefore \arcsin(\sin x) = -\pi + x$

多元微分

1、微分(单变量函数)

$dy = f'(x)dx$

全微分(多变量函数)

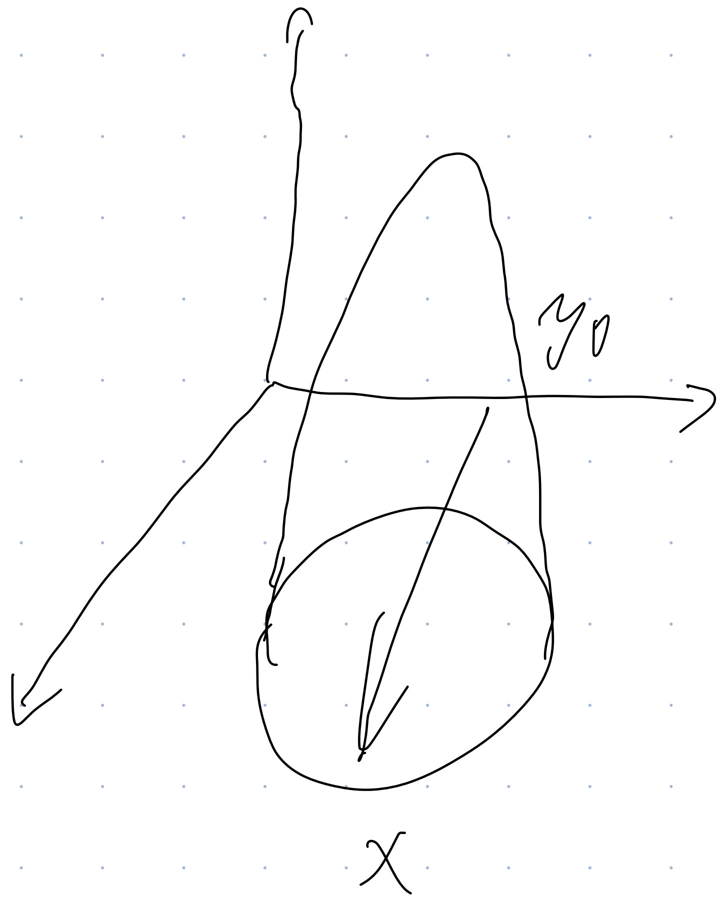
对于 $z = f(x,y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

definition:

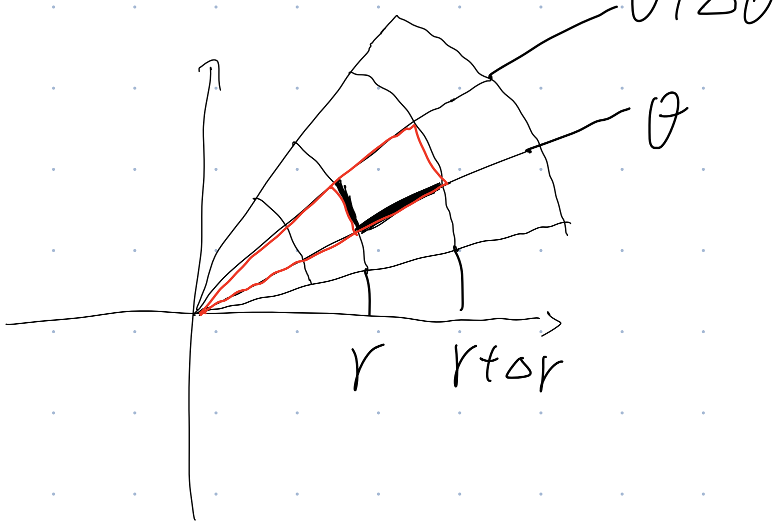
$$dz = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial z}{\partial x}(x-x_0) - \frac{\partial z}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

2. 偏导数 (只对 x 或 y 方向的导数)



