Algebra

Determinant

$$\begin{cases} 3x-2y=12 \\ 2x+y=1 \end{cases} \Rightarrow \begin{cases} a_{11}x_1 - a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3-(-4) = 7$$

$D=0$时 表示 1.有无穷多解 2.无解

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 12-(-2) = 14$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = 3-24 = -21$$

$$x_1 = \frac{D_1}{D} = \frac{14}{7} = 2$$

$$x_2 = \frac{D_2}{D} = \frac{-21}{7} = -3$$

逆序对

1. $t(32514) = 2+1+2 = 5$ 奇排列

   $t(23514) =$ 偶 排列 对换两数改变奇偶

2. $t(n, n-1, \cdots 1) = \frac{n(n-1)}{2}$

3. 判断 $a_{11}a_{23}a_{32}a_{44}$ 及 $a_{14}a_{23}a_{41}a_{32}$ 的符号

$t(1324)=1$ 负

$t(4321)=\frac{4\times3}{2}=6$ 正

D 性质

1. 行列地位等同, $|D|=|D^T|$
2. 交换两行/列, 反号
3. 可提取一行/列公因子到行列式外

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{23} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{vmatrix}$$

4. 可拆

5. 倍加

余子式，~~代数~~代数余子式

1) 1、$n!$ 项来自不同行列的代数和（每项符号由逆序对决定）

2、按某行/列的元素×该元素对应的 A_{ij} 的和

e.g. 设3阶D某行为a, $(a\neq 0)$, $D=1$, 求D中所有代数余子式和

求 $\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

解:

$$\begin{vmatrix} a & a & a \\ & & \\ & & \end{vmatrix} = 1$$

$$A_{11}+A_{12}+A_{13} = \begin{vmatrix} 1 & 1 & 1 \\ & & \\ & & \end{vmatrix} = \frac{1}{a}$$

$$A_{21}+A_{22}+A_{23} \begin{vmatrix} a & a & a \\ 1 & 1 & 1 \\ & & \end{vmatrix} = 0$$

$$A_{31}+A_{32}+A_{33} \begin{vmatrix} a & a & a \\ & & \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore$ 所有代数余子式之和为 $\frac{1}{a}$

拉普拉斯行列式

A与B均为方阵

$$\begin{vmatrix} A & O \\ O & B \end{vmatrix} = \begin{vmatrix} A & C \\ O & B \end{vmatrix} = \begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A||B|$$

$$\begin{vmatrix} O & A_{m\times m} \\ B_{n\times n} & O \end{vmatrix} = \begin{vmatrix} C & A_{m\times m} \\ B_{n\times n} & O \end{vmatrix} = \begin{vmatrix} O & A_{m\times m} \\ B_{n\times n} & C \end{vmatrix} = (-1)^{mn}|A||B|$$

e.g. D_4为4阶行列式 $= \begin{vmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{5} & 0 & 0 & 0 \end{vmatrix}$, 求$D_4$中所有代数余子式的和

解:

$$|A| = \begin{vmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{5} & 0 & 0 & 0 \end{vmatrix} = (-1)^{1\times 3}\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5} = -\frac{1}{120}$$

$$A^{-1} = \begin{pmatrix} O & B^{-1} \\ A^{-1} & O \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

$$\therefore \quad = -\frac{1}{120}\begin{pmatrix} 0 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} = -\frac{1}{120}(2+3+4+5) = -\frac{7}{60}$$

e.g. $A=(a_{ij})$ 为3阶矩阵, A_{ij} 为 a_{ij} 的代数余子式, 若 A 每行元素和均为2, 且 $|A|=3$, 则 $A_{11}+A_{21}+A_{31}$

法1:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^*A=|A|E$$

$$A^*A=2E$$

法2:

$$|A|=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}=\begin{pmatrix} 2 & a_{12} & a_{13} \\ 2 & a_{22} & a_{23} \\ 2 & a_{32} & a_{33} \end{pmatrix}$$

$$=2(A_{11}+A_{21}+A_{31})$$

$$\frac{3}{2}=A_{11}+A_{21}+A_{31}$$

e.g. $D_n = \begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{vmatrix}$

$= [x+(n-1)a] \begin{vmatrix} 1 & a & a & \cdots & a \\ 1 & x & a & \cdots & a \\ 1 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a & a & \cdots & x \end{vmatrix}$

$= [x+(n-1)a](x-a)^{n-1}$

行列和相等，提公因子

e.g. $D_n = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{vmatrix}$

$D_n = aA_{11} + bA_{n1}$

$= a \times (-1)^{1+1} \cdot a^{n-1} + b \times (-1)^{n+1} \times b^{n-1}$

$= a\times(-1)\times a + b\times(-1)\times b$

$= a^n + (-1)^{n+1} b$

// 型，按点展开

e.g. $\begin{vmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix} \xlongequal{r_2 \Leftrightarrow r_3} -\begin{vmatrix} 0 & a & b & 0 \\ 0 & c & d & 0 \\ a & 0 & 0 & b \\ c & 0 & 0 & d \end{vmatrix}$

