

Math

$$x \sim 0$$

$$\sin x \sim$$

$$1 - \cos x \sim$$

$$\tan x \sim$$

$$x - \ln(1+x) \sim$$

$$\arcsin x \sim$$

$$(1+x)^{\alpha} - 1 \sim$$

$$\arctan x \sim$$

$$a^x - 1 \sim x \ln a$$

$$\ln(1+x) \sim$$

$$e^x - 1 \sim$$

$$[\ln(x + \sqrt{1+x^2})]' = \frac{1}{\sqrt{1+x^2}}$$

$$x \sim 0 \text{ if } f(x) \sim 0$$

$$t \sim 1$$

$$\ln(t) \sim t - 1$$

$$x = \underbrace{e^y - e^{-y}}_2$$

泰勒级数公式 ( $x_0=0$ )

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\arcsin x = x + \frac{x^3}{3!}$$

$$\tan x = x + \frac{x^3}{3}$$

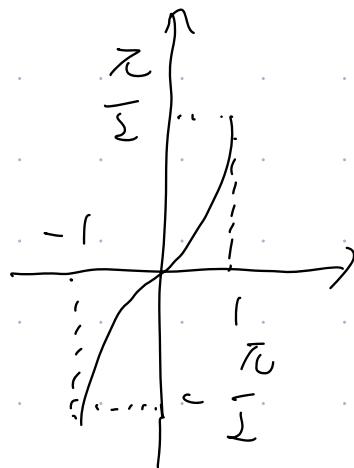
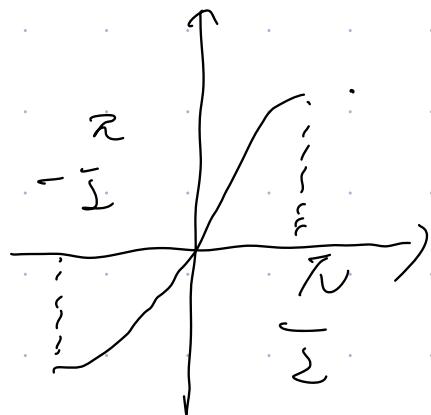
$$\arctan x = x - \frac{x^3}{3}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!} x^2$$



# 反函数

当  $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ,  $y = \sin x$  有反函数  $x = \arcsin y$ ,  $x \in [-1, 1]$



$y$  为自变量,  $\arcsin$  为, 输出  $x$  的值

e.g. 在下列式子中有意义的是

- A.  $\arcsin \sqrt{2}$     B.  $\arcsin(-\frac{\pi}{3})$     C.  $\sin(\arcsin 2)$     D.  $\arcsin(\sin 2)$

解:

$$\because y = \arcsin x, x \in [-1, 1]$$

$\therefore$  D

e.g.  $y = \sin x, x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ ,  $\not\exists y = f(x)$

$$y = \sin(\pi - x) = \sin x$$

$$\arcsin y = \pi - x$$

$$\therefore x = \pi - \arcsin y$$

$$y = \pi - \arcsin x$$

$$\text{e.g. } \arccos(\cos \frac{11\pi}{6}) = \underline{\hspace{2cm}}$$

$$\cos \frac{11\pi}{6} = \cos \left[ 2\pi - \frac{\pi}{6} \right] = \cos \frac{\pi}{6}$$