对x求偏导
则y不变

二重积分



$$\because S_{扇} = \frac{1}{2} r^2 \theta$$

$$\therefore \Delta \sigma = S_{大扇} - S_{小扇}$$

$$= \frac{1}{2}\left[(r+\Delta r)^2 \Delta\theta - r^2 \Delta\theta\right]$$

$$= \frac{1}{2}\Delta\theta\left[2r\Delta r + \Delta r^2\right]$$

$$\approx \frac{1}{2}\Delta\theta \cdot 2r\Delta r$$

$$= r\Delta r\Delta\theta$$

向量代数与几何

1、极限

七种未定式

$\frac{0}{0}$　$\frac{\infty}{\infty}$

0∞

$\frac{\infty}{\frac{1}{0}}$

$\frac{0}{\frac{1}{\infty}}$

$\infty-\infty$

通分

无分母提公因式

$0^{\infty}, 0^{0}$　$u^{v}=e^{v\ln u}$

1^∞ u^v, $u \to 1$ 则原式 $= e^{v(u-1)}$

1、 $x-1 < [x] \leq x$

2、抓大头

3、当 $x \to \infty$ 时，可考虑倒带换

4、 $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} - e$

$= e^{\frac{1}{x}\ln(1+x)} - e$

$= e[e^{\frac{1}{x}\ln(1+x)-1} - 1]$

对数运算法则：

$\log_a(MN) = \log_a M + \log_a N$

$\log_a(\frac{M}{N}) = \log_a M - \log_a N$

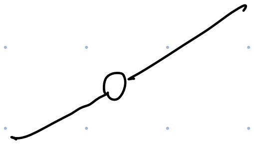
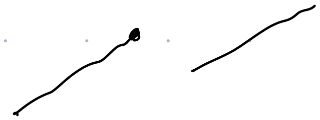
$$= e\left[\frac{1}{x}\ln(1+x) - 1\right]$$

$$= e\left[\frac{\ln(1+x) - x}{x}\right]$$

$$= e\left[\frac{-\frac{1}{2}x^2}{x}\right]$$

$$= -\frac{e}{2}x$$

连续 间断点

- 可去 
- 跳跃 
- 第二类(无穷, 振荡)

找无定义点，求左右limit

$$\lim_{x\to x_0+} f(x) = \lim_{x\to x_0-} f(x) \neq \lim_{x\to x_0} f(x) \text{ 为可去}$$

$$\lim_{x\to x_0+} f(x) \neq \lim_{x\to x_0-} f(x) \text{ 为跳跃}$$

2. Differential

Definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ 保持一致

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

证：f(x)可导偶，f'(x)为奇

$$f'(-x) = -f'(x)$$

$$f'(-x) = \lim_{\Delta x \to 0} \frac{f(-x+\Delta x)-f(-x)}{\Delta x}$$

$$= -\lim_{-\Delta x \to 0} \frac{f(x+(-\Delta x))-f(x)}{-\Delta x}$$

$$= -f'(x)$$

可导充分条件 $\left\{ \right.$ 1. 左导 = 右导

$|x-x_0|$ 在 x_0 处一定不可导

若 $f(x)\begin{cases}1.\text{在}x_0\text{处连续}\\2.\ f(x_0)=0\end{cases}$，则 $f(x)|x-x_0|$ 在 $x=x_0$ 处可导

e.g. 求 $f(x)=(x^2-1)|x^3+x^2-2x-2|$ 不可导点个数

$$=(x^2-1)|(x+1)(x^2-2)|$$

$$=(x+1)(x-1)|(x+1)(x+\sqrt{2})(x-\sqrt{2})|$$

不可导点：$-1,-\sqrt{2},\sqrt{2}$

$\because (x^2-1)\big|_{x=-1}=0$ 且在 $x=-1$ 处连续

$\therefore$ 有2个不可导点

property

1、求导改变奇偶性

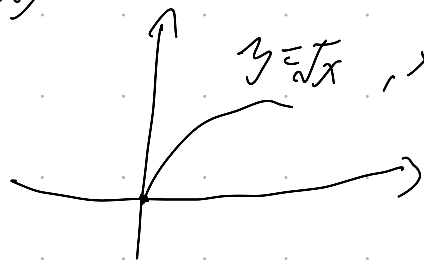
2、$f(x+T)=f(x) \Rightarrow f'(x+T)=f'(x)$

$f(x)$有界 $\cancel{\Longleftrightarrow}$ $f'(x)$有界

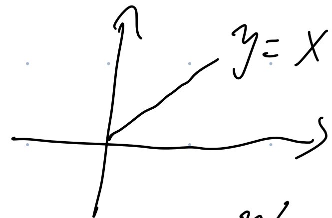
$f'(x)$在闭区间有界 $\rightarrow$ $f(x)$有界

y'在0处 ∞

$y=\sqrt{x}$, $x\in[0,1]$

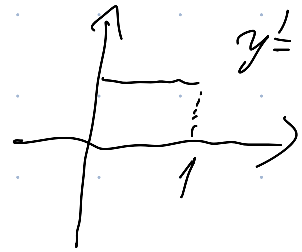


$y=x$



y'无界

$y'=1$



可微分

$$\Delta y = \underbrace{A\Delta x}_{\downarrow} + o(\Delta x)$$

$$dy\Big|_{x=x_0} = f'(x)\,\Delta x = f'(x)\,dx$$

反函数求导

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

$$\frac{d^2x}{dy^2} = \frac{d\left(\frac{dx}{dy}\right)}{dy} = \frac{d\left(\frac{dx}{dy}\right)}{dx\,dy} \cdot dx$$