$\xlongequal{c_2 \Leftrightarrow c_4} \begin{vmatrix} 0 & 0 & b & a \\ 0 & 0 & d & c \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{vmatrix} = (-1)^{2\times2}(bc-ad)(ad-bc) = -(ad-bc)^2$

e.g. $a_i \neq 0, i=1,2,3,4$, $\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix}$

$D = \begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ -a_1 & a_2 & 0 & 0 \\ -a_1 & 0 & a_3 & 0 \\ -a_1 & 0 & 0 & a_4 \end{vmatrix}$

$= \begin{vmatrix} 1+a_1+\frac{a_1}{a_4}+\frac{a_1}{a_3}+\frac{a_1}{a_2} & 1 & 1 & 1 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{vmatrix}$

$$\left| \begin{matrix} 0 & 0 & 0 & a_4 \end{matrix} \right.$$

$$\left(1+a_1+\frac{a_1}{a_4}+\frac{a_1}{a_3}+\frac{a_1}{a_2}\right)(a_2 \times a_3 \times a_4)$$

e.g.
$$\begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ x+1 & -1 & 1 & -1 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 0 & 0 & x \\ 1 & 0 & x & 0 \\ 1 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= x \cdot 1 \times (-1)^{4+1} \cdot (-1)^{\frac{3\times 2}{2}} x^3$$

$$= x^4$$

行列式性质2 Todo 9:29-40:30

矩阵 Matrix

运算

1. $AB \neq BA$ 无交换律

$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} =$

$\begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} =$

2. $AB = 0 \not\Rightarrow A = 0$ 或 $B = 0$ 无0因子

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} =$

3. $AB = AC$ 且 $A \neq 0 \not\Rightarrow B = C$ 无消去

$AB \neq BA$

$(A \pm B)^2 \neq (A^2 \pm 2AB + B^2)$

$(A \pm B)^3 \neq (A \pm B)(A^2 \mp AB + B^2)$

天然可交换，则所成立

1. E
2. $A^*A = AA^* = |A|E$
3. $AA^{-1} = A^{-1}A = E$

1、对称阵 $A^T=A$

反对称阵 $A^T=-A$

2、伴随矩阵 方

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix}$$

$A_{ij}=(-1)^{i+j}M_{ij}$ (代数余子式 $=(-1)^{i+j}$余子式)

M_{ij}为划去i行j列的)

$$A^*=\begin{pmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & & & & \vdots \\ A_{13} & & & & \vdots \\ \vdots & & & & \\ A_{1n} & \cdots & \cdots & \cdots & A_{nn} \end{pmatrix}$$

$$AA^*=A^*A=|A|E$$

1、求二阶A^*

e.g. $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

则 $A^*=\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

主对换,副变号

2、求 $A_{11}+A_{22}+\cdots+A_{nn}=tr(A^*)=\sum\lambda_{A^*}$

3. 逆矩阵 方

$AB=E$

$A^{-1}=B$

e.g. $ABC=E$

$AB=C^{-1}$

$\therefore CAB=E$

可逆性判定:

$|A|\neq 0 \Leftrightarrow r(A)=n \Leftrightarrow \exists A^{-1} \Leftrightarrow A$的$\lambda\neq 0$

满秩　可逆　A的特征值全$\neq 0$

$\because$ 行列式=所有λ的乘积

$\Leftrightarrow Ax=0$只有0解 $\Leftrightarrow Ax=\beta$有唯一解 $\Leftrightarrow A$的行/列向量组线性无关

$|A|=0 \Leftrightarrow r(A)<n \Leftrightarrow$ 不可逆

$\Leftrightarrow Ax=0$有非0解 $\Leftrightarrow$

$Ax=0$有∞多解

A^{-1} A^{*} A^{T}

1、带系数k

$(kA)^{-1} = \frac{1}{k}A^{-1}$

$(kA)^{*} = k^{n-1}A^{*}$

$(kA)^{T} = kA^{T}$

2.

$(A+B)^{-1} = $ ×

$(A+B)^{*} = $ ×

$(A+B)^{T} = A^{T}+B^{T}$

3.

