Math

$x \to 0$

$\sin x \sim$

$\tan x \sim$

$\arcsin x \sim$

$\arctan x \sim$

$\ln(1+x) \sim$

$e^x - 1 \sim$

$1 - \cos x \sim$

$x - \ln(1+x) \sim$

$(1+x)^\alpha - 1 \sim$

$a^x - 1 \sim x \ln a$

$$\left[\ln(x+\sqrt{1+x^2})\right]' = \frac{1}{\sqrt{1+x^2}}$$

$x \to 0$ 时 $f(x) \to 0$

$t \to 1$

$\ln(t) \sim t - 1$

$$x = \frac{e^y - e^{-y}}{2}$$

常见麦克劳林公式 ($x_0=0$)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\arcsin x = x + \frac{x^3}{3!}$$

$$\tan x = x + \frac{x^3}{3}$$

$$\arctan x = x - \frac{x^3}{3}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!} x^2$$

e.g. 求常数 a，使

解：分左右极限，再求之不难

$$\lim_{x\to 0}\int_{-x}^{x}\frac{1}{x}\left(1-\frac{|t|}{x}\right)\cos(a-t)\,dt$$ 存在，并求极限

奇

$$=\lim_{x\to 0^+}\int_{-x}^{x}\frac{1}{x}\left(1-\frac{|t|}{x}\right)(\cos a\cos t+\sin a\sin t)\,dt$$

$$=\lim_{x\to 0^+}\left[\int_{-x}^{x}\underbrace{\left(1-\frac{|t|}{x}\right)}^{\text{偶}}\frac{1}{x}\cos a\underbrace{\cos t}_{\text{偶}}\,dt+\int_{-x}^{x}\underbrace{\left(1-\frac{|t|}{x}\right)}^{\text{偶}}\frac{1}{x}\sin a\underbrace{\sin t}_{\text{奇}}\,dt\right]$$

对t 给只看t

$$=\lim_{x\to 0^+}2\cos a\int_{0}^{x}\left(1-\frac{|t|}{x}\right)\frac{1}{x}\cos t\,dt$$

$$=\lim_{x\to 0^+}2\cos a\int_{0}^{x}(x-|t|)\cos t\,dt\,/\,x^2$$

$$=\lim_{x\to 0^+} 2\cos a \frac{x\int_0^x \cos t\,dt - \int_0^x |t|\cos t\,dt}{x^2}$$

$$=\lim_{x\to 0^+} 2\cos a \frac{\int_0^x \cos t\,dt + x\cos x - x\cos x}{2x}$$

$$=\lim_{x\to 0^+} 2\cos a \frac{\cos x}{2}$$

$$=\lim_{x\to 0^+} \cos a$$

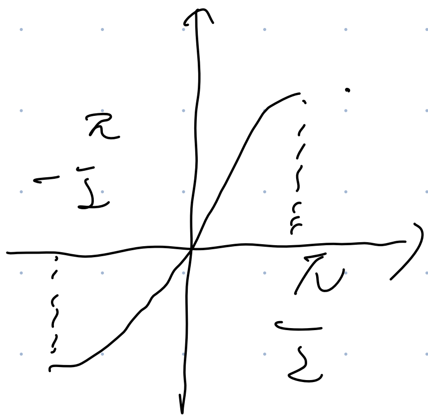
原式 $=\lim_{x\to 0^-} 2\cos a \int_0^x (x-|t|)\cos t\,dt \,/\, x^2$

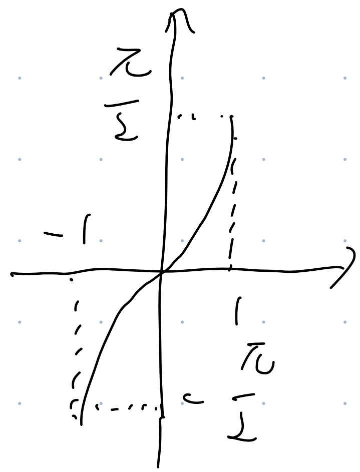
$=$

limit=0 $\quad a=\frac{\pi}{2}+k\pi \ (k\in\mathbb{R})$

反函数

当 $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$，$y = \sin x$ 有反函数 $x = \arcsin y$，$x \in [-1, 1]$





y为自变量，arc表弧，输出x弧度

e.g. 下列式子中有意义的是

A. $\arcsin\sqrt{2}$　B. $\arcsin(-\frac{\pi}{3})$　C. $\sin(\arcsin 2)$　D. $\arcsin(\sin 2)$

解:

$\because y=\arcsin x$, $x\in[-1,1]$

$\therefore$ D

e.g. $y=\sin x$, $x\in[\frac{\pi}{2},\frac{3\pi}{2}]$, 求 $y=f^{-1}(x)$

$y=\sin(\pi-x)=\sin x$

$\arcsin y=\pi-x$

$\therefore x=\pi-\arcsin y$

$y=\pi-\arcsin x$

e.g. $\arccos(\cos\frac{11\pi}{6}) =$ ______

$\cos\frac{11\pi}{6} = \cos[2\pi - \frac{\pi}{6}] = \cos\frac{\pi}{6}$

$\therefore$ 为 $\frac{\pi}{6}$

e.g. $x \in (\frac{\pi}{2}, \frac{3}{2}\pi)$ $\arcsin(\sin x) =$

$\because \sin(\pi - x) = \sin x$

$\therefore \arcsin(\sin x) = \pi - x$

e.g. $x \in (-\frac{3}{2}\pi, -\frac{\pi}{2})$ $\arcsin(\sin x) =$

$\because \sin(-\pi - x) = \sin x$

$\therefore \arcsin(\sin x) = -\pi + x$

多元微分

1、微分(单变量函数)

$$dy = f'(x)dx$$

全微分(多变量函数)

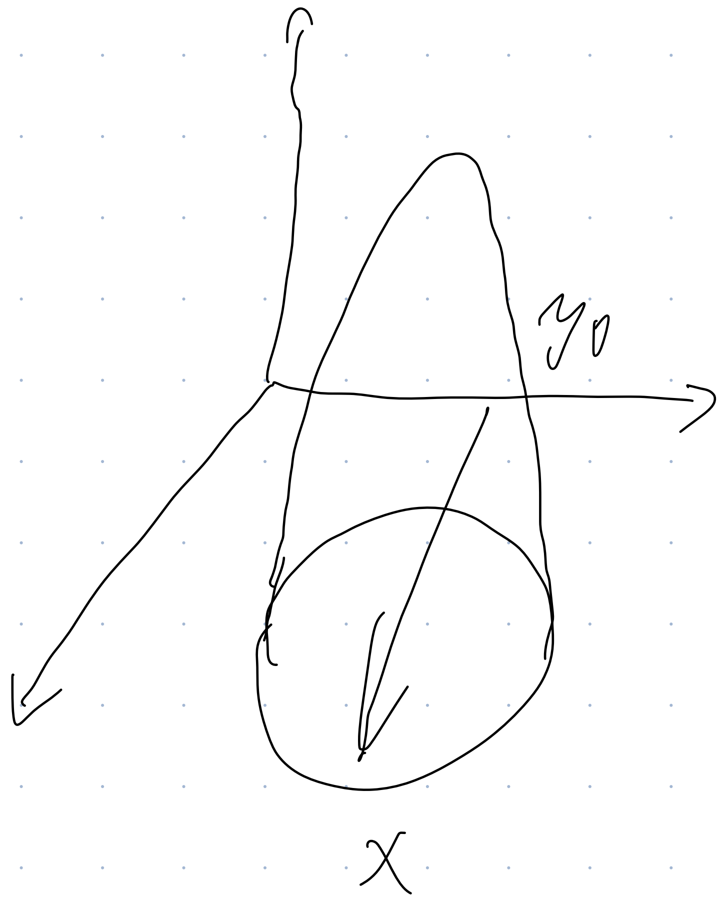
对于 $z = f(x,y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

definition:

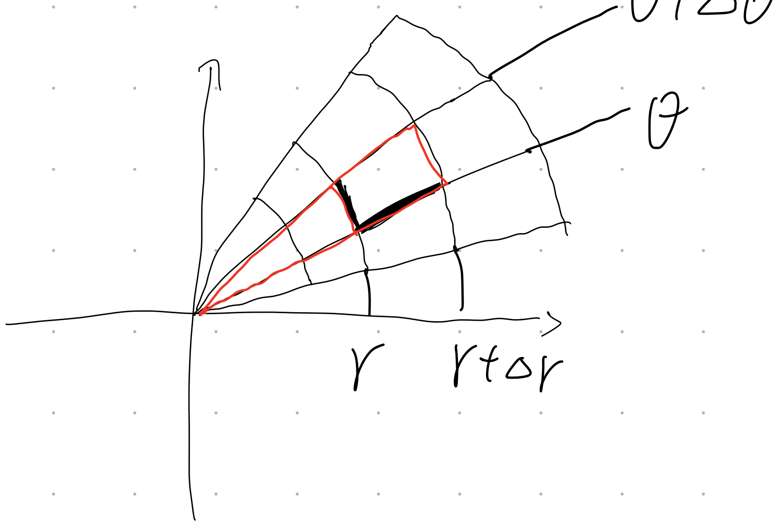
$$dz = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial z}{\partial x}(x-x_0) - \frac{\partial z}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