$$\therefore \text{为 } \frac{\pi}{6}$$

$$\text{e.g. } x \in (-\frac{\pi}{2}, \frac{3}{2}\pi) \quad \arcsin(\sin x) =$$

$$\because \sin(\pi - x) = \sin x$$

$$\therefore \arcsin(\sin x) = \pi - x$$

$$\text{e.g. } x \in (-\frac{3}{2}\pi, -\frac{\pi}{2}) \quad \arcsin(\sin x) =$$

$$\because \sin(-\pi - x) = \sin x$$

$$\therefore \arcsin(\sin x) = -\pi + x$$

# 多元微积分

## 1. 微分(单变量函数)

$$dy = f'(x)dx$$

## 全微分(多变量函数)

对于  $Z = f(x, y)$

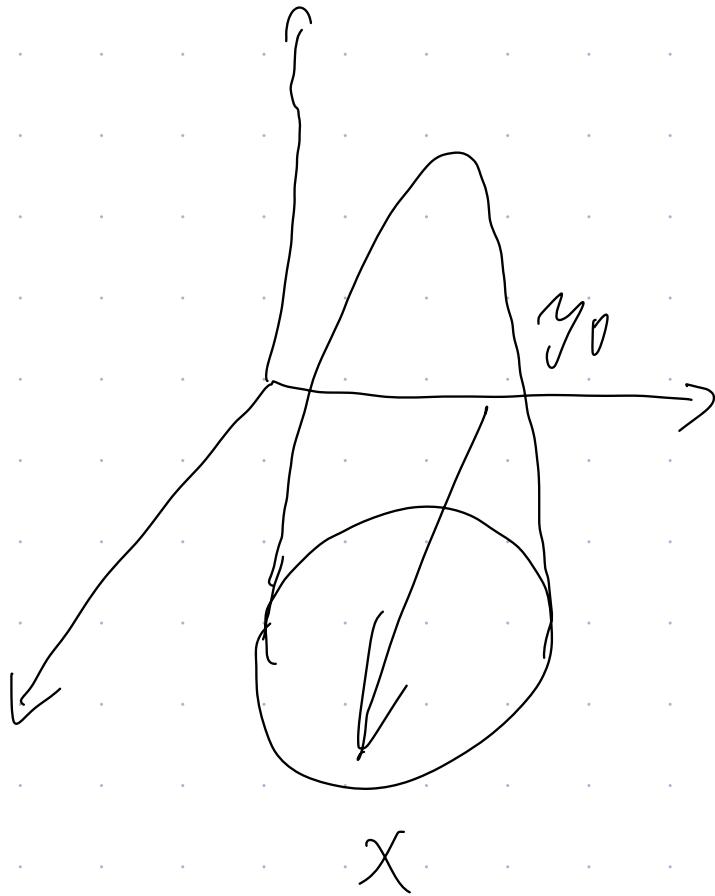
$$dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$$

definition:

$$dZ = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} \underbrace{f(x, y) - f(x_0, y_0)}_{\sqrt{(x-x_0)^2 + (y-y_0)^2}} - \frac{\partial Z}{\partial x}(x-x_0) - \frac{\partial Z}{\partial y}(y-y_0)$$

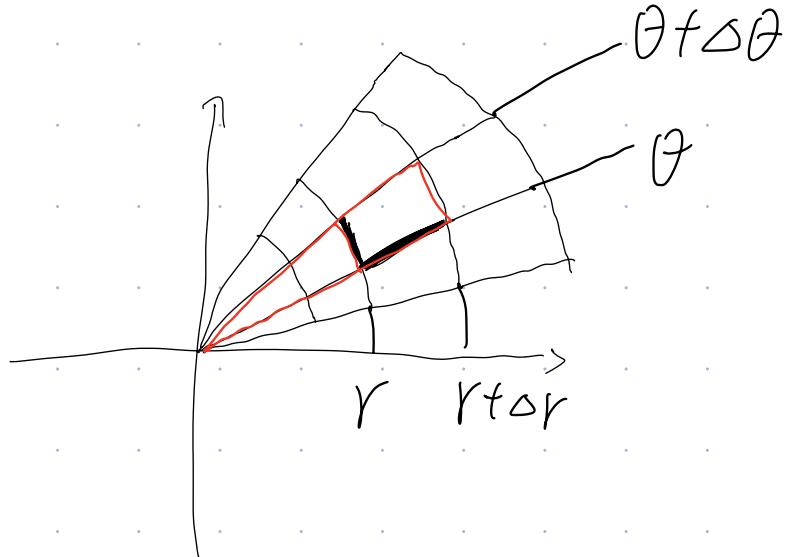


## 2. 偏导数 (只对 $X$ 或 $Y$ 方向的导数)



对  $X$  求偏导  
对  $Y$  不变

二重积分



$$\therefore S_{\vec{r}\vec{\theta}} = \frac{1}{2} r^2 \theta$$

$$\therefore \Delta G = S_{\vec{r}\vec{\theta}+d\vec{\theta}} - S_{\vec{r}\vec{\theta}}$$