$(A^{*})^{*}$

$(A^{-1})^{-1} = A$

$(A^{T})^{T} = A$

4、反交换

$(AB)^{-1}$

$(AB)^{*}$

$(AB)^{T}$

5、交换顺序

$(A^{-1})^{*} = (A^{*})^{-1}$

$(A^{*})^{T} = (A^{T})^{*}$

$(A^{T})^{-1} = (A^{-1})^{T}$

6.行列式

$|A^{-1}| = \frac{1}{|A|}$

$|A^{*}| = |A|^{n-1}$

$|A^{T}| = |A|$

AB为方阵

$|AB| = |A||B| = |B||A| = |BA|$

13.

$$\left|\left(\frac{1}{3}A^*\right)^{-1}-(2A^{-1})^*\right|$$

$$=\left|3(A^*)^{-1}-4(A^*)^{-1}\right|$$

$$=\left|-(A^*)^{-1}\right|$$

$$=-\left|(A^*)^{-1}\right|$$

$$=-\frac{1}{|A^*|}$$

$$=-\frac{1}{|A|^2}$$

$$=-\frac{1}{100}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{vmatrix}=(-1)^{\frac{3\times 2}{2}}\times 10=-10$$

14. $ABA^*=2BA^*+E$

$|A|=3$

$$AB|A|=2B|A|+A$$

$$3AB=6B+A$$

$$3AB-6B=A$$

$$3B(A-2E)=A$$

$$|3B|\,|A-2E|=|A|$$

$$27|B|\,|A-2E|=3$$

$$|B|=\frac{1}{9}$$

$$A-2E=\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$$|A-2E|=-1\times(0-1)=1$$

15. A为n阶方阵, $|A|=1$, α为n维列向量, 若 $\begin{vmatrix} A & \alpha \\ \alpha^T & y \end{vmatrix}=0$

求 $\begin{vmatrix} A & \alpha \\ \alpha^T & x \end{vmatrix}=$

解:

$$\begin{vmatrix} A & \alpha+0 \\ \alpha^T & y+x-x \end{vmatrix} = \begin{vmatrix} A & \alpha \\ \alpha^T & x \end{vmatrix} + \begin{vmatrix} A & 0 \\ \alpha^T & y-x \end{vmatrix}$$

$$0 = \begin{vmatrix} A & \alpha \\ \alpha^T & x \end{vmatrix} + |A|(y-x)$$

$$\therefore \begin{vmatrix} A & \alpha \\ \alpha^T & x \end{vmatrix} = x-y$$

16. 已知A是3阶矩阵 $\alpha_1\ \alpha_2\ \alpha_3$ 是3维线性无关列向量, 若 $A\alpha_1=\alpha_1-\alpha_2$, $A\alpha_2=\alpha_2-\alpha_3$, $A\alpha_3=\alpha_3+\alpha_1$, 则 $|A|=$ ______

解: $r(\alpha_1,\alpha_2,\alpha_3)=3$

$$(A\alpha_1, A\alpha_2, A\alpha_3) = (\alpha_1-\alpha_2, \alpha_2-\alpha_3, \alpha_3+\alpha_1)$$

$$\Updownarrow$$

$$A(\alpha_1,\alpha_2,\alpha_3) = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

$$= (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

法1: 令$(\alpha_1, \alpha_2, \alpha_3)$为P矩阵

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow A \overset{相似}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

法2: $$|A||\alpha_1, \alpha_2, \alpha_3| = |\alpha_1, \alpha_2, \alpha_3| \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$|A| = 2$$

若$\beta_1 \beta_2 \beta_3$能被$\alpha_1 \alpha_2 \alpha_3$表示，写为矩阵相乘形式

$$\beta_1 = x_{11}\alpha_1 + x_{12}\alpha_2 + x_{13}\alpha_3$$

$$\beta_2 = x_{21}\alpha_1 + x_{22}\alpha_2 + x_{23}\alpha_3$$

$$\beta_3 = x_{31}\alpha_1 + x_{32}\alpha_2 + x_{33}\alpha_3$$

$$(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}$$

17. $\alpha_1,\alpha_2,\alpha_3$为3维列向量，$A=(\alpha_1,\alpha_2,\alpha_3)$，$|A|=1$，若

$B=(\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3)$，则$|B|=$ ______

法1:

$$B=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}$$

$$|B|=\begin{vmatrix}1&1&1\\1&2&3\\1&4&9\end{vmatrix}=(2-1)(3-1)(3-2)=2$$

法2:

$$|B|=|\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+4\alpha_3,\ \alpha_1+3\alpha_2+9\alpha_3|$$

$$=\quad\sim$$

18. 设A、B均为3阶矩阵，$|A|=3$，$|B|=2$，$|A^{-1}+B|=2$，则 $|A+B^{-1}|=$ ____

解: $A+B^{-1}=EA+B^{-1}E$

$=B^{-1}BA+B^{-1}A^{-1}A$

$=B^{-1}(B+A^{-1})A$

$|A+B^{-1}|=|B^{-1}||B+A^{-1}||A|$

$=\frac{1}{2}\times 2\times 3$

$=3$

$A+B^{-1} \longleftrightarrow A^{-1}+B$

$A+B \longleftrightarrow A^{-1}+B^{-1}$

发生切换，加单位阵E

19. A.B均为正交矩阵，$|A|=-|B|$，$|A+B|=$ ____

$|A+B|=|EA+BE|=|BB^TA+BA^TA|$

$=|B(B^T+A^T)A|$

$=|B(A+B)^TA|$

$=|B||(A+B)^T||A|$

$= -|A+B|$

$\therefore |A+B| = 0$

强化20 设$\boldsymbol{A}$是n阶正交矩阵，且行列式$|\dot{\boldsymbol{A}}|<0$，则行列式$|\boldsymbol{A}+\boldsymbol{E}|=$______.