2. 偏导数（只对x或y方向的导数）



对x求偏导
则y不变

二重积分



$\because S_{扇} = \frac{1}{2} r^2 \theta$

$\therefore \Delta \sigma = S_{大扇} - S_{小扇}$

$= \frac{1}{2}\left[(r+\Delta r)^2 \Delta\theta - r^2 \Delta\theta\right]$

$= \frac{1}{2}\Delta\theta\left[2r\Delta r + \Delta r^2\right]$

$\approx \frac{1}{2}\Delta\theta \cdot 2r\Delta r$

$= r\Delta r\Delta\theta$

向量代数与几何

1、极限

七种未定式

$\frac{0}{0}$ $\frac{\infty}{\infty}$

$0 \cdot \infty$

- $\frac{\infty}{\frac{1}{0}}$
- $\frac{0}{\frac{1}{\infty}}$

$\infty - \infty$

- 通分
- 无分母提公因式

0^{∞}, 0^{0} $u^{v} = e^{v \ln u}$

1^{∞} u^{v}, $u \to 1$ 则原式 $= e^{v(u-1)}$

1、 $x-1 < [x] \le x$

2、抓大头

3、当 $x \to \infty$ 时，可考虑倒带换

4、 $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} - e$

$= e^{\frac{1}{x}\ln(1+x)} - e$

$= e[e^{\frac{1}{x}\ln(1+x)-1} - 1]$

对数运算法则：

$\log_a(MN) = \log_a M + \log_a N$

$\log_a(\frac{M}{N}) = \log_a M - \log_a N$

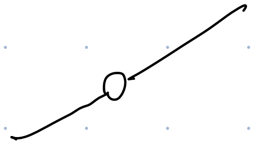
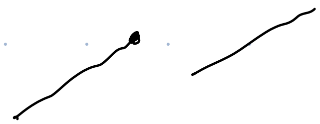
$$= e\left[\frac{1}{x}\ln(1+x) - 1\right]$$

$$= e\left[\frac{\ln(1+x) - x}{x}\right]$$

$$= e\left[\frac{-\frac{1}{2}x^2}{x}\right]$$

$$= -\frac{e}{2}x$$

连续 间断点

- 可去 
- 跳跃 
- 第二类(无穷,振荡)

找无定义点, 求左右limit

$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0} f(x)$ 为可去

$\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$ 为跳跃

2. Differential

Definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

保持一致

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

可导充分条件 $\begin{cases} 1. \text{左导} = \text{右导} \\ \\ \\ \end{cases}$

证：$f(x)$ 为偶, $f'(x)$ 为奇

$$f'(-x) = -f'(x)$$

$$f'(-x) = \lim_{\Delta x \to 0} \frac{f(-x+\Delta x)-f(-x)}{\Delta x}$$

$$= -\lim_{-\Delta x \to 0} \frac{f(x+(-\Delta x))-f(x)}{-\Delta x}$$

$$= -f'(x)$$

$|x-x_0|$在x_0处一定不可导

当$f(x)\begin{cases}1. 在x_0处连续 \\ 2. f(x_0)=0\end{cases}$，则$f(x)|x-x_0|$在$x=x_0$处可导

e.g. 求$f(x)=(x^2-1)|x^3+x^2-2x-2|$不可导点个数

$$=(x^2-1)|(x+1)(x^2-2)|$$

$$=(x+1)(x-1)|(x+1)(x+\sqrt{2})(x-\sqrt{2})|$$

不可导点：$-1,-\sqrt{2},\sqrt{2}$

$\because (x^2-1)|_{x=-1}=0$且在$x=-1$处连续

$\therefore$有2个不可导点

property

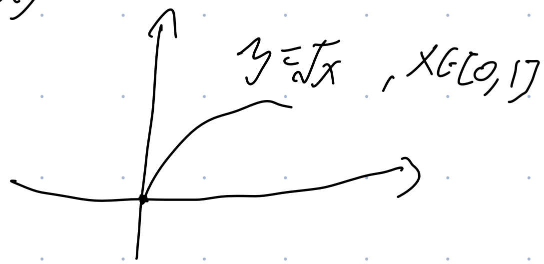
1、求导改变奇偶性

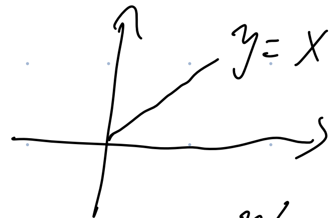
2、$f(x+T)=f(x) \Rightarrow f'(x+T)=f'(x)$