$$= \frac{1}{2} [(r+d\theta)^2 \Delta \theta - r^2 \Delta \theta]$$

$$= \frac{1}{2} \Delta \theta [2rdr + dr^2]$$

$$\approx \frac{1}{2} \Delta \theta \cdot 2rdr$$

$$= r dr \Delta \theta$$

# 向量代数与几何

L. f& Pk

七种未定式

$$\frac{0}{0} \quad \frac{\infty}{\infty}$$

$$0 \infty \quad \begin{array}{c} \swarrow \\ \frac{1}{0} \end{array} \quad \begin{array}{c} \uparrow \\ \frac{0}{1} \end{array}$$

$$\infty - \infty \quad \begin{array}{c} \swarrow \\ \text{通常} \end{array}$$

无分母提公因子

$$0^\infty, 0^0 \quad u^n = e^{v \ln u}$$

$$\int_1^\infty u^v, u \rightarrow 1 \text{ 时} \int \text{原式} = e^{v(u-1)}$$

$$1. x - 1 < [x] \leq x$$

2. 抓大头

3. 当  $x \rightarrow \infty$  时, 可考虑倒带换

$$\begin{aligned}
 4. \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} - e & \quad \text{对数运算法则:} \\
 &= e^{\frac{1}{x}(\ln(1+x))} - e \\
 &= e[e^{\alpha(\ln(1+x)-1)} - 1]
 \end{aligned}$$

$$\begin{aligned}
 \log_a(MN) &= \log_a M + \log_a N \\
 \log_a\left(\frac{M}{N}\right) &= \log_a M - \log_a N
 \end{aligned}$$

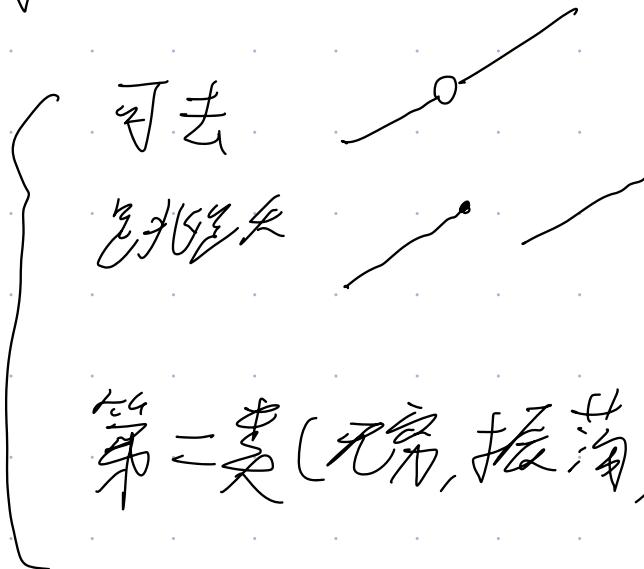
$$= e \left[ \frac{1}{x} \ln(1+x) - 1 \right]$$

$$= e \left[ \frac{\ln(1+x) - x}{x} \right]$$

$$= e \left[ -\frac{\frac{1}{2}x^2}{x} \right]$$

$$= -\frac{e}{2}x$$

连续与间断点



第二类(无穷, 振荡)

找无定义点求左右  $\lim$

$\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0} f(x)$  为可去

$\lim_{x \rightarrow x_0^+} f(x) \neq \lim_{x \rightarrow x_0^-} f(x)$  为跳跃

## 2. Differential

Definition:

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

保持一致

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

可导充分条件

$f(x)$  可导,  $f'(x)$  奇

$$f'(-x) = -f'(x)$$

$$f'(-x) = \lim_{\Delta x \rightarrow 0} \frac{f(-x+\Delta x) - f(-x)}{\Delta x}$$

$$\begin{aligned} &= - \lim_{-\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{-\Delta x} \\ &= -f'(x) \end{aligned}$$