$20. |A+E| = |A+AA^T| = |A(E^T+A^T)|$

$= |A|\,|(A+E)^T|$

$= |A|\,|A+E|$

$2|A+E| = 0$

$\therefore |A+E| = 0$

【改】设$\boldsymbol{A}=(a_{ij})$是3阶非零矩阵，$|\boldsymbol{A}|$为$\boldsymbol{A}$的行列式，A_{ij}为a_{ij}的代数余子式.

若$A_{ij}=a_{ij}$ $(i,j=1,2,3)$，则$|\boldsymbol{A}|=$______.

$A^* = A^T$

$|A|^2 = |A|$

$|A| = 0$或1

A为非0矩阵

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$>0 \quad \geq 0 \quad \geq 0$

$\therefore |A| > 0$

$a_{ij} = A_{ij} \Longleftrightarrow A^T = A^*$

当A为非0阵

$A^*A = A^{-1}A = A^TA$

强化21 设$A=(a_{ij})$是3阶非零矩阵，$|A|$为A的行列式，A_{ij}为a_{ij}的代数余子式. 若$a_{ij}+A_{ij}=0\ (i,j=1,2,3)$，则$|A|=$______.

21. $A^* = -A^T$

$|A|^2 = -|A|$

$|A| = -1$或0

$|A| = a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{13}$

$a^2 \quad a^2 \quad a^2$

$= -a_{11} - a_{22} - a_{33}$

$\quad >0 \quad \geq 0 \quad \geq 0$

< 0

$\therefore |A| = -1$

$r(A)=1$ 的性质

已知2个非0向量 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\beta = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\alpha^T\beta = (a_1b_1 + a_2b_2 + a_3b_3) = \beta^T\alpha$ ⟶ 一个数（内积）

$\alpha\beta^T = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$

⟶ 矩阵（外积）

$\beta\alpha^T = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{pmatrix} = (\alpha\beta^T)^T$

1、外积的 $tr(\alpha\beta^T)$ 为主对角元之和 = 内积

2、$r(\alpha\beta^T) \leq r(\alpha) = 1$

$r(A)=1$的充要条件

∃两个非0向量 α,β，使 $A=\alpha\beta^T$

$$\begin{pmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \\ 3 & 12 & 15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(1,4,5)$$

$r(A)=1$的特征值 λ

当 $r(A)=1$，A的 λ 为 $\operatorname{tr}(A), 0, 0 \cdots 0$

$r(A)=1$的相似对角化判定

2c.

1. $r(A)=1$ 性质

2. 正交矩阵

train

对角阵 Λ

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

1、$|\Lambda| = \lambda_1 \lambda_2 \cdots \lambda_n$

2、$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & & \\ & \frac{1}{b} & \\ & & \frac{1}{c} \end{pmatrix}$ $(abc \neq 0)$

$\begin{pmatrix} & & a \\ & b & \\ c & & \end{pmatrix}^{-1} = \begin{pmatrix} & & \frac{1}{c} \\ & \frac{1}{b} & \\ \frac{1}{a} & & \end{pmatrix}$ $(abc \neq 0)$

3、分块对角阵

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ O & B^{-1} \end{pmatrix}$$

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix}^{-1} = \begin{pmatrix} O & B^{-1} \\ A^{-1} & O \end{pmatrix}$$

正交矩阵

1、definition: $AA^T = A^TA = E$

2、$A^{-1} = A^T$

3、$|A| = 1$ 或 -1 $\qquad |A||A^T| = 1$

$\qquad |A|^2 = 1$

4、当 $A = (\alpha_1, \alpha_2 \cdots \alpha_n)$

$\sim$

5、若A为正交矩阵，则A^T，A^{-1}，A^*为正交阵

证：

$$(A^T)(A^T)^T = E$$

$$AA^T = E$$

$$A^{-1}(A^{-1})^T = E$$

$$A^TA = E$$

矩阵初等变换 $A \xrightarrow{r_i \times 2} B$

1、$r_i \leftrightarrow r_j$ $\quad c_i \longleftrightarrow c_j$ 对换

2、$k r_i$ $\quad k c_i$ $\quad (k \neq 0)$ 倍乘

3、$r_i + k r_j$ $\quad c_i + k c_j$ 倍加

初等矩阵

Definition：对E做一次初等变换

$$\begin{matrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}$$ 不是初等阵

e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} =$$

A为3阶矩阵，将A的第2列加到第1列得到矩阵B，再交换交换第2行与第3行得到E，记$P_1=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$，$P_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$