$$= \frac{d\left(\frac{dx}{dy}\right)}{dx} \cdot \frac{dx}{dy} = \left(\frac{1}{f'(x)}\right)' \cdot \frac{1}{f'(x)} = -\frac{f''(x)}{[f'(x)]^3}$$

e.g. 设$f(x)$单调可导，$f'(x)\neq 0$，且$f(x)$在$x=1$处二阶可导，$f(1)=-2$，$f'(1)=-\frac{\sqrt{2}}{2}$，$f''(1)=2$，记$y=f(x)$的反函数为$x=\varphi(y)$，求$\left.\frac{dx}{dy}\right|_{y=-2}$，$\left.\frac{d^2x}{dy^2}\right|_{y=-2}$

$$\left.\frac{dx}{dy}\right|_{y=-2}=\frac{1}{f'(1)}=\frac{1}{-\frac{\sqrt{2}}{2}}=-\sqrt{2}$$

$$\frac{d^2x}{dy^2}=\frac{d\left(\frac{dx}{dy}\right)}{dy}=\frac{d\left(\frac{dx}{dy}\right)}{dx}\cdot\frac{dx}{dy}=-\frac{f''(x)}{[f'(x)]^2}\cdot\left.\frac{1}{f'(x)}\right|_{x=1}$$

$$=-\frac{2}{\frac{1}{2}}\times(-\sqrt{2})=4\sqrt{2}$$

e.g.

$y=f(x)=3x^2+e^x$, $x=\varphi(y)$, 则 $\varphi''(3+e)=$ ______

解: $3+e=3x^2+e^x$ $\quad f'(x)\big|_{x=1}=6x+e^x\big|_{x=1}=6+e$

$\therefore x=1$ $\quad f''(x)\big|_{x=1}=6+e^x\big|_{x=1}=6+e$

$$\frac{d^2x}{dy^2}=\frac{d\frac{dx}{dy}}{dy}=-\frac{f''(x)}{[f'(x)]^2}\cdot\frac{1}{f'(x)}\Bigg|_{x=1}$$

$$=-\frac{6+e}{(6+e)^3}=-\frac{1}{(6+e)^2}$$

e.g. $y=y(x)$ $\begin{cases} x=\arctan t \\ 2y-ty^2+e^t=5 \end{cases}$ 求 $\frac{dy}{dx}$

$\frac{dy}{dt}=$ $\quad 2y'-y^2-2tyy'+e^t=0$

$y'=\frac{y^2-e^t}{2-2ty}$

$\frac{dx}{dt}=\frac{1}{1+t^2}$

$\frac{dy}{dx}=\frac{(y^2-e^t)(1+t^2)}{2-2ty}$

高阶导数

1、归纳

2、Taylor

e.g.

$f(x)=x^2\ln(1-x)$，当 $n\geq 3$，$f^{(n)}(0)=$

$\because \ln(1-x)=-x-\frac{x^2}{2}+\frac{-x^3}{3}+\cdots$

$\therefore f(x)=-x^3-\frac{x^4}{2}-\frac{x^5}{3}-\cdots$

$-\frac{n!}{n-2}$

3. 莱布尼茲

$$(uv)^{(n)} = C_n^0 u^{(n)} v^{(0)} + C_n^1 u^{(n-1)} v' + \dots C_n^n u^{(0)} v^{(n)}$$

e.g. 求 $f(x) = xe^x$，求 $f^{(n)}(x)$

$$(xe^x)^{(n)} = e^x x + C_n^1 e^x + 0$$

$$= xe^x + ne^x$$

中值定理

一元函数积分学

$F(x)$在某区间内处处可导，且 $F'(x) = f(x)$，则$F(x)$为$f(x)$的原函数

e.g.

例 8.2 函数 $f(x)=\begin{cases}\dfrac{1}{\sqrt{1+x^2}}, & x\leqslant 0,\\ (x+1)\cos x, & x>0\end{cases}$ 的一个原函数为（　　）.