$|x - x_0|$  在  $x_0$  处一定不可导

当  $f(x)$   $\left\{ \begin{array}{l} 1. 在 x_0 处连续 \\ 2. f(x_0) = 0 \end{array} \right.$ ,  $\exists \lim_{x \rightarrow x_0} f(x) | x - x_0 |$  在  $x = x_0$  处不成立

e.g. 求  $f(x) = (x^2 - 1)(x^3 + x^2 - 2x - 2)$  在  $\sqrt{2}$  处的个数

$$= (x^2 - 1) | (x+1)(x^2 - 2) |$$

$$= (x+1)(x-1) | (x+1)(x+\sqrt{2})(x-\sqrt{2}) |$$

不可导点:  $-1, -\sqrt{2}, \sqrt{2}$

$\therefore (x^2 - 1) |_{x=-1} = 0$  且在  $x = -1$  处连续

$\therefore$  有 2 个不可导点

property

1. 改变奇偶性

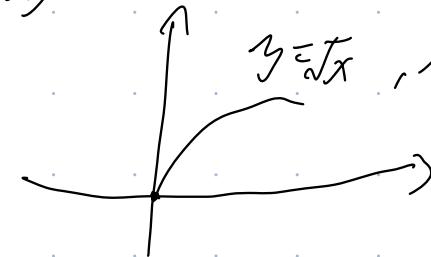
2.  $f(x+T) = f(x) \Rightarrow f'(x+T) = f'(x)$

$f(x)$  有界  $\Leftrightarrow f'(x)$  有界

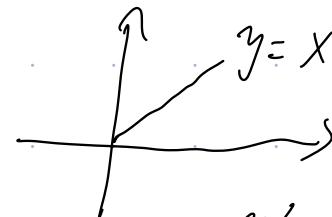
$f'(x)$  在闭区间有界  $\rightarrow f(x)$  有界

$y'_{x=0}$   $\infty$

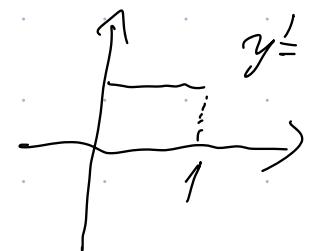
$y = \sqrt{x}, x \in [0, 1]$



$y'$  无界



$y' = 1$



可微性

$$\Delta y = \underbrace{A\Delta x}_\downarrow + O(\Delta x)$$

$$dy|_{x=x_0} = f'(x) \Delta x = f'(x) dx$$

反微分 求 $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

$$\frac{d^2x}{dy^2} = \frac{d\left(\frac{dx}{dy}\right)}{dy} = \frac{d\left(\frac{dx}{dy}\right)}{dx dy} \cdot dx$$

$$= \frac{d\left(\frac{dx}{dy}\right)}{dx} \cdot \frac{dx}{dy} = \left(\frac{1}{f'(x)}\right)' \cdot \frac{1}{f'(x)} = -\frac{f''(x)}{\left(f'(x)\right)^3}$$

e.g. 设  $f(x)$  单调可导,  $f'(x) \neq 0$ , 且  $f(x)$  在  $x=1$  处 = P<sub>f</sub> 可导,

$f(1) = -2$ ,  $f'(1) = -\frac{\sqrt{2}}{2}$ ,  $f''(1) = 2$ , 记  $y = f(x)$  的反函数

数为  $x = \varphi(y)$ , 求  $\frac{dx}{dy} \Big|_{y=-2}$ ,  $\frac{d^2x}{dy^2} \Big|_{y=-2}$

$$\frac{dx}{dy} \Big|_{y=-2} = \frac{1}{f'(1)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\frac{d^2x}{dy^2} = \frac{d(\frac{dx}{dy})}{dy} = \frac{d(\frac{dx}{dy})}{dx} \cdot \frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2} \cdot \frac{1}{f'(x)} \Big|_{x=1}$$

$$= -\frac{1}{2} x(-\sqrt{2}) \leq 4\sqrt{2}$$

e.g.

