

Algebra

Determinant

$$\begin{cases} 3x - 2y = 12 \\ 2x + y = 1 \end{cases} \Rightarrow \begin{cases} a_{11}x_1 - a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7$$

$D = 0$ 表示 1. 有无穷多解 2. 无解

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 12 - (-2) = 14$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = 3 - 24 = -21$$

$$x_1 = \frac{D_1}{D} = \frac{14}{7} = 2$$

$$x_2 = \frac{D_2}{D} = \frac{-21}{7} = -3$$

逆序对

$$1. f(3\ 2\ 5\ 1\ 4) = 2 + 1 + 2 = 5 \quad \text{奇数对}$$

$f(2\ 3\ 5\ 1\ 4) = 1 \quad \text{偶数对} \quad \text{对换两数改变奇偶}$

$$2. f(n, n-1, \dots, 1) = \frac{n(n-1)}{2}$$

3. 判断 a_1, a_2, a_3, a_4 及 $a_{14}, a_{23}, a_{41}, a_{32}$ 的符号

$$t(1324) = 1 \text{ 奇}$$

$$t(4321) = \frac{4 \times 3}{2} = 6 \text{ 偶}$$

D 性质

1. 行列式位数相同, $|D| = |D'|$

2. 交换两行/列, 反号

3. 可提取一行/列公因子到行列式外

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

4. 可扩

5. 倍加

余子式，代數余子式

$\text{tr}(A)$ 主对角元之和

设3阶D某行为 a , ($a \neq 0$), $D=1$, 则D中所有代数余子式为

$$\begin{vmatrix} a & a & a \end{vmatrix} = 1 \quad \text{及} \quad \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$A_{11} + A_{12} + A_{13} = \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} = \frac{1}{a}$$

$$A_{21} + A_{22} + A_{23} \begin{vmatrix} a & a & a \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$A_{31} + A_{32} + A_{33} \begin{vmatrix} a & a & a \\ 1 & 1 & 1 \end{vmatrix} = 0$$

\therefore 所有代数余子式之和为 $\frac{1}{a}$

- 1) { 1. n! 项来自不同行列的代数和 (每项由逆序对
2. 按某行/列的元素 x 该元素对应的 A_{ij} 的符号决定)

$$\begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix} (3 - 10) = \begin{pmatrix} 6 & -2 & 0 \\ 18 & -6 & 0 \\ 21 & -7 & 0 \end{pmatrix}$$

$$(1 \ 2 \ 3) \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = (3 + 2 + 6) = 11$$

1. $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} =$$

2. $AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} =$$

3. $AB = AC \not\Rightarrow A = 0 \not\Rightarrow B = C$

矩阵初等变换 $A \xrightarrow{V_i \times 2} B$

1. $V_i \leftrightarrow V_j$ $C_i \leftrightarrow C_j$ 又称

2. kV_i kC_i ($k \neq 0$) 倍乘

3. $V_i + kV_j$ $C_i + kC_j$ 倍加

初等矩阵

Definition: 又称三次一次初等变换

$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 不是初等矩阵

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} =$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} =$$

A 为 3 阶矩阵，将 A 的第 2 列加到第 1 列得到矩阵 B ，再交换交换第 2 行与第 3 行得到 E ，记 $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$R \setminus A =$$

解：

$$P_2 A P_1 = E$$

$$P_2^{-1} P_2 A P_1 P_1^{-1} = P_2^{-1} E P_1^{-1}$$

$$A = P_2^{-1} P_1^{-1}$$

P_2^{-1} 为行交换

$$P_2^{-1} = P_2$$

$$\therefore A = P_2 P_1^{-1}$$

线性方程组

1. 无解

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 - x_2 - x_3 = 3 \\ 2x_1 - 2x_2 - x_3 = 3 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ 2 & -2 & -1 & 3 \end{array} \right) \xrightarrow{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 2 & -2 & -1 & 3 \end{array} \right) \xrightarrow{r_3-2r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{r_2 \div 2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{array} \right) \quad r(A) < r(A, b)$$