则$A=$

解：

$$P_2AP_1=E$$

$$P_2^{-1}P_2AP_1P_1^{-1}=P_2^{-1}EP_1^{-1}$$

$$A=P_2^{-1}P_1^{-1}$$

$\because P_2^{-1}$为行交换

$\therefore P_2^{-1}=P_2$

$\therefore A=P_2P_1^{-1}$

线性方程组

1. 无解

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 - x_2 - x_3 = 3 \\ 2x_1 - 2x_2 - x_3 = 3 \end{cases}$$

$$\left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ 2 & -2 & -1 & 3 \end{array}\right) \xrightarrow[r_3-2r_1]{r_2+r_1} \left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[r_3-3r_2]{r_2\div 2} \left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right) \qquad r(A) < r(A,b)$$

2. 有唯一解

$$\left(\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 3 & 4 & 2 & 9 \\ -1 & -5 & 4 & 10 \\ 2 & 7 & 1 & 1 \end{array}\right) \xrightarrow[\substack{r_3+r_1 \\ r_4-2r_1}]{r_2-3r_1} \left(\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 0 & -5 & -1 & 3 \\ 0 & -2 & 5 & 12 \\ 0 & 1 & -1 & -3 \end{array}\right) \xrightarrow{r_3+2r_4} \left(\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 0 & -5 & -1 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

$$\xrightarrow{5r_4+r_2} \left(\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 0 & -5 & -1 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -6 & -12 \end{array}\right) \xrightarrow{r_4\times\frac{1}{2}+r_3} \left(\begin{array}{ccc:c} 1 & 3 & 1 & 2 \\ 0 & -5 & -1 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$r(A,b) = r(A) = 3 = n$$

3. 有无穷多解

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ 2 & -2 & -1 & 5 \end{pmatrix} \xrightarrow[r_3-2r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{pmatrix} \xrightarrow[r_2\div(-1)]{r_3-\frac{3}{2}r_2} \left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$r(A,b)=r(A)=2<3$$

有效方程个数　　未知数 n

Rank(秩)

1. 非0子式最高阶数
2. 有效方程数
3. 行阶梯形矩阵非0行行数
4. 独立向量个数

e.g. 求 $r(A)$, $r(B)$, $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -5\\ 4 & 7 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -5\\ 4 & 7 & 1\end{pmatrix} \Rightarrow \begin{vmatrix}2 & 3\\ 3 & -5\end{vmatrix} = -10-9=-19$

∵有2阶非0子式

$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -5\\ 4 & 7 & 1\end{pmatrix} \Rightarrow \begin{vmatrix}1 & 2 & 3\\ 2 & 3 & -5\\ 4 & 7 & 1\end{vmatrix} = 3-40+42-(36-35+4)$

$=0$

∵无3阶非0子式

∴ $r(A)=2$

秩的公式

转置不改变 $r(A)$

1、$r(A)=r(A^T)=r(AA^T)=r(A'A)$

2、$r(A)$ 越乘越小，越拼越大，分开加最大

$$r(AB)\le \left\{\begin{matrix} r(A)\\ \text{or}\\ r(B)\end{matrix}\right. \le \left\{\begin{matrix} r(A,B)\\ \text{or}\\ r\begin{pmatrix}A\\B\end{pmatrix}\end{matrix}\right. \le r(A)+r(B)$$

$$r(A)+r(B) \ge r(A+B)$$

3、$A_{m\times n}\times B_{n\times l}=0$，$r(A)+r(B)\le n$

4、初等变换不改变 $r(A)$

5、$0\le r(A_{m\times n})\le \min(m,n)$

向量组的线性表示

$\alpha_1=\begin{pmatrix}1\\1\\2\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\2\\1\\3\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\-1\\4\\0\end{pmatrix}$, $\beta=\begin{pmatrix}1\\0\\3\\1\end{pmatrix}$, 证$\beta$能由$\alpha_1,\alpha_2,\alpha_3$线性表示