(A) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}-x\right), & x\leqslant 0,\\ (x+1)\cos x-\sin x, & x>0\end{cases}$

(B) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}-x\right)+1, & x\leqslant 0,\\ (x+1)\cos x-\sin x, & x>0\end{cases}$

(C) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}+x\right), & x\leqslant 0,\\ (x+1)\sin x+\cos x, & x>0\end{cases}$

(D) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}+x\right)+1, & x\leqslant 0,\\ (x+1)\sin x+\cos x, & x>0\end{cases}$

∃原函数 ∴ 连续

A. $\lim\limits_{x\to 0^+} f(x) = 1$

$f(0) = 0$ ×

B. $f(0) = 1$ $\lim\limits_{x\to 0^-} f(x) = 1$

$\lim\limits_{x\to 0^+} f(x) = 1$ ×

$\cos x - (x+1)\sin x - \sin x$

$= \cos x - x\sin x$

C. $f(0)=0$ D. $f(0)=1$

$\lim\limits_{x \to 0^+} f(x)=1$

α

$\lim\limits_{x \to 0^+} f(x)=1$

$\lim\limits_{x \to 0^-} f(x)=1$ ✓

$\sin x + (x+1)\cos x - \sin x$

$\int f(x)dx = F(x)+C$，不定积分求 function cluster

定积分求面积，一个数

原函数∃定理

- $f(x)$连续 → 一定有原函数 ⇒ $\int f(x)dx = \int_a^x f(x)dx + C$
- $f(x)$不连续
  - 第一类间断点,无穷 ⇒ 无$F(x)$
  - 振荡 ⇒ 可能有

定积分∃定理

必要条件:区间有限,函数有界

充分条件:

$[a,b]$
- $f(x)$连续
- $f(x)$单调

| 有限个间断点

积分中值定理 $\int_a^b f(x)dx = f(\xi)(b-a)$

变限积分

表格法

$$\int_0^{\pi} e^{-t} \sin t \, dt$$

$\sin t$	$\cos t$	$-\sin t$
$+$	$-$	$+$
e^{-t}	$-e^{-t}$	e^{-t}

$$I = \left[-\sin t \, e^{-t} - e^{-t}\cos t\right]_0^{\pi} - \int_0^{\pi} \sin t \, e^{-t} \, dt$$

$$I = -\frac{1}{2} e^{-t}(\cos t + \sin t)\Big|_0^{\pi}$$

$$= -\frac{1}{2}\left(e^{-\pi}(\cos\pi - \sin\pi) - 1\right)$$

$$= \frac{1}{2}(e^{-\pi} + 1)$$

$$I = \int_0^{\pi^2} \sqrt{x} \cos\sqrt{x}\, dx$$

$$\sqrt{x} = t, \quad x = t^2, \quad dx = 2t\,dt$$

$$I = \int_0^{\pi} t \cos t \cdot 2t\, dt$$

$$= 2\int_0^{\pi} t^2 \cos t\, dt$$

$$\begin{array}{ccccccc} t^2 & & 2t & & 2 & & 0 \\ & + & & - & & + & \\ \cos t & & \sin t & & -\cos t & & -\sin t \end{array}$$

$$2\left(t^2 \sin t + 2t\cos t - 2\sin t \right)\Big|_0^{\pi}$$

$= 2(-2\pi) = -4\pi$

定积分

$$\int_a^b f(x)\,dx = \int f(a+b-t)\,dt$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin 2x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \begin{cases} \frac{4}{5}\cdot\frac{2}{3}\cdot 1, & n \in \text{odd} \end{cases}$$

$$\left\{ \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad n \in \text{even} \right.$$

Misc

$$1^2+2^2+3^2+\cdots n^2 = \frac{n(n+1)(2n+1)}{2}$$

$$S_{椭圆} = \pi ab$$

$$S_{扇} = \frac{1}{2} r^2 \theta$$

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 1-(-1) = 2$$

等比级数 | 等比数列

$\sum_{n=0}^{\infty} aq^n \ (a\neq 0)$

- 发散，$|q|\geq 1$
- 收敛，$|q|<1$

当收敛（$|q|<1$ 时）

$$S=\frac{a_1}{1-q}$$

$a_n = a_1 q^{n-1}$

$$S_n=\frac{a_1(1-q^n)}{1-q}$$

数列侧重于通项

series 重于求和

重要级数

1. P级数

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}发, & p\le 1\\ 收, & p>1\end{cases}$$

2. 等比级数

$$\sum^{\infty} \quad \begin{cases}发 & |q|\ge 1\end{cases}$$

$$\sum_{n=0} aq^n \quad (a\neq 0) \begin{cases} \quad \\ 收敛, |q|<1 \end{cases}$$

当收敛(|q|<1时)

$$S=\frac{a_1}{1-q}$$

3. 广义p

$$\sum^{\infty} \frac{1}{\quad^{\beta}} \begin{cases} 收敛, \alpha>1 \\ 发, \alpha<1 \end{cases}$$

$\overline{\quad}$

$n=1 \quad n^{\alpha}\ln^{\beta} n$

发, $\alpha < 1$

$\alpha = 1$, $\begin{cases} \beta > 1, \text{收} \\ \beta < 1, \text{发} \end{cases}$

$$y'' - 2y' + y = xe^x$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r_1 = r_2 = 1$$

設 $y^* = (Ax+B)e^x x^2$