2. 有唯一解

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 9 \\ 3 & 4 & 2 & 1 & 9 \\ -1 & -5 & 4 & 1 & 10 \\ 2 & 7 & 1 & 1 & 1 \end{array} \right) \xrightarrow{r_2-3r_1} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 9 \\ 0 & -5 & -1 & -1 & 0 \\ 0 & -2 & 5 & 1 & 12 \\ 0 & 1 & -1 & -3 & 1 \end{array} \right) \xrightarrow{r_3+2r_4} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 9 \\ 0 & -5 & -1 & -1 & 0 \\ 0 & 0 & 3 & 6 & 16 \\ 0 & 1 & -1 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{5r_4+r_2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 9 \\ 0 & -5 & -1 & -1 & 0 \\ 0 & 0 & 3 & 6 & 16 \\ 0 & 0 & -6 & -12 & 1 \end{array} \right) \xrightarrow{r_4 \times \frac{1}{2} + r_3} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 9 \\ 0 & -5 & -1 & -1 & 0 \\ 0 & 0 & 3 & 6 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$r(A, b) = r(A) = 3 = n$$

3. 有无穷多解

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ 2 & -2 & -1 & 5 \end{array} \right) \xrightarrow{r_2-r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 2 & -2 & -1 & 5 \end{array} \right) \xrightarrow{r_3-2r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{array} \right)$$

$$r(A, b) = r(A) = 2 < 3$$

↓
有效方程个数

↓
未知数个数

Rank (秩) $\left\{ \begin{array}{l} 1. \text{ 非零子式最高阶数} \\ 2. \text{ 有效方程数} \\ 3. \text{ 行阶梯形矩阵非零行数} \\ 4. \text{ 独立向量个数} \end{array} \right.$

e.g. $r(A), r(B)$, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -5 \\ 4 & 7 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -5 \\ 4 & 7 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} = -10 - 9 = -19$$

\because 有2阶非零子式

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -5 \\ 4 & 7 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -5 \\ 4 & 7 & 1 \end{vmatrix} = 3 - 40 + 42 - (36 - 35 + 4) = 0$$

\therefore 无3阶非零子式

$\therefore r(A) = 2$

秩的公式

转置不改变 $r(A)$

$$1. r(A) = r(A^T) = r(AA^T) = r(\bar{A}^T A)$$

2. $r(A)$ 越乘越大，越成块越大，分开加最大

$$r(AB) \leq \begin{cases} r(A) \\ \text{or} \\ r(B) \end{cases} \leq \begin{cases} r(AB) \\ \text{or} \\ r(A) + r(B) \end{cases} \leq r(A) + r(B)$$

$r(A+B)$

$$3. A_{m \times n} \times B_{n \times l} = 0, r(A) + r(B) \leq n$$

4. 初等变换不改变 $r(A)$

$$5. 0 \leq r(A_{m \times n}) \leq \min(m, n)$$

向量组的线性表示

$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}$, 证 β 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示

并写出表达式

解: 可被线性表示意思是方程 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \beta$ 有解

方程组有解 \Leftrightarrow 系数矩阵的秩 = 增广矩阵的秩 $\Leftrightarrow r(A) = r(A, b)$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 3 \\ 2 & 3 & 0 & 1 \end{array} \right) \xrightarrow{R_2-R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -3 & 6 & 3 \\ 0 & 2 & -4 & -2 \end{array} \right) \xrightarrow{R_3+3R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1-R_2} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} x_1 + 3x_3 = 2 \\ x_2 - 2x_3 = -1 \end{array}$$

$$\begin{cases} x_1 = 2 - 3x_3 \\ x_2 = -1 + 2x_3 \\ x_3 = x_3 \end{cases}, \text{令 } x_3 \text{ 为任意常数 } k \quad X = \begin{pmatrix} -3k+2 \\ 2k-1 \\ k \end{pmatrix}$$

$$\beta = (-3k+2)\alpha_1 + (2k-1)\alpha_2 + k\alpha_3$$

e.g. $\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, $\beta_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\beta_3 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}$, 证 α_1, α_2 与 $\beta_1, \beta_2, \beta_3$ 等价

两个向量组等价, 即可互相线性表示 $\Leftrightarrow r(I) = r(I, Z) = r(I, ZZ)$

若 I 能由 Z 线性表示, $r(I) = r(I, Z)$

则 $r(I) = r(I, I) = r(Z)$

$$\left(\begin{array}{cccc} 1 & 3 & 2 & 1 & 3 \\ -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2 \\ -1 & 3 & 1 & 2 & 0 \end{array} \right) \xrightarrow{r_2+r_1} \left(\begin{array}{ccccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 4 & 2 & 2 & 2 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 6 & 3 & 3 & 3 \end{array} \right) \xrightarrow{r_3+r_2} \left(\begin{array}{ccccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 6 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{r_3-3r_2} \left(\begin{array}{ccccc} 1 & 3 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\therefore V(\alpha_1, \alpha_2) = V(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) = V(\beta_1, \beta_2, \beta_3)$$