并写出表达式

解: 可被线性表示意思是方程 $x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=\beta$ 有解

方程组有解 $\Longleftrightarrow$ 系数矩阵的秩=增广矩阵 即 $r(A)=r(A,b)$

$$\left(\begin{array}{ccc|c}1&1&1&1\\1&2&-1&0\\2&1&4&3\\2&3&0&1\end{array}\right)\xrightarrow[\substack{r_3-2r_1\\r_4-r_2}]{r_2-r_1}\left(\begin{array}{ccc|c}1&1&1&1\\0&1&-2&-1\\0&-3&6&3\\0&2&-4&-2\end{array}\right)\xrightarrow[r_3\Leftrightarrow r_4]{r_3+3r_2}\left(\begin{array}{ccc|c}1&1&1&1\\0&1&-2&-1\\0&2&-4&-2\\0&0&0&0\end{array}\right)$$

$$\xrightarrow{r_3-2r_2}\left(\begin{array}{ccc|c}1&1&1&1\\0&1&-2&-1\\0&0&0&0\\0&0&0&0\end{array}\right)\xrightarrow{r_1-r_2}\left(\begin{array}{ccc|c}1&0&3&2\\0&1&-2&-1\\0&0&0&0\\0&0&0&0\end{array}\right)\therefore\ \begin{array}{l}x_1+3x_3=2\\x_2-2x_3=-1\end{array}$$

$$\begin{cases}x_1=2-3x_3\\x_2=-1+2x_3\\x_3=x_3\end{cases}$$, 令x_3为任意常数k $\quad X=\begin{pmatrix}-3k+2\\2k-1\\k\end{pmatrix}$

$$\beta=(-3k+2)\alpha_1+(2k-1)\alpha_2+k\alpha_3$$

e.g. $\alpha_1=\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}3\\1\\1\\3\end{pmatrix}$, $\beta_1=\begin{pmatrix}2\\0\\1\\1\end{pmatrix}$, $\beta_2=\begin{pmatrix}1\\1\\0\\2\end{pmatrix}$, $\beta_3=\begin{pmatrix}3\\-1\\2\\0\end{pmatrix}$, 证 $\alpha_1\alpha_2$ 与 $\beta_1\beta_2\beta_3$ 等价

两向量组等价,即可互相线性表示 $\Longleftrightarrow r(I)=r(II)=r(I,II)$

若 I能由II线性表示, $r(I)=r(II)$

则 $r(I)=r(II,I)=r(II)$

$$\left(\begin{array}{cc:ccc} 1 & 3 & 2 & 1 & 3 \\ -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2 \\ -1 & 3 & 1 & 2 & 0 \end{array}\right) \xrightarrow[\substack{r_3-r_1 \\ r_4+r_1}]{r_2+r_1} \left(\begin{array}{cc:ccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 4 & 2 & 2 & 2 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 6 & 3 & 3 & 3 \end{array}\right) \xrightarrow[\substack{r_3 \leftrightarrow r_4 \\ r_2 = r_2 \div 2}]{r_3+\frac{1}{2}r_2} \left(\begin{array}{cc:ccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 6 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_3-3r_2} \left(\begin{array}{cc:ccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore r(\alpha_1,\alpha_2)=r(\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3)=r(\beta_1,\beta_2,\beta_3)$

∴ 等价

向量组相关无关性

给定向量组: $\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_m$,

若∃不全为0的数 $k_1\ k_2\ k_3 \dots k_m$,

使 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3+\dots k_m\alpha_m=0$

则为线性相关

也就是齐次线性方程组仅有0解

则 $r(A)=r(A,b)<n$

给定向量组: $\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_m$,

当且仅当 $k_1=k_2=k_3=\dots k_m=0$

使 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3+\dots k_m\alpha_m=0$

则为线性无关

也就是齐次线性方程组仅有0解

即 $r(A)=r(A,b)=n$

e.g. $\alpha_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\2\\5\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\4\\7\end{pmatrix}$，试讨论 α_1,α_2 及 $\alpha_1\alpha_2\alpha_3$ 线性相关性

$$\begin{pmatrix}1&0&2\\1&2&4\\1&5&7\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-r_1}\begin{pmatrix}1&0&2\\0&2&2\\0&5&5\end{pmatrix}\xrightarrow[r_2\div 2]{r_3-\frac{5}{2}r_2}\begin{pmatrix}1&0&2\\0&1&1\\0&0&0\end{pmatrix}$$

$r(\alpha_1\alpha_2\alpha_3)=2<3$（向量个数）

$\therefore$ 日多余 vector

$\therefore$ 线性相关

e.g. 已知向量组 $\alpha_1\alpha_2\alpha_3$ 线性无关，$b_1=\alpha_1+\alpha_2$, $b_2=\alpha_2+\alpha_3$, $b_3=\alpha_3+\alpha_1$，证 $b_1b_2b_3$ 线性无关

解：设 $\exists\, x_1x_2x_3$ 使 $x_1b_1+x_2b_2+x_3b_3=0$

$$x_1(\alpha_1+\alpha_2)+x_2(\alpha_2+\alpha_3)+x_3(\alpha_3+\alpha_1)=0$$

$(x_1+x_3)\alpha_1 + (x_1+x_2)\alpha_2 + (x_2+x_3)\alpha_3 = 0$

$$\begin{cases} x_1 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

法1:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$D_2 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$\therefore x_1 = x_2 = x_3 = 0$

