Algorithms and Complexity I
Practical 1: Answers

Question 1. Solve the following fractional knapsack problem using the greedy algorithm we saw in Lecture 3: $W = 100$, $n = 5$ and

$$\begin{aligned} v_1 &= 25, & w_1 &= 5 \\ v_2 &= 120, & w_2 &= 60 \\ v_3 &= 50, & w_3 &= 50 \\ v_4 &= 25, & w_4 &= 50 \\ v_5 &= 60, & w_5 &= 40. \end{aligned}$$

Question 2. Redo this example, but this time for the 0-1 knapsack problem. What is the optimal solution in this case?

Question 3. Using the PARTITION* algorithm, modify the greedy algorithm for fractional knapsack to avoid sorting the list of ratios. For simplicity, you may assume that all value per weight ratios r_i are distinct.

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PARTITION*(A, q)
exchange A[q] ↔ A[n]
x ← A[n]
k ← 0
for j ← 1 to n − 1 do
      if A[j] ≤ x
            then k ← k + 1
                  exchange A[k] ↔ A[j]
exchange A[k + 1] ↔ A[n]
return k + 1
```

Remark 1. Although this strategy does not improve the worst-case running time, it can be shown that the average running time is $\Theta(n)$.

Question 4. Consider another kind of greedy algorithm for the (fractional or 0-1) knapsack: (we assume that no item weighs more than W)

- Step 1: Sort the item according to their value v_i in non-increasing order.
- Step 2: Take as much of the most valuable item as possible, then as much of the second most valuable item as possible, and so on until the knapsack is full.

Use this algorithm for the example above (both for fractional and 0-1). What is its running time?

Question 5. Denote the total value obtained by this modified greedy algorithm as M, and the optimal value as V. Prove the following:

1. For any fractional knapsack problem on n items, $V \leq nM$, or equivalently, $M \geq \frac{V}{n}$.
2. Conversely, for any $n \geq 2$ and any $\epsilon > 0$, there is a fractional knapsack problem on n items such that $V > nM - \epsilon$.

Remark 2. Taken separately, the original and the modified greedy strategies are poor for the 0-1 knapsack problem. However, combining them is actually pretty good! Indeed, consider the following clever greedy strategy: try the original greedy strategy, try to take the most valuable item only, and then take whichever is better. If we denote the total value of this strategy as C and the optimal strategy as V_{0-1}, then one can prove that $V_{0-1} \leq 2C$.

Question 6. Answer Exercises 1 and 2 of Lecture 2 (Slide 12).