∴ 等价

向量组相关无关性

线性向量组: $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m$,

若存在不全为 0 的数 $k_1, k_2, k_3, \dots, k_m$,

使 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \dots + k_m\alpha_m = 0$

则为线性相关

也就是齐次线性方程组仅有0解

即 $r(CA) = r(A, b) < n$

线性向量组: $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m$,

且仅当 $k_1 = k_2 = k_3 = \dots = k_m = 0$

使 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \dots + k_m\alpha_m = 0$

则为线性无关

也就是齐次线性方程组仅有0解

$$\text{rank}(A) = \text{rank}(A, b) = n$$

e.g. $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$, 试判断 α_1, α_2

$\alpha_1, \alpha_2, \alpha_3$ 线性相关性

$$\left(\begin{array}{ccc} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 1 & 5 & 7 \end{array} \right) \xrightarrow{r_2 - r_1} \left(\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 5 & 7 \end{array} \right) \xrightarrow{r_3 - r_1} \left(\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

$$\text{rank}(\alpha_1, \alpha_2, \alpha_3) = 2 < 3 \quad (\text{向量个数})$$

\therefore 有多余 vector

\therefore 线性相关

e.g. 已知向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, $b_1 = \alpha_1 + \alpha_2$, $b_2 = \alpha_2 + \alpha_3$,

$b_3 = \alpha_3 + \alpha_1$, 试证 b_1, b_2, b_3 线性无关

解: 设 $\exists x_1, x_2, x_3$ 使 $x_1 b_1 + x_2 b_2 + x_3 b_3 = 0$

$$x_1 (\alpha_1 + \alpha_2) + x_2 (\alpha_2 + \alpha_3) + x_3 (\alpha_3 + \alpha_1) = 0$$

$$(x_1+x_3)\alpha_1 + (x_1+x_2)\alpha_2 + (x_2+x_3)\alpha_3 = 0$$

$$\begin{cases} x_1+x_3=0 \\ x_1+x_2=0 \\ x_2+x_3=0 \end{cases}$$

解1:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\therefore x_1 = x_2 = x_3 = 0$$

解2:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

$$\therefore r(A) = r(A, b) = n$$

$x = \text{惟一解}$

\therefore 只有 0 解

两个 vector 内积 = 外积的迹 (这对角元)

$|A| \neq 0 \Leftrightarrow |A| \text{满秩} \Leftrightarrow A \text{可逆}$

$$A^{-1} = \frac{A^*}{|A|}$$

反交换

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & \vdots \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

$$A_{ij} = (-1)^{i+j} M_{ij} \quad (\text{代数余子式} = (-1)^{i+j} \text{余子式})$$

M_{ij} 为划去 i 行 j 列的 D

$$A^* = \begin{pmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & \vdots & & & \vdots \\ A_{13} & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ A_{1n} & \dots & \dots & \dots & A_{nn} \end{pmatrix}$$

$$AA^* = A^*A = |A|E$$

二阶 A^*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\therefore A^* = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

主对角，副反号

$P^{-1}AP = \Lambda$
 $R|A$ 可对角化

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, P^{-1}AP = \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, A^n$$

$$\text{证: } PP^{-1}APP^{-1} = P\Lambda P^{-1}$$

$$A = P\Lambda P^{-1}$$

$$\begin{aligned} A^n &= P\Lambda P^{-1} \underbrace{P\Lambda P^{-1}}_{= P\Lambda^n P^{-1}} \cdots \underbrace{P\Lambda P^{-1}}_{= P\Lambda^n P^{-1}} \\ P^{-1} &= \frac{P}{|P|} = \frac{\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}}{-1-1} = -\frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ &\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \right) \cdot \left(-\frac{1}{2} \right) \left(\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right) \\ &= -\frac{1}{2} \left(\begin{pmatrix} 1 & 3^n \\ 1 & -3^n \end{pmatrix} \right) \left(\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right) \\ &= -\frac{1}{2} \left(\begin{pmatrix} -1-3^n & -1+3^n \\ -1+3^n & -1-3^n \end{pmatrix} \right) \\ &= \frac{1}{2} \left(\begin{pmatrix} 1+3^n & 1-3^n \\ 1-3^n & 1+3^n \end{pmatrix} \right) \end{aligned}$$