法2:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

$\therefore r(A) = r(A,b) = n$

又$\because$为齐次

$\therefore$ 仅有0解

两个vector内积 = 外积的迹(主对角元

$|A| \neq 0 \Leftrightarrow |A|$ 满秩 $\Leftrightarrow A$ 可逆

$$A^{-1} = \frac{A^*}{|A|}$$

反交换

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

若 $P^{-1}AP=\Lambda$

则A可对角化

$P=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$，$A=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$，若 $P^{-1}AP=\Lambda=\begin{pmatrix}1&0\\0&3\end{pmatrix}$，求 A^n

解：$PP^{-1}APP^{-1}=P\Lambda P^{-1}$

$$A=P\Lambda P^{-1}$$

$$A^n=P\Lambda \underbrace{P^{-1}P}\Lambda \underbrace{P^{-1}\cdots P}\Lambda P^{-1}$$

$$=P\Lambda^n P^{-1}$$

$$P^{-1}=\frac{P^*}{|P|}=\frac{\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}}{-1-1}=-\frac{1}{2}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1^n&0\\0&3^n\end{pmatrix}\cdot\left(-\frac{1}{2}\right)\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$$

$$=-\frac{1}{2}\begin{pmatrix}1&3^n\\1&-3^n\end{pmatrix}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$$

$$=-\frac{1}{2}\begin{pmatrix}-1-3^n&-1+3^n\\-1+3^n&-1-3^n\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1+3^n&1-3^n\\1-3^n&1+3^n\end{pmatrix}$$

特征值与特征向量

$Ax=\lambda x \ (x\neq 0)$ A为系数矩阵 x为列向量 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}$

$Ax=\lambda E x$

$Ax-\lambda E x=0$

$(A-\lambda E)x=0$ 非齐次方程

又 $\because x\neq 0$ $\therefore$ 有无穷多解

$\therefore A-\lambda E$ 不满秩

$\therefore |A-\lambda E|=0 \Rightarrow$ 得出特征值 $\lambda_1 \lambda_2 \lambda_3$

$$|A-\lambda E|=0 \begin{cases} \lambda_1 \longrightarrow (A-\lambda_1 E)x=0 \longrightarrow \xi_1 \\ \lambda_2 \longrightarrow (A-\lambda_2 E)x=0 \longrightarrow \xi_2 \\ \lambda_3 \longrightarrow (A-\lambda_3 E)x=0 \longrightarrow \xi_3 \end{cases}$$

$$\Lambda=\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

--对应

$$P=(\xi_1, \xi_2, \xi_3)$$

e.g. 求 Matrix $A=\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ 的特征值和特征向量

1. 求特征值：$|A-\lambda E|=\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix}=(3-\lambda)^2-1=0$ $\begin{matrix}\lambda_1=2 \\ \lambda_2=4\end{matrix}$

2. 求特征向量：

当 $\lambda_1=2$

$(A-2E)x=0$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1-x_2=0 \\ x_2=k_1 \end{matrix} \Rightarrow x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} k_1 \\ k_1 \end{pmatrix}=k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

当 $\lambda_2=4$

$(A-4E)x=0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1+x_2=0 \\ x_2=k_2 \end{matrix} \Rightarrow X=\begin{pmatrix} -k_2 \\ k_2 \end{pmatrix} = k_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\therefore$ A的特征值为2,4

$k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$是A对应$\lambda=2$时的特征向量

$k_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ~

3、相似对角化

$$P=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Lambda=\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\therefore P^{-1}AP=\Lambda$$

3阶A的λ为2,3,λ,若$|2A|=-48$,求λ

$$|2A|=2^3|A|=8\times2\times3\lambda=-48$$

$$\therefore \lambda=-1$$

矩阵	A	$aA+bE$	A^n	A^{-1}	A^*	A^T
特征值	λ	$\lambda a+b$	λ^n	$\frac{1}{\lambda}$	$\frac{\lvert A\rvert}{\lambda}$	λ
特征向量	ξ	ξ	ξ	ξ	ξ	—