$f(x)$有界 $\not\Leftrightarrow$ $f'(x)$有界

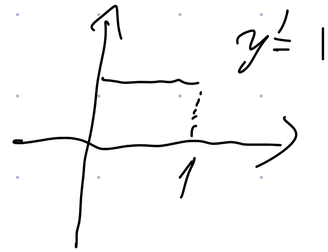
$f'(x)$在闭区间有界 $\rightarrow$ $f(x)$有界

y'在0处 ∞





y'无界



可微

$$\Delta y = \underbrace{A\Delta x} + o(\Delta x)$$

$$\downarrow$$

$$dy\Big|_{x=x_0} = f'(x)\Delta x = f'(x)dx$$

反函数求导

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

$$\frac{d^2x}{dy^2} = \frac{d\left(\frac{dx}{dy}\right)}{dy} = \frac{d\left(\frac{dx}{dy}\right)}{dx\,dy} \cdot dx$$

$$= \frac{d\left(\frac{dx}{dy}\right)}{dx} \cdot \frac{dx}{dy} = \left(\frac{1}{f'(x)}\right)' \cdot \frac{1}{f'(x)} = -\frac{f''(x)}{[f'(x)]^3}$$

e.g. 设$f(x)$单调可导，$f'(x)\neq 0$，且$f(x)$在$x=1$处二阶可导，$f(1)=-2$，$f'(1)=-\frac{\sqrt{2}}{2}$，$f''(1)=2$，记$y=f(x)$的反函数为$x=\varphi(y)$，求$\frac{dx}{dy}\Big|_{y=-2}$，$\frac{d^2x}{dy^2}\Big|_{y=-2}$

$$\frac{dx}{dy}\Big|_{y=-2}=\frac{1}{f'(1)}=\frac{1}{-\frac{\sqrt{2}}{2}}=-\sqrt{2}$$

$$\frac{d^2x}{dy^2}=\frac{d(\frac{dx}{dy})}{dy}=\frac{d(\frac{dx}{dy})}{dx}\cdot\frac{dx}{dy}=-\frac{f''(x)}{[f'(x)]^2}\cdot\frac{1}{f'(x)}\Big|_{x=1}$$

$$=-\frac{2}{\frac{1}{2}}\times(-\sqrt{2})=4\sqrt{2}$$

e.g.

$y=f(x)=3x^2+e^x$, $x=\varphi(y)$, 则 $\varphi''(3+e)=$ ______

解: $3+e=3x^2+e^x$ $\quad f'(x)\big|_{x=1}=6x+e^x\big|_{x=1}=6+e$

$\therefore x=1$ $\quad f''(x)\big|_{x=1}=6+e^x\big|_{x=1}=6+e$

$$\frac{d^2x}{dy^2}=\frac{d\frac{dx}{dy}}{dy}=-\frac{f''(x)}{[f'(x)]^2}\cdot\frac{1}{f'(x)}\Bigg|_{x=1}$$

$$=-\frac{6+e}{(6+e)^3}=-\frac{1}{(6+e)^2}$$

e.g. $y=y(x)$ $\begin{cases} x=\arctan t \\ 2y-ty^2+e^t=5 \end{cases}$ 求 $\dfrac{dy}{dx}$

$\dfrac{dy}{dt}=$ $\quad 2y'-y^2-2tyy'+e^t=0$

$$y'=\frac{y^2-e^t}{2-2ty}$$

$$\frac{dx}{dt}=\frac{1}{1+t^2}$$

$$\frac{dy}{dx}=\frac{(y^2-e^t)(1+t^2)}{2-2ty}$$

高阶导数

1、归纳

2、Taylor

e.g.