$$y = f(x) = 3x^2 + e^x, \quad x = \varphi(y), \quad \varphi'(3+e) = \underline{\hspace{2cm}}$$

if:  $3+e = 3x^2 + e^x$      $f'(x)|_{x=1} = 6x + e^x|_{x=1} = 6+e$

$$\therefore x = 1 \quad f''(x)|_{x=1} = 6+e^x|_{x=1} = 6+e$$

$$\frac{dx}{dy^2} = \frac{d\frac{dx}{dy}}{dy} = \frac{f''(x)}{(f'(x))^2} \Big|_{x=1}$$

$$= -\frac{6+e}{(6+e)^2} = -\frac{1}{(6+e)^2}$$

$$\text{e.g. } y = y(x) \begin{cases} x = \arctant \\ 2y - ty^2 + e^t = 5 \end{cases} \quad 2 \sqrt{\frac{dy}{dx}}$$

$$\frac{dy}{dt} = 2y' - y^2 - 2tyy' + e^t = 0$$

$$y' = \frac{y^2 - e^t}{2 - 2ty}$$

$$\frac{dx}{dt} = \frac{1}{1 + t^2}$$

$$\frac{dy}{dx} = \frac{(y^2 - e^t)(1 + t^2)}{2 - 2ty}$$

# 高阶导数

1. 例题

2. Taylor

e.g.

$$f(x) = x^2 \ln(1-x), \quad \forall n \geq 3, f^{(n)}(0) =$$

$$\ln(1-x) = -x - \frac{x^2}{2} + \frac{-x^3}{3} + \dots$$

$$\therefore f(x) = -x^3 - \frac{x^4}{2} - \frac{x^5}{3} - \dots$$

$$= \underbrace{\frac{n!}{n-2}}$$

### 3. 級数展開

$$(uv)^{(n)} = C_n^0 u^{(n)} v^{(0)} + C_n^1 u^{(n-1)} v^{(1)} + \dots + C_n^n u^{(0)} v^{(n)}$$

e.g.  $f(x) = xe^x$ ,  $f(x)$

$$(xe^x)^{(n)} = e^x + C_n^1 e^x + 0$$

$$= xe^x + ne^x$$

中值定理

# -元微积分学

$F(x)$  在某区间内处处可导, 且  $F'(x) = f(x)$ , 则  $F(x)$  为  $f(x)$  的原函数

e.g.

例 8.2 函数  $f(x) = \begin{cases} \frac{1}{\sqrt{1+x^2}}, & x \leq 0, \\ (x+1)\cos x, & x > 0 \end{cases}$  的一个原函数为 ( ) .

(A)  $F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x), & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0 \end{cases}$

(B)  $F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x) + 1, & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0 \end{cases}$

(C)  $F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x), & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0 \end{cases}$

(D)  $F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x) + 1, & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0 \end{cases}$

原函数 : 连续

A.  $\lim_{x \rightarrow 0^-} f(x) = 1$

$f(0) = 0$   $\times$

B.  $f(0) = 1$   $\lim_{x \rightarrow 0^+} f(x) = 1$

$\lim_{x \rightarrow 0^+} f(x) = 1$   $\times$

$\cos x - (x+1)\sin x - \sin x$

$= \cos x - x \sin x$

$$C. f(0) = 0$$

$$D. f(0) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

✓

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = 1 \quad \checkmark$$

$$\sin x + (x+1)\cos x - \sin x$$

$$\int f(x) dx = F(x) + C, \text{ 不连续的 function cluster}$$

连续分支函数，一个分支

## 原函数存在定理

$\left\{ \begin{array}{l} f(x) \text{ 连续} \rightarrow \text{一定有原函数} \Rightarrow \int f(x) dx = \int_a^x f(x) dx + C \\ f(x) \text{ 不连续} \left\{ \begin{array}{l} \text{第一类间断点, 无穷} \Rightarrow \text{无 } F(x) \\ \text{瑕点} \Rightarrow \text{可能有} \end{array} \right. \end{array} \right.$