特征值与特征向量

$$AX = \lambda X \quad (X \neq 0) \quad A \text{ 为系数矩阵 } X \text{ 为列向量} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}$$

$$AX = \lambda EX$$

$$AX - \lambda EX = 0$$

$$(A - \lambda E)X = 0 \quad \text{非齐次方程}$$

$\because X \neq 0 \therefore$ 有无数解

$\therefore A - \lambda E$ 不满秩

$\therefore |A - \lambda E| = 0 \Rightarrow$ 得出特征值 $\lambda_1, \lambda_2, \lambda_3$

$$|A - \lambda E| = 0 \quad \begin{cases} \lambda_1 \rightarrow (A - \lambda_1 E)X = 0 \rightarrow \xi_1 \\ \lambda_2 \rightarrow (A - \lambda_2 E)X = 0 \rightarrow \xi_2 \\ \lambda_3 \rightarrow (A - \lambda_3 E)X = 0 \rightarrow \xi_3 \end{cases}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

$$P = (\xi_1, \xi_2, \xi_3)$$

e.g. 求 Matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ 的特征值和特征向量

$$1. \text{求特征值: } |A - \lambda E| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0 \quad \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 4 \end{array}$$

2. 求特征向量:

$$\text{当 } \lambda_1 = 2$$

$$(A - 2E)X = 0$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 - x_2 = 0 \\ x_2 = k_1 \end{array} \Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{当 } \lambda_2 = 4$$

$$(A - 4E)X = 0$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_2 = k_2 \end{array} \Rightarrow X = \begin{pmatrix} -k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\therefore A$ 的特征值为 2, 4

$k_1(1)$ 是 A 对应 $\lambda=2$ 的特征向量

$k_2(-1)$ ~

3. 相似对角化

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\therefore P^{-1}AP = \Lambda$$

3 阶 A 的 λ 为 2, 3, 1, 且 $|2A| = -48$, $\neq 1$

$$|2A| = 2^3 |A| = 8 \times 2 \times 3 \lambda = -48$$

$$\therefore \lambda = -1$$

矩阵	A	$aA+bE$	A^n	A^{-1}	A^*	A^T
特征值	λ	$\lambda a+b$	λ^n	$\frac{1}{\lambda}$	$\frac{ A }{\lambda}$	λ
特征向量	ξ	ξ	ξ	ξ	ξ	-

e.g. 3阶 Matrix A , 特征值为 1, 2, 2, E 为阶单位阵, 求 $|4A^{-1} - E|$

A 的特征值: 1, 2, 2

A^{-1} 的特征值: 1, $\frac{1}{2}$, $\frac{1}{2}$

$4A^{-1}$: 4, 2, 2

$4A^{-1} - 1$: 3, 1, 1

$$\therefore |4A^{-1} - E| = 3 \times 1 \times 1 = 3$$

e.g. 设 3 阶 Matrix 特征值为 1, -1, 2, 求 $A^* + 3A - 2E$ 特征值

$$A: 1, -1, 2$$

$$A^*: |A|, -|A|, \frac{1}{2}|A|$$

$$3A: 3, -3, 6$$

$$-2E: -2, -2, -2$$

$$\therefore |A| = 1 \times (-1) \times 2 = -2$$

$$A^* + 3A - 2E: -2 + 3 - 2, 2 - 3 - 2, -1 + 6 - 2$$

$$-1 \quad -3 \quad 3$$

e.g. 3阶矩阵特征值为 0, -1, 2, 求 $A^* + 3A - 2E$ 的特征值

$$\varphi(A) = A^* + 3A - 2E \quad \varphi(\lambda) = \frac{|A|}{\lambda} + 3\lambda - 2$$

$$\varphi(0) = \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1} - 2 = -4$$

$$\varphi(-1) = \frac{0}{-1} - 3 - 2 = -5$$

$$\varphi(2) = 6 - 2 = 4$$

相似 definition

A, B 为 n 阶矩阵, 存在 可逆矩阵 P

使 $P^{-1}AP = B$, 则 A 与 B 相似

当 B 为 λ 时, 则 A 与 λ 对角化

properties: 相似

$$\begin{cases} \text{tr}(A) = \text{tr}(B) \\ |A| = |B| \\ r(A) = r(B) \\ \lambda_A = \lambda_B \\ |A - \lambda E| = |B - \lambda E| \end{cases}$$

e.g. $A = \begin{pmatrix} -2 & -2 & 1 \\ 2 & x & -2 \\ 0 & 0 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix}$ 试求 x, y