$f(x)=x^2\ln(1-x)$，当 $n\geq 3$，$f^{(n)}(0)=$

$\because \ln(1-x)=-x-\frac{x^2}{2}+\frac{-x^3}{3}+\cdots$

$\therefore f(x)=-x^3-\frac{x^4}{2}-\frac{x^5}{3}-\cdots$

$-\frac{n!}{n-2}$

3. 莱布尼兹

$$(uv)^{(n)} = C_n^0 u^{(n)} v^{(0)} + C_n^1 u^{(n-1)} v' + \dots C_n^n u^{(0)} v^{(n)}$$

e.g. 求 $f(x) = xe^x$，求 $f^{(n)}(x)$

$$(xe^x)^{(n)} = e^x x + C_n^1 e^x + 0$$

$$= xe^x + ne^x$$

中值定理

一元函数积分学

$F(x)$ 在某区间内处处可导，且 $F'(x) = f(x)$，则 $F(x)$ 为 $f(x)$ 的原函数

e.g.

例 8.2 函数 $f(x)=\begin{cases}\dfrac{1}{\sqrt{1+x^2}}, & x\leqslant 0,\\ (x+1)\cos x, & x>0\end{cases}$ 的一个原函数为（　　）.

(A) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}-x\right), & x\leqslant 0,\\ (x+1)\cos x-\sin x, & x>0\end{cases}$

(B) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}-x\right)+1, & x\leqslant 0,\\ (x+1)\cos x-\sin x, & x>0\end{cases}$

(C) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}+x\right), & x\leqslant 0,\\ (x+1)\sin x+\cos x, & x>0\end{cases}$

(D) $F(x)=\begin{cases}\ln\left(\sqrt{1+x^2}+x\right)+1, & x\leqslant 0,\\ (x+1)\sin x+\cos x, & x>0\end{cases}$

∃原函数 ∴ 连续

A. $\lim\limits_{x\to 0^+} f(x) = 1$

$f(0) = 0$ ×

B. $f(0) = 1$ $\lim\limits_{x\to 0^-} f(x) = 1$

$\lim\limits_{x\to 0^+} f(x) = 1$ ×

$\cos x - (x+1)\sin x - \sin x$

$= \cos x - x\sin x$

C. $f(0)=0$

$\lim\limits_{x\to 0^+} f(x)=1$

α

D. $f(0)=1$

$\lim\limits_{x\to 0^+} f(x)=1$

$\lim\limits_{x\to 0^-} f(x)=1$ ✓

$\sin x+(x+1)\cos x-\sin x$

$\int f(x)dx=F(x)+C$，不定积分求 function cluster

定积分求面积，一个数

原函数$\exists$定理

- $f(x)$连续 $\longrightarrow$ 一定有原函数 $\Rightarrow \int f(x)dx = \int_a^x f(x)dx + C$
- $f(x)$不连续
  - 第一类间断点，无穷 $\Rightarrow$ 无$F(x)$
  - 振荡 $\Rightarrow$ 可能有

定积分$\exists$定理

必要条件：区间有限，函数有界

充分条件：

$[a,b]$
- $f(x)$连续
- $f(x)$单调

| 有限个间断点

积分中值定理 $\int_a^b f(x)dx = f(\xi)(b-a)$

变限积分

表格法

$$\int_0^{\pi} e^{-t}\sin t\,dt$$

$$\begin{array}{ccccc}
\sin t & & \cos t & & -\sin t \\
 & + & & - \quad + & \\
e^{-t} & & -e^{-t} & & e^{-t}
\end{array}$$

$$I = \left[-\sin t\, e^{-t} - e^{-t}\cos t\right]_0^{\pi} - \int_0^{\pi} \sin t\, e^{-t}\,dt$$

$$I = -\frac{1}{2}e^{-t}(\cos t + \sin t)\Big|_0^{\pi}$$

$$= -\frac{1}{2}\left(e^{-\pi}[\cos\pi - \sin\pi] - 1\right)$$

$$= \frac{1}{2}(e^{-\pi}+1)$$

$$I=\int_0^{\pi^2}\sqrt{x}\cos\sqrt{x}\,dx$$

$$\sqrt{x}=t,\quad x=t^2,\quad dx=2t\,dt$$

$$I=\int_0^{\pi} t\cos t\cdot 2t\,dt$$

$$=2\int_0^{\pi} t^2\cos t\,dt$$

$$\begin{array}{ccccccc}
t^2 & & 2t & & 2 & & 0 \\
 & + & & - & & + & \\
\cos t & & \sin t & & -\cos t & & -\sin t
\end{array}$$

$$2\left(t^2\sin t+2t\cos t-2\sin t\right)\Big|_0^{\pi}$$

$= 2(-2\pi) = -4\pi$

定积分

$$\int_a^b f(x)\,dx = \int f(a+b-t)\,dt$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin 2x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \begin{cases} \frac{4}{5}\cdot\frac{2}{3}\cdot 1, & n \in \text{odd} \end{cases}$$

$$\left\{ \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \quad n \in \text{even} \right.$$