## 定积分存在定理

必要条件：区间有限，函数有界

充分条件：  
 $[a, b]$   $\left\{ \begin{array}{l} f(x) \text{ 连续} \\ f(x) \text{ 单调} \end{array} \right.$

| 有限区间定理

积分中值定理  $\int_a^b f(x) dx = f(\xi)(b-a)$

变限积分

表格法

$$\int_0^\pi e^{-t} \sin t dt$$

$$\begin{array}{ccccc} \sin t & & \cos t & & -\sin t \\ & f & & & -) + \\ e^{-t} & & -e^{-t} & & e^{-t} \end{array}$$

$$I = \left[ -\sin t e^{-t} - e^{-t} \cos t \right]_0^\pi - \int_0^\pi \sin t e^{-t} dt$$

$$I = -\frac{1}{2} e^{-t} (\cos t + \sin t) \Big|_0^\pi$$

$$= -\frac{1}{2} \left( e^{-\pi} (\cos \pi - \sin \pi) - 1 \right)$$

$$= \frac{1}{2} (e^{-\pi} + 1)$$

$$I = \int_0^{\pi^2} \sqrt{x} \cos \sqrt{x} dx$$

$$\sqrt{x} = t, x = t^2, dx = 2t dt$$

$$I = \int_0^{\pi} t \cos t \cdot 2t dt$$

$$= 2 \int_0^{\pi} t^2 \cos t dt$$

$$\begin{array}{ccccccccc} t^2 & & 2t & & 2 & & 0 \\ \swarrow + & & \searrow - & & \nearrow + & & \\ \cos t & & \sin t & & -\cos t & & -\sin t \end{array}$$

$$2 \left( t^2 \sin t + 2t \cos t - 2 \sin t \right) \Big|_0^{\pi}$$

$$= 2(-2\pi) = -4\pi$$

完稿

$$\int_a^b f(x) dx = \int f(a+b-t) dt$$

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\int_0^{\frac{\pi}{2}} x f(\sin x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin 2x) dx = \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{4}{5} \cdot \frac{2}{3} \cdot 1, & n \in \text{odd} \end{cases}$$

$$\left\{ \frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{\pi}{2}, n \text{ even} \right.$$

Mis C

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \sum_{k=1}^n k(k+1)$$

$$S_{\text{半圆}} = \pi a b$$

$$S_{\text{扇}} = \frac{1}{2} r^2 \theta$$

$$\int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(-1) = 2$$

等比級數

$$\sum_{n=0}^{\infty} aq^n \quad (a \neq 0)$$

$$\left\{ \begin{array}{l} \text{發散, } |q| \geq 1 \\ \text{收斂, } |q| < 1 \end{array} \right.$$

等比數列

$$a_n = a_1 q^{n-1}$$

$$S_n = \frac{a_1(1-q^n)}{1-q}$$

當收斂 ( $|q| < 1$  時)

$$S = \frac{a_1}{1-q}$$

數列的求和

series 累加求和

# 重要级数

## 1. p级数

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

发,  $p < 1$   
收,  $p > 1$

## 2. 等比级数

$\propto$

$$(等) |q| \geq 1$$

$$\sum_{n=0}^{\infty} aq^n \quad (a \neq 0) \quad \left. \begin{array}{l} \text{收斂} \\ |q| < 1 \end{array} \right\}$$

当收敛 ( $|q| < 1$ )

$$S = \frac{a_1}{1-q}$$

3.  $\bar{f} \times P$

$$\sum \quad \begin{array}{c} 1 \\ \hline B \end{array} \quad \left. \begin{array}{l} \text{收斂}, \alpha > 1 \\ \text{发散}, \alpha < 1 \end{array} \right\}$$

$\tilde{n} = n^\alpha \ln^\beta n$

$x, v \propto 1$

$\alpha = 1,$

$\beta > 1, \text{ 收}$

$\beta < 1, \text{ 发}$

$$y'' - 2y' + y = xe^x$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\nu_1 = \nu_2 = 1$$

$$\text{Ansatz } y^* = (Ax + b)e^x x^2$$

“ ”