解:

$$\begin{cases} \text{tr}(A) = \text{tr}(B) \\ |A| = |B| \end{cases} \quad \begin{cases} x - 4 = 1 + y \\ -2x(x-2)x(-2) = -2y \end{cases} \quad \begin{cases} x - 4 = 1 + y \\ 4x - 8 = -2y \end{cases}$$

$$\begin{cases} x = 3 \\ y = -2 \end{cases}$$

对角化 Definition

AB 为 n 阶矩阵，若存在可逆阵 $P = (\xi_1, \xi_2, \xi_3, \dots, \xi_n)$

使 $P^{-1}AP = \Lambda$, 则 A 可对角化

A 可相似对角化 $\Leftrightarrow A$ 有 n 个线性无关特征向量

e.g. 求齐次线性方程组

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - 5x_2 + 3x_3 + 2x_4 = 0 \\ 7x_1 - 2x_2 + 3x_3 + x_4 = 0 \end{cases}$$

的基本解系与通解

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -5 & 3 & 2 \\ 7 & -7 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -7 & 5 & 4 \\ 0 & -14 & 10 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -7 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -\frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{3}{7} \\ 0 & 1 & -\frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = \frac{2}{7}x_3 + \frac{3}{7}x_4$$

$$x_2 = \frac{5}{7}x_3 + \frac{4}{7}x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x = \begin{pmatrix} 2 \\ 5 \\ 1 \\ 0 \end{pmatrix} k_1 + \begin{pmatrix} 3 \\ 4 \\ 0 \\ 1 \end{pmatrix} k_2$$

$$k_1, k_2 \in \mathbb{R}$$

向量 Vector

1. 内积 (数量积) (点乘)

$$(x, y) = x \cdot y = |x| |y| \cos \theta$$

$$(x, y) = (x_1, x_2, x_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

2. 外积 (向量积) (叉乘)

正交(垂直)

$$(x, y) = x \cdot y = 0 \text{ 点积为 } 0$$

施密特正交化：

对于 $\alpha_1, \alpha_2, \alpha_3$, 令

$$\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_2 - \frac{(x_2, \beta_1)}{(x_1, \beta_1)} \beta_1 \\ \vdots \\ \beta_n = \alpha_n - \frac{(x_n, \beta_1)}{(x_1, \beta_1)} \beta_1 \end{cases}$$

内积

$$P_3 = K_3 - \frac{c}{(\beta_1, \beta_1)} P_2$$

$|A| \neq 0 \Leftrightarrow r(A) = n \Leftrightarrow \exists A^{-1} \Leftrightarrow A$ 的 $\lambda \neq 0$

满秩

可逆

A 的特征值全 $\neq 0$

$\Leftrightarrow Ax=0$ 只有 0 解 $\Leftrightarrow A$ 的行/列向量组线性无关

$AB=0$ ($A_{m \times n}$, $B_{n \times s}$), \mathbb{R}^s

1. $r(A) + r(B) \leq n$ n 为 A 的 column
2. B 的每一列 β_j 都是线性方程组 $AX=0$ 的解
3. B 的列与 A 的行两两正交

e.g. 二次型 $f=x^2 - 3z^2 - 4xy + yz$, 写出矩阵表达

$$f = x^2 - 3z^2 - 4xy + yz$$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & -2 & 0 \\ y & -2 & 0 & 1 \\ z & 0 & 1 & -3 \end{array}$$

$$\therefore f(x, y, z) = (x, y, z) \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

e.g. ~~不是~~ ~~不是~~ $f(x_1, x_2, x_3) = -2x_1x_2 + 2x_1x_3 + 2x_2x_3$ ~~的~~

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 0 & -1 & 1 \\ x_2 & -1 & 0 & 1 \\ x_3 & 1 & 1 & 0 \end{array} \quad A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0 - (-1) + (0 + 0 + 0) = -2 \neq 0$$

$$\therefore \text{rk}(A) = 3$$

标准二次型

$$f(x_1, x_2, x_3, \dots, x_n) = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + \dots + k_n x_n^2 \quad (\text{不含交叉项})$$

规范二次型

$$f(x_1, x_2, \dots, x_n) = d_1 x_1^2 + d_2 x_2^2 + \dots + d_n x_n^2 \quad (d = \{0, 1, -1\})$$