Misc

$$1^2+2^2+3^2+\cdots n^2 = \frac{n(n+1)(2n+1)}{2}$$

$$S_{椭圆} = \pi ab$$

$$S_{扇} = \frac{1}{2} r^2 \theta$$

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = 1-(-1) = 2$$

等比级数

$$\sum_{n=0}^{\infty} aq^n \quad (a \neq 0) \begin{cases} 发, & |q| \geq 1 \\ 收, & |q| < 1 \end{cases}$$

当收敛(|q|<1时)

$$S = \frac{a_1}{1-q}$$

等比数列

$$a_n = a_1 q^{n-1}$$

$$S_n = \frac{a_1(1-q^n)}{1-q}$$

数列侧重于通项

series重于求和

重要级数

1. P级数

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}发, & p\leq 1\\ 收, & p>1\end{cases}$$

2. 等比级数

∞

发 $|q|\geq 1$

$$\sum_{n=0}^{\infty} aq^{n} \quad (a\neq 0)\begin{cases} 发, & |q|\geq 1 \\ 收敛, & |q|<1 \end{cases}$$

当收敛（$|q|<1$ 时）

$$S=\frac{a_1}{1-q}$$

3. 广义 p

$$\sum^{\infty} \frac{1}{\quad} \begin{cases} 收敛, & \alpha>1 \\ 发, & \alpha\leq 1 \end{cases}$$

$\sum_{n=1} \frac{1}{n^{\alpha} \ln^{\beta} n}$ 发，$\alpha < 1$

$\alpha = 1$，$\begin{cases} \beta > 1，\text{收} \\ \beta < 1，\text{发} \end{cases}$

二重积分

1. D 关于 x-axis 对称
2. f(x, y) 关于 x 对称

当 $f(x,-y) = -f(x,y)$ 则 $\iint_D f(x,y)\,d\sigma = 0$

$$f(x,-y)=f(x,y) \text{ 则 } 2\iint_{D_1} f(x,y)\sigma$$

y同理

e.g. $D=\{(x,y)\mid x^2+y^2\le 1,\ (x-1)^2+y^2\ge 1\}$，求

$$\iint_D (5y^3+x^2+y^2-2x+y+1)\,dx\,dy$$

积分区域关于x轴对称

轮换对称

1、D关于 $y=x$ 对称

2、$$\iint\limits_D f(x,y)\,d\sigma = \iint f(y,x)\,d\sigma$$

$$\therefore \iint\limits_D f(x,y)\,d\sigma = \frac{1}{2}\left(\iint\limits_D f(x,y)\,d\sigma + \iint\limits_D f(y,x)\,d\sigma\right.$$

e.g. $z=f(x^y, y^x)$，则 $\frac{\partial z}{\partial x}=$

解：$z=f(u,v)$，$u=x^y$，$v=y^x$

$$\frac{\partial z}{\partial x}=f_1' y x^{y-1}+f_2' y^x \ln y$$

e.g. $f(x,y)$ 可微，$f(1,2)=2$，$f_x'(1,2)=3$，$f_y'(1,2)=3$，$z(x)=f(x, f(x,2x))$

求 $z'(1)$

$$z'(x)=f_1'(x,f(x,2x))+f_2'(x,f(x,2x))f_x'(x,2x)$$

$$=f_1'(x,f(x,2x))+f_2'(x,f(x,2x))[f_1'(x,2x)+f_2'(x,2x)\cdot 2]$$

$$=f_1'(1,f(1,2))+f_2'(1,f(1,2))[f_1'(1,2)+f_2'(1,2)\cdot 2]$$

$= f_1'(1,2) + f_2'(1,2)[f_1'(1,2) + 2f_2'(1,2)]$

$= 3 + 3 \times (3 + 2\times 3) = 30$

$f_1'(1,2) = f_x'(1,2)$

e.g. $f(xy, x+y) = x^2 + y^2 + xy$

1. 求 $z = f(x,y)$

2. 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$f(xy, x+y) = (x+y)^2 - xy$

$\frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = 2y$

$\therefore f(x,y) = y^2 - x$

e.g.