1 Solve the ODE

$$\frac{dr}{dt} = u(1 + z(1 - \frac{r(t)}{n})) - cr(t) \tag{1}$$
$$= u(1 + z) - r(t)(c + \frac{uz}{n}) \tag{2}$$

This is a linear ordinary differential equation and we know that $r(t)$ differentiates roughly into itself. So let's try a solution of the form ae^{bt}.

$$r(t) = ae^{bt} + d \tag{3}$$
$$\frac{dr}{dt} = bae^{bt} \tag{4}$$
$$= b(r(t) - d) \tag{5}$$
$$= br(t) - bd \tag{6}$$
$$\implies b = -(c + \frac{uz}{n}) \tag{7}$$
$$bd = -u(1 + z) \tag{8}$$
$$\implies d = \frac{u(1 + z)}{c + \frac{uz}{n}} \tag{9}$$
$$\implies r(t) = ae^{-t(c + \frac{uz}{n})} + \frac{un(1 + z)}{cn + uz} \tag{10}$$

Initial conditions give us

$$r(0) = a + \frac{un(1 + z)}{cn + uz} \tag{11}$$
$$a = r(0) - \frac{un(1 + z)}{cn + uz} \tag{12}$$
$$\implies r(t) = (r(0) - \frac{un(1 + z)}{cn + uz})e^{-t(c + \frac{uz}{n})} + \frac{un(1 + z)}{cn + uz} \tag{13}$$

2 Differentiate the Solution

Just in case.

$$r(t) = (r(0) - \frac{un(1 + z)}{cn + uz})e^{-t(c + \frac{uz}{n})} + \frac{un(1 + z)}{cn + uz} \tag{14}$$
$$\frac{dr}{dt} = -(c + \frac{uz}{n})(r(0) - \frac{un(1 + z)}{cn + uz})e^{-t(c + \frac{uz}{n})} \tag{15}$$
$$= -(c + \frac{uz}{n})(r(t) - \frac{un(1 + z)}{cn + uz}) \tag{16}$$
$$= u(1 + z) - r(t)(c + \frac{uz}{n}) \tag{17}$$