e.g. 3阶 Matrix A，特征值为 $1,2,2$，E 为阶单位阵，求 $|4A^{-1}-E|$

A 的特征值：$1,2,2$

A^{-1} 的特征值：$1,\frac{1}{2},\frac{1}{2}$

$4A^{-1}$：$4,2,2$

$4A^{-1}-1$：$3,1,1$

$\therefore |4A^{-1}-E| = 3\times1\times1=3$

e.g. 设3阶 Matrix 特征值为 $1,-1,2$，求 $A^*+3A-2E$ 特征值

A：$1,-1,2$

A^*：$|A|, -|A|, \frac{1}{2}|A|$

$3A$：$3,-3,6$

$-2E$：$-2,-2,-2$

$\because |A| = 1\times(-1)\times2=-2$

$A^*+3A-2E$：$-2+3-2,\ 2-3-2,\ -1+6-2$

$-1 \qquad -3 \qquad 3$

e.g. 3阶矩阵特征值为0,-1,2，求$A^*+3A-2E$的特征值

$\varphi(A)=A^*+3A-2E \quad \varphi(\lambda)=\frac{|A|}{\lambda}+3\lambda-2$

$\varphi(0)=\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1}-2=-4$

$\varphi(-1)=\frac{0}{-1}-3-2=-5$

$\varphi(2)=6-2=4$

相似 definition

AB为n阶矩阵，若∃可逆阵P

使$P^{-1}AP=B$，则A与B相似

当B为Λ时，则相似对角化

properties: 意相似

$$\begin{cases} tr(A)=tr(B) \\ |A|=|B| \\ r(A)=r(B) \\ \lambda_A=\lambda_B \\ |A-\lambda E|=|B-\lambda E| \end{cases}$$

e.g. $A=\begin{pmatrix} -2 & -2 & 1 \\ 2 & x & -2 \\ 0 & 0 & -2 \end{pmatrix}$, $B=\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix}$ 相似，求x, y

解:

$$\begin{cases} tr(A) = tr(B) \\ |A| = |B| \end{cases} \quad \begin{cases} x-4 = 1+y \\ -2\times(x-2)\times(-2) = -2y \end{cases} \quad \begin{cases} x-4=1+y \\ 4x-8=-2y \end{cases}$$

$$\begin{cases} x = 3 \\ y = -2 \end{cases}$$

对角化 Definition

A B 为 n 阶矩阵，若 $\exists$ 可逆阵 $P = (\xi_1, \xi_2, \xi_3 \cdots, \xi_n)$

使 $P^{-1}AP = \Lambda$，则 A 可相似对角化

A 可相似对角化 $\Longleftrightarrow$ A 有 n 个线性无关特征向量

e.g. 求齐次线性方程组 $\begin{cases} x_1+x_2-x_3-x_4=0 \\ 2x_1-5x_2+3x_3+2x_4=0 \\ 7x_1-7x_2+3x_3+x_4=0 \end{cases}$

的基础解系与通解

$$A=\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -5 & 3 & 2 \\ 7 & -7 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -7 & 5 & 4 \\ 0 & -14 & 10 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -7 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -\frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{3}{7} \\ 0 & 1 & -\frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = \frac{2}{7}x_3 + \frac{3}{7}x_4$$

$$x_2 = \frac{5}{7}x_3 + \frac{4}{7}x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$X=\begin{pmatrix} \frac{2}{7} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix} k_1 + \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} k_2$$

$k_1, k_2 \in R$

向量 vector

1、内积（数量积）（点乘）

$$(x,y) = x \cdot y = |x||y|\cos\theta$$

$$(x,y) = (x_1, x_2, x_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1y_1 + x_2y_2 + x_3y_3$$

2、外积（向量积）（叉乘）

正交（垂直）

$(x,y) = x \cdot y = 0$ 点乘为0

施密特正交化：

对于 $\alpha_1 \alpha_2 \alpha_3$、令

内积

$$\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_2 - \dfrac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 \end{cases}$$

$$\beta_3 = \alpha_3 - \frac{[\text{illegible}]}{(\beta_1, \beta_1)}\beta_2$$

.....

$AB=0$ ($A_{m\times n}$, $B_{n\times s}$)，则

1、$r(A)+r(B)\leq n$ n为A的column

2、B的每一列β_i都是线性方程组$AX=0$的解

3、B的列与A的行两两正交

e.g. 二次型 $f=x^2-3z^2-4xy+yz$，写出矩阵表达

$f = x^2 - 3z^2 - 4xy + yz$

$$\begin{array}{c|ccc} & x & y & z \\ x & 1 & -2 & 0 \\ y & -2 & 0 & \frac{1}{2} \\ z & 0 & \frac{1}{2} & -3 \end{array}$$

$$\therefore f(x,y,z) = (x,y,z)\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

e.g. 求二次型 $f(x_1,x_2,x_3) = -2x_1x_2 + 2x_1x_3 + 2x_2x_3$ 的秩

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 0 & -1 & 1 \\ x_2 & -1 & 0 & 1 \\ x_3 & 1 & 1 & 0 \end{array}$$

$$A = \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 - 1 - 1 + (0+0+0) = -2 \neq 0$$

$\therefore r(A) = 3$

标准二次型

$f(x_1,x_2,x_3 \cdots x_n) = k_1x_1^2 + k_2x_2^2 + k_3x^3 + \cdots + k_nx_n^2$ (不含交叉项)

规范二次型

$f(x_1,x_2,\cdots x_n) = d_1x_1^2 + d_2x_2^2 + \cdots + d_nx_n^2$ ($d = \{0, 1, -1\}$)